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A Lower Bound for a Condition Number Theorem of Variational Inequalities

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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Nonlinear variational inequalities in Banach spaces are considered. A suitable notion of condition number with respect to the right-hand side is introduced. A distance among variational inequalities is defined. Based on a new criterion about the Lipschitz property of set-valued mappings, it is shown that the distance to suitably restricted ill-conditioned variational inequalities is bounded from below by the reciprocal of the condition number. By using a similar upper bound of the companion paper <i>”An upper bound for a condition number theorem of variational inequalities”</i>, we obtain a full condition number theorem for variational inequalities. The particular case of convex minimization problems is considered. Known results dealing with optimization problems are thereby generalized.

Keywords: Variational inequalities, Lipschitz set-valued mappings, condition number theorems.

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1. Introduction

We consider variational inequalities in Banach spaces, defined by nonlinear operators acting on a fixed closed set, with right-hand side from a given region. Aim of this paper is to find a proper setting in order to obtain a partial version of a condition number theorem for variational inequalities. Starting with the condition number theorem of linear algebra (see [3, Theorem 1.7]), generalizations to scalar optimization problems have been obtained in [5, 6, 7, 8, 9, 10], often in the form of a pair of inequalities, linking the distance to ill-conditioning to the reciprocal of the condition number. We call condition number theorem any such pair of inequalities.

In analogy with such properties, the main result of this paper states that the (suitably defined) distance of a given variational inequality from a suitably restricted class of ill-conditioned variational inequalities is bounded from below by the reciprocal of its condition number. It is now appropriate to quote [2, p. 232]: “A very general theme in numerical analysis is a relationship between the condition number of a problem and the reciprocal of the distance to the set of ill-conditioned problems.”

In order to obtain a condition number theorem for variational inequalities, we need a definition of (absolute) condition number and a notion of distance. First we define here the (extended real-valued) distance between two variational inequalities as the uniform norm plus the Lipschitz norm of the difference between the corresponding

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operators. Next we consider the condition number with respect to the right-hand side, defined as the upper limit, as the perturbations vanish, of the ratio between the output and the input errors due to small perturbations of the right-hand side. This concept follows a standard procedure in numerical analysis when defining the condition number. Then we identify a suitable class of variational inequalities, within which we obtain a lower estimate as discussed above. Taking into account the upper bound we prove in the companion paper [11], we get a full condition number for variational inequalities. As a consequence, similar results are proved for convex optimization problems.

Such new results are of interest in the analysis of the computational complexity (see [2] and [3]), of the stability and sensitivity for variational inequalities, moreover in the evaluation of the performance of numerical methods.

The paper is organized as follows. After the preliminaries in section 2, we obtain in section 3 a lower estimate of the distance to ill-conditioning in terms of the reciprocal of the condition number. This result is based on a criterion of Lipschitz continuity for set-valued mappings, extending to the infinite-dimensional setting a result of [10]. The particular case of convex optimization problems is considered in section 4. In section 5 we recall the main results of the companion paper [11], to state a full condition number theorem for variational inequalities. The proofs are collected in section 6.

2. Notations, basic definitions and assumptions

We consider a real Banach space X with topological dual X^*, a fixed closed set $K \subset X$, and the set of all single-valued mappings

$$A\colon K \to X^*. \tag{1}$$

We assume that $0 \in K$ is a cluster point of K. We fix a set $\Omega \subset X^*$ which contains 0 as a cluster point. For every A as in (1) and $p \in \Omega$ we consider the variational inequality, to find $u \in K$ such that

$$< A(u), x-u > \geq < p, x-u > \text{ for every } x \in K, \tag{2}$$

where $< \cdot, \cdot >$ denotes the duality pairing between X^* and X. Given A we denote by $Q = (A, K, \Omega)$ the family of all inequalities (2) as $p \in \Omega$, and by $B = B(K, \Omega)$ the set of all variational inequalities Q as before. For each $p \in \Omega$ we denote by $S(p)$ the (possibly empty) set of all solutions to (2), so that $S\colon \Omega \to K$ is the set-valued *solution mapping* of Q.

We need first a notion of distance among variational inequalities. Given A_1, A_2 with corresponding variational inequalities $Q_1, Q_2 \in B$, consider

$$\hat{A} = A_1 - A_2$$

and define the distance between Q_1 and Q_2 as

$$\text{dist } (Q_1, Q_2) = \sup \{||\hat{A}(x)|| : x \in K\} + \sup \{\frac{||\hat{A}(x'') - \hat{A}(x')||}{||x'' - x'||} : x', x'' \in K\}. \tag{3}$$

In this way we obtain an extended real-valued distance on the set of all mappings (1), which induces a corresponding definition of distance among variational inequalities (2). Next we need a notion of condition number of any given variational inequality $Q \in B$. We consider the set-valued mapping $S\colon \Omega \to K$ corresponding to Q, and we define the (absolute) condition number of Q as

$$\text{cond } Q = \limsup_{p',p'' \to 0} \frac{\text{hausd } [S(p''), S(p')]}{||p'' - p'||} \tag{4}$$

where of course $p', p'' \in \Omega$, and hausd denotes the Pompeiu-Hausdorff distance (see [4, p.138] for the definition). Thus cond Q is the Lipschitz modulus at $p = 0$ of its solution mapping S.

The variational inequality $Q \in B$ will be called *well-conditioned* iff cond $Q < +\infty$, and the set of all well-conditioned variational inequalities (2) will be denoted by *WC*. The variational inequality $Q \in B$ will be called *ill-conditioned* iff cond $Q = +\infty$, and the corresponding set will be denoted by *IC*. Thus $Q \in WC$ iff Q has solutions for each sufficiently small $p \in \Omega$ and its set-valued solution mapping S is Lipschitz near $p = 0$ (according to the standard notion of Lipschitz continuity of set-valued mappings based on the Pompeiu-Hausdorff distance, see [4, p. 148]: the definition in the infinite-dimensional setting is identical to the one in finite dimensions). We have $Q \in IC$ if $S(p_n) = \emptyset$ for some sequence $p_n \in \Omega$ with $p_n \to 0$, or if $S(p) \neq \emptyset$ for each sufficiently small $p \in \Omega$ and S is not Lipschitz near $p = 0$.

Example 2.1. Suppose that X is Hilbert, $A\colon X \to X^*$ is Lipschitz on the convex set K, and A is strongly monotone with constant $\alpha > 0$, namely

$$< A(x) - A(y), x - y > \geq \alpha ||x - y||^2$$

for every $x, y \in K$. Then, as well-known, $S(p)$ is a singleton for each $p \in \Omega = X^*$ (see [1, Teorema 3.2]), and

$$||S(p'') - S(p')|| \leq \frac{1}{\alpha} ||p'' - p'||$$

hence $Q \in WC$ with cond $Q \leq 1/\alpha$. □

Example 2.2. Let $X = R, A = 0, \Omega = K = [0, 1]$. Then $S(0) = [0, 1]$, and $S(p) = \{1\}$ for $0 < p \leq 1$, hence Q is ill-conditioned. □

3. Lower bound

Aim of this section is to obtain a lower bound of the distance to ill-conditioning in terms of the reciprocal of the condition number. The lower bound we prove is based on the following criterion of Lipschitz continuity for set-valued mappings (generalizing to the infinite-dimensional setting Theorem 3.5 of [10]).

Theorem 3.1. *Let X, Y be real Banach spaces. Let $T \subset X$ be nonempty, and consider two set-valued mappings $S_1, S_2\colon T \to Y$.*

Suppose that

- *there exists a nonempty, closed and bounded set* $E \subset Y$ *such that*
$$E \subset S_1(T) \cap S_2(T); \tag{5}$$
- $S_2(p)$ *is closed for every* $p \in T$ *and*
$$S_1^{-1}, S_2^{-1} \text{ are single-valued on } E; \tag{6}$$
- *there exists a closed ball* P *of center* 0 *and radius* $\delta > 0$ *such that* $S_2(p) \neq \emptyset$ *for every* $p \in P$ *and*
$$S_1(P) \cup S_2(P) \subset E; \tag{7}$$
$$S_1 \text{ is Lipschitz on } P \text{ with constant } a; \tag{8}$$
$$S_1^{-1} - S_2^{-1} \text{ is Lipschitz on } E \text{ with constant } b \text{ such that } ab < 1; \tag{9}$$
$$2\,||S_1^{-1} - S_2^{-1}||_\infty < \delta, \text{ where } ||\cdot||_\infty \text{ is the uniform norm on } E. \tag{10}$$

Then the set-valued mapping S_2 *is Lipschitz near* 0.

Example 3.2. The assumption $ab < 1$ in (9) cannot be omitted in the statement of Theorem 3.1. Consider

$$X = Y = R,\ T = [-1,1] \text{ and } S_1(x) = x,\ S_2(x) = \sqrt[3]{x}.$$

By taking $E = P = T$ we see that

$$S_1(T) \cap S_2(T) = E = S_1(P) \cup S_2(P),$$

moreover $a = 1, b = 2$, and

$$2\,||S_1^{-1} - S_2^{-1}||_\infty = 2 \max \{|p - p^3| : |p| \leq 1\} = \frac{4}{3\sqrt{3}} < 1.$$

Thus all conditions but (9) in Theorem 3.1 are satisfied, however S_2 is not Lipschitz near 0. □

In the following example we show two variational inequalities whose solution mappings fulfill the assumptions of Theorem 3.1.

Example 3.3. Let $X = R, K = [-2,2], \Omega = [-5/4, 5/4]$. Consider the variational inequality: given $p \in \Omega$, find $u \in K$ such that

$$(u - p)(x - u) \geq 0 \text{ for every } x \in K. \tag{11}$$

Denote by $S_1(p)$ the set of all solutions to (11). Then $S_1(p) = p$ for every p, with $S_1^{-1}(u) = u$. Then consider the variational inequality: given $p \in \Omega$, find $u \in K$ such that

$$\left(\frac{4u}{5} - p\right)(x - u) \geq 0 \text{ for every } x \in K. \tag{12}$$

Let $S_2(p)$ denote the set of all solutions of (12), then

$$S_2(p) = \frac{5p}{4} \text{ and } S_2^{-1}(u) = \frac{4u}{5}.$$

If $T = [-5/4, 5/4] = E$ and $P = [-1, 1]$ we see that (5), (6), (7), (8), (9) are fulfilled with $a = 1, b = 1/5$, and

$$2\ ||S_1^{-1} - S_2^{-1}||_\infty = \frac{1}{2}$$

hence all conditions in Theorem 3.1 are satisfied by S_1 and S_2. □

In order to prove a lower estimate of the form discussed above, we need to restrict the class of variational inequalities we shall consider. Having fixed K and Ω, a given class D of variational inequalities $Q = (A, K, \Omega)$ with solution mapping S fulfills *property* Z if the following conditions are satisfied for each $Q \in D$. First, $S(p)$ is nonempty and closed for every sufficiently small $p \in \Omega$. Moreover, writing

$$K^* = \{u \in K : ||u|| \leq \delta^*\},$$

we require that there exists a sufficiently small $\delta^* > 0$ such that

$$A(K^*) \subset \Omega; \tag{13}$$

furthermore

$$S^{-1} \text{ is a single-valued function on } K^*; \tag{14}$$

finally, for every pair $Q_i = (A_i, K, \Omega) \in D$ with solution mappings $S_i, i = 1, 2$, we have $A_1(0) = A_2(0)$ and there exists a ball P of center 0 and radius $\delta > 0$ such that

$$S_1(P) \cup S_2(P) \subset K^*. \tag{15}$$

In general, uniqueness of the solution is unrelated to (14), as shown in the following example.

Example 3.4. (a) If X is Hilbert, K is convex and $\bar{p} \notin K$, for the unique orthogonal projection u of $\bar{p}$ on K we have $S(p) = u$ for each p of the segment with ends $\bar{p}$ and u, hence $S^{-1}(u)$ is not single-valued.
(b) Let $X = R^2$ and consider

$$K = \Omega = \{(0, q) \in R^2 : |q| \leq 1\}, A(x_1, x_2) = (2x_1, 0)$$

(for simpler notation we omit transpose). For the corresponding variational inequality (2) we have the solutions

$$S(0, q) = \{(0, 1)\} \text{ if } q > 0; S(0, 0) = \{(0, u_2) : |u_2| \leq 1\}; S(0, q) = \{(0, -1)\} \text{ if } q < 0.$$

Therefore S^{-1} is single-valued on K^* for every $\delta^* < 1$, however $S(0, 0)$ is not a singleton. □

Example 3.5. An important subclass fulfilling property Z is obtained by considering the variational inequalities corresponding to unconstrained minimization problems defined by convex quadratic forms. Suppose X is Hilbert, $K = X$ and $A_1, A_2 \colon X \to X^*$ are linear, bounded and self-adjoint operators such that, for some fixed $\alpha > 0$,

$$< A_i(x) - A_i(y), x - y > \geq\ \alpha\ ||x - y||^2 \text{ for every } x, y \in X \text{ and } i = 1, 2.$$

Let Ω be a ball centered at 0 with positive radius r. As is well-known, for each $p \in \Omega$, the unique solution of (A_i, K, Ω) is the unique minimizer u_i of

$$x \to (1/2) < A_i x, x > - < p, x >$$

on X, which verifies $A_i(u_i) = p$, so that $u_i = S_i(p)$ and $S_i^{-1} = A_i, i = 1, 2$. Moreover $||u_i|| \leq ||p||/\alpha, i = 1, 2$. Thus, if $||p|| \leq \alpha\delta^*$, then $S(p) \in K^*$, whence (15) is verified with $\delta = \alpha\delta^*$. Furthermore

$$A_i(K^*) \subset \Omega \text{ provided } \delta^* \leq r/||A_i||, i = 1, 2.$$

Then any subset D of convex quadratic forms fulfills all the conditions for property Z if we have

$$k = \sup \{||A|| : A \in D\} < +\infty$$

so that $\delta^* = r/k$. However, constrained minimization problems for convex quadratic forms correspond to variational inequalities which in general do not fulfill property Z, as shown in Example 3.4 (a). □

Let D fulfill property Z, let $Q \in D$ be with solution mapping S, let $p \in \Omega$ be sufficiently small and $u \in S(p)$. Then $u \in K^*$ for some δ^* by (15). Let $q = A(u)$, then $q \in \Omega$ by (13) and $u \in S(q)$. By (14) we see that $p = q \in S^{-1}(u)$ hence $p = A(u)$. It follows that the solutions to every variational inequality $(A, K, \Omega) \in D$ fulfilling property Z, and corresponding to sufficiently small right-hand sides p, are the solutions of the associated equation $A(u) = p, u \in K$.
In the next example we show however that, roughly speaking, classes possessing property Z do not contain only equations $A(u) = p$ associated with the variational inequalities (2).

Example 3.6. Let $X = R$ and consider the following variational inequality Q: given $p \in \Omega = [-2, 2]$, find $u \in K = [-2, 2]$ such that

$$A(u)(x - u) \geq p(x - u) \text{ for every } x \in K,$$

where $A \colon [-2, 2] \to R$ is the odd function given by

$$A(x) = -x - 3 \text{ if } -2 \leq x \leq -1, A(x) = 2x \text{ if } -1 \leq x \leq 0.$$

Then the solution is $S(p) = p/2$ provided $|p| < 1$, and $S^{-1}(u) = 2u$ is single-valued on K^* with $\delta^* = 1/2$. Moreover $A(K^*) \subset [-1, 1]$, and we have $S(P) \subset K^*$ if $P = [-1, 1]$. All conditions defining property Z are therefore fulfilled by $D = \{Q\}$. However we have $u = 2 \in S(3/2)$, but $A(2) \neq 3/2$. In passing, we note that $Q \in WC$. □

From the previous example we see that there are classes D fulfilling property Z such that $D \cap WC$ is not empty. The next example shows a class D satisfying Z such that $D \cap IC$ is not empty.

Example 3.7. Consider D given by the following variational inequality: for every $p \in \Omega = [-1, 1]$, find $u \in K = [-1, 1]$ such that

$$(u^3 - p)(x - u) \geq 0 \text{ for every } x \in [-1, 1].$$

Then $S(p) = \sqrt[3]{p}$ for every p, with $S^{-1}(u) = u^3$ which is single-valued on $K^* = K$. Of course $S(P) \subset K^*$ provided $P = K$. All conditions defining Z are fulfilled and $Q \in IC$. □

Example 3.2 shows that $D = \{Q_1, Q_2\}$, where $K = \Omega = [-1, 1]$, $Q_i = (A_i, K, \Omega)$, $i = 1, 2$, and $A_1(x) = x, A_2(x) = x^3$, fulfills property Z and both $D \cap WC, D \cap IC$ are not empty.

Given $Q \in B$ and any class $D \subset B$ of variational inequalities, we denote by $\text{dist}(Q, IC \cap D)$ the infimum of $\text{dist}(Q, \bar{Q})$ as $\bar{Q} \in IC \cap D$. Our lower bound is contained in the next result.

Theorem 3.8. *Assume that Ω is a ball centered at 0 with positive radius. Let $D \subset B$ fulfill property Z such that $D \cap IC \neq \emptyset$, and let $Q \in D \cap WC$ be such that cond $Q > 0$. Then*

$$\textit{dist }(Q, D \cap IC) \geq \frac{1}{\textit{cond } Q}. \tag{16}$$

Given K and Ω let $Z^* \subset B$ denote the set of all variational inequalities which fulfill property Z. As an immediate particular case of Theorem 3.8 we obtain the following.

Corollary 3.9. *Let Ω be a ball centered at 0 with positive radius. Let $Q \in WC \cap Z^*$ be such that cond $Q > 0$. Then*

$$\textit{dist }(Q, IC \cap Z^*) \geq \frac{1}{\textit{cond } Q}. \tag{17}$$

4. Convex optimization

In this section we assume that K is a bounded and convex subset of the reflexive Banach space X. Denote by T the set of all real-valued functions f defined on some open set containing K, which are lower semicontinuous and convex on K, and Gateaux differentiable. For each $f \in T$ we consider the optimization problems Q, to minimize

$$f(x) - < p, x >, \ x \in K \tag{18}$$

for each $p \in \Omega$, with solution mapping S. So we consider tilt perturbations of f constrained by K. As well-known, $u \in S(p)$ iff u solves (2) with $A = \nabla f$, the Gateaux gradient of f. Abusing notation, we denote by Q both the minimization problem (18) and the variational inequality (2) with $A = \nabla f$. We consider the condition number given by (4), and the distance between two optimization problems Q_1, Q_2 defined by (3). The above definition (3) induces a (infinite-valued) pseudo-metric on the space B_m of all optimization problems as in (18), defined by elements of T.

We remark that, if dist $(Q_1, Q_2) = 0$, then for the corresponding objective functions f_1, f_2 we have $f_1 = f_2+$ constant, hence the corresponding solution mappings $S_1 = S_2$, whence cond $Q_1 =$ cond Q_2. Denote again by B_m (abusing notation) the subset of B formed by all variational inequalities $(\nabla f, K, \Omega)$ as $f \in T$. Let us write

$$WC_m = B_m \cap WC, IC_m = B_m \cap IC.$$

The lower bound given by Theorem 3.8 holds in this new setting, namely we prove the following result.

Theorem 4.1. *Assume that* Ω *is a ball centered at* 0 *with positive radius. Let* $D \subset B_m$ *fulfill property* Z. *Let* $Q \in WC_m \cap D$ *be such that* $cond\ Q > 0$. *Then*

$$dist\ (Q, IC_m \cap D) \geq \frac{1}{cond\ Q}. \tag{19}$$

5. A condition number theorem

A condition number theorem for variational inequalities within Z^* can be obtained by coupling the lower bound of Corollary 3.9 with the main results of [11]. We assume that $0 \in \text{ int } K$. We suppose that X is a reflexive Banach space whose norm is strictly convex and Gateaux differentiable off the origin, with derivative $j(u) \in X^*$ at $u \neq 0$. Moreover we assume that there exists a constant $\alpha \geq 1$ such that the single-valued function $D\colon K \to X^*$ given by

$$D(0) = 0, D(u) = ||u||^{\alpha} j(u) \text{ if } u \neq 0, \text{ is Lipschitz on every bounded set.} \tag{20}$$

All these smoothness conditions about X are fulfilled if X is Hilbert. Finally we require the following about a given $Q \in WC$ with solution mapping S. There exists a positive constant C^* such that for each $p \in \Omega$ sufficiently small, every $u \in K$, every $w \in S(p)$ such that

$$||u - w|| \leq \text{ dist } [u, S(p)] + 1$$

and every $h \in X^*$ with $||h|| \leq 1$, there exists a (single-valued) selection z of $t \to S(p + th), t > 0$ sufficiently small, such that

$$z(0) = w, \quad ||z(t) - w|| \leq C^* t \ \text{ for every } t. \tag{21}$$

See [11] for examples about the existence of C^*.

Given $Q = (A, K, \Omega) \in WC$ with a bounded set K, consider

$$K_0 = \text{ sup } \{||x|| : x \in K\}; \ A_1 = \text{ sup } \{\frac{||A(x'') - A(x')||}{||x'' - x'||} : x', x'' \in K\};$$

$$D_1 = \text{ sup } \{\frac{||D(x'') - D(x')||}{||x'' - x'||} : x', x'' \in K\}$$

and

$$M = A_1 + (\alpha + 2)(K_0^{\alpha} + K_0 D_1),$$

where D and α are given by (20). Moreover let

$$L = \text{ sup } \{||(\alpha + 2)||x||D(x) - A(x)|| : x \in K\}.$$

We assume, without loss of generality, that if $0 < L, \ M < +\infty$, then

$$C^* > 1/(L + M).$$

Taking for granted the upper bound form [11, Corollary 10], we obtain the following result by Corollary 3.9.

Theorem 5.1. *Suppose that Ω is a ball centered at* 0 *with positive radius. Let K be bounded and convex. If* (20) *and* (21) *are satisfied, for every $Q \in WC \cap Z^*$ such that $cond\ Q > 0$ we have*

$$\frac{1}{cond\ Q} \leq \ dist\ (Q, IC \cap Z^*) \leq \frac{(L+M)C^*}{cond\ Q}.$$

A similar result holds within B_m for convex optimization problems.

6. Proofs

Proof of Theorem 3.1. By (10) we have

$$||S_1^{-1}(z) - S_2^{-1}(z)|| \leq \frac{\delta}{2} \text{ for every } z \in E. \tag{22}$$

Now $S_1(p)$ is closed for each $p \in P$ since S_1 is Lipschitz. Thus by (8), for each $q', q'' \in P$ and every $s' \in S_1(q')$ there exists $s'' \in S_1(q'')$ such that

$$||s' - s''|| \leq a\ ||q' - q''||. \tag{23}$$

We need to prove that there exist positive constants c_1, δ_1 such that for every p', p'' with $||p'|| \leq \delta_1, ||p''|| \leq \delta_1$ and for every $z' \in S_2(p')$ there exists

$$z'' \in S_2(p'') \text{ such that } ||z' - z''|| \leq c_1 ||p' - p''||. \tag{24}$$

Let
$$||p'|| \leq \frac{\delta}{2}, \quad ||p''|| \leq \frac{\delta}{2}, \quad z' \in S_2(p'),$$

then $p', p'' \in P$, and $z' \in E$ by (7). Remembering (6), let

$$H(z) = S_1[p'' - S_2^{-1}(z) + S_1^{-1}(z)], z \in E.$$

For every $z \in E$ we have by (22)

$$||p'' - S_2^{-1}(z) + S_1^{-1}(z)|| \leq ||p''|| + ||S_1^{-1}(z) - S_2^{-1}(z)|| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

then $p'' - S_2^{-1}(z) + S_1(z) \in P$. It follows that $H(z) \neq \emptyset, z \in E$, since $S_1(p) \neq \emptyset$ for every $p \in P$ because of (8). Assume that there exists $z'' \in H(z'')$ such that

$$||z'' - z'|| \leq c_1 ||p'' - p'||,$$

then $z'' \in E$ by (7) and

$$z'' \in S_1[p'' - S_2^{-1}(z'') + S_1^{-1}(z'')]$$

hence
$$S_1^{-1}(z'') = p'' - S_2^{-1}(z'') + S_1^{-1}(z'')$$

whence $p'' = S_2^{-1}(z'')$. Thus $z'' \in S_2(p'')$ and (24) is verified. We apply to the set-valued mapping $H \colon E \to Y$ the fixed point theorem [4, Theorem 5E.2] in order to prove the existence of z''. Let us check its assumptions. Suppose that u_n, z_n are given sequences and let u, z be given in E such that

$$u_n \in H(z_n), u_n \to u, z_n \to z.$$

Then by (8) and (9), for every n

$$\begin{aligned}\text{dist }[u, H(z)] &\leq ||u - u_n|| + \text{ hausd }[H(z_n), H(z)]\\ &\leq ||u - u_n|| + a\,||(S_2^{-1} - S_1^{-1})(z_n) - (S_2^{-1} - S_1^{-1})(z)||\\ &\leq ||u - u_n|| + ab\,||z_n - z||.\end{aligned}$$

Letting $n \to +\infty$ we get dist $[u, H(z)] = 0$, hence $u \in H(z)$ since S_2 has closed values by (8). This proves that the graph of H is closed. If $z \in E$ and $w \in H(z)$, then $w \in S_1(u)$ with some $u \in P$, hence $w \in E$ by (7), then E is invariant under H. Now let $z_1, z_2 \in E$ and $w_1 \in H(z_1)$. Then $w_1 \in S_1[p'' - S_2^{-1}(z_1) + S_1^{-1}(z_1)]$. By (23) there exists

$$w_2 \in S_1[p'' - S_2^{-1}(z_2) + S_1^{-1}(z_2)] = H(z_2)$$

such that

$$||w_1 - w_2|| \leq a\,||(S_1^{-1} - S_2^{-1})(z_1) - (S_1^{-1} - S_2^{-1})(z_2)|| \leq ab\,||z_1 - z_2||$$

by (9). Therefore, for every $w_1 \in H(z_1)$ we have

$$\text{dist }[w_1, H(z_2)] \leq ab\,||z_1 - z_2||$$

hence the excess (see [4, p. 138] for its definition)

$$e[H(z_1), H(z_2)] \leq ab\,||z_1 - z_2|| \;\text{ for every } z_1, z_2 \in E.$$

Since $z' \in S_2(p') \subset E$ by (7), we see that

$$z' \in S_1[p' - S_2^{-1}(z') + S_1^{-1}(z')], \;\text{ equivalently } p' = S_2^{-1}(z').$$

It follows that, by (22) and (8), if $\delta_1 \leq \delta/2$,

$$\begin{aligned}\text{dist }[z', H(z')] &\leq \text{ hausd }\{S_1[p' - S_2^{-1}(z') + S_1^{-1}(z')], S_1[p'' - S_2^{-1}(z') + S_1^{-1}(z')]\}\\ &\leq a\,||p' - p''||.\end{aligned}$$

Then all assumptions required by [4, Theorem 5E.2] are satisfied, and (24) is fulfilled provided

$$c_1 > \frac{a}{1 - ab}, \quad \delta_1 \leq \frac{\delta}{2}.$$ □

Proof of Theorem 3.8. Denote by S the solution mapping of $Q = (A, K, \Omega)$. Arguing by contradiction, suppose that

$$\text{dist }(Q, D \cap IC) < \frac{1}{\text{cond } Q}. \tag{25}$$

Then there exists $\bar{Q} = (\bar{A}, K, \Omega) \in D \cap IC$, with solution mapping $\bar{S}$, such that

$$\text{dist } (Q, \bar{Q}) \text{ cond } Q < 1. \tag{26}$$

We check the assumptions of Theorem 3.1. By (3) and (4), with $\hat{A} = A - \bar{A}$ we get from (26)

$$||\hat{A}(x'') - \hat{A}(x')|| \leq c_1 \, ||x'' - x'|| \tag{27}$$

for every $x', x'' \in K$, moreover

$$\text{hausd } [S(p''), S(p')] \leq c_2 \, ||p'' - p'|| \tag{28}$$

if $p', p'' \in \bar{P}$ a ball of center 0 and sufficiently small radius $\bar{\delta} \leq \delta$ the radius of P given by (15), and suitable positive constants c_1, c_2 such that

$$c_1 \, c_2 < 1. \tag{29}$$

By property Z, we can assume that $\delta^* < \bar{\delta}/2c_1$. Given $u \in K^*$ let $p = A(u) \in \Omega$ by (13), then $u \in S(p)$, and similarly for $\bar{Q}$. It follows that

$$K^* \subset S(\Omega) \cap \bar{S}(\Omega)$$

and (5) is fulfilled. Moreover by (14)

$$A(u) = S^{-1}(u), \text{ similarly } \bar{A}(u) = \bar{S}^{-1}(u), u \in K^*,$$

and (6) is fulfilled. By (15)

$$S(\bar{P}) \cup \bar{S}(\bar{P}) \subset S(P) \cup \bar{S}(P) \subset K^*$$

and (7) is verified with $E = K^*$. Moreover $S^{-1} - \bar{S}^{-1}$ is Lipschitz on K^* with constant c_1 by (27), and S is Lipschitz on $\bar{P}$ with constant c_2 by (28), hence (8) is fulfilled. Condition (9) comes from (27), (28) and (29). By property Z we have $\hat{A}(0) = 0$ and if $u \in K^*$, by (27)

$$2 \, ||\hat{A}(u)|| = 2 \, ||\hat{A}(u) - \hat{A}(0)|| \leq 2 \, c_1 \, ||u|| \leq 2 \, c_1 \, \delta^* < \bar{\delta}$$

so that (10) is verified. All assumptions of Theorem 3.1 are thereby met, hence $\bar{Q} \in WC$ which is a contradiction. Then (25) does not hold and the conclusion (17) is proved. □

Proof of Theorem 4.1. The conclusion is obvious from Theorem 3.8 since we have $D \cap IC_m \subset D \cap IC$. Hence by Theorem 3.8

$$\frac{1}{\text{cond } Q} \leq \text{ dist } (Q, D \cap IC) \leq \text{ dist } (Q, D \cap IC_m),$$

proving (19). □

References

[1] C. Baiocchi, A. Capelo: *Disequazioni Variazionali e Quasivariazionali. Applicazioni a Problemi di Frontiera Libera. Volume 1*, Pitagora, Bologna (1978).

[2] L. Blum, F. Cucker, M. Shub, S. Smale: *Complexity and Real Computation*, Springer, New York (1998).

[3] P. Bürgisser, F. Cucker: *Condition. The Geometry of Numerical Algorithms*, Springer, Berlin (2013).

[4] A. L. Dontchev, R. T. Rockafellar: *Implicit Functions and Solution Mappings*, Springer, New York (2009).

[5] J. Renegar: *Some perturbation theory for linear programming*, Math. Programming 65 (1994) 73–91.

[6] J. Renegar: *Linear programming, complexity theory and elementary functional analysis*, Math. Programming 70 (1995) 219–351.

[7] T. Zolezzi: *On the distance theorem in quadratic optimization*, J. Convex Analysis 9 (2002) 693–700.

[8] T. Zolezzi: *Condition number theorems in optimization*, SIAM J. Optim. 14 (2003) 507–516.

[9] T. Zolezzi: *A condition number theorem in convex programming*, Math. Programming Ser. A 149 (2015) 195–207.

[10] T. Zolezzi: *On condition number theorems in mathematical programming*, J. Optim. Theory Appl. 175 (2017) 597–623.

[11] T. Zolezzi: *An upper bound for a condition number theorem of variational inequalities*, submitted.