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Characterizing Optimality for a Class of Nonconvex Quadratic Robust Optimization Problems Bilaterally Quadratically Constrained Under Interval Uncertainty

Fabián Flores-Bazán, Ariel Pérez
Departamento de Ingeniería Matemática, Universidad de Concepción, Chile
fflores@ing-mat.udec.cl, arielperez@udec.cl

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This paper analyzes the following robust optimization problem:

$$\min\left\{\frac{1}{2}x^\top Ax+a^\top x:\ \alpha\leq\frac{1}{2}x^\top Bx+b^\top x+c\leq\beta,\ \forall\ (B,b)\in\mathcal{B}_0\right\},$$

where $\mathcal{B}_0\doteq\{B_1+\mu B_2:\mu\in[\mu_1,\mu_2]\}\times\{b_1+\delta b_2:\delta\in[\delta_1,\delta_2]\}$, with all the matrices involved are real symmetric, $a,b\in\mathbb{R}^n$ and $\alpha,\beta,\delta_1,\delta_2,\mu_1,\mu_2$ are given real numbers. To be more precise, we establish characterizations of the fulfillment of: (i) the robust alternative result; (ii) the robust S-lemma, and (iii) the robust optimality, to the problem above. To that purpose, we apply the convexity result proved by one of the authors valid for nonhomogeneous quadratic functions, instead of the Dines convexity theorem.

Keywords: Nonconvex quadratic programming under uncertainty, robust optimization, S-lemma, global optimality.

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1. Introduction

In this paper we will analyze the following robust optimization problem:

$$\min\left\{\frac{1}{2}x^\top Ax+a^\top x:\ \alpha\leq\frac{1}{2}x^\top Bx+b^\top x+c\leq\beta,\ \forall\ (B,b)\in\mathcal{B}_0\right\},\tag{1}$$

where $\mathcal{B}_0\doteq\{B_1+\mu B_2:\mu\in[\mu_1,\mu_2]\}\times\{b_1+\delta b_2:\delta\in[\delta_1,\delta_2]\}$, with all the matrices being real symmetric, $a,b\in\mathbb{R}^n$ and $\alpha,\beta,\delta_1,\delta_2,\mu_1,\mu_2$ are given real numbers. This model arises when a deterministic approach addresses an optimization problem under data uncertainty. In our case, the original problem may take the form:

$$\min\left\{\frac{1}{2}x^\top Ax+a^\top x:\ \alpha\leq\frac{1}{2}x^\top Bx+b^\top x+c\leq\beta\right\},\tag{2}$$

where the data B and b are uncertain, and, in our model such an uncertainty set is described by $\mathcal{B}_0$.

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It is known nowadays that quadratic functions besides playing an important role in modeling several situations in real-world problems, they provide a nice challenge in mathematics when analyzing analytical properties involving hidden convexity, for instance. Among classical works, we mention [1, 3, 5]. More recent developments and related to optimization were carried out in [2, 4, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], see also the references therein.

The purposes of this paper are to establish characterizations of the fulfillment of: (i) the robust alternative result; (ii) the robust S-lemma, and (iii) the robust optimality, to problem (1).

We point out that our study also cover the case $\alpha = -\infty$, that is, when only one single inequality constraint is considered. This particular situation was discussed in [11]; but there is a gap in the proof of Theorem 5.1: for details see [7], where problem (1) is also studied following the argument developed in [11]. In [11] and [7] the authors use the Dines theorem [3]; whereas the approach to be used presently is based on Theorem 2.2 (see [8, Theorem 4.19]). This use has, as a consequence, that the convexity imposed in those papers is on a set in $\mathbb{R}^5$ which is an image of $\mathbb{R}^{n+1}$ via a homogeneous quadratic function; whereas the convexity in the present paper refers to a set in $\mathbb{R}^5$ being the image of $\mathbb{R}^n$ via a (not necessarily homogeneous) quadratic function. For details, about a comparison of those assumptions, we refer to Section 4.

The paper is organized as follows. Section 2 recalls some basic definitions, notation and some preliminary results. In Section 3, the robust alternative result, the robust S-lemma, and the robust optimality, to problem (1), are established. It is also analyzed the weighted least squares problem. Section 4 discusses a comparison between our convexity assumption and that in [7, Theorem 3].

2. Basic definitions, notation and preliminaries results

Throughout this paper all the matrices considered have real entries. If a matrix A is symmetric it holds $A = A^\top$, that is, the matrix and its transpose coincide. The set of all symmetric matrices of order n is denoted by $\mathcal{S}^n$; elements in $\mathbb{R}^n$ are usually named by x, and they are considered as column vectors $x = (x_1, x_2, \dots, x_n)$, so that the transpose of x, $x^\top$, is the row vector $(x_1 \;\; x_2 \;\; \cdots \;\; x_n)$. By a quadratic form, or an homogeneous quadratic function, in $\mathbb{R}^n$, we mean a quadratic function of the type $x^\top Ax$ where A is a matrix of order n. It is of standard use that $\mathbb{R}$ denotes the real-line and $\mathbb{R}^n_+$ indicates the set of all nonnegative vectors in $\mathbb{R}^n$.

A symmetric matrix A of order n is *positive semidefinite* if $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$, and in such a case it is denoted by $A \succeq 0$. Given a set $K \subseteq \mathbb{R}^n$, int K stands for the (topological) interior of K. It is said that K is a cone if $tx \in K$ for all $t \geq 0$ and all $x \in K$.

In what follows we formulate two joint-range convexity results.

Theorem 2.1. [3] *Let A, B be any real matrices. Then, the set*

$$\{(x^\top Ax, x^\top Bx) : \; x \in \mathbb{R}^n\} \quad \textit{is convex.}$$

In the nonhomogeneous case, the next result arises.

Theorem 2.2. [8, Theorem 4.19] *Let us consider the functions*

$$f(x) = x^\top Ax + a^\top x \quad \text{and} \quad g(x) = x^\top Bx + b^\top x,$$

where A, B are any real matrices and a, b are any vectors in $\mathbb{R}^n$. If $Q \subseteq \mathbb{R}^2$ is a convex cone whose (topological) interior is nonempty, then the sets

$$\{(f(x), g(x)) : \ x \in \mathbb{R}^n\} + Q \quad \text{and} \quad \{(f(x), g(x)) : \ x \in \mathbb{R}^n\} + \operatorname{int} Q$$

are convex.

The following proposition will be applied frequently throughout this work and shows the convexity of an image set in a certain dimension implies the convexity of a related set in a lower dimension. The formulation is given for an arbitrary number of quadratic functions, although we will use it when $m = 2$.

Proposition 2.3. *Let $m \in \mathbb{N}$. For $i = 1, \dots, m$, let $A, B_i, \overline{B}_i \in \mathcal{S}^n$, $b_i, \overline{b}_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$ be given, along with $\mu_i, \overline{\mu}_i, \delta_i, \overline{\delta}_i \in \mathbb{R}$ with $\mu_i \leq \overline{\mu}_i$, and $\delta_i \leq \overline{\delta}_i$. Set*

$$f(x) \doteq \frac{1}{2} x^\top Ax + a^\top x,$$

and for $i \in \{1, \dots, m\}$: $$g_i(x, \mu, \delta) \doteq \frac{1}{2} x^\top (B_i + \mu \overline{B}_i)x + (b_i + \delta \overline{b}_i)^\top x + c_i.$$

Assume that the set

$$\Big\{ \Big(f(x), g_1(x, \mu_1, \delta_1), g_1(x, \mu_1, \overline{\delta}_1), g_1(x, \overline{\mu}_1, \delta_1), g_1(x, \overline{\mu}_1, \overline{\delta}_1), \dots,$$
$$g_m(x, \mu_m, \delta_m), g_m(x, \mu_m, \overline{\delta}_m), g_m(x, \overline{\mu}_m, \delta_m), g_m(x, \overline{\mu}_m, \overline{\delta}_m) \Big) : \ x \in \mathbb{R}^n \Big\} + \operatorname{int} \mathbb{R}^{4m+1}_+$$

is convex. Then, the set

$$\Omega_+ \doteq \Big\{ \Big(f(x), \max_{\mu_1 \leq \mu \leq \overline{\mu}_1} g_1(x, \mu, \delta_1), \max_{\mu_1 \leq \mu \leq \overline{\mu}_1} g_1(x, \mu, \overline{\delta}_1), \dots,$$
$$\max_{\mu_m \leq \mu \leq \overline{\mu}_m} g_m(x, \mu, \delta_m), \max_{\mu_m \leq \mu \leq \overline{\mu}_m} g_m(x, \mu, \overline{\delta}_m) \Big) : x \in \mathbb{R}^n \Big\} + \operatorname{int} \mathbb{R}^{2m+1}_+$$

is convex.

Proof. Take any $(y_1, z^1_1, \overline{z}^1_1, z^1_2, \overline{z}^1_2, \dots, z^1_m, \overline{z}^1_m)$, $(y_2, z^2_1, \overline{z}^2_1, z^2_2, \overline{z}^2_2, \dots, z^2_m, \overline{z}^2_m) \in \Omega_+$. Then, there exist $x_1, x_2 \in \mathbb{R}^n$ such that

$$\begin{aligned}
f(x_1) &< y_1, & f(x_2) &< y_2 \\
\max_\mu g_1(x_1, \mu, \delta_1) &< z^1_1, & \max_\mu g_1(x_2, \mu, \delta_1) &< z^2_1, \\
\max_\mu g_1(x_1, \mu, \overline{\delta}_1) &< \overline{z}^1_1, & \max_\mu g_1(x_2, \mu, \overline{\delta}_1) &< \overline{z}^2_1, \\
\max_\mu g_2(x_1, \mu, \delta_2) &< z^1_2, & \max_\mu g_2(x_2, \mu, \delta_2) &< z^2_2, \\
\max_\mu g_2(x_1, \mu, \overline{\delta}_2) &< \overline{z}^1_2, & \max_\mu g_2(x_2, \mu, \overline{\delta}_2) &< \overline{z}^2_2, \\
&\vdots & &\vdots \\
\max_\mu g_m(x_1, \mu, \overline{\delta}_m) &< \overline{z}^1_m, & \max_\mu g_m(x_2, \mu, \overline{\delta}_1) &< \overline{z}^2_m.
\end{aligned}$$

Since, for all x, the functions $\mu \mapsto g_i(x,\mu,\overline{\delta}_i)$ and $\mu \mapsto g_i(x,\mu,\delta_i)$ are linear, their maximun values are attained in either the extreme point $\mu = \mu_i$ or $\mu = \overline{\mu}_i$. Hence, the previous inequalities imply

$$\begin{aligned}
f(x_1) < y_1, &\quad f(x_2) < y_2\\
g_1(x_1,\mu_1,\delta_1) < z_1^1, &\quad g_1(x_1,\overline{\mu}_1,\delta_1) < z_1^1,\\
g_1(x_2,\mu_1,\delta_1) < z_1^2, &\quad g_1(x_2,\overline{\mu}_1,\delta_1) < z_1^2,\\
g_1(x_1,\mu_1,\overline{\delta}_1) < \overline{z}_1^1, &\quad g_1(x_1,\overline{\mu}_1,\overline{\delta}_1) < \overline{z}_1^1,\\
g_1(x_2,\mu_1,\overline{\delta}_1) < \overline{z}_1^2, &\quad g_1(x_2,\overline{\mu}_1,\overline{\delta}_1) < \overline{z}_1^2,\\
\vdots &\quad \vdots\\
g_m(x_1,\mu_m,\overline{\delta}_m) < \overline{z}_m^1, &\quad g_m(x_1,\overline{\mu}_m,\overline{\delta}_m) < \overline{z}_m^1,\\
g_m(x_2,\mu_m,\overline{\delta}_m) < \overline{z}_m^2, &\quad g_m(x_2,\overline{\mu}_m,\overline{\delta}_m) < \overline{z}_m^2.
\end{aligned}$$

Thus, we infer that the points

$$u \doteq (y_1, z_1^1, z_1^1, \overline{z}_1^1, \overline{z}_1^1, z_2^1, z_2^1, \overline{z}_2^1, \overline{z}_2^1, \ldots, \overline{z}_m^1, \overline{z}_m^1)$$
$$v \doteq (y_2, z_1^2, z_1^2, \overline{z}_1^2, \overline{z}_1^2, z_2^2, z_2^2, \overline{z}_2^2, \overline{z}_2^2, \ldots, \overline{z}_m^2, \overline{z}_m^2)$$

belong to the set

$$\Big\{\Big(f(x), g_1(x,\mu_1,\delta_1), g_1(x,\overline{\mu}_1,\delta_1), g_1(x,\mu_1,\overline{\delta}_1), g_1(x,\overline{\mu}_1,\overline{\delta}_1), \ldots,$$
$$g_m(x,\mu_m,\overline{\delta}_m), g_m(x,\overline{\mu}_m,\overline{\delta}_m)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}_+^{4m+1}.$$

By the convexity assumption, for all $\lambda \in \ ]0,1[$, one has that $\lambda u + (1-\lambda)v$ belongs to the same set. Then, there exists $x_3 \in \mathbb{R}^n$ satisfying

$$\begin{aligned}
f(x_3) &< \lambda y_1 + (1-\lambda) y_2,\\
g_1(x_3,\mu_1,\delta_1) < \lambda z_1^1 + (1-\lambda) z_1^2, &\quad g_1(x_3,\overline{\mu}_1,\delta_1) < \lambda z_1^1 + (1-\lambda) z_1^2,\\
g_1(x_3,\mu_1,\overline{\delta}_1) < \lambda \overline{z}_1^1 + (1-\lambda) \overline{z}_1^2, &\quad g_1(x_3,\overline{\mu}_1,\overline{\delta}_1) < \lambda \overline{z}_1^1 + (1-\lambda) \overline{z}_1^2,\\
\vdots &\quad \vdots\\
g_m(x_3,\mu_m,\overline{\delta}_m) < \lambda \overline{z}_m^1 + (1-\lambda) \overline{z}_m^2, &\quad g_m(x_3,\overline{\mu}_m,\overline{\delta}_m) < \lambda \overline{z}_m^1 + (1-\lambda) \overline{z}_m^2.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
f(x_3) &< \lambda y_1 + (1-\lambda) y_2,\\
\max_\mu g_1(x_3,\mu,\delta_1) < \lambda z_1^1 + (1-\lambda) z_1^2, &\quad \max_\mu g_1(x_3,\mu,\overline{\delta}_1) < \lambda \overline{z}_1^1 + (1-\lambda) \overline{z}_1^2,\\
\max_\mu g_2(x_3,\mu,\delta_2) < \lambda z_2^1 + (1-\lambda) z_2^2, &\quad \max_\mu g_2(x_3,\mu,\overline{\delta}_2) < \lambda \overline{z}_2^1 + (1-\lambda) \overline{z}_2^2\\
\vdots &\quad \vdots\\
\max_\mu g_m(x_3,\mu,\delta_m) < \lambda z_m^1 + (1-\lambda) z_m^2, &\quad \max_\mu g_m(x_3,\mu,\overline{\delta}_m) < \lambda \overline{z}_m^1 + (1-\lambda) \overline{z}_m^2.
\end{aligned}$$

Hence $\lambda u + (1-\lambda)v \in \Omega_+$, proving the convexity of Ω_+. $\square$

When the functions g_i are quadratic homogeneous, or more generally when $\delta_i = \overline{\delta}_i$ for all i, we can consider the sets in a lower dimension, as the next corollary shows.

Corollary 2.4. *With the same data as in Proposition* 2.3*, and assuming that* $\delta_i = \overline{\delta}_i$ *for all* $i \in \{1, \dots, m\}$ (*which includes the case when the functions* g_i *are quadratic forms*)*, one obtains that the convexity of the set*

$$\Big\{\Big(f(x), g_1(x, \mu_1, \delta_1), g_1(x, \overline{\mu}_1, \delta_1), g_2(x, \mu_2, \delta_2), g_2(x, \overline{\mu}_2, \delta_2), \dots,$$
$$g_m(x, \mu_m, \delta_m), g_m(x, \overline{\mu}_m, \delta_m)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}^{2m+1}_+$$

implies the convexity of the set

$$\Big\{\Big(f(x), \max_{\mu_1 \le \mu \le \overline{\mu}_1} g_1(x, \mu, \delta_1), \max_{\mu_2 \le \mu \le \overline{\mu}_2} g_2(x, \mu, \delta_2), \dots,$$
$$\max_{\mu_m \le \mu \le \overline{\mu}_m} g_m(x, \mu, \delta_m)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}^{m+1}_+.$$

Proof. It is straightforward once one looks at the proof of the preceding proposition since $\delta_i = \overline{\delta}_i$ for all i. □

In case we consider homogeneous quadratic functions, an important class of problems where the convexity assumption in Corollary 2.4 holds, is that when the A is a Z-matrix and the other matrices B_i are diagonal. To be more precise, we obtain the following result

Proposition 2.5. *Let the same data be as in Proposition 2.3, and assume that* $a = 0, b_i = \overline{b}_i = 0$ *for all* $i = 1, 2, \dots, m$*. If* A *is a Z-matrix and* B_i, $\overline{B}_i$ *are diagonal matrices, then the set*

$$\Omega^h \doteq \Big\{\Big(f(x), g_1(x, \mu_1, 0), g_1(x, \overline{\mu}_1, 0), g_2(x, \mu_2, 0), g_2(x, \overline{\mu}_2, 0), \dots,$$
$$g_m(x, \mu_m, 0), g_m(x, \overline{\mu}_m, 0)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}^{2m+1}_+$$

is convex, and so is the set

$$\Big\{\Big(f(x), \max_{\mu_1 \le \mu \le \overline{\mu}_1} g_1(x, \mu, 0), \max_{\mu_2 \le \mu \le \overline{\mu}_2} g_2(x, \mu, 0), \dots,$$
$$\max_{\mu_m \le \mu \le \overline{\mu}_m} g_m(x, \mu, 0)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}^{m+1}_+.$$

Proof. We simply observe that the matrices $B_i + \mu_i \overline{B}_i$ $(i = 1, 2, \dots, m)$, being diagonal, are Z-matrices, and so by Theorem 5.1 in [10], Ω^h is convex. The second part follows from the previous corollary.

□

3. The main results

In this section we present the fundamental results under bounded interval uncertainty: the robust versions of the alternative theorem, the S-lemma and characterization of optimality, all of them refering to problem (1). As we will see, Theorem

2.2 will play an important role in deriving our first main result. In the first part, we consider the nonhomogeneous situation and in the second part, the homogeneous case is discussed.

The functions involved throughout this section are the following

$$f(x) \doteq \frac{1}{2}x^\top Ax + a^\top x \tag{3}$$

$$g(x,\mu,\delta) \doteq \frac{1}{2}x^\top(B_1+\mu B_2)x + (b_1+\delta b_2)^\top x + c, \tag{4}$$

where A, B_1, B_2 are real symmetric matrices of order n; $a, b_1, b_2 \in \mathbb{R}^n$ and c, μ, δ are real numbers.

As expected, we will apply Proposition 2.3 to the functions

$$g_1(x,\mu,\delta) \doteq g(x,\mu,\delta);\ \ g_2(x,\mu,\delta) \doteq -g(x,\mu,\delta). \tag{5}$$

3.1. The inhomogeneous case

With the previous notation, the robust optimization problem is the following

$$\min\Big\{f(x):\ \ \alpha \le g(x,\mu,\delta) \le \beta,\ \forall\ \mu \in [\mu_1,\mu_2], \forall\ \delta \in [\delta_1,\delta_2]\Big\}. \tag{6}$$

Here, $\alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2$ are fixed real numbers satisfying $\alpha < \beta$, $\mu_1 < \mu_2$, $\delta_1 < \delta_2$. In view of (6), the sets appearing in the formulation of Proposition 2.3, and which will play a central role in our analysis, become (see (5))

$$\Omega_0(\mu_1,\mu_2,\delta_1,\delta_2) \doteq \Big\{\Big(f(x), g(x,\mu_1,\delta_1), g(x,\mu_1,\delta_2), g(x,\mu_2,\delta_1), g(x,\mu_2,\delta_2),$$
$$-g(x,\mu_1,\delta_1), -g(x,\mu_1,\delta_2), -g(x,\mu_2,\delta_1), -g(x,\mu_2,\delta_2)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}^9_+, \tag{7}$$

and
$$\Omega_\mu(\delta_1,\delta_2) \doteq \Big\{\Big(f(x), \max_{\mu\in[\mu_1,\mu_2]} g(x,\mu,\delta_1), \max_{\mu\in[\mu_1,\mu_2]} g(x,\mu,\delta_2),$$
$$-\min_{\mu\in[\mu_1,\mu_2]} g(x,\mu,\delta_1), -\min_{\mu\in[\mu_1,\mu_2]} g(x,\mu,\delta_2)\Big) : x \in \mathbb{R}^n\Big\} + \operatorname{int} \mathbb{R}^5_+. \tag{8}$$

We establish the first main result concerning the robust version of the alternative theorem.

Theorem 3.1. (A robust alternative result) *Let the data be as described above. If the set $\Omega_\mu(\delta_1,\delta_2)$ is convex, then exactly one of the two following assertions hold:*

(a) $\exists\ x \in \mathbb{R}^n,\ \forall\ \mu \in [\mu_1,\mu_2],\ \forall\ \delta \in [\delta_1,\delta_2]:$
$\frac{1}{2}x^\top Ax + a^\top x + \gamma < 0, \quad \alpha < \frac{1}{2}x^\top(B_1+\mu B_2)x + (b_1+\delta b_2)^\top x + c < \beta.$

(b) $\exists\ (\lambda_0,\lambda_1,\lambda_2) \in \mathbb{R}^3_+\backslash\{0\}, \exists\ \mu_\alpha, \mu_\beta \in [\mu_1,\mu_2], \exists\ \delta_\alpha, \delta_\beta \in [\delta_1,\delta_2]:\ \forall\ x \in \mathbb{R}^n$
$\lambda_0\Big(\frac{1}{2}x^\top Ax + a^\top x + \gamma\Big) + \lambda_1\Big(\frac{1}{2}x^\top(B_1+\mu_\beta B_2)x + (b_1+\delta_\beta b_2)^\top x + c - \beta\Big)$
$+\lambda_2\Big(\alpha - \Big(\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x + (b_1+\delta_\alpha b_2)^\top x + c\Big)\Big) \ge 0,$
where $\mu_\alpha + \mu_\beta = \mu_1 + \mu_2$.

In addition, we observe that (b) *may be written equivalently as*

$$\begin{aligned}
&\lambda_0 A + \lambda_1(B_1 + \mu_\beta B_2) - \lambda_2(B_1 + \mu_\alpha B_2) \succeq 0, \text{ and} \\
&\exists\, \overline{x} \in \mathbb{R}^n : \Big(\lambda_0 A + \lambda_1(B_1 + \mu_\beta B_2) - \lambda_2(B_1 + \mu_\alpha B_2)\Big)\overline{x} \\
&\qquad = -\lambda_0 a - \lambda_1(b_1 + \delta_\beta b_2) + \lambda_2(b_1 + \delta_\alpha b_2).
\end{aligned}$$

Proof. It is obvious that both statements (a) and (b) cannot be fulfilled simultaneously. Thus, we must check that if (a) does not hold, (b) does.

Step 1: If (a) does not hold, then there exists no $x \in \mathbb{R}^n$ such that for all $\mu \in [\mu_1, \mu_2]$ and all $\delta \in [\delta_1, \delta_2]$

$$\begin{gathered}
\frac{1}{2}x^\top Ax + a^\top x + \gamma < 0, \quad -\Big(\frac{1}{2}x^\top(B_1 + \mu B_2)x + (b_1 + \delta b_2)^\top x + c\Big) < -\alpha, \\
\frac{1}{2}x^\top(B_1 + \mu B_2)x + (b_1 + \delta b_2)^\top x + c < \beta.
\end{gathered}$$

By setting $\mathcal{B}_0 \doteq \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}$, the previous expression is equivalent to the nonexistence of $x \in \mathbb{R}^n$ such that

$$\begin{gathered}
\frac{1}{2}x^\top Ax + a^\top x + \gamma < 0, \quad -\min\Big\{\frac{1}{2}x^\top Bx + b^\top x + c : (B, b) \in \mathcal{B}_0\Big\} < -\alpha, \\
\max\Big\{\frac{1}{2}x^\top Bx + b^\top x + c : (B, b) \in \mathcal{B}_0\Big\} < \beta.
\end{gathered}$$

For fixed $x \in \mathbb{R}^n$: due to the linearity with respect to (μ, δ), both maximum and minimum values are attained in some of the extreme points of the rectangle $[\mu_1, \mu_2] \times [\delta_1, \delta_2]$. Thus, the following system has no solution in $\mathbb{R}^n$:

$$\begin{gathered}
\frac{1}{2}x^\top Ax + a^\top x + \gamma < 0, \quad -\min_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_1) < -\alpha, \quad -\min_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_2) < -\alpha, \\
\max_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_1) < \beta, \quad \max_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_2) < \beta.
\end{gathered}$$

This means that $(0, \beta, \beta, -\alpha, -\alpha) \notin \Omega_\mu(\delta_1, \delta_2)$.

Step 2: (A first use of separation of convex sets)
We have the existence of $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^5_+ \backslash\{0\}$ such that for all $x \in \mathbb{R}^n$

$$\begin{aligned}
&\lambda_0(f(x) + \gamma) + \lambda_1\Big(\max_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_1) - \beta\Big) + \lambda_2\Big(\max_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_2) - \beta\Big) \\
&\quad + \lambda_3\Big(\alpha - \min_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_1)\Big) + \lambda_4\Big(\alpha - \min_{\mu\in[\mu_1,\mu_2]} g(x, \mu, \delta_2)\Big) \geq 0.
\end{aligned}$$

This can be written equivalently as: there exists $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^5_+ \backslash\{0\}$ such that for all $x \in \mathbb{R}^n$

$$\begin{aligned}
&\lambda_0(f(x) + \gamma) + \lambda_1\Big(\max\{g(x, \mu_1, \delta_1), g(x, \mu_2, \delta_1)\} - \beta\Big) \\
&+ \lambda_2\Big(\max\{g(x, \mu_1, \delta_2), g(x, \mu_2, \delta_2)\} - \beta\Big) + \lambda_3\Big(\alpha - \min\{g(x, \mu_1, \delta_1), g(x, \mu_2, \delta_1)\}\Big) \\
&+ \lambda_4\Big(\alpha - \min\{g(x, \mu_1, \delta_2), g(x, \mu_2, \delta_2)\}\Big) \geq 0.
\end{aligned}$$

Thus, the following system has no solution in $\mathbb{R}^n$:

$$\lambda_0(f(x)+\gamma)+\lambda_1\Big(g(x,\mu_1,\delta_1)-\beta\Big)+\lambda_2\Big(g(x,\mu_1,\delta_2)-\beta\Big) \\ +\lambda_3\Big(\alpha-g(x,\mu_2,\delta_1)\Big)+\lambda_4\Big(\alpha-g(x,\mu_2,\delta_2)\Big)<0 \quad \text{and}$$

$$\lambda_0(f(x)+\gamma)+\lambda_1\Big(g(x,\mu_2,\delta_1)-\beta\Big)+\lambda_2\Big(g(x,\mu_2,\delta_2)-\beta\Big) \\ +\lambda_3\Big(\alpha-g(x,\mu_1,\delta_1)\Big)+\lambda_4\Big(\alpha-g(x,\mu_1,\delta_2)\Big)<0.$$

By Theorem 2.2 (see [8, Theorem 4.19]), the set

$$\Omega\doteq\Big\{(\lambda_0(f(x)+\gamma)+\lambda_1\Big(g(x,\mu_1,\delta_1)-\beta\Big)+\lambda_2\Big(g(x,\mu_1,\delta_2)-\beta\Big)+\lambda_3\Big(\alpha-g(x,\mu_2,\delta_1)\Big) \\ +\lambda_4\Big(\alpha-g(x,\mu_2,\delta_2)\Big),\lambda_0 f(x)+\lambda_1\Big(g(x,\mu_2,\delta_1)-\beta\Big)+\lambda_2\Big(g(x,\mu_2,\delta_2)-\beta\Big) \\ +\lambda_3\Big(\alpha-g(x,\mu_1,\delta_1)\Big)+\lambda_4\Big(\alpha-g(x,\mu_1,\delta_2)\Big):x\in\mathbb{R}^n\Big\}+\operatorname{int}\mathbb{R}^2_+$$

is convex, and from the previous remark, $(0,0)\notin\Omega$.

Step 3: (A second use of separation of convex sets and final conclusion)
We again obtain the existence of $(\xi_1,\xi_2)\in\mathbb{R}^2_+\backslash\{0\}$ such that for all $x\in\mathbb{R}^n$,

$$\xi_1\Big[\lambda_0(f(x)+\gamma)+\lambda_1\Big(g(x,\mu_1,\delta_1)-\beta\Big)+\lambda_2\Big(g(x,\mu_1,\delta_2)-\beta\Big)+\lambda_3\Big(\alpha-g(x,\mu_2,\delta_1)\Big) \\ +\lambda_4\Big(\alpha-g(x,\mu_2,\delta_2)\Big)\Big]+\xi_2\Big[\lambda_0 f(x)+\lambda_1\Big(g(x,\mu_2,\delta_1)-\beta\Big)+\lambda_2\Big(g(x,\mu_2,\delta_2)-\beta\Big) \\ +\lambda_3\Big(\alpha-g(x,\mu_1,\delta_1)\Big)+\lambda_4\Big(\alpha-g(x,\mu_1,\delta_2)\Big)\Big]\geq 0.$$

Thus, by setting

$$\overline{\lambda}_0=\lambda_0(\xi_1+\xi_2),\quad \overline{\lambda}_1=(\lambda_1+\lambda_2)(\xi_1+\xi_2),\quad \overline{\lambda}_2=(\lambda_3+\lambda_4)(\xi_1+\xi_2),$$
$$\mu_\beta=\frac{\xi_1\mu_1+\xi_2\mu_2}{\xi_1+\xi_2},\ \mu_\alpha=\frac{\xi_1\mu_2+\xi_2\mu_1}{\xi_1+\xi_2},\ \delta_\beta=\frac{\lambda_1\delta_1+\lambda_2\delta_2}{\lambda_1+\lambda_2},\ \delta_\alpha=\frac{\lambda_3\delta_2+\lambda_4\delta_1}{\lambda_3+\lambda_4},$$

we get $(\overline{\lambda}_0,\overline{\lambda}_1,\overline{\lambda}_2)\in\mathbb{R}^3_+\backslash\{0\}$, $\mu_\beta,\mu_\alpha\in[\mu_1,\mu_2]$, $\delta_\beta,\delta_\alpha\in[\delta_1,\delta_2]$ and

$$\overline{\lambda}_0\Big(\frac{1}{2}x^\top Ax+a^\top x+\gamma\Big)+\overline{\lambda}_1\Big(\frac{1}{2}x^\top(B_1+\mu_\beta B_2)x+(b_1+\delta_\beta b_2)^\top x+c-\beta\Big) \\ +\overline{\lambda}_2\left(\alpha-\Big(\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x+(b_1+\delta_\alpha b_2)^\top x+c\Big)\right)\geq 0\quad\forall\ x\in\mathbb{R}^n.$$

It is easy to verify that $\mu_\alpha+\mu_\beta=\mu_1+\mu_2$, and the proof is completed. □

We are now in a position to establish the robust version of the S-lemma for our optimization problem (6).

Theorem 3.2. (A robust S-lemma) *Let the data be as described above. Assume that the set $\Omega_\mu(\delta_1,\delta_2)$, defined in* (8)*, is convex, and the Slater-type condition: there exists $x_0 \in \mathbb{R}^n$:*

$$\alpha < \frac{1}{2}x_0^\top(B_1+\mu B_2)x_0 + (b_1+\delta b_2)^\top x_0 + c < \beta, \quad \forall\, \mu\in[\mu_1,\mu_2],\ \forall\,\delta\in[\delta_1,\delta_2], \tag{9}$$

is satisfied. Then, the following assertions are equivalent:

(a) $\Big[\alpha \le \frac{1}{2}x^\top(B_1+\mu B_2)x + (b_1+\delta b_2)^\top x + c \le \beta,\ \forall\,\mu\in[\mu_1,\mu_2],\ \forall\,\delta\in[\delta_1,\delta_2]\Big]$
$\Longrightarrow \frac{1}{2}x^\top Ax + a^\top x + \gamma \ge 0.$

(b) $\exists\ (\lambda_1,\lambda_2)\in\mathbb{R}^2_+,\ \exists\ \mu_\alpha,\mu_\beta\in[\mu_1,\mu_2], \exists\ \delta_\alpha,\delta_\beta\in[\delta_1,\delta_2] : \forall\ x\in\mathbb{R}^n,$

$$\Big(\frac{1}{2}x^\top Ax + a^\top x + \gamma\Big) + \lambda_1\Big(\frac{1}{2}x^\top(B_1+\mu_\beta B_2)x + (b_1+\delta_\beta b_2)^\top x + c - \beta\Big)$$
$$+\lambda_2\Big(\alpha - \Big(\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x + (b_1+\delta_\alpha b_2)^\top x + c\Big)\Big) \ge 0,$$

where $\mu_\alpha+\mu_\beta = \mu_1+\mu_2$. Moreover, notice that (b) *can be written equivalently as:*

$$A + \lambda_1(B_1+\mu_\beta B_2) - \lambda_2(B_1+\mu_\alpha B_2) \succeq 0 \quad \text{and}$$

$$\exists\,\overline{x}\in\mathbb{R}^n\colon (A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2))\overline{x}+a+\lambda_1(b_1+\delta_\beta b_2)-\lambda_2(b_1+\delta_\alpha b_2) = 0.$$

Proof. This is a consequence of Theorem 3.1, where (9) forces to get $\lambda_0 > 0$. □

In what follows a characterization of robust optimality to problem (6) is established.

Theorem 3.3. (Characterizing robust optimality) *Let the data be as described above. Assume that the set $\Omega_\mu(\delta_1,\delta_2)$, defined in* (8)*, is convex, and the Slater-type condition* (9) *holds. Then, a feasible point $\overline{x}$ to problem* (6) *is optimal if, and only if there exists $(\lambda_1,\lambda_2)\in\mathbb{R}^2_+$, $\mu_\alpha,\mu_\beta\in[\mu_1,\mu_2]$, δ_α, $\delta_\beta\in[\delta_1,\delta_2]$ such that the following conditions are satisfied*

(a) $\Big(A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\Big)\overline{x} = -\Big(a+\lambda_1(b_1+\delta_\beta b_2)-\lambda_2(b_1+\delta_\alpha b_2)\Big);$

(b) $\lambda_1\Big(\frac{1}{2}\overline{x}^\top(B_1+\mu_\beta B_2)\overline{x} + (b_1+\delta_\beta b_2)^\top\overline{x} + c - \beta\Big) = 0;$
$\lambda_2\Big(\alpha - \Big(\frac{1}{2}\overline{x}^\top(B_1+\mu_\alpha B_2)\overline{x} + (b_1+\delta_\alpha b_2)^\top\overline{x} + c\Big)\Big) = 0;$

(c) $A + \lambda_1(B_1+\mu_\beta B_2) - \lambda_2(B_1+\mu_\alpha B_2) \succeq 0.$

Proof. The necessary condition follows from Theorem 3.2(b) by taking $\gamma = -f(\overline{x})$ since that part (b) implies clearly (c) and, putting $x=\overline{x}$, also (b). The sufficiency part is already standard since (c) implies the convexity of the function

$$h(x) \doteq \frac{1}{2}x^\top Ax + a^\top x - f(\overline{x}) + \lambda_1\Big(\frac{1}{2}x^\top(B_1+\mu_\beta B_2)x + (b_1+\delta_\beta b_2)^\top x + c - \beta\Big)$$
$$+\lambda_2\Big(\alpha - \Big(\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x + (b_1+\delta_\alpha b_2)^\top x + c\Big)\Big),$$

and (a) and (b) allow us to prove that $\overline{x}$ is in fact a solution to problem (6). □

Remark 3.4. We observe that from Theorem 3.3 every feasible point $\overline{x}$ to problem (6) satisfying (a), (b) and (c) is also a solution to the problem

$$\begin{aligned} \min\quad & \frac{1}{2}x^\top Ax + a^\top x \\ \text{s.t.}\quad & \frac{1}{2}x^\top(B_1+\mu_\beta B_2)x + (b_1+\delta_\beta b_2)^\top x + c \leq \beta \\ & -\Big(\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x + (b_1+\delta_\alpha b_2)^\top x + c\Big) \leq -\alpha. \end{aligned} \tag{10}$$

3.2. The homogeneous case

In this section we particularize problem (6) to the case when all the functions are homogeneous, that is, we consider the quadratic robust optimization problem:

$$\min\ \frac{1}{2}x^\top Ax \ \text{ s.t. } \ \alpha \leq \frac{1}{2}x^\top(B_1+\mu B_2)x \leq \beta \quad \forall\,\mu\in[\mu_1,\mu_2], \tag{11}$$

where A, B_1, B_2 are real symmetric matrices, $\mu_1,\mu_2,\alpha,\beta\in\mathbb{R}$ satisfy $\mu_1<\mu_2$, $\alpha<\beta$.

Due to Corollary 2.4, we realize that the sets arising for problem (11) reduce to

$$\begin{aligned} \Omega_0^h(\mu_1,\mu_2) = \Big\{\Big(& x^\top Ax, x^\top(B_1+\mu_1B_2)x, x^\top(B_1+\mu_2B_2)x, \\ & -x^\top(B_1+\mu_1B_2)x, -x^\top(B_1+\mu_2B_2)x\Big) : x\in\mathbb{R}^n\Big\} + \operatorname{int}\mathbb{R}^5_+. \end{aligned}$$

and

$$\Omega_\mu^h \doteq \Big\{\Big(x^\top Ax, \max_{\mu_1\leq\mu\leq\mu_2} x^\top(B_1+\mu B_2)x,\ -\min_{\mu_1\leq\mu\leq\mu_2} x^\top(B_1+\mu B_2)x\Big) : x\in\mathbb{R}^n\Big\} + \operatorname{int}\mathbb{R}^3_+,$$

Due to their importance, in what follows we single out the homogeneous versions of Theorems 3.1, 3.2 and 3.3. For their proofs simply take into account Corollary 2.4.

Theorem 3.5. (A homogeneous robust alternative result) *Let A, B_1, B_2 be real symmetric matrices or order n and $\gamma,\alpha,\beta,\mu_1,\mu_2\in\mathbb{R}$, with $\mu_1<\mu_2$ and $\alpha<\beta$. Let f and g be the quadratic functions as described above. If the set Ω_μ^h is convex, then exactly one of the two following assertions hold:*

(a) $\exists\ x\in\mathbb{R}^n : \frac{1}{2}x^\top Ax+\gamma<0,\ \ \alpha<\frac{1}{2}x^\top(B_1+\mu B_2)x<\beta,\ \ \forall\ \mu\in[\mu_1,\mu_2]$.

(b) $\exists\ (\lambda_0,\lambda_1,\lambda_2)\in\mathbb{R}^3_+\backslash\{0\},\ \exists\,\mu_\alpha,\mu_\beta\in[\mu_1,\mu_2] :\ \forall\,x\in\mathbb{R}^n$

$$\lambda_0\Big(\frac{1}{2}x^\top Ax+\gamma\Big)+\lambda_1\Big(\frac{1}{2}x^\top(B_1+\mu_\beta B_2)x-\beta\Big)+\lambda_2\Big(\alpha-\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x\Big)\geq 0,$$

where $\mu_\alpha+\mu_\beta=\mu_1+\mu_2$.

In addition, we observe that (b) *may be written equivalently as*

$$\lambda_0A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\succeq 0$$

and $$\exists\ \overline{x}\in\mathbb{R}^n : \Big(\lambda_0A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\Big)\overline{x}=0.$$

Theorem 3.6. (A homogeneous robust S-lemma) *Let the same data be as in Theorem* 3.5. *Assume that the set* Ω_μ^h *is convex, and the Slater-type condition: there exists* $x_0 \in \mathbb{R}^n$ *such that*

$$\alpha < \frac{1}{2}x_0^\top(B_1+\mu B_2)x_0 < \beta, \quad \forall\ \mu \in [\mu_1,\mu_2], \tag{12}$$

is satisfied. Then, the following assertions are equivalent:

(a) $\Big[\alpha \le \frac{1}{2}x^\top(B_1+\mu B_2)x \le \beta,\ \forall\ \mu\in[\mu_1,\mu_2]\Big] \Longrightarrow \frac{1}{2}x^\top Ax+\gamma \ge 0.$

(b) $\exists\,(\lambda_1,\lambda_2)\in\mathbb{R}^2_+,\ \exists\,\mu_\alpha,\mu_\beta\in[\mu_1,\mu_2]\colon\forall\ x\in\mathbb{R}^n$ *we have*
$\Big(\frac{1}{2}x^\top Ax+\gamma\Big)+\lambda_1\Big(\frac{1}{2}x^\top(B_1+\mu_\beta B_2)x-\beta\Big)+\lambda_2\Big(\alpha-\frac{1}{2}x^\top(B_1+\mu_\alpha B_2)x\Big)\ge 0,$

where $\mu_\alpha+\mu_\beta=\mu_1+\mu_2$. *Moreover, notice that* (b) *can be written equivalently as:*

$$A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\succeq 0$$

and $$\exists\,\overline{x}\in\mathbb{R}^n : \Big(A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\Big)\overline{x}=0.$$

We end this section by establishing a characterization of robust optimality.

Theorem 3.7. (Characterizing homogeneous robust optimality) *Let the same data be as in Theorem* 3.5. *Assume that the set* Ω_μ^h *is convex, and the Slater-type condition* (12) *holds. Then, a feasible point* $\overline{x}$ *to problem* (11) *is optimal if, and only if there exist* $(\lambda_1,\lambda_2)\in\mathbb{R}^2_+$, $\mu_\alpha,\mu_\beta\in[\mu_1,\mu_2]$ *such that the following conditions are satisfied*

(a) $\Big(A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\Big)\overline{x}=0;$

(b) $\lambda_1\Big(\frac{1}{2}\overline{x}^\top(B_1+\mu_\beta B_2)\overline{x}-\beta\Big)=0;\ \lambda_2\Big(\alpha-\Big(\frac{1}{2}\overline{x}^\top(B_1+\mu_\alpha B_2)\overline{x}\Big)\Big)=0;$

(c) $A+\lambda_1(B_1+\mu_\beta B_2)-\lambda_2(B_1+\mu_\alpha B_2)\succeq 0.$

We end this section by dealing with the weighted least squares problem.

An application: robust solution of weighted least squares

For $i=1,\dots,n$, let $v_i\in\mathbb{R}$ and $\underline{w}_i$, $\overline{w}_i$, α and β are real numbers satisfying $\underline{w}_i\le\overline{w}_i$, $\alpha<\beta$. We consider the following uncertain weighted least squares problem:

$$\min \quad \frac{1}{2}\sum_{i=1}^n v_i x_i^2 \quad \text{s.t.} \quad \alpha \le \frac{1}{2}\sum_{i=1}^n w_i x_i^2 \le \beta, \tag{13}$$

where the data w_i, $i=1,\dots,n$ are uncertain and each w_i varies in the uncertainty set $\mathcal{V}_i=[\underline{w}_i,\overline{w}_i]$ for all $i\in\{1,\dots,n\}$.

Problem (13) may be written equivalently as in the form (11) with data

$$A=\operatorname{diag}(v_1,\dots,v_n),\quad B_1=\operatorname{diag}\left(\frac{\overline{w}_1+\underline{w}_1}{2},\dots,\frac{\overline{w}_n+\underline{w}_n}{2}\right),$$
$$B_2=\operatorname{diag}\left(\frac{\overline{w}_1-\underline{w}_1}{2},\dots,\frac{\overline{w}_n-\underline{w}_n}{2}\right),\quad \mu_1=-1,\ \mu_2=1.$$

The following optimality result is obtained.

Theorem 3.8. *For the above data, assume that the existence of $x_0 \in \mathbb{R}^n$ such that $\alpha < \frac{1}{2}\sum_{i=1}^{n} w_i x_{0_i}^2 < \beta$ for all $w_i \in [\underline{w}_i, \overline{w}_i]$. Then, a feasible point $\overline{x}$ to problem (13) is a robust optimal solution if, and only if there exist $(\lambda_1, \lambda_2) \in \mathbb{R}^2_+ \setminus \{0\}$ and $w_{\alpha_i}, w_{\beta_i} \in [\underline{w}_i, \overline{w}_i]$, $i = 1, \dots, n$, such that*

(a) $\displaystyle\sum_{i=1}^{n}(v_i + \lambda_1 w_{\beta_i} - \lambda_2 w_{\alpha_i})\overline{x}_i = 0;$

(b) $\displaystyle\lambda_1\left(\frac{1}{2}\sum_{i=1}^{n} w_{\beta_i}\overline{x}_i^2 - \beta\right) + \lambda_2\left(\frac{1}{2}\sum_{i=1}^{n} w_{\alpha_i}\overline{x}_i^2\right) = 0;$

(c) $v_i + \lambda_1 w_{\beta_i} - \lambda_2 w_{\alpha_i} \geq 0.$

Proof. By taking the above formulation, we apply Proposition 2.5 and Theorem 3.7 to conclude with the required result. □

Finally we provide an example.

Example 3.9. Let us consider the nonconvex problem:

$$\min\ x_1^2 - x_2^2 + \frac{3}{2}x_3^2 \quad \text{s.t.} \quad 20 \leq a_1 x_2^2 + a_2 x_2^2 + a_3 x_3^2 \leq 100, \tag{14}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ are uncertain data. Suppose that we know that the nominal value of a_1, a_3 are 1 with the possible error of a_3 are ± 0.5; whereas the nominal value of a_2 is -1 with the possible error of ± 1.

The robust counterpart of this problem is

$$\min\ \frac{1}{2}x^\top A x \quad \text{s.t.}\ \ 20 \leq \frac{1}{2}x^\top(B_1 + \mu B_2)x \leq 100, \quad \forall\, \mu \in [\mu_1, \mu_2], \tag{15}$$

where $$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix},\ B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha = 20$, $\beta = 100$, $\mu_1 = -1$, $\mu_2 = 1$.

All the matrices being diagonal, the set $\Omega_0^h(\mu_1, \mu_2)$ is convex. Moreover, the point $x_0 = (5, 1, 5)$ satisfies the Slater-type condition:

$$20 < \frac{1}{2}x_0^\top(B_1 + \mu B_2)x_0 < 100, \quad \forall\ \mu \in [\mu_1, \mu_2].$$

By setting $\overline{x} = \begin{pmatrix} 0 \\ \frac{2\sqrt{15}}{3} \\ \frac{10\sqrt{6}}{3} \end{pmatrix}$, $\mu_\beta = 1$, $\mu_\alpha = -1$, $\lambda_1 = \frac{7}{6}$, $\lambda_2 = \frac{1}{2}$, one can check that

$$A + \lambda_1(B_1 + \mu_\beta B_2) - \lambda_2(B_1 + \mu_\alpha B_2) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \succeq 0,$$

which implies that (c) holds.

It turns out that (a) also fulfills; (b) follows since

$$\frac{1}{2}\overline{x}^\top(B_1+\mu_\beta B_2)\overline{x}-\beta=0=\alpha-\frac{1}{2}\overline{x}^\top(B_1+\mu_\alpha B_2)\overline{x}.$$

This proves that $\overline{x}$ is a robust solution.

4. Comparison between the convexity assumption imposed in Theorem 3.3 and [7, Theorem 3]

Although the optimality conditions (a), (b) and (c) of Theorem 3.3 and those in [7, Theorem 3] coincide, they were obtained under different convexity assumptions. First of all, notice that Theorem 3.3 requires the convexity of a set (see (8)) in $\mathbb{R}^5$ which is an image of $\mathbb{R}^n$ via a (not necessarily homogeneous) quadratic function; whereas the convexity in [7, Theorem 3] is imposed on a set in $\mathbb{R}^5$ being an image of $\mathbb{R}^{n+1}$ via a homogeneous quadratic function. To be more precise, let us define the matrices

$$H_0 \doteq \begin{pmatrix} A & a \\ a^\top & 2\gamma \end{pmatrix},\ W(\delta,\lambda) \doteq \begin{pmatrix} B_1 & b_1+\delta b_2 \\ (b_1+\delta b_2)^\top & -2\lambda \end{pmatrix},\ W_2 \doteq \begin{pmatrix} B_2 & 0 \\ 0^\top & 0 \end{pmatrix},$$

and set $W_{1_\beta} \doteq W(\delta_1,\beta)$; $W_{2_\beta} \doteq W(\delta_2,\beta)$; $W_{1_\alpha} \doteq W(\delta_1,\alpha)$; $W_{2_\alpha} \doteq W(\delta_2,\alpha)$.

Then, the convexity assumption in [7, Theorem 3] is imposed on the set

$$\Omega_W \doteq \Big\{\Big(\frac{1}{2}y^\top H_0 y, \max_{\mu\in[\mu_1,\mu_2]}\frac{1}{2}y^\top(W_{1_\beta}+\mu W_2)y), \max_{\mu\in[\mu_1,\mu_2]}\frac{1}{2}y^\top(W_{2_\beta}+\mu W_2)y,$$
$$-\min_{\mu\in[\mu_1,\mu_2]}\frac{1}{2}y^\top(W_{1_\alpha}+\mu W_2)y, -\min_{\mu\in[\mu_1,\mu_2]}\frac{1}{2}y^\top(W_{2_\alpha}+\mu W_2)y\Big) : y\in\mathbb{R}^{n+1}\Big\}+\text{int }\mathbb{R}^5_+.$$

By Corollary 1, Ω_W is convex if the set

$$\Omega_\mu \doteq \Big\{\Big(y^\top H_0 y, y^\top(W_{1_\beta}+\mu_1 W_2)y, y^\top(W_{2_\beta}+\mu_1 W_2)y, y^\top(W_{1_\beta}+\mu_2 W_2)y,$$
$$y^\top(W_{2_\beta}+\mu_2 W_2)y, -y^\top(W_{1_\alpha}+\mu_1 W_2)y, -y^\top(W_{2_\alpha}+\mu_1 W_2)y, -y^\top(W_{1_\alpha}+\mu_2 W_2)y,$$
$$-y^\top(W_{2_\alpha}+\mu_2 W_2)y\Big) : y\in\mathbb{R}^{n+1}\Big\}+\text{int }\mathbb{R}^9_+$$

is so, with $\gamma=-f(\overline{x})$ in H_0 and $\overline{x}$ is a feasible point of the robust optimization problem (1), which is either to be supposed becoming optimal solution once (a)–(c) hold (sufficient condition), or to be optimal for deriving conditions (a)–(c) (necessary condition).

Certainly, the fact that Ω_W involves the image of $\mathbb{R}^{n+1}$ via an homogeneous quadratic functions is because the authors in [7] – following [11] – homogenize problem (1) in order to apply the Dines convexity theorem valid for quadratic forms. Instead, we apply the convexity Theorem 2.2 (see [8, Theorem 4.19]) valid for inhomogeneous quadratic functions. Moreover, observe that the matrix H_0 depends on the reference point $\overline{x}$; whereas there is no dependence of $\overline{x}$ in the set on which our convexity assumption is required.

Finally, we were unable to show any relationship between our convexity assumption and that in [7, Theorem 3]. This is an open question.

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