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The Extended Exterior Sphere Condition

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We prove that the complement of a closed set S satisfying an *extended* exterior sphere condition is nothing but the union of closed balls with *common* radius. This generalizes Theorem 3 of F. Nacry and L. Thibault [*Distance function associated to a prox-regular set*, Set-Valued Var. Analysis 30 (2022) 731–750] where the set S is assumed to be prox-regular, a property *stronger* than the extended exterior sphere condition. We also provide a sufficient condition for the equivalence between prox-regularity and the extended exterior sphere condition that generalizes previous work of C. Nour, R. J. Stern and J. Takche [*Proximal smoothness and the exterior sphere condition*, J. Convex Analysis 16/2 (2009) 501–514] to the case in which S is not necessarily regular closed.

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1. Introduction

Let $S \subset \mathbb{R}^n$ be a closed set. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal* to S at x if there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle \zeta, s - x\rangle \leq \sigma \|s - x\|^2, \quad \forall s \in S, \tag{1}$$

where $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ denote the standard inner product and Euclidean norm, respectively. The relation (1) is commonly referred to as the *proximal normal inequality.* Note that for $\zeta \neq 0$ and $\sigma \neq 0$, the proximal normal inequality (1) is equivalent to

$$B\left(x + \frac{1}{2\sigma}\frac{\zeta}{\|\zeta\|}; \frac{1}{2\sigma}\right) \cap S = \emptyset,$$

where $B(y;\rho)$ denotes the open ball of radius ρ centered at y. In that case, we say that ζ is *realized* by a $\frac{1}{2\sigma}$-sphere. On the other hand, for $\zeta \neq 0$ and $\sigma = 0$, the proximal normal inequality (1) is equivalent to $B(x + \rho\zeta; \rho) \cap S = \emptyset$ for all $\rho > 0$. In that case, ζ is realized by a ρ-sphere for any $\rho > 0$. Now, in view of (1), the set of all proximal normals to S at x is a convex cone containing 0, and we denote it by $N_S^P(x)$. Since no nonzero ζ satisfying (1) exists if $x \in \operatorname{int} S$ (the interior of S), we deduce that $N_S^P(x) = \{0\}$ for all $x \in \operatorname{int} S$. This may also occur for $x \in \operatorname{bdry} S$, the boundary of S, as is the case when S is the epigraph of the function $f(z) = -|z|$ and x is the origin. More information about $N_S^P(\cdot)$ and proximal analysis can be found in [6, 10, 19, 22].

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Now we fix $r > 0$. We recall that the set S is said to be *r-prox-regular* if for any $x \in \operatorname{bdry} S$ and *for any* $0 \neq \zeta \in N_S^P(x)$, ζ is realized by an r-sphere. For more information about prox-regularity, and related properties such as *positive reach*, *proximal smoothness*, *p-convexity* and *φ_0-convexity*, see [1, 7, 8, 9, 20, 23]. On the other hand, the set S is said to be satisfying the *exterior r-sphere condition* if for any $x \in \operatorname{bdry} S$, *there exists* $0 \neq \zeta \in N_S^P(x)$ such that ζ is realized by an r-sphere. Clearly if S is r-prox-regular, then it satisfies the exterior r-sphere condition. For the converse, it does not hold in general as it is shown via counterexamples in [13]. In this latter, Nour, Stern and Takche provided sufficient condition for the equivalence between prox-regularity and the exterior sphere condition. More precisely, they proved that the two properties are equivalent if S is *epi-Lipschitz* (or *wedged*) and has compact boundary, see [13, Corollary 3.12].[1] Recall that a closed set S is said to be epi-Lipschitz at a point $x \in S$ if the set S can be viewed near x, after application of an orthogonal matrix, as the epigraph of a Lipschitz continuous function. If this holds for all $x \in S$, then we simply say that S is epi-Lipschitz. This geometric definition was introduced by Rockafellar in [21]. The epi-Lipschitz property of S at x is also characterizable in terms of the nonemptiness of the topological interior of the Clarke tangent cone of S at x which is also equivalent to the *pointedness* of the Clarke normal cone of S at x, see [5, Theorem 7.3.1].

In their recent paper [11], where a general Hilbert space is considered, Nacry and Thibault proved in [11, Theorem 3] that if S is r-prox-regular, then the complement of S, denoted by S^c, is the union of closed balls with *common* radius. What is remarkable in this result is that the set S can be any closed set, including the sets that are not *regular closed* (that is, $S \neq \operatorname{cl}(\operatorname{int} S)$ the closure of the interior of S).

In this paper, we generalize, in finite-dimensional setting, [11, Theorem 3] by replacing the r-prox-regularity of S by *weaker* conditions, including the exterior r-sphere condition if S is regular closed, and a new *extended* version of the exterior r-sphere condition if S is not necessarily regular closed.

When S is regular closed, the exterior r-sphere condition of S coincides with the *interior r-sphere condition* of $(\operatorname{int} S)^c$, a well known condition in control theory, see e.g., [2, 3, 4, 12]. [2] Using this fact and [16, Conjecture 1.2],[3] we obtain the following theorem which is the first main result of this paper.

Theorem 1.1. *Let $S \subset \mathbb{R}^n$ be a closed set such that $\operatorname{cl}(\operatorname{int} S)$ satisfies the exterior r-sphere condition for some $r > 0$. Then $(\operatorname{int} S)^c$ is the union of closed $\frac{r}{2}$-balls. If in addition S is regular closed, then S^c is the union of closed r'-balls for any $r' < \frac{r}{2}$.*

Note that the "In addition" part of Theorem 1.1 is not necessarily true when S is not regular closed, see Example 2.4. On the other hand, the assumption "$\operatorname{cl}(\operatorname{int} S)$ satisfies the exterior r-sphere condition" is *weaker* than the assumption "S satisfies the exterior r-sphere condition", see Proposition 2.1 and Example 2.2.

[1] This equivalence result is generalized to the variable radius case in [15], see also [17].

[2] In these references, the interior sphere condition is used to study the regularity of the unilateral and the bilateral minimal time functions associated to a control system.

[3] This conjecture is introduced in [13, 14], and proved in [16]. A generalization of this result to the variable radius case is given in [16, Theorem 3.1], see also [18].

In the following theorem, which is our second main result, we prove that a similar result to [11, Theorem 3] can be obtained if the prox-regularity is replaced by a weaker property, namely the following *extended* exterior r-sphere condition. We say that S satisfies the extended exterior r-sphere condition if:

- For any $x \in \operatorname{bdry}(\operatorname{int} S) \subset \operatorname{bdry} S$, there exists $0 \neq \zeta \in N_S^P(x)$ such that ζ is realized by an r-sphere.
- For any $x \in (\operatorname{bdry} S) \setminus \operatorname{bdry}(\operatorname{int} S)$ and for any $0 \neq \zeta \in N_S^P(x)$, ζ is realized by an r-sphere.

Theorem 1.2. *Let $S \subset \mathbb{R}^n$ be a closed set satisfying the extended exterior r-sphere condition for some $r > 0$. Then S^c is the union of closed $\frac{r}{2}$-balls.*

One can easily see that if S satisfies the extended exterior r-sphere condition then S satisfies the exterior r-sphere condition. The converse, which does not hold in general (see Example 2.2), is valid if S is regular closed. Note that the extended exterior r-sphere condition of Theorem 1.2 cannot be replaced by the exterior r-sphere condition as it will be shown in Example 2.5. On the other hand, clearly if S is r-prox-regular then S satisfies the extended exterior r-sphere condition. The converse does not hold in general, even if S is regular closed, as it is proved in the counterexamples of [13, Section 2]. In the following theorem which is our third and last main result, we provide a sufficient condition for the equivalence between prox-regularity and the extended exterior sphere condition. It generalizes [13, Corollary 3.12] to the case in which S is not necessarily regular closed.

Theorem 1.3. *Let $S \subset \mathbb{R}^n$ be a closed set such that $\operatorname{cl}(\operatorname{int} S)$ is epi-Lipschitz and has compact boundary. Then S is r-prox-regular for some $r > 0$ if and only if S satisfies the extended exterior r'-sphere condition for some $r' > 0$.*

The details of the proofs of Theorems 1.1, 1.2 and 1.3 will be presented in the next section after giving a brief description of the notation used in this paper, and providing some auxiliary results.

2. Proofs of the main results

We denote by $\|\cdot\|$, $\langle \cdot, \cdot \rangle$, B and $\bar{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\bar{B}(x;\rho) := x + \rho \bar{B}$. For a set $S \subset \mathbb{R}^n$, S^c, $\operatorname{int} S$, $\operatorname{bdry} S$ and $\operatorname{cl} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of S, respectively. The closed segment joining two points x and y in $\mathbb{R}^n$ is denoted by $[x, y]$. The distance from a point x to a set S is denoted by $d_S(x)$.

We also denote by $\operatorname{proj}_S(x)$ the set of closest points in S to x, that is, the set of points $s \in S$ which satisfy $d_A(x) = \|s - x\|$. For A and B two subsets of $\mathbb{R}^n$, $d(A, B)$ denotes the distance between A and B, that is,

$$d(A, B) := \inf\{\|a - b\| : (a, b) \in A \times B\}.$$

For $S \subset \mathbb{R}^n$ closed, the following proposition provides some useful properties of the set $\operatorname{cl}(\operatorname{int} S)$.

Proposition 2.1. *Let $S \subset \mathbb{R}^n$ be closed. Then we have the following*:

(i) $\operatorname{cl}(\operatorname{int} S) \subset S$, $\operatorname{int}(\operatorname{cl}(\operatorname{int} S)) = \operatorname{int} S$, *and*
$$\operatorname{bdry}(\operatorname{cl}(\operatorname{int} S)) = \operatorname{bdry}(\operatorname{int} S) \subset \operatorname{bdry} S.$$

(ii) $S = [\operatorname{cl}(\operatorname{int} S)] \cup [(\operatorname{bdry} S) \setminus \operatorname{bdry}(\operatorname{int} S)]$ *with*
$$[\operatorname{cl}(\operatorname{int} S)] \cap [(\operatorname{bdry} S) \setminus \operatorname{bdry}(\operatorname{int} S)] = \emptyset.$$

(iii) *If $S = \operatorname{cl} O$, where O is open, then S is regular closed.*

(iv) *If for $r > 0$, S satisfies the exterior r-sphere condition, then* $\operatorname{cl}(\operatorname{int} S)$ *also satisfies the exterior r-sphere condition.*

Proof. The properties (i)–(iii) are well known in metric topology. For (iv), let $r > 0$ and assume that S satisfies the exterior r-sphere condition. Let $x \in \operatorname{bdry}(\operatorname{cl}(\operatorname{int} S))$. By (i) we have that $x \in \operatorname{bdry} S$, and hence there exists $0 \neq \zeta \in N_S^P(x)$ such that ζ is realized by an r-sphere. This gives using the proximal normal inequality that
$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2, \quad \forall s \in S.$$
Since $\operatorname{cl}(\operatorname{int} S) \subset S$, we obtain that
$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2, \quad \forall s \in \operatorname{cl}(\operatorname{int} S).$$
Therefore, $\zeta \in N^P_{\operatorname{cl}(\operatorname{int} S)}(x)$ and is realized by an r-sphere. □

Example 2.2. The converse of Proposition 2.1(iv) is not necessarily true. Indeed, consider in $\mathbb{R}^2$, $S := \bar{B}(0;1) \cup [(1,0),(2,0)]$. Clearly S does not satisfy the exterior r-sphere condition for any $r > 0$, but $\operatorname{cl}(\operatorname{int} S) = \bar{B}(0;1)$ satisfies the exterior r-sphere condition for any $r > 0$.

In the following proposition we list some well known characterizations and consequences of prox-regular and epi-Lipschitz properties. For the proofs, see [5, 20, 22].

Proposition 2.3. *Let $S \subset \mathbb{R}^n$ be closed. Then we have the following*:

(i) *If S is prox-regular, then for each $x \in \operatorname{bdry} S$ we have*
$$N_S^P(x) = N_S^L(x) = N_S(x),$$
where $N_S^L(x)$ and $N_S(x)$ denotes the Mordukhovich (*or* limiting) *and the* Clarke *normal cones to S at x, respectively.*

(ii) *For $x \in \operatorname{bdry} S$, S is epi-Lipschitz at x if and only if $N_S(x)$ is pointed, that is, $N_S(x) \cap -N_S(x) = \{0\}$.*

(iii) *If for $x \in \operatorname{bdry} S$, S is epi-Lipschitz at x, then $x \in \operatorname{cl}(\operatorname{int} S)$.*

2.1. Proof of Theorem 1.1

Let $S \subset \mathbb{R}^n$ be a closed set such that $\operatorname{cl}(\operatorname{int} S)$ satisfies the exterior r-sphere condition for some $r > 0$. Since by Proposition 2.1(*iii*) we have that $\operatorname{cl}(\operatorname{int} S)$ is regular closed, we get that $(\operatorname{int}(\operatorname{cl}(\operatorname{int} S))^c = (\operatorname{int} S)^c$ satisfies the interior r-sphere condition. Hence by [16, Conjecture 1.2] we obtain that $(\operatorname{int} S)^c$ is the union of closed $\frac{r}{2}$-balls.

We proceed to prove the "In addition" part of Theorem 1.1. We assume that S is regular closed, and we consider $r' < \frac{r}{2}$. Let $x \in S^c$. Since $S^c \subset (\operatorname{int} S)^c$ and $(\operatorname{int} S)^c$ is the union of closed $\frac{r}{2}$-balls, there exists $y_x \in (\operatorname{int} S)^c$ such that

$$x \in \bar{B}\left(y_x; \frac{r}{2}\right) \subset (\operatorname{int} S)^c. \tag{2}$$

If $\|y_x - x\| < r'$, then using (2) we get that

$$x \in \bar{B}(y_x; r') \subset B\left(y_x; \frac{r}{2}\right) \subset \bar{B}\left(y_x; \frac{r}{2}\right) \subset (\operatorname{int} S)^c.$$

This gives that $x \in \bar{B}(y_x; r') \subset \operatorname{int}((\operatorname{int} S)^c) = (\operatorname{cl}(\operatorname{int} S))^c = S^c$.

Now we assume that $\|y_x - x\| \geq r'$. Then by (2) and since $x \in S^c$, we have that

$$\begin{aligned} x \in \bar{B}\left(x + r' \frac{y_x - x}{\|y_x - x\|}; r'\right) &\subset B\left(y_x; \|y_x - x\|\right) \cup \{x\} \\ &\subset B\left(y_x; \frac{r}{2}\right) \cup \{x\} \subset \operatorname{int}((\operatorname{int} S)^c) \cup S^c = S^c. \end{aligned}$$

Therefore, S^c is the union of closed r'-balls. This terminated the proof of the "In addition" part, and hence the proof of Theorem 1.1. $\square$

In the following example, we prove that the "In addition" part of Theorem 1.1 is not necessarily true when S is not regular closed.

Example 2.4. Let $S := \{(x, e^x) : x \in \mathbb{R}\} \cup \{(x, -e^x) : x \in \mathbb{R}\}$.
Clearly S is not regular closed, and satisfies the exterior $\frac{3\sqrt{3}}{2}$-sphere condition. On the other hand, S^c is not the union of closed r-balls for *any* $r > 0$. Note that S does not satisfy the extended exterior r-sphere condition for any $r > 0$. $\square$

2.2. Proof of Theorem 1.2

Let $S \subset \mathbb{R}^n$ be a closed set satisfying the extended exterior r-sphere condition for some $r > 0$. We consider $x \in S^c$. Let $s_0 \in \operatorname{proj}_S(x)$ and let $r_0 := \|x - s_0\|$. Clearly we have $s_0 \in \operatorname{bdry} S$ and $s_0 \neq x$, and hence $r_0 \neq 0$. Moreover, we have

$$B(x; r_0) \cap S = \emptyset. \tag{3}$$

There are three cases to consider.

Case 1: $r_0 > \frac{r}{2}$.

Then using (3), we obtain that $x \in \bar{B}\left(x; \frac{r}{2}\right) \subset B(x; r_0) \subset S^c$.

Case 2: $r_0 \leq \frac{r}{2}$ and $s_0 \notin \operatorname{bdry}(\operatorname{int} S)$.

Then the proximal normal vector $\zeta_0 := x - s_0 \in N_S^P(s_0)$ is realized by an r-sphere.

This gives that
$$B\left(s_0 + r\frac{\zeta_0}{\|\zeta_0\|}; r\right) \subset S^c. \tag{4}$$

We claim that
$$\bar{B}\left(x + \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}; \frac{r}{2}\right) \subset B\left(s_0 + r\frac{\zeta_0}{\|\zeta_0\|}; r\right). \tag{5}$$

Indeed, for $y \in \bar{B}\left(x + \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}; \frac{r}{2}\right)$ we have

$$\begin{aligned}\left\|y - s_0 - r\frac{\zeta_0}{\|\zeta_0\|}\right\| &= \left\|\left(y - x - \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}\right) + \left(x - s_0 - \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}\right)\right\| \\ &\leq \left\|y - x - \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}\right\| + \left\|x - s_0 - \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}\right\| \\ &\leq \frac{r}{2} + \left\|r_0\frac{\zeta_0}{\|\zeta_0\|} - \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}\right\| = \frac{r}{2} + \frac{r}{2} - r_0 = r - r_0 < r.\end{aligned}$$

Now combining (4) and (5), we obtain that

$$x \in \bar{B}\left(x + \frac{r}{2}\frac{\zeta_0}{\|\zeta_0\|}; \frac{r}{2}\right) \subset B\left(s_0 + r\frac{\zeta_0}{\|\zeta_0\|}; r\right) \subset S^c.$$

Case 3: $r_0 \leq \frac{r}{2}$ and $s_0 \in \operatorname{bdry}(\operatorname{int} S)$.
Then for every $\varepsilon > 0$, we have $B(s_0;\varepsilon) \cap \operatorname{int} S \neq \emptyset$. Let $z_\varepsilon \in B(s_0;\varepsilon) \cap \operatorname{int} S$. We denote by $\xi_\varepsilon := \frac{z_\varepsilon - x}{\|z_\varepsilon - x\|}$ and we consider $s_\varepsilon \in [z_\varepsilon, x] \cap \operatorname{bdry} S$, where this latter intersection is nonempty since $z_\varepsilon \in \operatorname{int} S$ and $x \in S^c$. Let $\zeta_\varepsilon \in N_S^P(s_\varepsilon)$ be the unit vector realized by an r-sphere. Hence, for $y_\varepsilon := s_\varepsilon + r\zeta_\varepsilon$, we have

$$B(y_\varepsilon; r) \cap S = \emptyset. \tag{6}$$

If $y_\varepsilon = x$, then $x \in \bar{B}\left(x; \frac{r}{2}\right) \subset B(y_\varepsilon; r) \subset S^c$.

If $0 < \|y_\varepsilon - x\| < r$, then for $w \in \bar{B}\left(x + \frac{r}{2}\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}; \frac{r}{2}\right)$, we have

$$\begin{aligned}\|w - y_\varepsilon\| &\leq \left\|w - x - \frac{r}{2}\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}\right\| + \left\|x + \frac{r}{2}\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|} - y_\varepsilon\right\| \\ &\leq \frac{r}{2} + \left\|\frac{r}{2}\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|} - (y_\varepsilon - x)\right\| = \frac{r}{2} + \left|\frac{r}{2} - \|y_\varepsilon - x\|\right| < r.\end{aligned}$$

This gives using (6) that $x \in \bar{B}\left(x + \frac{r}{2}\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}; \frac{r}{2}\right) \subset B(y_\varepsilon; r) \subset S^c$.

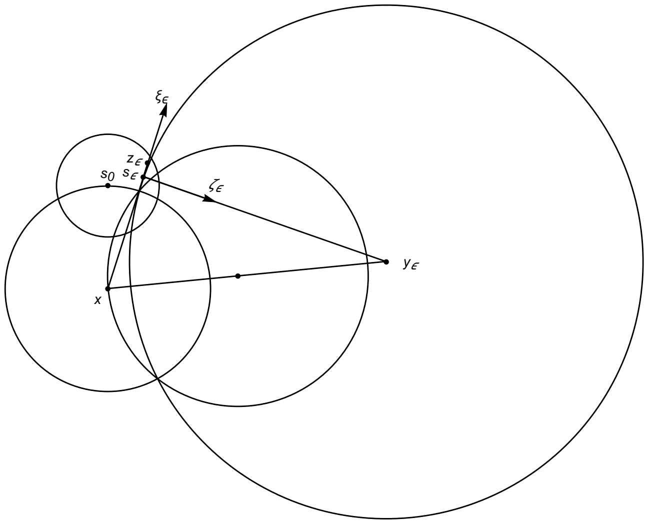
Now we assume that $\|y_\varepsilon - x\| \geq r$ and we denote by

$$r_\varepsilon := \frac{r_0^2\|y_\varepsilon - x\|}{\|y_\varepsilon - x\|^2 + r_0^2 - r^2} > 0. \tag{7}$$

Claim 1. $B\left(x + r_\varepsilon\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}; r_\varepsilon\right) \subset B(x; r_0) \cup B(y_\varepsilon; r)$.

To prove this claim, let $w \in B\left(x + r_\varepsilon\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}; r_\varepsilon\right)$ and assume that $w \notin B(x; r_0)$.

Then
$$\|w - x\| < 2r_\varepsilon\left\langle\frac{w - x}{\|w - x\|}, \frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}\right\rangle. \tag{8}$$



Figur 1: Case 3: $\|y_\varepsilon - x\| \geq r$

We have
$$\begin{aligned}
\|w-y_\varepsilon\|^2 &= \|(w-x)-(y_\varepsilon - x)\|^2 \\
&= \|w-x\|^2 + \|y_\varepsilon - x\|^2 - 2\langle w-x, y_\varepsilon - x\rangle \\
&\overset{(8)}{<} \|w-x\|^2 + \|y_\varepsilon - x\|^2 - \frac{\|y_\varepsilon - x\|}{r_\varepsilon}\|w-x\|^2 \\
&= \|y_\varepsilon - x\|^2 + \|w-x\|^2\left(1-\frac{\|y_\varepsilon - x\|}{r_\varepsilon}\right) \\
&\overset{(7)}{=} \|y_\varepsilon - x\|^2 + \|w-x\|^2\left(1-\frac{\|y_\varepsilon - x\|^2 + r_0^2 - r^2}{r_0^2}\right) \\
&= \|y_\varepsilon - x\|^2 + \underbrace{\|w-x\|^2}_{\geq r_0^2}\underbrace{\left(\frac{r^2 - \|y_\varepsilon - x\|^2}{r_0^2}\right)}_{\leq 0} \\
&\leq \|y_\varepsilon - x\|^2 + r^2 - \|y_\varepsilon - x\|^2 = r^2.
\end{aligned}$$

This gives that $w \in B(y_\varepsilon; r)$ which terminates the proof of the claim.

Claim 2. $\langle \xi_\varepsilon, \zeta_\varepsilon\rangle < \frac{\varepsilon}{2r}$.

To prove this claim, first we remark that since $z_\varepsilon \in \operatorname{int} S$, we deduce from (6) that $z_\varepsilon \notin \bar{B}(y_\varepsilon; r) = \bar{B}\left(s_\varepsilon + r\zeta_\varepsilon; r\right)$. Hence, using the equality $z_\varepsilon - s_\varepsilon = \|z_\varepsilon - s_\varepsilon\|\xi_\varepsilon$, we obtain that $\|z_\varepsilon - s_\varepsilon\| > 2r\,\langle \xi_\varepsilon, \zeta_\varepsilon\rangle$.

Therefore, using that $s_0 \in \operatorname{proj}_S(x)$, we get that

$$\begin{aligned}
\langle \xi_\varepsilon, \zeta_\varepsilon\rangle < \frac{1}{2r}\|z_\varepsilon - s_\varepsilon\| &= \frac{1}{2r}\left(\|z_\varepsilon - x\| - \|s_\varepsilon - x\|\right) \\
&\leq \frac{1}{2r}\left(\|z_\varepsilon - s_0\| + \underbrace{\|s_0 - x\| - \|s_\varepsilon - x\|}_{\leq 0}\right) \\
&\leq \frac{1}{2r}\|z_\varepsilon - s_0\| < \frac{\varepsilon}{2r}.
\end{aligned}$$

The proof of the claim is complete.

Claim 3. If $\varepsilon \in \left(0, \frac{r_0^3}{4r^2}\right]$ then $r_\varepsilon > \frac{r}{2}$.

To prove the claim, first we remark using (7) that $r_\varepsilon > \frac{r}{2}$ is equivalent to

$$P(\|y_\varepsilon - x\|) := r\|y_\varepsilon - x\|^2 - 2r_0^2\|y_\varepsilon - x\| + r(r_0^2 - r^2) < 0. \tag{9}$$

Since the discriminant Δ' of $P(\|y_\varepsilon - x\|)$ is $(r^2 - r_0^2)^2 + r^2 r_0^2 > 0$ and the product of the roots is $r_0^2 - r^2 < 0$, we conclude that (9) is equivalent to

$$\|y_\varepsilon - x\| < \frac{r_0^2 + \sqrt{\Delta'}}{r}. \tag{10}$$

We have

$$\begin{aligned}
\|y_\varepsilon - x\|^2 &= \|y_\varepsilon - s_\varepsilon\|^2 + \|s_\varepsilon - x\|^2 + 2\langle y_\varepsilon - s_\varepsilon, s_\varepsilon - x\rangle \\
&= r^2 + \|s_\varepsilon - x\|^2 + 2r\|s_\varepsilon - x\| \, \langle \zeta_\varepsilon, \xi_\varepsilon\rangle \\
&\overset{\text{Claim 2}}{<} r^2 + \|s_\varepsilon - x\|^2 + \varepsilon\|s_\varepsilon - x\| \\
&\overset{s_\varepsilon \in [z_\varepsilon, x]}{\leq} r^2 + \|z_\varepsilon - x\|^2 + \varepsilon\|z_\varepsilon - x\| \\
&\leq r^2 + (\|z_\varepsilon - s_0\| + \|s_0 - x\|)^2 + \varepsilon(\|z_\varepsilon - s_0\| + \|s_0 - x\|) \\
&\leq r^2 + (\varepsilon + r_0)^2 + \varepsilon(\varepsilon + r_0) \;=\; r^2 + r_0^2 + 2\varepsilon^2 + 3\varepsilon r_0 \\
&\overset{\varepsilon \in \left(0, \frac{r_0^3}{4r^2}\right]}{\leq} r^2 + r_0^2 + \frac{r_0^6}{8r^4} + \frac{3r_0^4}{4r^2} = r^2 + r_0^2 + \frac{r_0^4}{r^2}\left(\frac{r_0^2}{8r^2} + \frac{3}{4}\right) \\
&\overset{r_0 \leq \frac{r}{2}}{<} r^2 + r_0^2 + \frac{r_0^4}{r^2}.
\end{aligned} \tag{11}$$

On the other hand, we have

$$\begin{aligned}
(r_0^2 + \sqrt{\Delta'})^2 &= 2r_0^4 + r^4 - r^2 r_0^2 + 2r_0^2\sqrt{\left(r^2 - \frac{1}{2}r_0^2\right)^2 + \frac{3r_0^4}{4}} \\
&> 2r_0^4 + r^4 - r^2 r_0^2 + 2r_0^2\left(r^2 - \frac{1}{2}r_0^2\right) \\
&= r_0^4 + r^4 + r^2 r_0^2.
\end{aligned}$$

This gives that $\left(\frac{r_0^2 + \sqrt{\Delta'}}{r}\right)^2 > r^2 + r_0^2 + \frac{r_0^4}{r^2}$.

Combining this latter with (11), we obtain inequality (10), and this terminates the proof of Claim 3.

Now we fix $\varepsilon \in \left(0, \frac{r_0^3}{4r^2}\right]$. By Claim 3 we have that $\frac{r}{2} < r_\varepsilon$. Hence, using Claim 1, (3) and (6), we get that

$$x \in \bar{B}\left(x + \frac{r}{2}\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}; \frac{r}{2}\right) \subset B\left(x + r_\varepsilon\frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}; r_\varepsilon\right) \cup \{x\} \subset B(x; r_0) \cup B(y_\varepsilon; r) \subset S^c.$$

In the three cases above, we have shown that x belongs to a closed $\frac{r}{2}$-ball included in S^c. The proof of Theorem 1.2 is terminated. $\square$

In the following example, we prove that the extended exterior r-sphere condition assumption of Theorem 1.2 cannot be replaced by the exterior r-sphere condition.

Example 2.5. In Figure 2, S is the set that consists of both curves and the dashed region between them. The set S satisfies the exterior r-sphere condition, but it does not satisfy the extended exterior r-sphere condition since some of the boundary points on the right-hand side have two normal vectors where only one of them is realized by an r-sphere. Clearly, S^c is not the union of r'-balls for any $r' > 0$. □

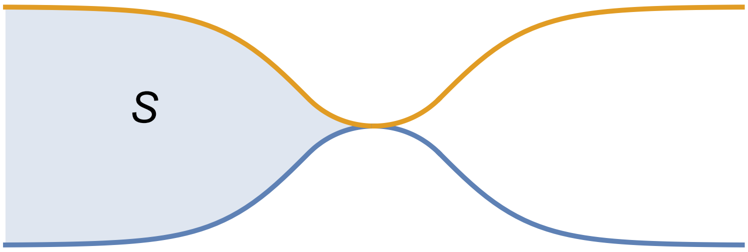


Figure 2: Example 2.5

Remark 2.6. From Theorem 1.2, we deduce that if S is regular closed and satisfies the exterior r-sphere condition for some $r > 0$, then S^c is the union of closed $\frac{r}{2}$-balls, which is stronger than the "In addition" part of Theorem 1.1. Of course we can adjust the proof of the "In addition" part of Theorem 1.1 and obtain the radius $\frac{r}{2}$ instead of $r' < \frac{r}{2}$. We did not do this to not complicate the proof of Theorem 1.1, and to show how its proof, when $r' < \frac{r}{2}$, is a *direct* consequence of the union of closed balls conjecture [16, Conjecture 1.2]. Note that since $\frac{r}{2} < \frac{nr}{2\sqrt{n^2-1}}$ for all $n \geq 2$, the same techniques used in the proof of Theorem 1.1 combined with the strong version of the union of closed balls conjecture [16, Conjecture 1.3], gives that S^c is the union of closed r'-balls for any $r' < \frac{nr}{2\sqrt{n^2-1}}$, including $r' = \frac{r}{2}$. Unfortunately, this latter result cannot be confirmed since the strong version of the union of closed balls conjecture remains till now an open question.

Remark 2.7. The radius $\frac{r}{2}$ of Theorem 1.2 is the *largest* radius that works in *all* the spaces $\mathbb{R}^n$ for *any* $n \geq 2$. Indeed, assume that the radius $\frac{r}{2}$ of Theorem 1.2 is replaced by an $r' > 0$. We claim that $r' \leq \frac{r}{2}$. Indeed, inspired by [17, Example 2], we consider in $\mathbb{R}^n$ the set S to be the complement of the $(n+1)$ open r-balls centered at C_i, $i = 0, \ldots, n$, where for $(e_i)_{i=1}^n$ the standard basis of $\mathbb{R}^n$ and $e_0 := 0$, we have

$$\begin{aligned} C_i &:= \frac{-nr}{\sqrt{n(n-1)}} \left(\sqrt{\frac{i}{i+1}}\, e_i + \sum_{k=i+1}^{n} \frac{-1}{\sqrt{k(k+1)}} e_k \right) \\ &= \frac{-nr}{\sqrt{n(n-1)}} \left(0, \ldots, 0, \frac{\sqrt{i}}{\sqrt{i+1}}, \frac{-1}{\sqrt{(i+1)(i+2)}}, \frac{-1}{\sqrt{(i+2)(i+3)}}, \ldots, \frac{-1}{\sqrt{n(n+1)}} \right). \end{aligned}$$

The set S satisfies the extended exterior r-sphere condition (in fact, it is regular closed and satisfies the exterior r-sphere condition), but the largest closed balls contained in S^c and containing the origin $0 \in S^c$ is of radius $\frac{nr}{2\sqrt{n^2-1}}$, see Figure 3 for

$n = 2$ and $n = 3$. Hence, $r' \leq \frac{nr}{2\sqrt{n^2-1}}$ for all $n \geq 2$. Taking $n \longrightarrow \infty$ in this latter, we conclude that $r' \leq \frac{r}{2}$.

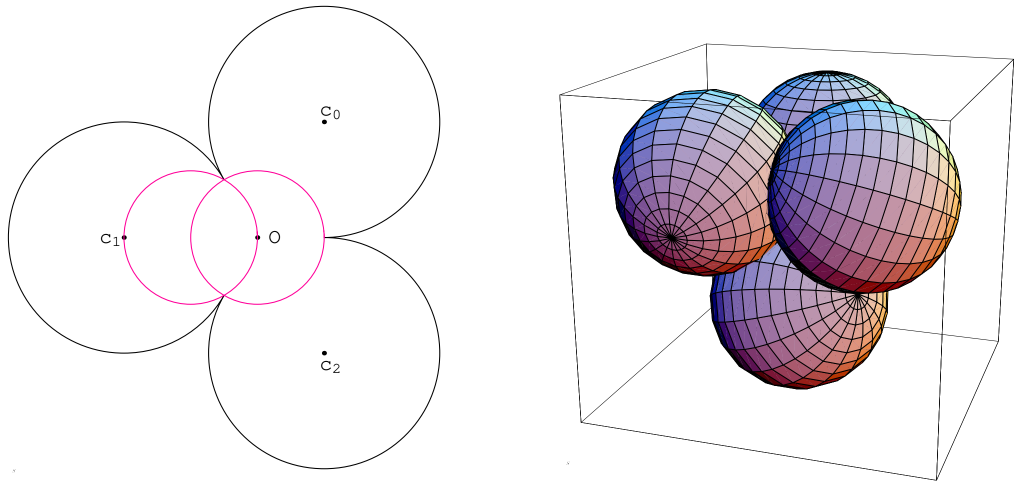


Figure 3: Example of Remark 2.7

2.3. Proof of Theorem 1.3

Let $S \subset \mathbb{R}^n$ be a closed set such that $\mathrm{cl}\,(\mathrm{int}\, S)$ is epi-Lipschitz and has compact boundary. Since the first implication is straightforward (take $r' = r$), we focus on the second implication. So, we assume that S satisfies the extended exterior r'-sphere condition for some $r' > 0$. Then S satisfies the exterior r'-sphere condition, and hence by Proposition 2.3(iv), $\mathrm{cl}\,(\mathrm{int}\, S)$ satisfies the exterior r'-sphere condition. Now using [13, Corollary 3.12], applied to $\mathrm{cl}\,(\mathrm{int}\, S)$, we get that $\mathrm{cl}\,(\mathrm{int}\, S)$ is ρ-prox-regular for some $\rho > 0$.

Claim 1. $d(\mathrm{bdry}\,(\mathrm{int}\, S), (\mathrm{bdry}\, S) \setminus \mathrm{bdry}\,(\mathrm{int}\, S)) = r_S > 0$.

If not, then there exist two sequences x_n and y_n such that

$$x_n \in \mathrm{bdry}\,(\mathrm{int}\, S),\;\; y_n \in (\mathrm{bdry}\, S) \setminus \mathrm{bdry}\,(\mathrm{int}\, S) \;\text{ and }\; \|x_n - y_n\| \leq \frac{1}{n},\;\; \forall n \geq 1.$$

Since $\mathrm{cl}\,(\mathrm{int}\, S)$ has compact boundary, we get the existence of $\bar{x} \in \mathrm{bdry}\,(\mathrm{int}\, S)$ and a subsequence of y_n, we do not relabel, such that $y_n \longrightarrow \bar{x}$. Since $y_n \notin \mathrm{cl}\,(\mathrm{int}\, S)$, we have from Proposition 2.3(iii) that S is not epi-Lipschitz at y_n. This gives using Proposition 2.3(ii) the existence of unit vector ζ_n such that

$$\zeta_n \in N_S(y_n) \cap -N_S(y_n). \tag{12}$$

Now since $y_n \in (\mathrm{bdry}\, S) \setminus \mathrm{bdry}\,(\mathrm{int}\, S)$ and since S satisfies the extended exterior r'-sphere condition, we have that S is r'-prox-regular near y_n, that is, in a $(\mathrm{bdry}\, S)$-neighborhood of y_n. Hence, Proposition 2.3(i) and (12) yield that

$$\zeta_n \in N_S^P(y_n) \cap -N_S^P(y_n).$$

Using the proximal normal inequality and the fact that ζ_n is unit and realized by an r'-sphere, we obtain that $(\zeta_n)_n$ has a subsequence, we do not relabel, that converges to a unit vector ζ satisfying

$$\zeta \in N_S^P(\bar{x}) \cap -N_S^P(\bar{x}).$$

Since $N_S^P(\bar{x}) \subset N^P_{\mathrm{cl}\,(\mathrm{int}\,S)}(\bar{x})$, we deduce that

$$\zeta \in N^P_{\mathrm{cl}\,(\mathrm{int}\,S)}(\bar{x}) \cap -N^P_{\mathrm{cl}\,(\mathrm{int}\,S)}(\bar{x}).$$

This yields using the prox-regularity of $\mathrm{cl}\,(\mathrm{int}\,S)$ and Proposition 2.3(i), that

$$\zeta \in N_{\mathrm{cl}\,(\mathrm{int}\,S)}(\bar{x}) \cap -N_{\mathrm{cl}\,(\mathrm{int}\,S)}(\bar{x}),$$

which contradicts the fact that $\mathrm{cl}\,(\mathrm{int}\,S)$ is epi-Lipschitz. The proof of the claim is terminated.

Now we prove that S is r-prox-regular for $r := \min\{\rho, r', \frac{r_S}{4}\}$. Take some $x \in \mathrm{bdry}\,S$. If $x \in (\mathrm{bdry}\,S) \setminus \mathrm{bdry}\,(\mathrm{int}\,S)$, then from the definition of the extended r'-sphere condition, we know that any $0 \neq \zeta \in N_S^P(x)$ is realized by an r'-sphere. This gives that any $0 \neq \zeta \in N_S^P(x)$ is realized by an r-sphere. Now we assume that $x \in \mathrm{bdry}\,(\mathrm{int}\,S)$. We know that $\mathrm{cl}\,(\mathrm{int}\,S)$ is ρ-prox-regular. This implies that for any $0 \neq \zeta \in N_S^P(x) \subset N^P_{\mathrm{cl}\,(\mathrm{int}\,S)}(x)$, we have

$$B\left(x + \rho\frac{\zeta}{\|\zeta\|}; \rho\right) \cap \mathrm{cl}\,(\mathrm{int}\,S) = \emptyset.$$

This gives that for any $0 \neq \zeta \in N_S^P(x)$, we have

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap \mathrm{cl}\,(\mathrm{int}\,S) = \emptyset. \tag{13}$$

On the other hand, since $r < \frac{r_S}{4}$ and using Claim 1, we have for any $0 \neq \zeta \in N_S^P(x)$ that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \subset B\left(x; \frac{r_S}{2}\right) \subset [(\mathrm{bdry}\,S) \setminus \mathrm{bdry}\,(\mathrm{int}\,S)]^c.$$

Combining this latter result with Proposition 2.1(ii) and equation (13), we get for any $0 \neq \zeta \in N_S^P(x)$ that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S^c = \emptyset.$$

This terminates the proof of Theorem 1.3. □

References

[1] A. Canino, *On p-convex sets and geodesics*, J. Diff. Equations 75/1 (1988) 118–157.

[2] P. Cannarsa, H. Frankowska: *Interior sphere property of attainable sets and time optimal control problems*, ESAIM: Control Optim. Calc. Var. 12 (2006) 350–370.

[3] P. Cannarsa, C. Sinestrari: *Convexity properties of the minimum time function*, Calc. Var. 3 (1995) 273–298.

[4] P. Cannarsa, C. Sinestrari: *Semiconcave Functions, Hamiton-Jacobi Equations and Optimal Control*, Birkhäuser, Boston (2004).

[5] F. H. Clarke: *Optimization and Nonsmooth Analysis*, John Wiley, New York (1983).

[6] F. H. Clarke, Yu. Ledyaev, R. J. Stern, P. R. Wolenski: *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics 178, Springer, New York (1998).

[7] F. H. Clarke, R. J. Stern, P. R. Wolenski: *Proximal smoothness and the lower-C^2 property*, J. Convex Analysis 2/1-2 (1995) 117–144.

[8] G. Colombo, A. Marigonda: *Differentiability properties for a class of non-convex functions*, Calc. Var. 25 (2005) 1–31.

[9] H. Federer: *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959) 418–491.

[10] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Springer, Berlin (2006).

[11] F. Nacry, L. Thibault: *Distance function associated to a prox-regular set*, Set-Valued Var. Analysis 30 (2022) 731–750.

[12] C. Nour, R. J. Stern: *Semiconcavity of the bilateral minimal time function: the linear case*, Systems & Control Letters 57 (2008) 863–866.

[13] C. Nour, R. J. Stern, J. Takche: *Proximal smoothness and the exterior sphere condition*, J. Convex Analysis 16/2 (2009) 501–514.

[14] C. Nour, R. J. Stern, J. Takche: *The union of uniform closed balls conjecture*, Control and Cybernetics 38/4B (2009) 1525–1534.

[15] C. Nour, R. J. Stern, J. Takche: *The θ-exterior sphere condition, φ-convexity and local semiconcavity*, Nonlinear Analysis 73/2 (2010) 573–589.

[16] C. Nour, R. J. Stern, J. Takche: *Validity of the union of uniform closed balls conjecture*, J. Convex Analysis 18/2 (2011) 589–600.

[17] C. Nour, R. J. Stern, J. Takche: *Generalized exterior sphere conditions and φ-convexity*, Discrete Contin. Dyn. Systems 29/2 (2011) 615–622.

[18] C. Nour, J. Takche: *On the union of closed balls property*, J. Optim. Theory Appl. 155 (2012) 376–389.

[19] J.-P. Penot: *Calculus Without Derivatives*, Graduate Texts in Mathematics 266, Springer, New York (2013).

[20] R. A. Poliquin, R. T. Rockafellar, L. Thibault: *Local differentiability of distance functions*, Trans. Amer. Math. Soc. 352 (2000) 5231–5249.

[21] R. T. Rockafellar: *Clarke's tangent cones and the boundaries of closed sets in R^n*, Nonlinear Analysis Theory Math. Appl. 3 (1979) 145–154.

[22] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften 317, Springer, Berlin (1998).

[23] A. S. Shapiro: *Existence and differentiability of metric projections in Hilbert spaces*, SIAM J. Optim. 4 (1994) 231–259.