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The Centroid Banach-Mazur Distance between the Parallelogram and the Triangle

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Let C and D be convex bodies in the Euclidean space E^d. We define the centroid Banach-Mazur distance $\delta_{BM}^{\mathrm{cen}}(C,D)$ similarly to the classic Banach-Mazur distance $\delta_{BM}(C,D)$, but with the extra requirement that the centroids of C and an affine image of D coincide. We prove that for the parallelogram P and the triangle T in E^2 we have $\delta_{BM}^{\mathrm{cen}}(P,T)=\frac{5}{2}$.

Keywords: Banach-Mazur distance, centroid Banach-Mazur distance, convex body, centroid, parallelogram, triangle.

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1. Introduction

The classical definition of the Banach-Mazur distance of centrally symmetric convex bodies of the Euclidean d-space E^d is given by Banach [2] in behalf of him and Mazur over nine decades ago. For over four decades this definition is considered also for arbitrary convex bodies of E^d. Namely, for convex bodies C, D of E^d this extended Banach-Mazur distance reads as follows

$$\delta_{\mathrm{BM}}(C,D) = \inf_{a,\,h_\lambda}\big\{\lambda;\quad a(D)\subset C\subset h_\lambda a(D)\big).$$

where a stands for an affine transformation and h_λ denotes a homothety with a positive ratio λ. For the relationship between them see Claim of [9].

A survey on the Banach-Mazur distance is given in the book [11] by Tomczak-Jaegerman. Moreover, in Sections 3.2 and 3.3 of the book [12] by Toth, and in Section 4.1 of the book [1] by Aubrun and Szarek.

Here is the notion of *centroid Banach-Mazur distance* of convex bodies C, D of E^d:

$$\delta_{\mathrm{BM}}^{\mathrm{cen}}(C,D) = \inf_{a,\,h_\lambda}\{\lambda;\ a(D)\subset C\subset h_\lambda a(D) \text{ and } \mathrm{cen}(a(D)) = \mathrm{cen}(C)\},$$

where a again stands for an affine transformation, but h_λ means a homothety with the ratio $\lambda \geq 1$ whose center is at the centroids $\mathrm{cen}(a(D)) = \mathrm{cen}(C)$ of $a(D)$ and C. Observe that the centroids of C and D in this notion take over the roles of the centers of the centrally-symmetric bodies in the original definition of Banach-Mazur

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distance from [2]. We easily show that $\delta^{\text{cen}}_{\text{BM}}(C,D) = \delta^{\text{cen}}_{\text{BM}}(D,C)$ for every C and D. Recall that pioneer research on the centroid was provided by Neumann [10].

In our theorem below we prove that $\delta^{\text{cen}}_{\text{BM}}(P,T) = \frac{5}{2}$ for the parallelogram P and the triangle T in E^2. Our effort is put in order to show that $\delta^{\text{cen}}_{\text{BM}}(P,T) \geq \frac{5}{2}$ since the opposite inequality immediately follows from easy examples.

At the end of the paper we present a few remarks. The first concerns the positions of our triangle with respect to the parallelogram for which the ratio $\frac{5}{2}$ is realized. The second comments the dual version of our theorem. The third shows the following generalization of Theorem for a centrally-symmetric convex body M in place of P: for every triangle T inscribed in M with the common centroid we have $M \subset 3T$. We also propose a more general task to consider an arbitrary convex body instead of M. Finally, we ask about a generalization of theorem for E^d.

As usual, by $\text{int}(A)$ and $\text{bd}(A)$ we denote the *interior* and *boundary* of a set A.

2. The distance between the parallelogram and the triangle is $\frac{5}{2}$

Theorem. *For the parallelogram P and the triangle T we have $\delta^{\text{cen}}_{BM}(P,T) = \frac{5}{2}$.*

Proof. In the proof, by λC we mean the homothetic image of a set C with a positive ratio λ and the center at the origin o of E^2. By S denote the square with vertices $(1,\pm 1)$ and $(-1,\pm 1)$.

We rephrase our theorem as the combination of the two following statements:

(*) for every triangle $\Delta \subset S$ with the centroid at the center o of S the interior of the triangle $\frac{5}{2}\Delta$ does not contain S,

(**) there exists a triangle $\Delta_0 \subset S$ with the centroid in the center of S for which $\frac{5}{2}\Delta_0$ contains S.

Taking as Δ_0 the triangle with vertices $(1,\frac{1}{2}), (-1,\frac{1}{2})$ and $(0,-1)$ we see that (**) is true. Later, our aim is to show that (*) holds true.

The following obvious fact is applied twice in the proof:

($\triangledown$) *If $(*)$ is true for a triangle centered at o and containing Δ, then $(*)$ is true for Δ.*

By ($\triangledown$) we may assume that $\Delta \subset S$ has at least one vertex in the boundary of S. Still if all the vertices of the triangle are in the interior of $\text{int}(S)$, we can increase this triangle by a homothety with center o such that at least one of its vertices “arrives” at the boundary of S.

Without losing the generality assume that such a vertex a is $(1,\alpha)$, where $0 \leq \alpha \leq 1$.

Denote by b and c the endpoints of the opposite side of Δ such that a, b, c are in the positive order. The midpoint of bc is $d = (-\frac{1}{2}, -\frac{1}{2}\alpha)$. The reason is that the segment connecting d with a passes through o which is the centroid of Δ.

We may assume that b or c is in $\text{bd}(S)$. Still if b and c are in $\text{int}(S)$, we do not loose the generality by properly enlarging Δ. Namely, we increase bc by the homothety at its center d as long as an endpoint attains the boundary of S, but clearly, the other must remain in S. This follows by ($\triangledown$) and the observation that the center of the increased triangle is o again.

Case 1. When the vertex b attains the boundary of S not later than c.

Subcase 1.1. When $\alpha \in (0, 1]$.

Since $\alpha > 0$, we have $-\frac{1}{2}\alpha < 0$. Thus d is below the axis $y = 0$. Since d is the midpoint of bc, this and the fact that c is over or on the straight line $y = -1$ imply that b must be below the straight line $y = 1$. Hence b does not belong to the side $(1, 1)(-1, 1)$. Obviously, it also does not belong to the sides $(-1, -1)(1, -1)$ and $(1, -1)(1, 1)$. So b belongs to the side $(-1, 1)(-1, -1)$ and has the form $b = (-1, \beta)$. See Figure 1. Clearly, $-1 \leq \beta \leq 1$.

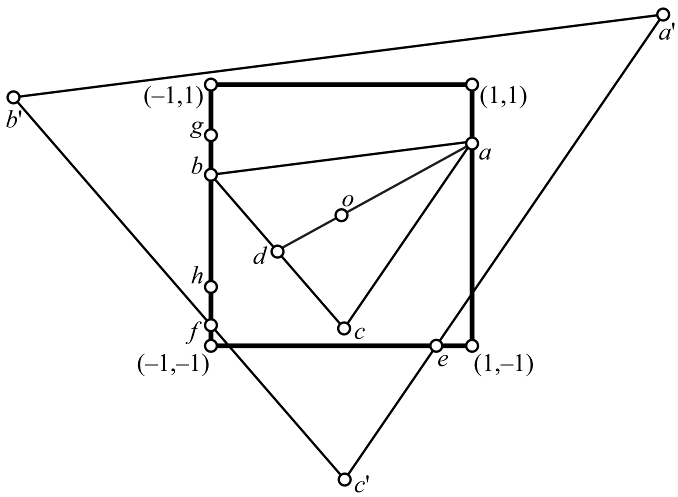

Fig. 1. Positions of a, b, c in Case 1

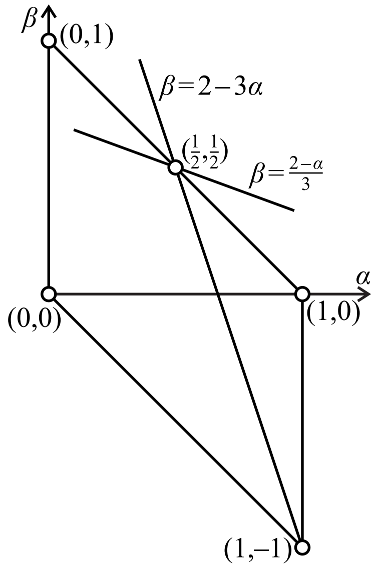

Fig. 2. Region V

Applying again the fact that $d = (-\frac{1}{2}, -\frac{1}{2}\alpha)$ is the middle of bc, we conclude that $c = (0, -\alpha - \beta)$. Since the second coordinate of c is at least -1, we get $\alpha - \beta \geq -1$ which means that $1 - \alpha \geq \beta$. Since the order of vertices a, b, c of Δ is positive, we get $\beta \geq -\alpha$ (for $\beta = -\alpha$ our triangle Δ degenerates to a segment). Resuming, $-\alpha \leq \beta \leq 1 - \alpha$ and thus $b \in gh$, where $g = (-1, 1 - \alpha)$ and $h = (-1, -\alpha)$.

Take into consideration the triangle $\frac{5}{2}\Delta$. Its corresponding vertices are $a' = (\frac{5}{2}, \frac{5}{2}\alpha)$, $b' = (-\frac{5}{2}, \frac{5}{2}\beta)$ and $c' = (0, -\frac{5}{2}\alpha - \frac{5}{2}\beta)$. Here are the equations of the straight lines $\ell_{a'c'}$ and $\ell_{b'c'}$ containing the sides $a'c'$ and $b'c'$, respectively:

$\ell_{a'c'}$: $y - \frac{5}{2}\alpha = (2\alpha + \beta)(x - \frac{5}{2})$,

$\ell_{b'c'}$: $y + \frac{5}{2}\alpha + \frac{5}{2}\beta = (-\alpha - 2\beta)x$.

We intend to show that for every $\alpha \in (0, 1]$ and $\beta \in [-\alpha, 1 - \alpha]$ at least one of the straight lines $\ell_{a'c'}$, $\ell_{b'c'}$ intersects the square S. This would mean that the interior of $\frac{5}{2}\Delta$ does not contain the whole S. Consequently (*) would hold true.

Consider the parallelogram $V = \{(\alpha, \beta);\ 0 < \alpha \leq 1, \alpha \geq \beta \geq 1 - \alpha\}$ in the coordinate system $o\alpha\beta$, where $o = (0, 0)$, without the side $(0, 0)(0, 1)$. See Figure 2.

The consideration below shows that for every $(\alpha, \beta) \in R$ at least one of the following two sentences is true in the system Oxy:

$\ell_{a'c'}$ intersects the side $(-1,-1)(1,-1)$,

$\ell_{b'c'}$ intersects the side $(-1,-1)(-1,1)$.

We intersect $\ell_{a'c'}$ with the line $y=-1$ containing the side $(-1,-1)(1,-1)$. The point of intersection is $e = \left(\frac{2-5\alpha+5\beta}{4\alpha+2\beta}, -1\right)$. If the line $\ell_{a'c'}$ intersects the side $(-1,-1)(1,-1)$, from $x \le 1$ we obtain $\beta \le \frac{2-\alpha}{3}$. As a result we see the region $V(\ell_{a'c'})$ of points (α,β) in the coordinate system $o\alpha\beta$ for which the line $\ell_{a'c'}$ intersects the side $(-1,-1)(1,-1)$; it is the part of R not above the straight line $\beta = \frac{2-\alpha}{3}$.

The straight line $\ell_{b'c'}$ intersects the one $x=-1$ containing the side $(-1,1)(-1,-1)$. The point f of intersection is $(-1, -\frac{3}{2}\alpha - \frac{1}{2}\beta)$. If the line $\ell_{a'c'}$ intersects the side $(-1,1)(-1,-1)$, then from $y \ge -1$ we conclude that $\beta \le 2-3\alpha$. The region $V(\ell_{b'c'})$ of points (α,β) in the system $o\alpha\beta$ for which the line $\ell_{b'c'}$ intersects the side $(-1,-1)(-1,1)$ is the part of R not above the straight line $\beta = 2-3\alpha$.

These two straight lines $\beta = \frac{2-\alpha}{3}$ and $\beta = 2-3\alpha$ intersect at the point $(\frac{1}{2}, \frac{1}{2})$ of the system $o\alpha\beta$ which is in the side $(1,0)(0,1)$ of R. From this and the two preceding paragraphs we see that $V \subset \ell_{a'c'} \cup V(\ell_{b'c'})$. Hence (*) holds true.

Subcase 1.2. When $\alpha = 0$.

We have $d = (-\frac{1}{2}, 0)$ which implies that $b \in (0,1)(-1,1)$ and $c \in (-1,-1)(0,-1)$ are symmetric with respect to d. We let the reader to check that the interior of the triangle $\frac{5}{2}\Delta$ does not contain S, so (*) holds true.

Case 2. When the vertex c attains the boundary of S not later than b.

Subcase 2.1. When $\alpha \in (0,1]$.

Observe that c must be in the side $(-1,-1)(1,-1)$ (see Figure 3). So c has the form $(\gamma,-1)$, where $-1 \le \gamma \le 1$. From $b \in S$ and the fact that d has the first coordinate $-\frac{1}{2}$ we see that $-1 \le \gamma \le 0$. Since d is the midpoint of bc, we have $b = (-1-\gamma, 1-\alpha)$.

Take into account the triangle $\frac{5}{2}\Delta$. Its corresponding vertices are $a' = (\frac{5}{2}, \frac{5}{2}\alpha)$, $b' = (-\frac{5}{2} - \frac{5}{2}\gamma, \frac{5}{2} - \frac{5}{2}\alpha)$ and $c' = (\frac{5}{2}\gamma, -\frac{5}{2})$ (see Figure 3).

Here are the equations of the straight lines containing the sides of the triangle $a'b'c'$.

$\ell_{a'b'} : y - \frac{5}{2}\alpha = \frac{-1+2\alpha}{2+\gamma}(x - \frac{5}{2})$,

$\ell_{b'c'} : y + \frac{5}{2} = \frac{2-\alpha}{-1-2\gamma}(x - \frac{5}{2}\gamma)$,

$\ell_{a'c'} : y - \frac{5}{2}\alpha = \frac{\alpha+1}{1-\gamma}(x - \frac{5}{2})$.

We intend to show that for every $\gamma \in [-1,0]$ at least one of these lines $\ell_{a'b'}$, $\ell_{b'c'}$ and $\ell_{a'c'}$ intersects the square S. This would mean that (*) holds true.

Now let us deal with the set $W = \{(\alpha,\gamma) : 0 < \alpha \le 1, -1 \le \gamma \le 0\}$ of points in the coordinate system $o\alpha\gamma$ (see Figure 4), where $o = (0,0)$.

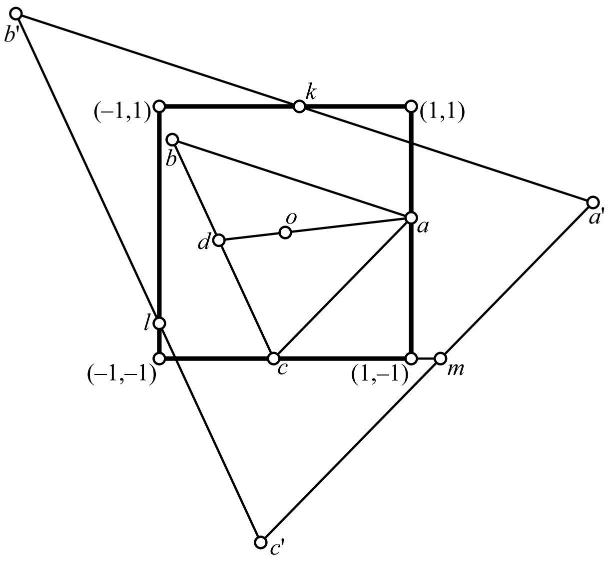

Fig. 3. Positions of a, b, c in Case 2

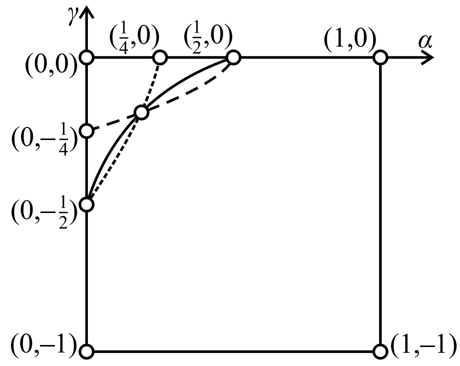

Fig. 4. Region W

The point k of intersection of $\ell_{a'b'}$ with the straight line $x = 1$ containing the side $(1,1)(1,-1)$ has the second coordinate $y = -\frac{3}{2} \cdot \frac{-1+2\alpha}{2+\gamma} + \frac{5}{2}\alpha$. If $\ell_{a'b'}$ intersects this side, then from $y \leq 1$ we get $-\frac{3}{2} \cdot \frac{-1+2\alpha}{2+\gamma} + \frac{5}{2}\alpha \leq 1$. Equivalently, $\gamma \geq \frac{1-4\alpha}{-2+5\alpha}$. So the curve $\gamma = \frac{1-4\alpha}{-2+5\alpha}$ in the coordinate system $o\alpha\gamma$ intersects the axis $o\alpha$ at $(\frac{1}{4}, 0)$ and the axis $o\gamma$ at $(0, -\frac{1}{2})$. This permits to see the subregion $W(\ell_{a'b'})$ of W of points for which $\ell_{a'b'}$ intersects S. It is bounded by the piece of $\gamma = \frac{1-4\alpha}{-2+5\alpha}$ for $0 \leq \alpha \leq \frac{1}{4}$ (marked by the dotted line in Figure 4) and the segments $(0,0)(0,\frac{1}{4})$ and $(0,0)(0,-\frac{1}{2})$ in the system $o\alpha\gamma$.

We find the intersection of $\ell_{b'c'}$ with the straight line $y = -1$ containing the side $(-1,-1)(1,-1)$ at a point $l = (x, -1)$, where x fulfills $\frac{3}{2} = \frac{2-\alpha}{-1-2\gamma}(x - \frac{5}{2}\gamma)$, this is $x = \frac{-3-6\gamma}{4-2\alpha} + \frac{5}{2}\gamma$. If $\ell_{b'c'}$ intersects the side $(1,-1)(-1,-1)$, then from $x \geq -1$ we obtain $-1 \leq \frac{-3-6\gamma}{4-2\alpha} + \frac{5}{2}\gamma$ which (for $\alpha \in [0, \frac{1}{2}]$) is equivalent to $\gamma \geq \frac{1-2\alpha}{-4+5\alpha}$. We easily check that the curve $\gamma = \frac{1-2\alpha}{-4+5\alpha}$ in the coordinate system $o\alpha\gamma$ intersects the axis $o\alpha$ at $(\frac{1}{2}, 0)$ and the axis $o\gamma$ at $(0, -\frac{1}{4})$. In Figure 4 we see the subregion $W(\ell_{b'c'})$. It is bounded by the piece of $\gamma = \frac{1-2\alpha}{-4+5\alpha}$ for $0 \leq \alpha \leq \frac{1}{2}$ (marked by the dashed line in Figure 4) and the segments $(0,0)(0,\frac{1}{2})$ and $(0,0)(0,-\frac{1}{4})$ in the coordinate system $o\alpha\gamma$.

The intersection of $\ell_{a'c'}$ with the straight line $y = -1$ is at a point $m = (x, -1)$, where x fulfills $-1 - \frac{5}{2}\alpha = \frac{\alpha+1}{1-\gamma}(x - \frac{5}{2})$, this is $x = \frac{5}{2} + \frac{-2-5\alpha}{2+2\alpha}(1-\gamma)$. If $\ell_{a'c'}$ intersects the side $(-1,-1)(1,-1)$, than from $x \leq 1$ we obtain $\frac{3}{2} \leq \frac{2+5\alpha}{2+\alpha}(1-\gamma)$ which, for our positive α is equivalent to $\gamma \leq \frac{-1+2\alpha}{2+5\alpha}$. We easily check that the curve $\gamma = \frac{-1+2\alpha}{2+5\alpha}$ in the coordinate system $o\alpha\gamma$ intersects the axis $o\alpha$ at $(\frac{1}{2}, 0)$ and the axis $o\gamma$ at $(0, -\frac{1}{2})$. In Figure 4 we see this subregion $W(\ell_{a'c'})$ of W of points for which $\ell_{a'c'}$ intersects S. This subregion is bounded by the piece of the curve $\gamma = \frac{-1+2\alpha}{2+5\alpha}$ for $0 < \alpha \leq \frac{1}{2}$ (marked by the solid line in Figure 4) and by the segments connecting the succeeding pairs of points $(0, -\frac{1}{2}), (0, -1), (1, -1), (1, 0)$ and $(\frac{1}{2}, 0)$ in the coordinate system $o\alpha\gamma$. Clearly, here the first of these segments is not in $W(\ell_{a'c'})$.

Observe that the point $(\frac{1}{5}, -\frac{1}{5})$ belongs to the three pieces of curves bounding our three considered subregions. Moreover,

$$\frac{-1+2\alpha}{2+5\alpha} \geq \frac{1-4\alpha}{-2+5\alpha} \quad \text{for} \quad 0 < \alpha \leq \frac{1}{5} \quad \text{and} \quad \frac{-1+\alpha}{2+5\alpha} \geq \frac{1-2\alpha}{-4+5\alpha} \quad \text{for} \quad \frac{1}{5} \leq \alpha \leq \frac{1}{2}$$

(see Figure 4). These facts and the three preceding paragraphs imply that

$$W \subset W(\ell_{a'b'}) \cup W(\ell_{b'c'}) \cup W(\ell_{a'c'}).$$

Subcase 2.2. For $\alpha = 0$.

Now we have $d = (-\frac{1}{2}, 0)$. This implies that $c \in (-1,-1)(0,-1)$ and $b \in (0,1)(-1,1)$ are symmetric with respect to d. It is easy to check that the interior of the triangle $\frac{5}{2}\Delta$ does not contain S. So (*) holds true.

From Cases 1 and 2 we see that always $\text{int}(\frac{5}{2}\Delta)$ does not contain S, i.e., (*) is fulfilled. □

The fact proved in the above theorem is claimed without a proof at the bottom of p. 259 of [4].

3. Final remarks

Let us comment the positions of the triangle Δ_0 in S (drawn by a thick line in Figure 5) from the proof of Theorem, for which (**) holds true. As it follows from our considerations, the only two such triangles $\Delta_0 \subset S$ (up to symmetric positions) are the following. The first has vertices $a_1 = (1, \frac{1}{2})$, $b_1 = (-1, \frac{1}{2})$ and $c_1 = (0, -1)$ (by the way, two symmetric positions are also seen for $\alpha = 0$ in Subcases 1.2 and 2.2 of the proof of Theorem). The second has vertices $a_2 = (1, \frac{1}{5})$, $b_2 = (-\frac{4}{5}, \frac{4}{5})$ and $c_2 = (-\frac{1}{5}, -1)$. We see both and also corresponding $\frac{5}{2}\Delta_0$ in Figure 5.

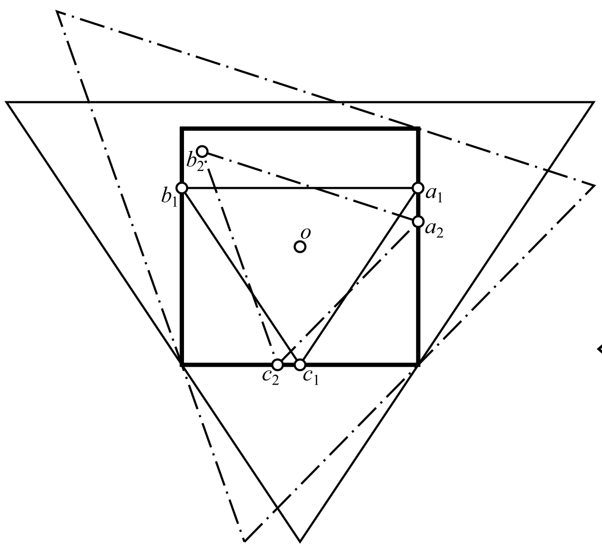

Fig. 5. Extreme positions of Δ in S

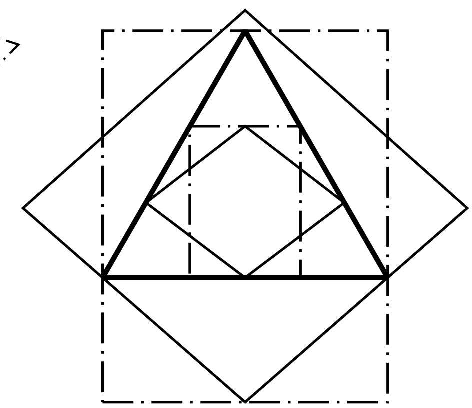

Fig. 6. Extreme positions of P in T

Look at the dual situation. Putting $C = P$ and $D = T$ in $\delta^{\text{cen}}_{\text{BM}}(C, D) = \delta^{\text{cen}}_{\text{BM}}(D, C)$, by Theorem we get $\delta^{\text{cen}}_{\text{BM}}(T, P) = \frac{5}{2}$. In particular, for a given equilateral triangle

T (marked by a thick line in Figure 6) there are only two extreme positions (up to symmetric ones) of a parallelogram P_0 for which $\frac{5}{2}P_0$ contains T.

In connection to our theorem in the present paper, recall that the theorem in [7] shows that every centrally symmetric convex body $M \subset E^2$ permits to inscribe a triangle Δ whose centroid is at the center of symmetry of M such that $M \subset \frac{5}{2}\Delta$ (our present Theorem explains that this ratio cannot be lessened when M is a parallelogram.) Here is a claim which considers any inscribed triangle.

Claim. *Let $M \subset E^2$ be a centrally-symmetric convex body. For every inscribed triangle Δ in M with the common centroid we have $M \subset 3\Delta$.*

Proof. Let $\Delta = abc$. Take the symmetric triangle $\Delta_s = a_s b_s c_s$ with respect to o. The convex hull H of $\Delta \cup \Delta_s$ is an affine-regular hexagon inscribed in M. Prolonging the sides ab_s, bc_s, ca_s we get three points of intersection. Denote by $S(H)$ the star being union of the triangle with these three vertices and the symmetric triangle with respect to o. By the convexity of M we conclude that $M \subset S(H)$. Moreover, observe that $S(H) \subset 3\Delta$. Consequently, $M \subset 3\Delta$. $\square$

By the way, Figure 3 of [5] shows an analogous situation of extreme homothetic parallelograms $S \subset \Delta$ and $(\sqrt{2}+1)S \supset \Delta$ without the requirement that centroids of S and Δ coincide. The ratio 3 in Claim cannot be lessened as it shows the example of the square with vertices $(-1, \pm 1), (1, \pm 1)$ and the triangle with vertices $(1, 1), (-1, 0), (0, -1)$.

Theorem of [7] says the following. *Let $M \subset E^2$ be a centrally symmetric convex body. In M it can be inscribed a triangle Δ whose centroid is the center of symmetry of M such that $M \subset \frac{5}{2}\Delta$.* From our Theorem it follows that this ratio $\frac{5}{2}$ cannot be lessened.

What about considering an arbitrary convex body in place of M in the above claim? The author conjectures that for every planar convex body C and an arbitrary inscribed triangle Δ with the common centroid we have $C \subset 4\Delta$. Possibly the result of Neumann [10] would be a good tool. The the ratio 4 cannot be lessened for C being a triangle with the centroid at o and $\Delta = -\frac{1}{4}C$.

The above remarks can be seen in the wider context of approximation by triangles in [3].

Finally, let us ask about a higher dimensional generalization of Theorem. In E^3 the paralellotope Q permits to inscribe (thus also put inside) a simplex S whose centroid is in the center of Q and $Q \subset 3S$. Just put the vertices of S at some four non-neighboring vertices of Q. Thus $\delta_{BM}^{\text{cen}}(Q, P) \leq 3$. The author believes that the equality holds true here, but possible evaluation seems to be complicated. For higher dimensions the task of finding or at least estimating $\delta_{BM}^{\text{cen}}(Q, P)$ remains open. From [8] we only conclude that $\delta_{BM}^{\text{cen}}(Q, P) \leq 2d - 1$. Since $\delta_{BM}^{\text{cen}}(P, Q) = \delta_{BM}^{\text{cen}}(Q, P)$, the same also follows from the last paragraph of [6].

References

[1] G. Aubrun, S. J. Szarek: *Alice and Bob Meet Banach. The interface of asymptotic geometric analysis and quantum information theory*, Mathematical Surveys and Monographs 223, American Mathematical Society, Providence (2017).

[2] S. Banach: *Théorie des Opérations Linéaires*, Monografie Matematyczne 1, Polish Scientific Publishers, Warszawa (1932); English translation: *Theory of Linear Operations*, North-Holland Mathematical Library 38, North-Holland, Amsterdam (1987).

[3] P. Brass, M. Lassak: *Problems on approximation by triangles*, Geombinatorics 10 (2001) 103–115.

[4] B. Grünbaum: *Measures of symmetry for convex sets*, in: *Convexity*, Proc. Symp. Pure Math. 7, American Mathematical Socity, Providence (1963) 233–270.

[5] M. Lassak: *Approximation of plane convex bodies by centrally symmetric bodies*, J. London Math. Soc. (2) 40 (1989) 369–377.

[6] M. Lassak: *Approximation of convex bodies by parallelotopes*, Bull. Polish Acad. Sci. Math. 39 (1991) 219–223.

[7] M. Lassak: *Approximation of convex bodies by triangles*, Proc. Amer. Math. Soc. 115 (1992) 207–210.

[8] M. Lassak: *Approximation of convex bodies by centrally symmetric bodies*, Geom. Dedicata **72** (1998), 63–68.

[9] M. Lassak: *Banach-Mazur distance from the parallelogram to the affine-regular hexagon and other affine-regular even-gons*, Results Math. 76 (2021) 76–82.

[10] B. H. Neumann: *On some affine invariants of closed convex regions*, J. London Math. Soc. 14 (1939) 262–272.

[11] N. Tomczak-Jaegerman: *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics 38, Longman, Harlow (1989).

[12] G. Toth: *Measures of Symmetry for Convex Sets and Stability*, Universitext, Springer, Cham (2015).