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# Perturbed Degenerate State-Dependent Sweeping Processes with Convex and Nonconvex Sets

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We prove by means of the Moreau-Yosida regularization an existence result for perturbed degenerate state-dependent sweeping processes with convex and nonconvex moving sets. The moving sets vary in a Lipschitz continuous way with respect to the truncated Hausdorff distance. The theoretical results are applied to obtain the well-posedness for the online mirror descent method. The novelty of this work lies in the introduction of new local arguments for the Moreau-Yosida regularization that allow us to obtain the existence of solutions in small intervals and then pass to any interval.

*Keywords:* Sweeping Processes, normal cone, mirror descent method, subsmooth sets, Moreau-Yosida regularization.

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## 1. Introduction

The sweeping process is a first-order dynamical system involving normal cones to moving sets. It was introduced and deeply studied in the convex case by Moreau as a model for an elastoplastic mechanical system (see, e.g., [32, 33, 34, 35]). Since then, it has been used as a model for several phenomena (see, e.g. [1, 8, 12, 14, 22, 28]). Moreau's work was the depart of a series of articles regarding differential inclusions involving normal cones. Among them, we can mention state-dependent sweeping processes [19, 43], second-order sweeping processes [3, 4, 18], implicit sweeping processes [20, 44, 45, 46], generalized sweeping processes [17, 18], and degenerate sweeping processes [2, 21, 24, 25, 26]. We refer to [9] for a complete review on the subject.

The state-dependent sweeping process corresponds to the case where the moving set also depends on the state. This differential inclusion has been motivated by quasi-variational inequalities arising, e.g., in quasistatic evolution problems with friction, the evolution of sandpiles and micromechanical damage models for iron materials, among others (see [27] and the references therein). We refer to [19] for a review on the subject.

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The degenerate sweeping process corresponds to the case where an operator is added "inside" the normal cone on the sweeping process. This dynamic was proposed by Kunze and Monteiro-Marques as a model for quasistatic elastoplasticity (see, e.g., [24]). Since then, the degenerate sweeping process has been studied by several authors in the context of convex and prox-regular sets (see, e.g., [2, 21, 24, 25, 26]).

Recently, in [37], we unified the theory of state-dependent sweeping processes and degenerate sweeping processes by introducing a differential inclusion that generalizes both processes in the unperturbed case.

This paper aims to continue the work [37] by considering perturbed versions of degenerate state-dependent sweeping processes. Indeed, we consider single-valued perturbations of degenerate state-dependent sweeping processes, which complement the results obtained in [2, 7, 21, 24, 25, 26]. Moreover, we prove the existence of solutions for convex and nonconvex (prox-regular, subsmooth and positively $\alpha$-far) moving sets which vary in a Lipschitz way with respect to the truncated Hausdorff distance. We prove our result by means of the Moreau-Yosida regularization (see, e.g., [19, 23, 29, 30, 31, 32, 40, 41]) and by using an appropriate existence result for differential inclusions (see Theorem 4.1 below). The Moreau-Yosida regularization technique consists of approaching a given differential inclusion by a penalized one, depending on a parameter, whose existence is easier to obtain (for example, by using the classical Cauchy-Lipschitz Theorem), and then to pass to the limit as the parameter goes to zero. We refer to [36] for a detailed survey on the subject.

The paper is organized as follows. After some preliminaries, in Section 3, we list the hypotheses used throughout the paper. In Section 4, we establish some technical results used to prove the main result of the paper. Next, in Section 5, we present the main result of the paper (Theorem 5.5), namely, the convergence (up to a subsequence) of the Moreau-Yosida regularization for the degenerate state-dependent sweeping process under the Lipschitzianity of the moving sets with respect to the truncated Hausdorff distance. Finally, in Section 6, we apply our theoretical results to the existence of solutions for the online mirror descent method.

## 2. Preliminaries

In this paper, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ stands for a separable Hilbert with norm $\|\cdot\|$. We denote by $\mathbb{B}(x,\rho)$ the closed ball centered at $x$ with radius $\rho$. The closed unit ball is denoted by $\mathbb{B}$. We write $\mathcal{H}_w$ for the space $\mathcal{H}$ endowed with the weak topology and $x_n \rightharpoonup x$ stands for the weak convergence in $\mathcal{H}$ of a sequence $(x_n)_n$ to $x$.

A vector $h \in \mathcal{H}$ belongs to the *Clarke tangent cone* $T_C(S;x)$, when for every sequence $(x_n)_n$ in $S$ converging to $x$ and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in $\mathcal{H}$ converging to $h$ such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex and its negative polar is the *Clarke normal cone* to $S$ at $x \in S$, that is,

$$N(S;x) = \{v \in \mathcal{H}\colon\ \langle v, h\rangle \leq 0 \text{ for all } h \in T_C(S;x)\}.$$

As usual, $N(S;x) = \emptyset$ if $x \notin S$. Through the Clarke normal cone, we define the *Clarke subdifferential* of a function $f\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ as

$$\partial f(x) := \{v \in \mathcal{H}\colon (v,-1) \in N(\operatorname{epi} f, (x, f(x)))\},$$

where epi $f := \{(y,r) \in \mathcal{H} \times \mathbb{R} \colon f(y) \leq r\}$ is the *epigraph* of $f$. When the function $f$ is finite and locally Lipschitzian around $x$, the Clarke subdifferential is characterized (see [11, Proposition 2.1.5]) in the following way

$$\partial f(x) = \{v \in \mathcal{H} \colon \langle v, h\rangle \leq f^\circ(x;h) \text{ for all } h \in \mathcal{H}\},$$

where
$$f^\circ(x;h) := \limsup_{(t,y)\to(0^+,x)} t^{-1}\left[f(y+th) - f(y)\right],$$

is the *generalized directional derivative* of the locally Lipschitzian function $f$ at $x$ in the direction $h \in \mathcal{H}$. The function $f^\circ(x;\cdot)$ is in fact the *support* of $\partial f(x)$, that is, $f^\circ(x;h) = \sup\{\langle v,h\rangle \colon v \in \partial f(x)\}$. This characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function has the important property of *upper semicontinuity from* $\mathcal{H}$ *into* $\mathcal{H}_w$.

For $S \subset \mathcal{H}$, the *distance function* is defined by $d_S(x) := \inf_{y\in S}\|x-y\|$ for $x \in \mathcal{H}$. We denote $\operatorname{Proj}_S(x)$ as the set (possibly empty) of points which attain this infimum. The equality (see [11, Proposition 2.5.4])

$$N(S;x) = \operatorname{cl}^w\left(\mathbb{R}_+\partial d_S(x)\right) \quad \text{for } x \in S,$$

gives an expression of the Clarke normal cone in terms of the distance function. As usual, it will be convenient to write $\partial d(x,S)$ in place of $\partial d(\cdot,S)(x)$.

**Remark 2.1.** In the present paper, we will calculate the Clarke subdifferential of the distance function to a moving set. By doing so, the subdifferential will always be calculated with respect to the variable involved in the distance function by assuming that the set is fixed. More explicitly, $\partial d_{C(t,y)}(x)$ means the subdifferential of the function $d_{C(t,y)}(\cdot)$ (here $C(t,y)$ is fixed) is calculated at the point $x$, i.e., $\partial\left(d_{C(t,y)}(\cdot)\right)(x)$. In the same way, $\partial d_{C(t,x)}(x)$ means the subdifferential of the function $d_{C(t,x)}(\cdot)$ (here $C(t,x)$ is a fixed set) is calculated at the point $x$, i.e., $\partial\left(d_{C(t,x)}(\cdot)\right)(x)$. □

Let $f \colon \mathcal{H} \to \mathbb{R}\cup\{+\infty\}$ be a proper and lower semicontinuous (lsc, for short) function and $x \in \operatorname{dom} f$. An element $\zeta$ belongs to the *proximal subdifferential* (see [11, Chapter 1]) $\partial_P f(x)$ of $f$ at $x$ if there exist two positive numbers $\sigma$ and $\eta$ such that
$$f(y) \geq f(x) + \langle \zeta, y-x\rangle - \sigma\|y-x\|^2 \quad \forall y \in B(x;\eta).$$

For a closed set $S \subset \mathcal{H}$, the *proximal normal cone* of $S$ at $x \in S$ is defined as $N^P(S;x) = \partial_P I_S(x)$, where $I_S$ is the indicator function of the set $S$. The indicator function $I_S$ is defined as $I_S(x) = 0$ if $x \in S$, and $I_S(x) = +\infty$ otherwise.

Now, we recall the definition of the class of positively $\alpha$-far sets, introduced in [15] and widely studied in [17].

**Definition 2.2.** Let $\alpha \in ]0,1]$ and $\rho \in ]0,+\infty]$. Let $S$ be a nonempty closed subset of $\mathcal{H}$ with $S \neq \mathcal{H}$. We say that the Clarke subdifferential of the distance function $d(\cdot,S)$ keeps the origin *$\alpha$-far-off* on the open $\rho$-tube around $S$, which is defined as $U_\rho(S) := \{x \in \mathcal{H} \colon 0 < d(x,S) < \rho\}$, provided

$$0 < \alpha \leq \inf_{x\in U_\rho(S)} d(0, \partial d(\cdot,S)(x)). \tag{1}$$

Moreover, if $E$ is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is *positively $\alpha$-far* if every $S(t)$ satisfies (1) with the same $\alpha \in ]0,1]$ and $\rho > 0$. □

Several characterizations of this notion and examples were given in [17]. We refer to [17] for more details.

The notion of positively $\alpha$-far sets includes strictly the notion of uniformly subsmooth sets (see Proposition 2.5 below) and the notion of uniformly prox-regular sets (see [17]).

**Definition 2.3.** ([38]) For a fixed $r > 0$, the set $S$ is said to be *$r$-uniformly prox-regular* if for any $x \in S$ and $\zeta \in N_S^P(x)$ with $\|\zeta\| < 1$ one has $x = \text{proj}_S(x + r\zeta)$. □

It is known that $S$ is $r$-uniformly prox-regular if and only if every nonzero proximal vector $\zeta \in N_S^P(x)$ to $S$ at any point $x \in S$ can be realized by an $r$-ball, that is, $S \cap B\big(x + r\frac{\zeta}{\|\zeta\|}\big) = \emptyset$, which is equivalent to

$$\langle \zeta, y - x\rangle \leq \frac{\|\zeta\|}{2r}\|y - x\|^2 \quad \text{for all } y \in S.$$

**Definition 2.4.** Let $S$ be a closed subset of $\mathcal{H}$. We say that $S$ is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -\varepsilon\|x_1 - x_2\| \tag{2}$$

holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S; x_i) \cap \mathbb{B}$ for $i = 1, 2$. Also, if $E$ is a given nonempty set, we say that the family $(S(t))_{t\in E}$ is equi-uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (2) holds for each $t \in E$ and all $x_1, x_2 \in S(t)$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(S(t); x_i) \cap \mathbb{B}$ for $i = 1, 2$. □

**Proposition 2.5.** ([17]) *Assume that $S$ is uniformly subsmooth. Then, for all $\varepsilon \in ]0,1[$ there exists $\rho \in ]0,+\infty[$ such that*

$$\sqrt{1-\varepsilon} \leq \inf_{y\in U_\rho(S)} d(0, \partial d(y, S)).$$

**Remark 2.6.** The class of positively $\alpha$-far sets contains strictly that of uniformly subsmooth sets. We refer to [17] for more details. □

Let $A$ be a bounded subset of $\mathcal{H}$. We define the *Kuratowski measure of non-compactness* of $A$, $\alpha(A)$, as

$$\alpha(A) = \inf\{d > 0\colon A \text{ admits a finite cover by sets of diameter } \leq d\},$$

and the *Hausdorff measure of non-compactness* of $A$, $\beta(A)$, as

$$\beta(A) = \inf\{r > 0\colon A \text{ can be covered by finitely many balls of radius } r\}.$$

The following result gives the main properties of the Kuratowski and Hausdorff measure of non-compactness (see [13, Proposition 9.1 from Section 9.2]).

**Proposition 2.7.** *Let $\mathcal{H}$ be a Hilbert space and $B, B_1, B_2$ be bounded subsets of $\mathcal{H}$. Let $\gamma$ be the Kuratowski or the Hausdorff measure of non-compactness. Then,*

(i) $\gamma(B) = 0$ *if and only if* $\operatorname{cl}(B)$ *is compact.*

(ii) $\gamma(\lambda B) = |\lambda|\gamma(B)$ *for every* $\lambda \in \mathbb{R}$.

(iii) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.

(iv) $B_1 \subset B_2$ *implies* $\gamma(B_1) \leq \gamma(B_2)$.

(v) $\gamma(\operatorname{conv} B) = \gamma(B)$.

(vi) $\gamma(\operatorname{cl}(B)) = \gamma(B)$.

(vii) *If* $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ *is a Lipschitz map of constant* $M \geq 0$, *then* $\gamma(\mathcal{A}(B)) \leq M\gamma(B)$.

Given an interval $I \subset \mathbb{R}$, we denote by $L^1(I;\mathcal{H})$ the space of $\mathcal{H}$-valued Lebesgue integrable functions over $I$. Whenever $\mathcal{H} = \mathbb{R}$, we simply write $L^1(I)$. Moreover, we say that $x\colon I \to \mathcal{H}$ is absolutely continuous (denoted by $x \in \operatorname{AC}(I;\mathcal{H})$) if there exists $f \in L^1(I;\mathcal{H})$ and $x_0 \in \mathcal{H}$ and $t_0 \in I$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s)ds \text{ for all } t \in I.$$

The following result provides a compactness criterion for absolutely continuous functions (see [19, Lemma 2.2]).

**Lemma 2.8.** *Let $(x_n)_n$ be a sequence of absolutely continuous functions from the interval $I := [0,T]$ into $\mathcal{H}$ with $x_n(0) = x_0^n$. Assume that for all $n \in \mathbb{N}$*

$$\|\dot{x}_n(t)\| \leq \psi(t) \text{ a.e } t \in I,$$

*where $\psi \in L^1(I)$ and that $x_0^n \to x_0$ as $n \to \infty$. Then, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and $x \in \operatorname{AC}(I;\mathcal{H})$ such that*

(i) $x_{n_k}(t) \rightharpoonup x(t)$ *in* $\mathcal{H}$ *as* $k \to +\infty$ *for all* $t \in I$.

(ii) $x_{n_k} \rightharpoonup x$ *in* $L^1(I;\mathcal{H})$ *as* $k \to +\infty$.

(iii) $\dot{x}_{n_k} \rightharpoonup \dot{x}$ *in* $L^1(I;\mathcal{H})$ *as* $k \to +\infty$.

(iv) $\|\dot{x}(t)\| \leq \psi(t)$ *a.e.* $t \in I$.

### Truncated Hausdorff distance

Let $r \in ]0,+\infty]$ be a given extended real and let $A$ and $B$ be nonempty subsets of $\mathcal{H}$. We define the *$r$-truncated excess* of $A$ over $B$ as the extended real

$$\exp_r(A,B) := \sup_{x \in A \cap r\mathbb{B}} d(x,B).$$

It is clear that under the usual convention $r\mathbb{B} = \mathcal{H}$ for $r = +\infty$, the $r$-truncated excess of $A$ over $B$ is the usual *excess* of $A$ over $B$, i.e.,

$$\operatorname{exc}_\infty(A,B) = \sup_{x \in A} d(x,B) =: \operatorname{exc}(A,B).$$

In this paper, our results are based on the following notions of distance between sets.

- $\mathrm{haus}_r(A,B) := \max\{\mathrm{exc}_r(A,B), \mathrm{exc}_r(B,A)\}$
- $\widehat{\mathrm{haus}}_r(A,B) := \sup_{z\in r\mathbb{B}} |d(z,A) - d(z,B)|$.

The quantity $\mathrm{haus}_r(A,B)$ receives the name of *$r$-truncated Hausdorff distance.* It is important to emphasize that

$$\mathrm{haus}_r(A,B) \leq \widehat{\mathrm{haus}}_r(A,B).$$

Moreover, for any extended real $r' \geq 2r+\max\{d_A(0), d_B(0)\}$, the following inequality holds (see, e.g., [39, Proposition 4.37])

$$\widehat{\mathrm{haus}}_r(A,B) \leq \mathrm{haus}_{r'}(A,B).$$

As a consequence of the last two inequalities, we obtain the following result.

**Proposition 2.9.** *Let $C\colon I \rightrightarrows \mathcal{H}$ be a set-valued map with nonempty and closed sets. Then, the following assertions are equivalent.*

(a) *for all $r \geq 0$, there exists $\kappa_r \geq 0$ such that*

$$\widehat{\mathrm{haus}}_r(C(t), C(s)) \leq \kappa_r|t-s| \text{ for all } s,t \in I.$$

(b) *for all $r \geq 0$, there exists $\kappa_r \geq 0$ such that*

$$\mathrm{haus}_r(C(t), C(s)) \leq \kappa_r|t-s| \text{ for all } s,t \in I.$$

Therefore, we use $(a)$ and $(b)$ indistinctly.

**Definition 2.10.** We say that a set-valued map $C\colon I \rightrightarrows \mathcal{H}$ is

(i) *Lipschitz with respect to the Hausdorff distance* if there exists $\kappa \geq 0$ such that

$$\mathrm{haus}(C(t), C(s)) := \sup_{z\in\mathcal{H}} |d(z,C(t)) - d(z,C(s))| \leq \kappa|t-s| \text{ for all } s,t \in I.$$

(ii) *Lipschitz with respect to the truncated Hausdorff distance* if for all $r \geq 0$, there exists $\kappa_r \geq 0$ such that

$$\widehat{\mathrm{haus}}_r(C(t), C(s)) \leq \kappa_r|t-s| \text{ for all } s,t \in I. \qquad \square$$

It is clear that (i) implies (ii), but the reverse implication is not true. Moreover, as discussed in [36, 42], the Hausdorff distance can be too restrictive for practical applications. We refer to [37] for more details.

## 3. Main hypotheses

In this section, we list the main hypotheses used throughout the paper.

**Hypotheses on the map $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$:**

$\mathcal{A}$ is a (possibly) nonlinear operator satisfying the following conditions:

$(\mathcal{H}_{\mathcal{A}}^1)$ There exists $m > 0$ such that

$$\langle \mathcal{A}(x) - \mathcal{A}(y), x-y\rangle \geq m\|x-y\|^2 \text{ for all } x,y \in \mathcal{H}.$$

$(\mathcal{H}_{\mathcal{A}}^2)$ There exists $M > 0$ such that

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq M\|x-y\| \text{ for all } x,y \in \mathcal{H}.$$

**Hypotheses on $C\colon I \times \mathcal{H} \rightrightarrows \mathcal{H}$:**

$C$ is a set-valued map with closed and nonempty values. Moreover, the following conditions will be used in the paper.

($\mathcal{H}_1^x$) For all $r \geq 0$, there exist $\kappa_r \geq 0$ such that for $s, t \in I$ and $x, y \in \mathcal{H}$

$$\sup_{z \in r\mathbb{B}} |d(z, C(t,x)) - d(z, C(s,y))| \leq \kappa_r |t-s| + L\|x-y\|,$$

where $L \in [0, m[$ is independent of $r$ and $m$ is the constant given by ($\mathcal{H}_\mathcal{A}^1$).

($\mathcal{H}_2^x$) There exist constants $\alpha \in ]0,1]$ and $\rho \in ]0, +\infty]$ such that for every $y \in \mathcal{H}$

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t,y))} d\left(0, \partial d(\cdot, C(t,y))(x)\right) \qquad \text{a.e. } t \in I,$$

where $U_\rho\left(C(t,y)\right) = \{x \in \mathcal{H}\colon 0 < d(x, C(t,y)) < \rho\}$.

($\mathcal{H}_3^x$) The family $\{C(t,v)\colon (t,v) \in I \times \mathcal{H}\}$ is equi-uniformly subsmooth.

($\mathcal{H}_4^x$) There exists $k \in L^1(I)$ such that for every $t \in I$, every $r > 0$ and every bounded set $B \subset \mathcal{H}$,

$$\gamma\left(C(t,B) \cap r\mathbb{B}\right) \leq k(t)\gamma(\mathcal{A}(B)),$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness (see Proposition 2.7), $k(t) < 1$ for a.e. $t \in I$ and $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is an operator satisfying ($\mathcal{H}_\mathcal{A}^1$) and ($\mathcal{H}_\mathcal{A}^2$).

**Remark 3.1.** (i) Let $L \in [0, m[$. Under ($\mathcal{H}_3^x$) for every $\alpha \in ]\sqrt{\frac{L}{m}}, 1]$ there exists $\rho > 0$ such that ($\mathcal{H}_2^x$) holds. This follows from Proposition 2.5.

(ii) The condition $L \in [0, m[$ in ($\mathcal{H}_\mathcal{A}^2$) is sharp, as it is shown in [25]. $\square$

**Hypotheses on $C\colon I \rightrightarrows \mathcal{H}$:**

$C$ is a set-valued map with closed and nonempty values. Moreover, the following condition will be used in the paper.

($\mathcal{H}_1$) For all $r \geq 0$, there exists $\kappa_r \geq 0$ such that for all $s, t \in I$

$$\sup_{z \in r\mathbb{B}} |d(z, C(t)) - d(z, C(s))| \leq \kappa_r |t-s|.$$

($\mathcal{H}_2$) There exist two constants $\alpha \in ]0,1]$ and $\rho \in ]0, +\infty]$ such that

$$0 < \alpha \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in I,$$

where $U_\rho\left(C(t)\right) = \{x \in \mathcal{H}\colon 0 < d(x, C(t)) < \rho\}$.

($\mathcal{H}_3$) For a.e. $t \in I$ the set $C(t)$ is ball-compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in $\mathcal{H}$.

($\mathcal{H}_4$) For a.e. $t \in I$ the set $C(t)$ is $r$-uniformly prox-regular for some $r > 0$.

**Hypotheses on $F\colon I \times \mathcal{H} \rightrightarrows \mathcal{H}$:**

$F$ is a set-valued map with nonempty closed and convex values satisfying

($\mathcal{H}_1^F$) For every $x \in \mathcal{H}$, $F(\cdot, x)$ is measurable.

($\mathcal{H}_2^F$) For a.e. $t \in I$, $F(t, \cdot)$ is upper semicontinuous from $\mathcal{H}$ into $\mathcal{H}_w$.

($\mathcal{H}_3^F$) There exist $\bar{c}, \bar{d} \in L^1(I)$ such that

$$d(0, F(t,v)) := \inf\{\|w\| \colon w \in F(t,v)\} \leq \bar{c}(t)\|v\| + \bar{d}(t),$$

for all $v \in \mathcal{H}$ and a.e. $t \in [0,T]$.

($\mathcal{H}_4^F$) For a.e. $t \in I$ and $A \subset \mathcal{H}$ bounded, $\gamma(F(t,A)) \leq k(t)\gamma(A)$, for some $k \in L^1(I)$ with $k(t) < +\infty$ for all $t \in I$, where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non-compactness.

**Hypotheses on $g \colon I \times \mathcal{H} \to \mathcal{H}$:**

$g$ is a function satisfying

($\mathcal{H}_1^g$) For every $x \in \mathcal{H}$, the map $t \mapsto g(t,x)$ is measurable.

($\mathcal{H}_2^g$) For all $r > 0$, there exists $\mu_r \geq 0$ such that the map $x \mapsto g(t,x)$ is Lipschitz of constant $\mu_r$ on $r\mathbb{B}$ for a.e. $t \in I$.

($\mathcal{H}_3^g$) There are constants $c, d \geq 0$ such that for a.e. $t \in I$, $\|g(t,x)\| \leq c\|x\| + d$ for $x \in \mathcal{H}$.

($\mathcal{H}_4^g$) For a.e. $t \in I$, the map $x \mapsto g(t,x)$ is continuous.

## 4. Preparatory lemmas

In this section, we give some preliminary results that will be used in the following sections. They are related to set-valued maps and properties of the distance function.

The following result is used to prove the existence of the Moreau-Yosida regularization scheme.

**Theorem 4.1.** *Let $\mathcal{H}$ be a separable Hilbert space and $I = [0,T]$ for some $T > 0$. Assume that ($\mathcal{H}_1^F$)–($\mathcal{H}_4^F$) and ($\mathcal{H}_1^g$)–($\mathcal{H}_3^g$) hold. Then, the differential inclusion*

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) + g(t,x(t)) & a.e.\ t \in I, \\ x(0) = x_0 \in \mathcal{H}, \end{cases} \tag{3}$$

*has at least one solution absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption ($\mathcal{H}_2^g$) can be weakened to ($\mathcal{H}_4^g$).*

**Proof.** Set
$$r(t) := \left(\|x_0\| + \int_0^t \tilde{d}(s)ds\right) \exp\left(\int_0^t \tilde{c}(s)ds\right),$$

where $\tilde{c}(t) := \bar{c}(t) + c$ and $\tilde{d}(t) := \bar{d}(t) + d$ (see ($\mathcal{H}_3^F$) and ($\mathcal{H}_3^g$)). Let us consider the problem:

$$\begin{cases} \dot{x}(t) \in \tilde{F}(t,x) + \tilde{g}(t,x(t)) & \text{a.e. } t \in I, \\ x(0) = x_0 \in \mathcal{H}, \end{cases} \tag{4}$$

where $\tilde{F}(t,x) := F(t,x) \cap (\bar{c}(t)\|x\| + \bar{d}(t))\mathbb{B}$, $\tilde{g}(t,x) = g(t, P_{r(t)}(x))$ and

$$P_{r(t)}(x) = \begin{cases} x & \text{if } \|x\| \leq r(t) \\ r(t)\frac{x}{\|x\|} & \text{elsewhere} \end{cases},$$

is the retraction onto the ball $r(t)\mathbb{B}$.

It is clear that $P_{r(t)}$ is 2-Lipschitz and, thus, by virtue of $(\mathcal{H}_2^g)$, for a.e. $t \in [0,T]$, the map $x \mapsto \tilde{g}(t,x)$ is Lipschitz continuous on $\mathcal{H}$. Indeed, let $x, y \in \mathcal{H}$, then

$$\|\tilde{g}(t,x) - \tilde{g}(t,y)\| \leq \mu_{r(T)}\|P_{r(t)}(x) - P_{r(t)}(y)\| \leq 2\mu_{r(T)}\|x-y\|,$$

where $\mu_{r(T)}$ is the Lipschitz constant given by $(\mathcal{H}_2^g)$. Therefore we conclude that the map $G(t,x) = \tilde{F}(t,x) + \tilde{g}(t,x)$ satisfies the assumption of [6, Theorem 4]. Hence, the differential inclusion (4) admits at least one absolutely continuous solution $x\colon [0,T] \to \mathcal{H}$. Finally, by virtue of Gronwall's inequality, we can prove that $x(t) \in r(t)\mathbb{B}$, which proves that $x\colon [0,T] \to \mathcal{H}$ is a solution of (3). □

## 5. Perturbed degenerate state-dependent sweeping processes

This section aims to extend the results from [19, 37, 43] to perturbed degenerate state-dependent sweeping processes. For $T > 0$, let us consider the problem:

$$(\mathcal{SP}) \qquad \begin{cases} \dot{x}(t) \in -N_{C(t,x(t))}\left(\mathcal{A}(x(t))\right) + g(t,x(t)) & \text{a.e. } t \in I := [0,T], \\ x(0) = x_0 \in \mathcal{A}^{-1}(C(0,x_0)). \end{cases}$$

We use the Moreau-Yosida regularization technique from [19, 37, 43] to deal with $(\mathcal{SP})$.

Let $\lambda > 0$ and consider the following differential inclusion: Find $x_\lambda\colon I \to \mathcal{H}$ with $x_\lambda(0) = x_0$ and such that for a.e. $t \in I$

$$(\mathcal{P}_\lambda) \qquad \dot{x}_\lambda(t) \in -\frac{1}{2\lambda}\partial d^2_{C(t,x_\lambda(t))}\left(\mathcal{A}(x_\lambda(t))\right) + g(t,x_\lambda(t)),$$

where $\mathcal{A}(x_0) \in C(0,x_0)$. It follows from Theorem 4.1 and [37, Proposition 4.1] that the following existence result for the approached problem $(\mathcal{P}_\lambda)$ holds.

**Proposition 5.1.** *Assume, in addition to $(\mathcal{H}_\mathcal{A}^2)$, $(\mathcal{H}_1^x)$, $(\mathcal{H}_3^x)$ and $(\mathcal{H}_4^x)$, that the conditions $(\mathcal{H}_1^g)$–$(\mathcal{H}_3^g)$ hold. Then, for every $\lambda > 0$ there exists at least one absolutely continuous solution $x_\lambda$ of $(\mathcal{P}_\lambda)$.*

Let us define $\varphi_\lambda(t) := d_{C(t,x_\lambda(t))}(\mathcal{A}(x_\lambda(t)))$ for $t \in I$.

**Remark 5.2.** Recall that under condition $(\mathcal{H}_3^x)$, according to Proposition 2.5, for every $\alpha \in ]\sqrt{\frac{L}{m}}, 1]$ there exists $\rho > 0$ such that $(\mathcal{H}_2^x)$ holds. □

The following proposition establishes that trajectories of $(\mathcal{P}_\lambda)$ stay uniformly close (with respect to $\lambda$) to the moving sets in a small interval of $I = [0,T]$. The proof is based on ideas from [37].

**Proposition 5.3.** *Assume that, in addition to the hypotheses of Proposition 5.1, $(\mathcal{H}_\mathcal{A}^1)$ holds. Then, for every $R > 0$ there exists $\tau_R \in ]0,T]$ (independent of $\lambda$) such that if*

$$\lambda < \lambda_0 := \frac{(m\alpha^2 - L)}{\tilde{\kappa}_R + (L+M)(c(R + \|x_0\|) + d)}\,\rho,$$

*then* $$\dot{\varphi}_\lambda(t) \leq \tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d) - \frac{m\alpha^2 - L}{\lambda}\varphi_\lambda(t) \quad \text{a.e. } t \in [0,\tau_R], \tag{5}$$

*where* $\alpha \in ]\sqrt{\frac{L}{m}}, 1]$ *and* $\rho > 0$ *are given by Remark* 5.2, $\tilde{\kappa}_R := \kappa_{\|\mathcal{A}(x_0)\|+MR}$ *is the constant given by* $(\mathcal{H}_1^x)$ *and* $c, d \geq 0$ *are the constants given by* $(\mathcal{H}_3^g)$. *Moreover,*

$$\varphi_\lambda(t) \leq \frac{\tilde{\kappa}_R + (L+M)(c(R + \|x_0\|) + d)}{m\alpha^2 - L}\lambda \quad \text{for all } t \in [0, \tau_R]. \tag{6}$$

**Proof.** Fix $R > 0$ and define the set

$$\Omega_\lambda := \{t \in I \colon \|x_\lambda(t) - x_0\| > R\}.$$

On the one hand, if $\Omega_\lambda = \emptyset$, then

$$x_\lambda(t) \in \mathbb{B}(x_0, R) \text{ for all } t \in I.$$

On the other hand, if $\Omega_\lambda \neq \emptyset$, then we can define

$$\tau_\lambda := \inf\{t \in I \colon \|x_\lambda(t) - x_0\| > R\} > 0.$$

Therefore, in what follows, we can assume that

$$x_\lambda(t) \in \mathbb{B}(x_0, R) \text{ for all } t \in [0, \tau_\lambda], \tag{7}$$

where we set $\tau_\lambda := T$ if $\Omega_\lambda = \emptyset$. Thus, by virtue of (7) and $(\mathcal{H}_1^x)$, it follows that

$$\|\mathcal{A}(x_\lambda(t))\| \leq \|\mathcal{A}(x_0)\| + MR \text{ for all } t \in [0, \tau_\lambda]. \tag{8}$$

Moreover, according to Proposition 5.1, $x_\lambda$ is absolutely continuous and, due to (8) and $(\mathcal{H}_\mathcal{A}^2)$, for every $s, t \in [0, \tau_\lambda]$,

$$|\varphi_\lambda(t) - \varphi_\lambda(s)| \leq (L+M)\|x_\lambda(t) - x_\lambda(s)\| + \tilde{\kappa}_R|t - s|,$$

where $\tilde{\kappa}_R := \kappa_{\|\mathcal{A}(x_0)\|+MR}$ is the constant given by $(\mathcal{H}_1^x)$. Hence, $\varphi_\lambda$ is absolutely continuous.

Let $\lambda < \lambda_0$. Then, by using similar arguments to the given in [37, Proposition 5.2], it follows that

$$\dot{\varphi}_\lambda(t) \leq \tilde{\kappa}_R + \frac{L - m\alpha^2}{\lambda}\varphi_\lambda(t) + (L+M)\|g(t, x_\lambda(t))\| \quad \text{a.e. } t \in [0, \tau_\lambda],$$

which implies (6) in the interval $[0, \tau_\lambda]$. It remains to prove that there exists some $\tau_R \in ]0, T]$ such that $\tau_R \leq \tau_\lambda$ for all $\lambda < \lambda_0$. Indeed, for a.e. $t \in [0, \tau_\lambda]$

$$\begin{aligned}
\|\dot{x}_\lambda(t)\| &\leq \frac{1}{\lambda}\varphi_\lambda(t) + \|g(t, x_\lambda(t))\| \\
&\leq \frac{\tilde{\kappa}_R + (L+M)(c(R + \|x_0\|) + d)}{m\alpha^2 - L} + (c(R + \|x_0\|) + d).
\end{aligned}$$

Hence, by the continuity of $x_\lambda$ and the definition of $\tau_\lambda$,

$$\begin{aligned}
R \leq \|x_\lambda(\tau_\lambda) - x_0\| &\leq \int_0^{\tau_\lambda} \|\dot{x}_\lambda(s)\| ds \\
&\leq \Big(\frac{\tilde{\kappa}_R + (L+M)(c(R + \|x_0\|) + d)}{m\alpha^2 - L} + (c(R + \|x_0\|) + d)\Big)\tau_\lambda,
\end{aligned}$$

which implies the existence of $\tau_R \in ]0, T]$ such that $\tau_R \leq \tau_\lambda$ for all $\lambda < \lambda_0$. So we have finally proved that (5) and (6) hold for the interval $[0, \tau_R]$, which completes the proof. $\square$

As a consequence of the last proposition, we obtain that $x_\lambda$ is Lipschitz continuous on $[0, \tau_R]$ for $\lambda > 0$ small enough.

**Corollary 5.4.** *For $\lambda < \lambda_0$ the function $x_\lambda$ is Lipschitz continuous of constant*

$$C_R := \frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L} + (c(R+\|x_0\|)+d) \quad \text{on } [0, \tau_R].$$

Let $(\lambda_n)_n$ be a sequence converging to 0. The next result shows the existence of a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $\left(x_{\lambda_{n_k}}\right)_k$ converges (in the sense of Lemma 2.8) to a solution of $(\mathcal{SP})$ over all the interval $I$. We perform the analysis in the interval $[0, \tau_R]$ for some $R > 0$ and then we extend iteratively the solution to all the interval.

The next theorem extends all known results on Perturbed State-Dependent Sweeping Processes and Degenerate Sweeping Processes.

**Theorem 5.5.** *Assume, in addition to $(\mathcal{H}^1_{\mathcal{A}})$, $(\mathcal{H}^2_{\mathcal{A}})$, $(\mathcal{H}^x_1)$, $(\mathcal{H}^x_3)$ and $(\mathcal{H}^x_4)$, that $(\mathcal{H}^g_1)$, $(\mathcal{H}^g_2)$ and $(\mathcal{H}^g_3)$ hold. Then, the problem $(\mathcal{SP})$ admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}^g_2)$ can be weakened to $(\mathcal{H}^g_4)$.*

**Proof.** Fix $R > 0$ and $\tau_R$ from Proposition 5.3. Then, by virtue of Proposition 5.3, the sequences $(x_{\lambda_n})_n$ and $(\mathcal{A}(x_{\lambda_n}))_n$ satisfy the hypotheses of Lemma 2.8 over the interval $[0, \tau_R]$ with

$$\psi(t) := \frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L} + (c(R+\|x_0\|)+d) \ \text{ and } \ \tilde{\psi}(t) := M\psi(t),$$

respectively. Therefore, there exist subsequences $(x_{\lambda_{n_k}})_k$ and $(\mathcal{A}(x_{\lambda_{n_k}}))_k$ of $(x_{\lambda_n})_n$ and $(\mathcal{A}(x_{\lambda_n}))_n$, respectively, and maps $x\colon [0, \tau_R] \to \mathcal{H}$ and $\mathcal{A}x\colon [0, \tau_R] \to \mathcal{H}$ satisfying the hypotheses (i)–(iv) of Lemma 2.8. For simplicity, we write $x_k$ and $\mathcal{A}(x_k)$ instead of $x_{\lambda_{n_k}}$ and $\mathcal{A}(x_{\lambda_{n_k}})$, respectively.

**Claim 1.** $(\mathcal{A}(x_k(t)))_k$ and $(x_k(t))_k$ are relatively compact in $\mathcal{H}$ $\forall t \in [0, \tau_R]$.

Proof. We observe that, according to $(\mathcal{H}^x_4)$, for any $t \in I$ and $u \in \mathcal{H}$, the set $C(t, u)$ is ball compact. Indeed, fix $t \in I$ and consider $B = \{u\}$ in $(\mathcal{H}^x_4)$. Then, for all $r > 0$,

$$\gamma(C(t,u) \cap r\mathbb{B}) \leq k(t)\gamma(\{\mathcal{A}(u)\}) = 0,$$

which implies that $C(t, u)$ is ball compact. Therefore, for all $t \in I$ and $u, v \in \mathcal{H}$, the projection $\operatorname{Proj}_{C(t,u)}(v) \neq \emptyset$.

Let $t \in [0, \tau_R]$. Consider $y_k(t) \in \operatorname{Proj}_{C(t,x_k(t))}(\mathcal{A}(x_k(t)))$. Then, $\|\mathcal{A}(x_k(t)) - y_k(t)\| = d_{C(t,x_k(t))}(\mathcal{A}(x_k(t)))$. Thus,

$$\begin{aligned}
\|y_k(t)\| &\leq d_{C(t,x_k(t))}(\mathcal{A}(x_k(t))) + \|\mathcal{A}(x_k(t))\| \\
&\leq \left[\frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L}\right]\lambda_{n_k} + M\|x_k(t) - x_0\| + \|\mathcal{A}(x_0)\| \\
&\leq \tilde{r}(t) := \left[\frac{\tilde{\kappa}_R + (L+M)(c(R+\|x_0\|)+d)}{m\alpha^2 - L}\right]\lambda_{n_k} + MC_R t + \|\mathcal{A}(x_0)\|.
\end{aligned}$$

Also, since $(\mathcal{A}(x_k(t)) - y_k(t))$ converges to 0,

$$\gamma\left(\{\mathcal{A}(x_k(t))\colon k \in \mathbb{N}\}\right) = \gamma\left(\{y_k(t)\colon k \in \mathbb{N}\}\right).$$

Therefore, if $A := \{\mathcal{A}(x_k(t))\colon k \in \mathbb{N}\}$,

$$\gamma(A) = \gamma(\{y_k(t)\colon k \in \mathbb{N}\}) \leq \gamma\left(C\left(t, \{x_k(t)\colon k \in \mathbb{N}\}\right) \cap \tilde{r}(t)\mathbb{B}\right) \leq k(t)\gamma(A),$$

where we have used $(\mathcal{H}_4^x)$. Finally, since $k(t) < 1$, we obtain that $\gamma(A) = 0$, which shows that $(\mathcal{A}(x_k(t)))_k$ is relatively compact in $\mathcal{H}$. Finally, $(x_k(t))_k$ is relatively compact in $\mathcal{H}$ by virtue of $(\mathcal{H}_{\mathcal{A}}^2)$.

**Claim 2.** $\mathcal{A}(x(t)) \in C(t, x(t))$ for all $t \in [0, \tau_R]$.

Proof. As a result of Claim 1 and the weak convergence $x_k(t) \rightharpoonup x(t)$ for all $t \in [0, \tau_R]$ (due to (i) of Lemma 2.8), we obtain the strong convergence of $(x_k(t))_k$ to $x(t)$ for all $t \in [0, \tau_R]$. Therefore, due to $(\mathcal{H}_{\mathcal{A}}^2)$,

$$\begin{aligned} d_{C(t,x(t))}(\mathcal{A}(x(t))) &\leq \liminf_{k\to\infty} \left(d_{C(t,x_k(t))}\left(\mathcal{A}(x_k(t))\right) + (M+L)\|x_k(t) - x(t)\|\right) \\ &\leq \liminf_{k\to\infty} \left(\left[\frac{\tilde{\kappa}_R + (L+M)(c(R + \|x_0\|) + d)}{m\alpha^2 - L}\right]\lambda_{n_k} + (M+L)\|x_k(t) - x(t)\|\right) \\ &= 0, \end{aligned}$$

which proves the claim.

The proof of $x$ is a solution of $(\mathcal{SP})$ over the interval $[0, \tau_R]$ can be done in a similar way to [37, Theorem 5.1]. Finally, to extend the solution over all the interval $I$, we can proceed as in [17, Lemma 5.7]. $\square$

### 5.1. Perturbed degenerate sweeping processes

In this subsection, we provide existence results for the perturbed degenerate sweeping process:

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}\left(\mathcal{A}(x(t))\right) + g(t, x(t)) & \text{a.e. } t \in I := [0, T], \\ x(0) = x_0 \in \mathcal{A}^{-1}(C(0)). \end{cases} \tag{9}$$

The latter differential inclusion can be seen as a particular case of $(\mathcal{SP})$ for state-independent moving sets.

The following theorem extends the result from [37, Theorem 5.2]. Moreover, it is well-known that $(\mathcal{H}_4)$ implies $(\mathcal{H}_2)$ with $\alpha = 1$. Thus, the following result includes the existence of solutions for regular (convex/prox-regular) sets.

**Theorem 5.6.** *Assume, in addition to $(\mathcal{H}_{\mathcal{A}}^1)$, $(\mathcal{H}_{\mathcal{A}}^2)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, that $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$ hold. Then, the problem (9) admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.*

**Proof.** According to the proof of Theorem 5.5, we note that $(\mathcal{H}_3^x)$ was used to obtain $(\mathcal{H}_2^x)$ and the upper semicontinuity of $\partial d_{C(t,\cdot)}(\cdot)$ from $\mathcal{H}$ into $\mathcal{H}_w$ for all $t \in I$.

Since in the state-independent case these two properties hold under $(\mathcal{H}_2)$ (see [37, Proposition 4.2]), it is sufficient to repeat the proof of Theorem 5.5 to get the result. $\square$

## 6. An application to online optimization

Let $\phi\colon \mathcal{H} \to \mathbb{R}$ be a convex smooth function. We say that $\phi$ is $\beta$-strongly convex if $\phi - (\beta/2)\|\cdot\|^2$ is convex (see, e.g., [5, Chapter 10]). It is well-known (see, e.g., [5, Exercise 17.5]) that $\phi$ is $\beta$-strongly convex if and only if

$$\langle \nabla\phi(x) - \nabla\phi(y), x - y\rangle \geq \beta\|x - y\|^2 \text{ for all } x, y \in \mathcal{H}.$$

We denote by $\phi^*$ its convex conjugate. It is well-known that for strongly convex functions $\phi\colon \mathcal{H} \to \mathbb{R}$ the operator $(\nabla\phi)^{-1}\colon \mathcal{H} \to \mathcal{H}$ is well-defined and the following formula holds:

$$(\nabla\phi)^{-1}(x) = \nabla\phi^*(x) \text{ for all } x \in \mathcal{H}.$$

Moreover, $\phi^*$ is $1/\beta$-strongly convex if and only if $\nabla\phi$ is $\beta$-Lipschitz continuous (see, e.g., [5, Theorem 18.15]).

Assume that $\phi$ is twice differentiable over $\mathcal{H}$ and consider the problem: Find $x\colon [0, T] \to \mathcal{H}$ such that $x(0) = x_0$ and

$$\nabla^2\phi(x(t))\dot{x}(t) \in -N_{\mathcal{K}}(x(t)) + g(t, x(t)) \text{ a.e. } t \in [0, T], \tag{10}$$

where $\mathcal{K} \subset \mathcal{H}$ is a nonempty and closed set. The latter dynamic, called *online mirror descent*, was considered in [10] for convex and compact sets $\mathcal{K}$ to obtain a $O((\log k)^2)$-competitive randomized algorithm for the $k$-server problem on hierarchically separated trees (see [10, Theorem 2.2]). The system (10) also includes the so-called *oblique projected dynamical systems* considered in [16] as a way of preserving monotonicity in projected dynamical systems.

We observe that (10) is formally equivalent to the following perturbed degenerate sweeping process:

$$\dot{z}(t) \in -N_{\mathcal{K}}(\nabla\phi^*(z(t))) + g(t, \nabla\phi^*(z(t))) \text{ a.e. } t \in [0, T]. \tag{11}$$

Indeed, $x\colon [0, T] \to \mathcal{H}$ is a solution of (10) if and only if $z(t) = \nabla\phi(x(t))$ is a solution of (11). Hence, by using the above observation and Theorem 5.5, we obtain the following result that generalizes [10, Theorem 2.2].

**Theorem 6.1.** *Let $\mathcal{K}$ be a closed, ball-compact and positively $\alpha$-far set (see Definition 2.2). Let $\phi\colon \mathcal{H} \to \mathbb{R}$ be a strongly convex function with Lipschitz continuous gradient. Assume that $g\colon [0, T] \times \mathcal{H} \to \mathcal{H}$ satisfies $(\mathcal{H}_1^g)$, $(\mathcal{H}_2^g)$ and $(\mathcal{H}_3^g)$. Then, for any $x_0 \in \mathcal{H}$, the problem (10) admits at least one absolutely continuous solution. Moreover, if $\mathcal{H}$ is finite-dimensional, then assumption $(\mathcal{H}_2^g)$ can be weakened to $(\mathcal{H}_4^g)$.*

The latter result offers a significant improvement to [10, Theorem 2.2] because we relax the compactness and the geometry of the set $\mathcal{K}$ (we recall that any convex or uniformly prox-regular set is positively $\alpha$-far with $\alpha = 1$). Moreover, the problem (11) offers an equivalent formulation to (10), where no second differentiability of $\phi$

is involved. Finally, by means of Theorem 5.5, it is straightforward to consider the system (10) with moving sets, which can be useful for studying the $k$-server problem with time-dependent servers.

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