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Compactness in Locally Convex Cones

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We investigate notions of compactness with respect to the relative topologies of a locally convex cone. Since these topologies are generally not Hausdorff and the uniform structures induced by the upper and lower relative topologies are not symmetric, we introduce suitable generalizations for the concepts of precompactness and completeness, which nonetheless coincide with the usual ones in the symmetric case, that is in particular for the example of locally convex topological vector spaces. Locally compact cones comprise the topic for the final part of this paper.

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1. Locally convex cones

In order to keep this paper largely self-contained, we begin with a brief introduction of locally convex cones as developed in [4], [11] and other papers. We just provide definitions, results and examples and refer to the sources for details and proofs. Section 2 deals with compactness and suitable concepts of precompactness and completeness for the relative topologies of locally convex cones. The special case of the weak topology induced by a dual pair is considered in Section 3. In the final Section 4 we investigate local compactness and its implications. This yields non-trivial generalizations of the same concepts for subcones of locally convex topological vector spaces.

A *cone* is a set $\mathcal{P}$ endowed with an addition $(a,b) \mapsto a+b$ and a scalar multiplication $(\alpha, a) \mapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. The *cancellation law*, stating that $a + c = b + c$ implies $a = b$, is not required in general. It holds if and only if the cone $\mathcal{P}$ can be embedded into a real vector space.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. Anti-symmetry is however not required. Equality on $\mathcal{P}$ is obviously such an order.

The theory of locally convex cones uses order theoretical concepts to introduce a quasiuniform topological structure on an ordered cone. In a first approach, the resulting topological neighborhoods themselves will be considered to be elements of the cone. In this vein, a *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$

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that contains an *abstract neighborhood system* $\mathcal{V}$, that is a subset of positive (i.e. greater or equal to zero) elements which is downward directed, closed for addition and multiplication by scalars $\alpha > 0$. The elements v of $\mathcal{V}$ define *upper* resp. *lower neighborhoods* for the elements $a \in \mathcal{P}$ by

$$v(a) = \{ b \in \mathcal{P} \mid b \leq a + v \} \quad \text{resp.} \quad (a)v = \{ b \in \mathcal{P} \mid a \leq b + v \},$$

creating the *upper* resp. *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology* generated by the neighborhoods $v^s(a) = v(a) \cap (a)v$. All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda \geq 0$. They need however not be bounded above. An element $a \in \mathcal{P}$ is called *bounded (above)* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $a \leq \lambda v$. The presence of unbounded elements represents the main difference between locally convex cones and subcones of locally convex vector spaces and accounts for much of the richness and subtlety of this setting.

Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered topological vector space is a locally convex cone in this sense, as it may be canonically embedded into a full locally convex cone (see Example 1.1(c) below and I.2.7 in [4]). The subsets $\{(a,b) \mid a, b \in \mathcal{P},\ a \leq b + v\}$ of $\mathcal{P}^2$, for all $v \in \mathcal{V}$ form a *convex quasiuniform structure* on $\mathcal{P}$. Conversely, it is shown in Chapter I.5.2 of [4] how a convex quasiuniform structure on a cone can be used to construct a full locally convex cone which contains the given one as a subcone and induces the given quasiuniform structure. This yields a second, equivalent approach to locally convex cones. We shall recollect a few examples. Many more can be found in [11].

Examples 1.1. (a) In the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\epsilon \in \mathbb{R} \mid \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \epsilon]$ are the upper and the intervals $[a - \epsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point. It is finer than the usual topology of $\overline{\mathbb{R}}$ where the intervals $[a, +\infty]$ for $a \in \mathbb{R}$ are the neighborhoods of $+\infty$.

(b) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the singleton neighborhood system $\mathcal{V} = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below even with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone. For $a \in \overline{\mathbb{R}}$ the intervals $(-\infty, a]$ and $[a, +\infty]$ are the only upper and lower neighborhoods, respectively. The symmetric topology is the discrete topology on $\overline{\mathbb{R}}_+$ and 0 is its only bounded element.

(c) Let $(E, \mathcal{V}, \leq)$ be a locally convex ordered topological vector space, where $\mathcal{V}$ is a basis of closed, convex, balanced and order convex neighborhoods of the origin in E. Because equality is an order relation, this example will cover locally convex vector spaces in general. In order to interpret E as a locally convex cone we shall embed it into a larger full cone. This is done in a canonical way: Let Conv be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A \quad \text{and} \quad b \in B\}$ for $A, B \in \mathrm{Conv}$ and $\alpha \geq 0$. We define the order on Conv by

$$A \leq B \qquad \text{if for every } a \in A \text{ there is } b \in B \text{ such that } a \leq b.$$

The requirements for an ordered cone are easily checked. The neighborhood system in Conv is given by the neighborhood basis $\mathcal{V} \subset \mathrm{Conv}$, that is

$$A \leq B + V \quad \text{if for every } a \in A \text{ there are } b \in B \text{ and } v \in V \text{ such that } a \leq b + v.$$

for $A, B \in \mathrm{Conv}$ and $V \in \mathcal{V}$. We observe that for every $A \in \mathrm{Conv}$ and $V \in \mathcal{V}$ there is $\rho > 0$ such that $\rho V \cap A \neq \emptyset$. This yields $0 \in A + \rho V$. Therefore $\{0\} \leq A + \rho V$, and every element $A \in \mathrm{Conv}$ is indeed bounded below. Thus $\big(\mathrm{Conv}, \mathcal{V}\big)$ is a full locally convex cone. Via the embedding $x \mapsto \{x\} : E \to \mathrm{Conv}$ the space E itself is a subcone of Conv. This embedding preserves the order structure of E, and on its image the symmetric topology of Conv coincides with the given vector space topology of E. Thus E is indeed a locally convex cone, but not a full cone.

(d) Given a locally convex cone $(\mathcal{P}, \mathcal{V})$ the preceding procedure can be applied to generate the locally convex cone $(Conv(\mathcal{P}), \mathcal{V})$ of all non-empty convex subsets of $\mathcal{P}$. Subcones of $Conv(\mathcal{P})$ that merit further investigation are those of all closed, closed and bounded, or compact convex sets in $Conv(\mathcal{P})$, respectively. Details on these and further related examples can be found in [4] and [11].

(e) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, X a set and let $\mathcal{F}(X, \mathcal{P})$ be the cone of all $\mathcal{P}$-valued functions on X, endowed with the pointwise operations and order. If $\bar{\mathcal{P}}$ is a full cone containing both $\mathcal{P}$ and $\mathcal{V}$, then we may identify the elements $v \in \mathcal{V}$ with the constant functions $x \mapsto v$ for all $x \in X$, hence $\mathcal{V}$ is a subset and a neighborhood system for $\mathcal{F}(X, \bar{\mathcal{P}})$. A function $f \in \mathcal{F}(X, \bar{\mathcal{P}})$ is uniformly bounded below, if for every $v \in \mathcal{V}$ there is $\rho \geq 0$ such that $0 \leq f + \rho v$. These functions form a full locally convex cone $\big(\mathcal{F}_b(X, \bar{\mathcal{P}}), \mathcal{V}\big)$, carrying the topology of uniform convergence. As a subcone, $\big(\mathcal{F}_b(X, \mathcal{P}), \mathcal{V}\big)$ is a locally convex cone. The case $\mathcal{P} = \overline{\mathbb{R}}$ is of particular interest. Non-constant neighborhood functions $x \mapsto v(x)$ lead to more general neighborhood systems for cone-valued functions, modelling pointwise or setwise or weighted convergence (see Example I.1.4(e) of [11]).

(f) For $x \in \overline{\mathbb{R}}$ denote $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. For $1 \leq p \leq +\infty$ and a sequence $(x_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ let $\|(x_i)\|_p$ denote the usual l^p norm, that is

$$\|(x_i)\|_p = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^{(1/p)} \in \overline{\mathbb{R}} \quad \text{for} \quad p < +\infty$$

and $\|(x_i)\|_\infty = \sup\{|x_i| \mid i \in \mathbb{N}\} \in \overline{\mathbb{R}}$. Now let $\mathcal{C}^p$ be the cone of all sequences $(x_i)_{i \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ such that $\|(x_i^-)\|_p \leq +\infty$. We use the pointwise order in $\mathcal{C}^p$ and the neighborhood system $\mathcal{V}_p = \{\rho v_p \mid \rho > 0\}$, where

$$(x_i)_{i \in \mathbb{N}} \leq (y_i)_{i \in \mathbb{N}} + \rho v_p$$

means that $\|(x_i - y_i)^+\|_p \leq \rho$. (In this expression the l^p norm is evaluated only over the indexes $i \in \mathbb{N}$ for which $y_i < +\infty$.) It can be easily verified that $(\mathcal{C}^p, \mathcal{V}_p)$ is a locally convex cone. In fact $(\mathcal{C}^p, \mathcal{V}_p)$ can be embedded into a full cone following a procedure analogous to that in 1.1(c). $\square$

For cones $\mathcal{P}$ and $\mathbb{Q}$ a mapping $T : \mathcal{P} \to \mathbb{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ holds for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $\mathcal{P}$ and $\mathbb{Q}$ are ordered, then T is called *monotone,* if $a \leq b$ implies $T(a) \leq T(b)$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathbb{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$.

It is immediate from this definition that uniform continuity implies and combines continuity for the operator $T : \mathcal{P} \to \mathbb{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathbb{Q}$, respectively. Continuous operators are monotone, even though in a slightly weakened sense (Lemma II.1.4 in [4]).

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all *polars* v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \leq \mu(b) + 1$, whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. Continuity implies that a linear functional μ is monotone, and for a full cone $\mathcal{P}$ it requires just that $\mu(v) \leq 1$ holds for some $v \in \mathcal{V}$ in addition. We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$.

For this purpose $\overline{\mathbb{R}}$ is endowed with its usual topology, that is the Euclidean topology on $\mathbb{R}$ with the neighborhoods $[\alpha, +\infty]$ for the element $+\infty$. As in locally convex topological vector spaces, the polar v° of a neighborhood $v \in \mathcal{V}$ is seen to be $w(\mathcal{P}^*, \mathcal{P})$-compact and convex (Theorem II.2.4 in [4]). A variety of Hahn-Banach type extension and separation theorems for linear functionals is available and can be found in [10] and [11]. These are essential for the development of an effective duality theory for locally convex cones.

Examples 1.2. Revisiting the preceding Examples 1.1, we observe that the dual cone $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ (see 1.1(a)) consists of $\mathbb{R}_+$, that is all positive reals $\alpha \geq 0$ (via the usual multiplication), and the singular functional $\bar{0}$ such that $\bar{0}(a) = 0$ for all $a \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$. Likewise, in 1.1(b), the continuous linear functionals on $\overline{\mathbb{R}}_+$ endowed with the neighborhood system $\mathcal{V} = \{0\}$ are the positive reals together with $\bar{0}$, but further include the element $+\infty$, acting as $+\infty(0) = 0$ and $+\infty(a) = +\infty$ for all $0 \neq a \in \overline{\mathbb{R}}_+$. This functional is obviously contained in the polar of the neighborhood $0 \in \mathcal{V}$. In 1.1(c), (d) and (e) on the other hand, due to the generality of the settings, a complete description for the respective dual cones is not readily available. We can, however, identify some of their elements: In 1.1(d), let $\mu \in \mathcal{P}^*$. Then the mapping

$$A \mapsto \sup\{\mu(a) \mid a \in A\} : \mathit{Conv}(\mathcal{P}) \to \overline{\mathbb{R}}$$

is seen to be an element of $\mathit{Conv}(\mathcal{P})^*$. In 1.1(e), for $\mu \in \mathcal{P}^*$ and $x \in X$, the mapping $\mu_x : \mathcal{F}_b(X, \mathcal{P}) \to \overline{\mathbb{R}}$ such that

$$\mu_x(f) = \mu\big(f(x)\big) \qquad \text{for all} \quad f \in \mathcal{F}_b(X, \mathcal{P})$$

is a continuous linear functional on $\mathcal{F}_b(X, \mathcal{P})$. In 1.1(f) for $p < +\infty$ the dual cone of $\mathcal{C}^p$ consists of all sequences $(y_i)_{i \in \mathbb{N}}$ such that $y_i \geq 0$ for all $i \in \mathbb{N}$ and $\|(y_i)\|_q < +\infty$, where q is the conjugate index of p. $\square$

We also consider a (topological and linear) closure of the given order on a locally convex cone, called the *weak preorder* $\preceq$ which is defined as follows (see I.3 in [11]):

We set $a \preceq b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if for every $\epsilon > 0$ there is $1 \leq \gamma \leq 1 + \epsilon$ such that $a \leq \gamma b + (1+\epsilon)v$, and set $a \preceq b$ if $a \preceq b + v$ for all $v \in \mathcal{V}$. That is $a \preceq b$ if for every $v \in \mathcal{V}$ and $\epsilon > 0$ there is $1 \leq \gamma \leq 1 + \epsilon$ such that $a \leq \gamma b + \epsilon v$. The weak preorder is weaker than the given order in the sense that $a \leq b$ or $a \leq b + v$ implies $a \preceq b$ or $a \preceq b + v$. Importantly, the weak preorder on a locally convex cone $\mathcal{P}$ is entirely determined by its dual cone $\mathcal{P}^*$, that is $a \preceq b$ holds if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*$, and $a \preceq b + v$ if and only $\mu(a) \leq \mu(b) + 1$ for all $\mu \in v^\circ$ (Corollaries I.4.31 and I.4.34 in [11]). If endowed with the weak preorder, $(\mathcal{P}, \mathcal{V})$ is again a locally convex cone with the same dual $\mathcal{P}^*$.

Unfortunately, the previously introduced upper, lower and symmetric locally convex cone topologies for $\mathcal{P}$ are not sufficiently compatible with its algebraic operations. While the addition $(a, b) \mapsto a + b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is continuous, the scalar multiplication $(\alpha, a) \mapsto \alpha a : [0, +\infty) \times \mathcal{P} \to \mathcal{P}$ is generally discontinuous at points where a is unbounded (Proposition I.1.1 in [11]. The latter proves particularly disadvantageous when dealing with continuous $\mathcal{P}$-valued functions. It is remedied using the coarser (but somewhat cumbersome) associated *relative* topologies on $\mathcal{P}$ instead. These topologies correspond to the weak preorder on $\mathcal{P}$. The *upper, lower and symmetric relative topologies* on a locally convex cone $(\mathcal{P}, \mathcal{V})$ are generated by the neighborhoods $v_\epsilon(a)$, $(a)v_\epsilon$ and $v_\epsilon^s(a) = v_\epsilon(a) \cap (a)_\epsilon$, respectively, for $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon > 0$, where

$$v_\epsilon(a) = \{b \in \mathcal{P} \mid b \leq \gamma a + \epsilon v \quad \text{for some} \quad 1 \leq \gamma \leq 1 + \epsilon\},$$
$$(a)v_\epsilon = \{b \in \mathcal{P} \mid a \leq \gamma b + \epsilon v \quad \text{for some} \quad 1 \leq \gamma \leq 1 + \epsilon\}.$$

We have $a \preceq b$ for $a, b \in \mathcal{P}$ if and only if $a \in v_\epsilon(b)$ for all $\epsilon > 0$. Propositions I.4.2 and I.4.4 from [11] yield that the respective relative topologies are the finest topologies on $\mathcal{P}$ with the following properties:

Proposition 1.3. *The upper, lower or symmetric relative topologies of $\mathcal{P}$ each satisfy the following:*

(a) *The relative topology is locally convex, coarser than the given topology but locally coincides with it at bounded elements.*

(b) *The mapping $(a, b) \mapsto a + b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is continuous.*

(c) *The mapping $(\alpha, a) \mapsto \alpha a : [0, +\infty) \times \mathcal{P} \to \mathcal{P}$ is continuous at all points $(\alpha, a) \in [0, +\infty) \times \mathcal{P}$ such that either $\alpha > 0$ or $a \in \mathcal{P}$ is bounded. (In this context the interval $[0, +\infty)$ is considered in the usual topology of* $\mathbb{R}$.)

Part (a) states that the relative topologies are locally convex, that is every element admits a neighborhood basis consisting of convex sets. These are however not necessarily locally convex cone topologies in our sense since the resulting uniformity need not be convex. Continuity of the scalar multiplication (Part (c) above) at points $(0, a)$ where $a \in \mathcal{P}$ is unbounded would contradict Part (a), that is the equivalence of the relative and the given neighborhood systems of $0 \in \mathcal{P}$.

Theorem I.4.30 in [11] provides further evidence for the suitability of the relative topologies due to their compatibility with the algebraic structure of a locally convex cone. It states that the closure of a convex subset A of $\mathcal{P}$ with respect to the upper and lower relative topologies, that is $\overline{A}^u$ and $\overline{A}^l$, respectively, is determined by the dual cone $\mathcal{P}^*$, that is

$$b \in \overline{A}^u \quad \text{if and only if} \quad \mu(b) \geq \inf\{\mu(a) \mid a \in A\} \quad \text{for all} \quad \mu \in \mathcal{P}^*,$$
$$b \in \overline{A}^l \quad \text{if and only if} \quad \mu(b) \leq \sup\{\mu(a) \mid a \in A\} \quad \text{for all} \quad \mu \in \mathcal{P}^*.$$

The closed convex hull with respect to the symmetric relative topology, that is $\overline{A}^s$, is a subset but does generally not coincide with $\overline{A}^l \cap \overline{A}^u$, as simple examples can demonstrate.

For these reasons we shall use the relative rather than the given topologies in our forthcoming investigations about compactness in locally convex cones. For detailed examples illustrating the interplay of these topologies we refer to Examples I.4.37 in [11]. We note in particular that in some important settings the relative and the corresponding given topologies coincide, for example if $\mathcal{P}$ is a locally convex vector space (Proposition1.3(a)), if $\mathcal{P} = \overline{\mathbb{R}}$ (I.4.37(a) in [11]) or if $\mathcal{P}$ is endowed with the weak topology of a duality $(\mathcal{P}, \mathbb{Q})$ (Proposition 3.2 below).

For $\mathcal{P} = \overline{\mathbb{R}}_+$ (Example 1.1(b)) the symmetric relative topology coincides with the Euclidean topology on $(0, +\infty)$, but renders 0 and $+\infty$ as isolated points.

Importantly, a linear functional $\mu \in \mathcal{P}^*$ is also continuous with respect to the upper, lower and symmetric relative topologies in $\mathcal{P}$ and the corresponding given topologies in $\overline{\mathbb{R}}$ (Proposition I.4.5 in [11]). As mentioned before, the given and relative topologies for $\overline{\mathbb{R}}$ coincide.

We shall use the following technical observations:

Lemma 1.4. *Let $a, b \in \mathcal{P}$, $v \in \mathcal{V}$, $\lambda \geq 0$ and $\epsilon, \delta > 0$.*

(a) *If $a \in v_\epsilon(b)$ and $0 \leq b + \lambda v$ for $\lambda \geq 0$, then $a \leq (1+\epsilon)b + \epsilon(1+\lambda)v$.*

(b) *If $a \in v_\epsilon(b)$, then $v_\delta(a) \subset v_{(\epsilon+\delta+\epsilon\delta)}(b)$. In particular, $v_\epsilon(a) \subset v_{3\epsilon}(b)$, provided that $\epsilon \leq 1$.*

(c) *$v_\delta(a)$ is contained in the topological interior of $v_\epsilon(a)$ for all $0 < \delta < \epsilon$.*

Similar statements hold for the lower and symmetric neighborhoods.

Proof. We shall prove the case for the upper neighborhoods. For Part (a) let $a \in v_\epsilon(b)$ and $0 \leq b + \lambda v$ for $\lambda \geq 0$. Then $a \leq \gamma b + \epsilon v$ for $1 \leq \gamma \leq 1 + \epsilon$ implies $a \leq \gamma b + \epsilon v + (1 + \epsilon - \gamma)(b + \lambda v) \leq (1+\epsilon)b + (\epsilon + \epsilon\lambda)v$, as claimed.

For Part (b) let $a \in v_\epsilon(b)$ and $c \in v_\delta(a)$. Then $a \leq \gamma b + \epsilon v$ and $c \leq \gamma' a + \delta v$, hence $c \leq \gamma'\gamma b + (\delta + \gamma'\epsilon)v$ for $1 \leq \gamma \leq 1+\epsilon$ and $1 \leq \gamma' \leq 1 + \delta$. Our claim follows since $1 \leq \gamma'\gamma \leq (1+\delta)(1+\epsilon) = 1 + (\epsilon + \delta + \epsilon\delta)$ and $\delta + \gamma'\epsilon \leq \delta + (1+\delta)\epsilon = \epsilon + \delta + \epsilon\delta$. If in particular $\delta = \epsilon \leq 1$, then $\epsilon + \delta + \epsilon\delta \leq 3\epsilon$. This verifies the second statement of Part (b).

Part (c) is an obvious consequence of Part (b). The arguments for the lower and symmetric neighborhoods are similar. □

Lemma 1.5. *Let $a, b \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon, \delta > 0$. Then*

(a) $\alpha\big(v_\epsilon(a)\big) = \big(\alpha v\big)_\epsilon(\alpha a)$ *for all* $\alpha > 0$.

(b) $a + v_\epsilon(b) \subset v_{\epsilon(1+\lambda)}(a+b)$ *provided that* $0 \le a + \lambda v$.

(c) $v_\epsilon(a) + v_\delta(b) \subset v_{(\epsilon+\delta)(1+\lambda)}(a+b)$ *provided that* $0 \le a + \lambda v$ *and* $0 \le b + \lambda v$.

Similar statements hold for the lower and symmetric neighborhoods.

Proof. Part (a) is obvious. For Part (b) let $a + c \in a + v_\epsilon(b)$, that is $c \le \gamma b + \epsilon v$ for $1 \le \gamma \le 1 + \epsilon$. Then

$$a + c \le \gamma b + a + \epsilon v \le \gamma b + a + \epsilon v + (\gamma - 1)(a + \lambda v) \le \gamma(a+b) + \epsilon(1+\lambda)v.$$

This shows $a + c \in v_{\epsilon(1+\lambda)}(a+b)$, hence our claim. For (c) let $c + d \in v_\epsilon(a) + v_\delta(b)$, that is $c \le \gamma a + \epsilon v$ and $d \le \gamma' b + \delta v$ for $1 \le \gamma \le 1+\epsilon$ and $1 \le \gamma' \le 1 + \delta$. Thus $c + d \le \gamma a + \gamma' b + (\epsilon+\delta)v$. If $\gamma \ge \gamma'$, we add the inequality $0 \le (\gamma - \gamma')(b + \lambda v)$ to the preceding one and obtain

$$\begin{aligned} c + d &\le \gamma(a+b) + (\epsilon+\delta)v + (\gamma-\gamma')\lambda v \le \gamma(a+b) + (\epsilon\delta + \epsilon\lambda v) \\ &\le \gamma(a+b) + (\epsilon+\delta)(1+\lambda v). \end{aligned}$$

The case $\gamma < \gamma'$ yields the same result. We have $c + d \in v_{(\epsilon+\delta)(1+\lambda)}(a+b)$ as claimed. The arguments for the lower and symmetric neighborhoods are similar. □

The symmetric relative topology is known to be Hausdorff if and only if the weak preorder on $\mathcal{P}$ is antisymmetric (Proposition I.4.8 in [11]). If endowed with the weak preorder, $(\mathcal{P}, \mathcal{V})$ is again a locally convex cone with the same dual $\mathcal{P}^*$ and equivalent relative neighborhoods. More precisely: Consider the relative neighborhoods in $\mathcal{P}$ that are defined using the weak preorder rather than the given order, that is for $a \in \mathcal{P}$, $\epsilon > 0$ and $v \in \mathcal{V}$ let

$$\tilde{v}_\epsilon(a) = \{b \in \mathcal{P} \mid b \preceq \gamma a + \epsilon v \quad \text{for some} \quad 1 \le \gamma \le 1 + \epsilon\}.$$

Clearly $v_\epsilon(a) \subset \tilde{v}_\epsilon(a)$. Equality does not hold true, but we observe the following.

Lemma 1.6. *For $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon > 0$ we have $\tilde{v}_\epsilon(a) = \bigcap_{\sigma>\epsilon} v_\sigma(a) = \bigcap_{\sigma>\epsilon} \tilde{v}_\sigma(a)$. Similar statements hold for the lower and symmetric neighborhoods.*

Proof. Let $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon > 0$. If $b \in \tilde{v}_\epsilon(a)$, then $b \preceq \gamma a + \epsilon v$ for some $1 \le \gamma \le 1 + \epsilon$. Given $\sigma > \epsilon$ set $\delta = (\sigma - \epsilon)/(1+\epsilon)$, that is $1 + \delta = (1+\sigma)/(1+\epsilon)$. There is $1 \le \rho \le 1 + \delta$ such that

$$b \le \rho(\gamma a) + (1+\delta)(\epsilon v).$$

Since $1 \le \rho\gamma \le (1+\delta)(1+\epsilon) = 1 + \sigma$ and $(1+\delta)\epsilon = \big((1+\sigma)/(1+\epsilon)\big)\epsilon \le (\sigma/\epsilon)\epsilon = \sigma$, we infer that $b \in v_\sigma(a)$. Thus

$$\tilde{v}_\epsilon(a) \subset \bigcap_{\sigma>\epsilon} v_\sigma(a) \subset \bigcap_{\sigma>\epsilon} \tilde{v}_\sigma(a).$$

For the reverse inclusion, suppose that $b \in \tilde{v}_\sigma(a)$ for all $\sigma > \epsilon$ and consider a downward directed net $(\sigma)_{i\in\mathcal{I}}$ converging to ϵ. We have $b \preceq \gamma_i a + \sigma_i v$ for $1 \le \gamma_i \le 1 + \sigma_i$ and may assume that the net $(\gamma_i)_{i\in\mathcal{I}}$ converges to some $1 \le \gamma \le 1 + \epsilon$. Thus for every $\mu \in v^\circ$ we have $\mu(b) \le \gamma\mu(a) + \epsilon$, verifying that $b \preceq \gamma a + \epsilon v$ (Corollary I.4.34 in [11]). This yields $b \in \tilde{v}_\epsilon(a)$ as claimed. The arguments for the lower and symmetric neighborhoods are similar. $\square$

Proposition 1.7. *Suppose that $\mathcal{P}$ is endowed with its weak preorder. Then for $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon > 0$ the upper, lower and symmetric neighborhoods $v_\epsilon(a)$, $(a)v_\epsilon$ and $v^s_\epsilon(a)$ are closed in the lower, upper and symmetric relative topologies, respectively.*

Proof. We consider the case of the upper neighborhoods: Let $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon > 0$. Given $b \in \overline{v_\epsilon(a)}^l$ and $\sigma > \epsilon$ we choose $\delta = (\sigma - \epsilon)/(1 + \epsilon)$. There is $c \in (b)v_\delta \cap v_\epsilon(a)$. Now $b \in v_\delta(c)$ and $c \in v_\epsilon(a)$ implies that $b \in v_{(\epsilon+\delta+\epsilon\delta)}(a) = v_\sigma(a)$ by Lemma 1.4(b). Hence $b \in v_\epsilon(a)$ by Lemma 1.6. The arguments for the lower and symmetric neighborhoods are similar. $\square$

Lemma 1.8. *Suppose that $\mathcal{P}$ is endowed with its weak preorder, let $v \in \mathcal{V}$ and $\epsilon > 0$. Let $a, b, c \in \mathcal{P}$ such that a is bounded, that is $a \le \lambda v$ for some $\lambda \ge 0$. If $a + c \in v_\epsilon(a + b)$, then $c \in v_{\epsilon(1+\lambda)}(b)$. Similar statements hold for the lower and symmetric neighborhoods.*

Proof. We consider the case of the upper neighborhoods. Let $a, b, c \in \mathcal{P}$, $v \in \mathcal{V}$, $\lambda \ge 0$ and $\epsilon > 0$ as stated. Let $a + c \in v_\epsilon(a + b))$, that is $a + c \le \gamma(a + b) + \epsilon v$ for some $1 \le \gamma \le 1 + \epsilon$. For $\mu \in v^\circ$ this yields $\mu(a) + \mu(c) \le \gamma(\mu(a) + \mu(b)) + \epsilon$, hence

$$\mu(c) \le \mu(\gamma b) + (\gamma - 1)\mu(a) + \epsilon \le \mu(\gamma b) + \epsilon\lambda + \epsilon \le \mu(\gamma b) + \epsilon(1 + \lambda)$$

since $\mu(a) < +\infty$. Because $\mathcal{P}$ carries the weak preorder, this yields $c \le \gamma b + \epsilon(1 + \lambda)v$, hence $c \in v_{\epsilon(1+\lambda)}(b)$ as claimed. The arguments for the lower and symmetric neighborhoods are similar. $\square$

The statement of Lemma 1.8 holds for all $\epsilon > 0$ and furnishes a cancellation property for bounded elements of $\mathcal{P}$. That is, if $a + c \preceq a + b$ holds for $a, b, c \in \mathcal{P}$ such that a is bounded, then $c \preceq b$. This observation will be strengthened in Proposition 2.6 below.

We shall at times assume that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is endowed with its weak preorder rather than the given one. This assumption does not interfere with its topologies, uniform structures, its duality or any concepts of boundedness. Moreover, the second iteration of the weak preorder, that is the second weak preorder generated by the first one, is known to coincide with the first one (Lemma I.3.2 in [11]).

2. Compactness, precompactness and completeness in the relative topologies of a locally convex cone

Compactness is defined in general topological spaces, but the notions of precompactness and completeness require an uniform structure, which in the case of the upper and lower relative topologies of a locally convex cone is generally not symmetric. There are different approaches to this situation (see for example [3]).

In [5] Motallebi defined and investigated precompactness and completeness and their relation to compactness for the given topologies of a locally convex cone. We shall use the relative topologies instead which are better compatible with the algebraic operations. We also apply different notions of precompactness and completeness. In the case of a symmetric uniformity they all coincide. For the sake of simplicity we shall omit the explicit reference to the relative topologies in the upcoming notations.

We shall say that a non-empty subset A of a locally convex cone $(\mathcal{P}, \mathcal{V})$ is *upper, lower* or *symmetric precompact (with respect to the relative topologies)* if for every $v \in \mathcal{V}$ and $\epsilon > 0$ there are $a_1, \ldots, a_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} v_\epsilon(a_i), \qquad A \subset \bigcup_{i=1}^{n} (a_i)v_\epsilon \qquad \text{or} \qquad A \subset \bigcup_{i=1}^{n} v_\epsilon^s(a_i)$$

respectively. We note that according to this definition every subset B of a symmetric precompact set A is again symmetric precompact. Indeed, given $v \in \mathcal{V}$ and $\epsilon > 0$ set $\delta = \min\{1, \epsilon/3\}$. There are $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{i=1}^{n} v_\delta^s(a_i)$. Choose $b_i \in v_\delta^s(a_i) \cap B$, provided that this intersection is not empty. For these indexes i, Lemma 1.4(b) yields that $v_\delta^s(a_i) \subset v_{3\delta}^s(b_i) \subset v_\epsilon^s(b_i)$. Hence the union of these neighborhoods $v_\epsilon^s(b_i)$ contains B. However, simple examples (see 2.4 below) show that a subset of an upper or lower precompact set is not necessarily again upper or lower precompact. A symmetric precompact set is both upper and lower precompact, but a set which is both upper and lower precompact need not be symmetric precompact.

Let A be a subset of $\mathcal{P}$. A net $(a_i)_{i \in \mathcal{I}}$ in A is an *upper, lower or symmetric Cauchy net (with respect to the relative topologies) in A* if for every $v \in \mathcal{V}$ and $\epsilon > 0$ there is $a \in A$ and $i_0 \in \mathcal{I}$ such that

$$a_i \in v_\epsilon(a), \qquad a_i \in (a)v_\epsilon \qquad \text{or} \qquad a_i \in v_\epsilon^s(a) \qquad \text{for all} \quad i \geq i_0,$$

respectively. Note that these definitions involve and depend on the containing set, that is in the upper and lower cases. Obviously, a net with a limit in A is a Cauchy net in A, and a subnet of a Cauchy net is again a Cauchy net. In the symmetric case we recover the usual definition of a Cauchy net, not depending on the containing set.

Proposition 2.1. *Let A be a subset of $\mathcal{P}$. A net $(a_i)_{i \in \mathcal{I}}$ in A is a symmetric Cauchy net if and only if for every $v \in \mathcal{V}$ and $\epsilon > 0$ there is $i_0 \in \mathcal{I}$ such that $a_i \in v_\epsilon^s(a_k)$ for all $i, k \geq i_0$.*

Proof. Clearly, the condition of this proposition implies that $(a_i)_{i \in \mathcal{I}}$ is a symmetric Cauchy net according to our definition. On the other hand, if $(a_i)_{i \in \mathcal{I}}$ satisfies this definition, that is, given $v \in \mathcal{V}$ and $\epsilon > 0$ we set $\delta = \min\{1, \epsilon/3\}$ and find $a \in A$ such that $a_k \in v_\delta^s(a)$ for all $k \geq i_0$. According to Lemma 1.4(b), $a \in v_\delta^s(a_k)$ implies that $v_\delta^s(a) \subset v_\epsilon^s(a_k)$. Thus $a_i \in v_\epsilon^s(a_k)$ for all $i, k \geq i_0$, as claimed. □

Proposition 2.2. *A subset A of a locally convex cone $(\mathcal{P}, \mathcal{V})$ is upper, lower or symmetric precompact if and only if every ultranet in A is an upper, lower or symmetric Cauchy net, respectively.*

Proof. We shall discuss the upper case. Suppose that A is upper precompact and let $(a_i)_{i \in \mathcal{I}}$ be an ultranet in A. Given $v \in \mathcal{V}$ and $\epsilon > 0$ there are $b_1, \ldots, b_n \in A$

such that $A \subset \bigcup_{k=1}^{n} v_\epsilon(b_k)$. As an ultranet $(a_i)_{i\in\mathcal{I}}$ is residually contained in one of these upper neighborhoods $v_\epsilon(b_{k_0})$. Thus there is $i_0 \in \mathcal{I}$ such that $a_i \in v_\epsilon(b_{k_0})$ for all $i \geq i_0$, hence $(a_i)_{i\in\mathcal{I}}$ is indeed an upper Cauchy net in A. For the reverse, assume that every ultranet is upper Cauchy in A, but A is not upper precompact. Then there are $v \in \mathcal{V}$ and $\epsilon > 0$ such that for every finite subset $\mathfrak{i} = \{a_1, \ldots, a_n\}$ the union $\bigcup_{k\in\mathfrak{i}} v_\epsilon(a_k)$ does not cover A. Consider the index set $\mathcal{I}$ consisting of all finite subsets $\mathfrak{i}$ of A and ordered by set inclusion. For each $\mathfrak{i} = \{a_1, \ldots, a_n\} \in \mathcal{I}$ we choose an element $a_\mathfrak{i} \in A \setminus \bigcup_{k\in\mathfrak{i}} v_\epsilon(a_k)$. Let $(b_j)_{j\in\mathcal{J}}$ be an ultra subnet of $(a_\mathfrak{i})_{\mathfrak{i}\in\mathcal{I}}$, defined by the mapping $\varphi : \mathcal{J} \to \mathcal{I}$. By our assumption $(b_j)_{j\in\mathcal{J}}$ is upper Cauchy, that is there are $b \in A$ and $j_0 \in \mathcal{J}$ such that $b_j \in v_\epsilon(b)$ for all $j \geq j_0$. But for $\mathfrak{i} = \{b\}$ there is $j \in \mathcal{J}$ such that both $\varphi(j) \geq \mathfrak{i}$ and $j \geq j_0$, that is $b_j \notin v_\epsilon(b)$, a contradiction. The arguments in the lower and symmetric cases are similar. □

A subset A of $\mathcal{P}$ is called *upper, lower or symmetric complete (with respect to the relative topologies)* if every upper, lower or symmetric Cauchy net in A admits a subnet that is convergent towards a limit in A in the upper, lower or symmetric relative topology; or equivalently, if every upper, lower or symmetric Cauchy ultranet is convergent towards a limit in A in the upper, lower or symmetric relative topology.

In case of the symmetric relative topology ist is straightforward to verify that a Cauchy net which admits a convergent subnet is itself convergent. A set which is both upper and lower complete need not be symmetric complete (see 2.4 below). The following theorem recovers a corresponding equivalence in locally convex topological vector spaces (see for example Theorem III.5 in [9]). Again, the term *compact* will refer to the relative topologies of a locally convex cone.

Theorem 2.3. *A subset A of a locally convex cone $(\mathcal{P}, \mathcal{V})$ is upper, lower or symmetric compact if and only if it is both upper, lower or symmetric precompact and upper, lower or symmetric complete, respectively.*

Proof. We shall use the fact that a set A is compact if and only if every ultranet in A has a limit in A (see Theorem 17.4 in [13]). Let A be upper compact. Then every ultranet in A is convergent, hence upper Cauchy in A, and therefore A is upper precompact by Proposition 2.2. Likewise, every upper Cauchy net $(a_i)_{i\in\mathcal{I}}$ in A admits a convergent ultrasubnet, hence A is complete. Conversely, suppose that A is both upper precompact and complete, and let $(a_i)_{i\in\mathcal{I}}$ be an ultranet in A. Upper precompactness implies that $(a_i)_{i\in\mathcal{I}}$ is upper Cauchy. It is therefore convergent due to the upper completeness of A. This verifies that A is upper compact. The arguments in the lower and symmetric cases are similar. □

Every subset of a locally convex cone that contains a maximal or a minimal element is obviously upper or lower compact, since every upper or lower neighborhood of this element contains the whole set, respectively.

Example 2.4. The locally convex cone $\overline{\mathbb{R}}$ (see 1.1(a)) is itself upper compact, hence upper precompact and complete.

In general, a subset A of $\overline{\mathbb{R}}$ is upper precompact if and only if it either contains $+\infty$ or has an upper bound in $\mathbb{R}$. $A \subset \overline{\mathbb{R}}$ is upper complete if and only if the set $A \cap \mathbb{R}$ is either unbounded above or contains a maximal element. Consequently, $A \subset \overline{\mathbb{R}}$ is upper compact if and only if either $+\infty \in A$ and $A \cap \mathbb{R}$ is either unbounded above or contains a maximal element, or $A \subset \mathbb{R}$ contains a maximal element.

Similarly, a subset A of $\overline{\mathbb{R}}$ is lower precompact if and only if it is bounded below, and A is lower complete if and only if it is either unbounded below or contains a minimal element. Hence $A \subset \overline{\mathbb{R}}$ is lower compact if and only it contains a minimal element.

The symmetric relative topology on $\overline{\mathbb{R}}$ coincides with the usual one on $\mathbb{R}$, isolating the point $+\infty$. A subset A of $\mathcal{P}$ is therefore symmetric precompact if and only if $A \cap \mathbb{R}$ is bounded, and A is symmetric complete if and only if $A \cap \mathbb{R}$ is closed in $\mathbb{R}$. Thus $A \subset \overline{\mathbb{R}}$ is symmetric compact if and only if $A \cap \mathbb{R}$ is compact in the usual topology of $\mathbb{R}$.

Note that the respective notions for upper and lower precompactness, completeness and compactness do not necessarily combine into the symmetric ones. The set $[0, +\infty]$, for example, is both upper and lower compact, but not symmetric precompact. The set $[0, 1) \cup (1, 2]$ is both upper and lower compact, but not symmetric complete. □

Recall that a linear functional $\mu \in \mathcal{P}^*$ is also continuous with respect to the upper, lower and symmetric relative topologies in $\mathcal{P}$ and the corresponding given, hence also the relative topologies of $\overline{\mathbb{R}}$. Using the characterization of the compact subsets of $\overline{\mathbb{R}}$ in the preceding example we infer:

Proposition 2.5. *If $A \subset \mathcal{P}$ is upper or lower compact, then every linear functional $\mu \in \mathcal{P}^*$ attains its maximum or minimum value on A, respectively. If $A \subset \mathcal{P}$ is symmetric compact, then the set $\mu(A) \cap \mathbb{R}$ is compact in the usual topology of $\mathbb{R}$. In particular, μ attains both its maximum and minimum values on A.*

We proceed to investigate the relationship of our concepts of compactness and precompactness with some corresponding notions of boundedness in locally convex cones. Due to the intricacies of these concepts we will have to make use of a variety definitions. There are notions of boundedness referring to a particular neighborhood $v \in \mathcal{V}$ and corresponding concepts involving the whole neighborhood system. We commence introducing the boundedness components of a locally convex cone.

Given a neighborhood $v \in \mathcal{V}$ the *upper, lower and symmetric v-boundedness components* of an element $a \in \mathcal{P}$ are defined in Section I.4 of [11] as

$$\mathcal{B}_v(a) = \bigcup_{\epsilon>0} v_\epsilon(a), \qquad (a)\mathcal{B}_v = \bigcup_{\epsilon>0} (a)v_\epsilon \qquad \text{and} \qquad \mathcal{B}_v^s(a) = \bigcup_{\epsilon>0} v_\epsilon^s(a),$$

respectively. We note that $\mathcal{B}_v^s(a) = (a)\mathcal{B}_v \bigcap \mathcal{B}_v(a)$ and that all v-boundedness components are convex and contain positive multiples of their elements. The upper ones are indeed subcones of $\mathcal{P}$. The upper and lower v-boundedness components are known to be open in the upper and lower, closed in the lower and upper relative topologies, respectively (Proposition I.4.18 in [11]). The symmetric v-boundedness components are both open, closed and connected in the symmetric relative topology and furnish a partition of $\mathcal{P}$ into disjoint convex subsets (Proposition I.4.19 in [11]).

Moreover, the v-boundedness components of $\mathcal{P}$ are entirely determined by the functionals in $v^\circ \subset \mathcal{P}^*$, that is we have $b \in \mathcal{B}_v(a)$ if and only if $\mu(a) < +\infty$ for $\mu \in v^\circ$ implies that $\mu(a) < +\infty$, we have $b \in (a)B_v$ if and only if $\mu(a) = +\infty$ for $\mu \in v^\circ$ implies that $\mu(a) = +\infty$ and we have $b \in \mathcal{B}_v^s(a)$ if and only if $\mu(a) = +\infty$ for $\mu \in v^\circ$ if and only if $\mu(a) = +\infty$ (Propositions I.4.10, I.4.12 and I.4.14 in [11]).

The *upper, lower and symmetric global boundedness components* of $a \in \mathcal{P}$ are the sets

$$\mathcal{B}(a) = \bigcap_{v \in \mathcal{V}} \mathcal{B}_v(a), \qquad (a)\mathcal{B} = \bigcap_{v \in \mathcal{V}} (a)\mathcal{B}_v \qquad \text{and} \qquad \mathcal{B}^s(a) = \bigcap_{v \in \mathcal{V}} \mathcal{B}_v^s(a),$$

respectively. We note that $\mathcal{B}^s(a) = (a)\mathcal{B} \bigcap \mathcal{B}(a)$. The upper and lower global boundedness components are known to be closed in the lower and upper relative topologies, respectively. (Propositions I.4.11 and I.4.13 in [11]). The symmetric global boundedness components are closed and connected in the symmetric relative topology and furnish a partition of $\mathcal{P}$ into disjoint convex subsets (Proposition I.4.17 in [11]). The global boundedness components of $\mathcal{P}$ are entirely determined by the functionals in $\mathcal{P}^*$, that is we have $b \in \mathcal{B}(a)$ if and only if $\mu(a) < +\infty$ for $\mu \in \mathcal{P}^*$ implies that $\mu(a) < +\infty$, we have $b \in (a)B$ if and only if $\mu(a) = +\infty$ for $\mu \in \mathcal{P}^*$ implies that $\mu(a) = +\infty$ and we have $b \in \mathcal{B}^s(a)$ if and only if $\mu(a) = +\infty$ for $\mu \in \mathcal{P}^*$ if and only if $\mu(a) = +\infty$ (Propositions I.4.11, I.4.12 and I.4.24 in [11]). Note that $(a)\mathcal{B} = \mathcal{P}$ whenever the element $a \in \mathcal{P}$ is bounded.

We shall employ the following version of the cancellation law in $\mathcal{P}$. Its formulation uses the weak preorder which was introduced and characterized in Section 1.

Proposition 2.6. *Suppose that $a + c \preceq b + c$ for $a, b, c \in \mathcal{P}$.*

(a) *If $c \in \mathcal{B}_v(b)$ for $v \in \mathcal{V}$, then $a \preceq b + \epsilon v$ for all $\epsilon > 0$.*

(b) *If $c \in \mathcal{B}(b)$, then $a \preceq b$.*

Proof. We shall make use of the fact that the weak preorder is determined by the elements of the dual cone. Thus $a + c \preceq b + c$ for $a, b, c \in \mathcal{P}$ implies that $\mu(a) + \mu(c) \le \mu(b) + \mu(c)$ holds for all $\mu \in \mathcal{P}^*$. For (a) suppose that $c \in \mathcal{B}_v(b)$. We shall verify that $\mu(a) \le \mu(b)$ for all $\mu \in v^\circ$, hence $a \preceq b + \epsilon v$ for all $\epsilon > 0$ as stated.

Indeed, if $\mu(b) = +\infty$, this claim is trivial. If $\mu(b) < +\infty$, then $\mu(c) < +\infty$ as well since $\mu \in v^\circ$ and $c \in \mathcal{B}_v(b)$. Thus $\mu(a) \le \mu(b)$ holds in any case. For (b) suppose that $c \in \mathcal{B}(b)$. For every $\mu \in \mathcal{P}^*$ we have either $\mu(b) = +\infty$ or both $\mu(b), \mu(c) < +\infty$. Thus $\mu(a) \le \mu(b)$ holds in both cases. This yields $a \preceq b$ as claimed. □

If the weak preorder on $\mathcal{P}$ is antisymmetric, that is if the symmetric relative topology is Hausdorff, Part (b) shows that $a + c = b + c$ for $a, b, c \in \mathcal{P}$ such that $c \in \mathcal{B}(b)$ implies that $a = b$.

The relationship between precompactness and boundedness turns out to be significantly more delicate in locally convex cones than in topological vector spaces. We shall use the following notions of boundedness:

Let A be a subset of a locally convex cone $(\mathcal{P}, \mathcal{V})$ and let $v \in \mathcal{V}$.

(i) A is *v-bounded above relative to* $b \in \mathcal{P}$ if there are $\lambda, \rho \ge 0$ such that $a \le \rho b + \lambda v$ for all $a \in A$.

(ii) A is *v-bounded below relative to* $b \in \mathcal{P}$ if there are $\lambda, \rho \geq 0$ such that $b \leq \rho a + \lambda v$ for all $a \in A$.

(iii) A is *relatively v-bounded* if there are $\lambda, \rho \geq 0$ such that $a \leq \rho b + \lambda v$ for all $a, b \in A$.

The corresponding global notions of boundedness do not refer to a particular neighborhood: A subset $A \subset \mathcal{P}$ is *bounded above or below relative to* $b \in \mathcal{P}$ if it is v -bounded above or below relative to b for all $v \in \mathcal{V}$. Likewise, A is *relatively bounded* if it is relatively v-bounded for all $v \in \mathcal{V}$. Note that v-boundedness below relative to some $b \in \mathcal{P}$ implies v-bounded below relative to $0 \in \mathcal{P}$. Similarly,v-boundedness above relative to $0 \in \mathcal{P}$ implies v-boundedness above relative to all $b \in \mathcal{P}$. We shall use the terms *v-bounded above, v-bounded below, bounded above and bounded below* if we refer to the case $b = 0$. In case of a locally convex vector space with the equality as order, all the above notions of boundedness coincide with the usual one.

Proposition 2.7. *Let A be a subset of $\mathcal{P}$ and let $v \in \mathcal{V}$.*

(a) *If A is upper precompact, then A is v-bounded above relative to some element in its convex hull.*

(b) *If A is lower precompact and $A \subset (b)\mathcal{B}_v$ for $b \in \mathcal{P}$, then A is v-bounded below relative to b.*

(c) *If A is symmetric precompact and $A \subset \mathcal{B}_v^s(b)$ for $b \in \mathcal{P}$, then A is relatively v -bounded.*

Proof. For Part (a) let $A \subset \mathcal{P}$ be upper precompact. Given $v \in \mathcal{V}$ and $\epsilon = 1$ there are $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{i=1}^n v_\epsilon(a_i)$. There is $\lambda \geq 0$ such that $0 \leq a_i + \lambda v$ for all $i = 1, \ldots, n$. Set $b = (1/n)(a_1 + \ldots + a_n)$, which contained in the convex hull of A. Then for every $a \in A$ we have $a \in v_\epsilon(a_{i_0})$ for some i_0, that is $a \leq \gamma a_{i_0} + v$ for some $1 \leq \gamma \leq 2$. Hence

$$a \leq \gamma a_{i_0} + v + (2 - \gamma)(a_{i_0} + \lambda v) \leq 2a_{i_0} + (1 + 2\lambda)v.$$

We have $0 \leq 2a_i + 2\lambda v$ for all $i = 1, \ldots, n$. Thus adding the latter inequalities for all $i \neq i_0$ to the former results in

$$a \leq 2(a_1 + \ldots + a_n) + \big((1 + 2\lambda) + 2\lambda(n - 1)\big)v = (2n)\, b + (1 + 2\lambda n)v,$$

verifying our claim. For Part (b) suppose that $A \subset (b)B_v$ for some $b \in \mathcal{P}$ and that A is lower precompact. Given $v \in \mathcal{V}$ and $\epsilon = 1$ there are $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{i=1}^n (a_i)v_\epsilon$. There are $\lambda \geq 0$ such that $0 \leq a_i + \lambda v$ and $\alpha, \beta \geq 0$ such that $b \leq \alpha a_i + \beta v$ for all $i = 1, \ldots, n$. Then for every $a \in A$ we have $a \in (a_{i_0})v_\epsilon$ for some i_0, that is $a_{i_0} \leq \gamma a + v$ for some $1 \leq \gamma \leq 2$. Hence $0 \leq a_{i_0} + \lambda v \leq \gamma a + (1 + \lambda)v$ and $0 \leq a + (\lambda + 1)v$. Thus $0 \leq (2 - \gamma)a + (2 - \gamma)(\lambda + 1)v \leq (2 - \gamma)a + (1 + \lambda)v$. Adding this inequality to the above yields

$$a_{i_0} \leq (\gamma a + v) + (2 - \gamma)a + (\lambda + 1)v = 2a + (2 + \lambda)v.$$

Thus $b \leq (2\alpha)a + \big(\alpha(2 + \lambda) + \beta\big)v$ for all $a \in A$, as claimed. For Part (c) suppose that $A \subset (b)B_v^s$ for some $b \in \mathcal{P}$ and that A is symmetric precompact. According to Part (a) there is $c \in (b)B_v^s$ and $\alpha, \beta \geq 0$ such that $a \leq \alpha c + \beta v$ for all $a \in A$.

Moreover, according to Part (b) there are $\lambda, \rho \geq 0$ such that $c \leq \rho d + \lambda v$ for all $d \in A$. Combining these inequalities yields $a \leq (\alpha\rho)d + (\alpha\lambda + \beta)v$ for all $a, d \in A$, our claim. $\square$

Since $(0)\mathcal{B}_v = \mathcal{P}$ for every $v \in \mathcal{V}$, Part (b) implies in particular that every lower precompact subset of $\mathcal{P}$ is bounded below.

We obtain a canonical partition for symmetric precompact and symmetric compact sets:

Theorem 2.8. *Given $v \in \mathcal{V}$, a symmetric precompact subset A of $\mathcal{P}$ can be expressed as a finite union of disjoint subsets $A_1, \ldots, A_n$, each of which is relatively v-bounded and both open and closed in the trace of the symmetric relative topology on A. If A is compact, then the subsets A_i are also compact.*

Proof. Let $A \subset \mathcal{P}$ and $v \in \mathcal{V}$. For every $a \in A$ and $\epsilon > 0$ the symmetric relative neighborhood $v_\epsilon^s(a)$ is contained in $\mathcal{B}_v^s(a)$. Thus if A is symmetric precompact, only finitely many of the intersections of A with the symmetric v-boundedness components of $\mathcal{P}$ can be non-empty. Therefore A can be expressed as a finite union of disjoint subsets $A_1, \ldots, A_n$, each of which is the intersections of A with a symmetric v-boundedness component of $\mathcal{P}$. These components are both symmetric open and closed in $\mathcal{P}$, hence the sets A_i are themselves both open and closed in the trace of the symmetric relative topology on A. They are indeed symmetric compact, if A itself is symmetric compact. Furthermore, the subsets A_i of A are also symmetric precompact and therefore relatively v-bounded by Proposition 2.7(c). $\square$

We shall illustrate these attributes in the following examples.

Examples 2.9. If the neighborhood system $\mathcal{V}$ of a locally convex consists of multiples of a single neighborhood, then its v-boundedness components coincide with the global ones.

(a) In $\overline{\mathbb{R}}$ the symmetric boundedness components are $\mathcal{B}^s(a) = \mathbb{R}$ if $a \in \mathbb{R}$ and $\mathcal{B}^s(+\infty) = \{+\infty\}$. Theorem 2.8 thus confirms the observations from Example 2.4.

(b) The symmetric boundedness components in $\overline{\mathbb{R}}_+$ (see Ex. 1.1(b)) are $\mathcal{B}^s(0) = \{0\}$, $\mathcal{B}^s(a) = (0, +\infty)$ if $a \in (0, +\infty)$ and $\mathcal{B}^s(+\infty) = \{+\infty\}$. A subset A of $\overline{\mathbb{R}}_+$ is symmetric compact if and only if $A \cap (0, +\infty)$ is compact in the Euclidean topology.

(c) Let $\mathcal{P}$ be the cone of all $\overline{\mathbb{R}}$-valued functions on a set X endowed with the topology of pointwise convergence. That is, the neighborhood system $\mathcal{V}$ consists of the positive multiples of the functions v_Y, where $Y$is a finite subset of X and $v_Y(x) = 1$ if $x \in Y$, and $v_Y(x) + \infty$ else (see Example 1.1(e)). In fact, $\mathcal{P}$ is the product cone $\overline{\mathbb{R}}^X$, endowed with the upper, lower and symmetric relative product topologies of $\overline{\mathbb{R}}$. According to Tychonoff's theorem $\mathcal{P}$ itself is upper compact. The symmetric v_Y-boundedness component of a function $f \in \mathcal{P}$ is given by

$$\mathcal{B}_{v_Y}^s(f) = \{g \in \mathcal{P} \mid g(x) = +\infty \quad \text{for } x \in Y \text{ if and only if } \quad f(x) = +\infty\}.$$

It is both open and closed in the symmetric relative topology of $\mathcal{P}$. The global boundedness component of $f \in \mathcal{P}$ is given by

$$\mathcal{B}^s(f) = \{g \in \mathcal{P} \mid g(x) = +\infty \quad \text{for } x \in X \text{ if and only if } \quad f(x) = +\infty\}.$$

It is closed but not necessarily open. Thus the symmetric v_Y-boundedness components and the global boundedness components of $\mathcal{P}$ are characterized by subsets of Y and of X, respectively. Let us construct a symmetric compact subset of $\mathcal{P}$. For every $x \in X$ let S_x be a closed and bounded subset of $\mathbb{R}$ adjoined by the element $+\infty$. Then S_x is symmetric compact in $\overline{\mathbb{R}}$. Let

$$A = \{f \in \mathcal{P} \mid f(x) \in S_x \quad \text{for all} \quad x \in X\}.$$

Hence A is a symmetric compact subset of $\mathcal{P}$. Let $v_Y \in \mathcal{V}$. The finite decomposition of A into both open and closed subsets (in the trace of the symmetric relative topology on A) as in Theorem 2.8 is characterized by the subsets of Y : For each $Z \subset Y$ the corresponding component of A is

$$A_Z = \{f \in A \mid f(x) = +\infty \quad \text{for } x \in Z \text{ and } f(x) < +\infty \text{ for } x \in Y \setminus Z\}.$$

The disjoint union of these compact sets renders A. □

We shall conclude this section investigating how precompact, complete and compact subsets transfer within a locally convex cone. If the symmetric relative topology is Hausdorff, that is if the weak preorder of $\mathcal{P}$ is antisymmetric, the standard results about precompactness, completeness and compactness apply (see for example Ch. 3 in [9] or Ch. 6 and 9 in [13]):

Proposition 2.10. *Suppose that the weak preorder of $\mathcal{P}$ is antisymmetric. Then*

(a) *Every subset of a symmetric precompact set in $\mathcal{P}$ is again symmetric precompact.*

(b) *Every symmetric complete set in $\mathcal{P}$ is symmetric closed. A subset of a symmetric complete set in $\mathcal{P}$ is itself symmetric complete if and only if it is symmetric closed.*

(c) *Every symmetric compact set in $\mathcal{P}$ is symmetric closed. A subset of a symmetric compact set in $\mathcal{P}$ is itself symmetric compact if and only if it is symmetric closed.*

Relations involving the non-symmetric topologies tend to be more delicate. For our upcoming arguments we may always assume that $\mathcal{P}$ is endowed with its weak preorder since this assumption does not affect any of the above concepts. We shall list a few observations, the first of which is obvious:

Proposition 2.11. *A finite union of upper, lower or symmetric precompact, complete or compact subsets of $\mathcal{P}$ is again upper, lower or symmetric precompact, complete or compact, respectively.*

Using the formula $\alpha\big(v_\epsilon(a)\big) = \big(\alpha v\big)_\epsilon(\alpha a)$ from Lemma 1.5(a) for precompact and complete sets and Proposition 1.3(c) for compact sets yields immediately:

Proposition 2.12. *Let A be a subset of $\mathcal{P}$.*

(a) *If A is upper, lower or symmetric precompact or complete and $\alpha > 0$, then the set αA is also upper, lower or symmetric precompact or complete, respectively.*

(b) *If A is upper, lower or symmetric compact, and Λ is a compact subset of $(0, +\infty)$, then the set $A_\Lambda = \{\lambda a \mid a \in A,\ \lambda \in \Lambda\}$ is also upper, lower or symmetric compact, respectively.*

We use Lemma 1.5(b) and (c) and Proposition 1.3(b) for the following.

Proposition 2.13. *Let A and B be subsets $\mathcal{P}$ and let a be a bounded element of $\mathcal{P}$.*

(a) *If A and B are bounded below and upper, lower or symmetric precompact, then $A+B$ is also upper, lower or symmetric precompact.*

(b) *If B is upper, lower or symmetric complete, then $a+B$ is also upper, lower or symmetric complete.*

(c) *If A and B are upper, lower or symmetric compact, then $A+B$ is also upper, lower or symmetric compact.*

Proof. Let $A, B \subset \mathcal{P}$ and $a \in \mathcal{P}$ be as stated. For Part (a) suppose that A and B are bounded below and upper precompact. Given $v \in \mathcal{V}$ and $\epsilon > 0$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$ and $0 \leq b + \lambda v$ for all $a \in A$ and $b \in B$. We choose $\delta = \epsilon/(2+2\lambda)$ and find $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in B$ such that $A \subset \bigcup_{i=1}^{n} v_\delta(a_i)$ and $B \subset \bigcup_{k=1}^{m} v_\delta(b_k)$. Then

$$A + B \subset \bigcup \{v_\epsilon(a_i + b_k) \mid i = 1, \ldots, n, \quad k = 1, \ldots, m\}.$$

Indeed, for $a + b \in A + B$ we have $a \in v_\delta(a_i)$ and $b \in v_\delta(b_k)$ for some i and k. Thus $a + b \in v_\epsilon(a_i + b_k)$ by Lemma 1.5(c). For Part (b) suppose that B is upper complete and let $(a+b_i)_{i\in\mathcal{I}}$ be an upper Cauchy net in $a+B$. Then $(b_i)_{i\in\mathcal{I}}$ is an upper Cauchy net in B. Indeed, given $v \in \mathcal{V}$ and $\epsilon > 0$ let $\lambda \geq 0$ such that $a \leq \lambda v$.

Set $\delta = \epsilon/(1+\lambda)$. There is $a + b \in a + B$ and $i_0 \in \mathcal{I}$ such that $a + b_i \in v_\delta(a + b)$, hence $b_i \in v_\epsilon(b)$ by Lemma 1.8, for all $i \geq i_0$. Since B is upper complete, $(b_i)_{i\in\mathcal{I}}$ admits a subnet $(b_j)_{j\in\mathcal{J}}$ that has a limit $b \in B$ in the upper topology. We conclude that $(a+b_j)_{j\in\mathcal{J}}$ converges to $a+b \in a+B$. Indeed, given $v \in \mathcal{V}$ and $\epsilon > 0$ let $\lambda \geq 0$ such that $0 \leq a + \lambda v$. Set $\delta = \epsilon/(1+\lambda)$. There is $j_0 \in \mathcal{J}$ such that $b_j \in v_\delta(b)$ for all $j \geq j_0$. We have $a+v_\delta(b) \subset v_\epsilon(a+b)$ by Lemma 1.5(b), hence $a+b_j \in v_\epsilon(a+b)$ for all $j \geq j_0$. Obviously, $(a + b_j)_{j\in\mathcal{J}}$ is a subnet of $(a + b_i)_{i\in\mathcal{J}}$. Thus every upper Cauchy net in $a + B$ is has a convergent subnet, and $a + B$ is upper complete as claimed. Part (c) follows from the continuity of the mapping $(a, b) \mapsto a + b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ (Proposition 1.3(b)). The cases of the lower and symmetric relative topologies are argued in a similar way. □

We shall use Lemmas 1.5 and 1.8 for the following.

Proposition 2.14. *Let B be a subset and let a be a bounded element of $\mathcal{P}$. Then*

(a) *If $a + B$ is upper, lower or symmetric precompact, then B is upper, lower or symmetric precompact.*

(b) *If $a + B$ is upper, lower or symmetric complete, then B is upper, lower or symmetric complete.*

(c) *If $a + B$ is upper, lower or symmetric compact, then B is upper, lower or symmetric compact.*

Proof. We shall prove the statements for the upper relative topology. Let $B \subset \mathcal{P}$ and $a \in \mathcal{P}$ be as stated. For (a) suppose that $a + B$ is upper precompact, let $v \in \mathcal{V}$ and $\epsilon > 0$. There is $\lambda \geq 0$ such that $a \leq \lambda v$. For $\delta = \epsilon/(1 + \lambda)$ there are

$b_1, \ldots, b_n \in B$ such that $a + B \subset \bigcup_{i=1}^{n} v_\delta(a + b_i)$. Thus for every $b \in B$ we have $a + b \in v_\delta(a + b_k)$ for some $0 \leq k \leq n$. Hence $b \in v_\epsilon(b_k)$ according to Lemma 1.8.

This yields $B \subset \bigcup_{i=1}^{n} v_\epsilon(b_i)$. For Part (b) suppose that $a + B$ is upper complete and let $(b_i)_{i\in\mathcal{I}}$ be an upper Cauchy net in B. Then $(a + b_i)_{i\in\mathcal{I}}$ is an upper Cauchy net in $a + B$. Indeed, given $v \in \mathcal{V}$ and $\epsilon > 0$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$.

Set $\delta = \epsilon/(1 + \lambda)$. There is $b \in B$ and $i_0 \in \mathcal{I}$ such that $b_i \in v_\delta(b)$ for all $i \geq i_0$. We have $a + v_\delta(b) \subset v_\epsilon(a + b)$ by Lemma 1.5(b), hence $a + b_i \in v_\epsilon(a + b)$ for all $i \geq i_0$. The net $(a+b_i)_{i\in\mathcal{I}}$ therefore admits a convergent subnet $(a+b_j)_{j\in\mathcal{J}}$ with limit $a + b \in a + B$. We conclude that $(b_j)_{j\in\mathcal{J}}$ converges to $b \in B$. Indeed, given $v \in \mathcal{V}$ and $\epsilon > 0$ let $\lambda \geq 0$ such that $a \leq \lambda v$. Set $\delta = \epsilon/(1 + \lambda)$. There is $j_0 \in \mathcal{I}$ such that $b_j \in v_\delta(b)$ for all $j \geq j_0$. Hence $a+b_j \in v_\delta(a+b)$ and $b_j \in v_\delta(b)$ by Lemma 1.8 for all $j \geq j_0$. Thus every upper Cauchy net in B has an upper convergent subnet, and B is upper complete as claimed. Part (c) follows from Parts (a) and (b) together with Theorem 2.3. The cases of the lower and symmetric relative topologies are argued in a similar way. □

Proposition I.4.14(vii) in [11] states that for elements a and b of the same symmetric global boundedness component the mapping $\alpha \mapsto \alpha a + (1 - \alpha)b \ : \ [0, 1] \to \mathcal{P}$ is continuous with respect to the usual topology of $[0, 1]$ and the symmetric relative topology of $\mathcal{P}$. We prove a generalization of this result:

Lemma 2.15. *For a symmetric global boundedness component $\mathcal{B}^s(c)$ of $\mathcal{P}$ the map*

$$(a, b, \alpha) \mapsto \alpha a + (1 - \alpha)b \ : \ \mathcal{B}^s(c) \times \mathcal{B}^s(c) \times [0, 1] \to \mathcal{B}^s(c)$$

is continuous with respect to the usual topology of $[0, 1]$ *and each of the upper, lower or symmetric relative topologies on* $\mathcal{B}^s(c)$.

Proof. We shall argue the case that $\mathcal{B}^s(c)$ carries the upper relative topology. Let $(a_0, b_0, \alpha_0) \in \mathcal{B}^s(c)\times\mathcal{B}^s(c)\times[0, 1]$. Given $v \in \mathcal{V}$ and $0 < \epsilon \leq 1$, according to I.4.14(vii) in [11], that is the continuity of the mapping $\alpha \mapsto \alpha a_0 + (1-\alpha)b_0 \ : \ [0, 1] \to \mathcal{P}$ with respect to the symmetric relative topology, there is a neighborhood I of $\alpha_0 \in [0, 1]$ such that $\alpha \in I$ implies that $\alpha a_0 + (1 - \alpha)b_0 \in v^s_{\epsilon/3}\big(\alpha_0 a_0 + (1 - \alpha_0)b_0\big)$. Let $\lambda \geq 0$ such that both $0 \leq a_0 + \lambda v$ and $0 \leq b_0 + \lambda v$. We set $\sigma = \epsilon/3(1+\lambda)$. Then $a \in v_\sigma(a_0)$ and $b \in v_\sigma(b_0)$ implies

$$a \leq (1 + \sigma)a_0 + \frac{\epsilon}{3}v \quad \text{and} \quad b \leq (1 + \sigma)b_0 + \frac{\epsilon}{3}v$$

by Lemma 1.4(a), hence

$$\alpha a + (1 - \alpha)b \leq (1 + \sigma)\big(\alpha a_0 + (1 - \alpha)b_0\big) + \frac{\epsilon}{3}v$$

for all $\alpha \in I$. Since $\sigma \leq \epsilon/3 \leq 1$, this yields

$$\alpha a + (1 - \alpha)b \in v_{\epsilon/3}\big(\alpha a_0 + (1 - \alpha)b_0\big) \subset v_\epsilon\big(\alpha_0 a_0 + (1 - \alpha_0)b_0\big)$$

according to Lemma 1.4(b). The latter holds whenever $(a, b, \alpha) \in v_\sigma(a_0)\times v_\sigma(b_0)\times I$, proving continuity with respect to the upper relative topology on $\mathcal{B}^s(c)$ for our mapping at $(a_0, b_,\alpha_0)$. Similarly one argues the cases of the lower and symmetric topologies. □

This yields immediately:

Proposition 2.16. *If two upper, lower or symmetric compact convex subsets of $\mathcal{P}$ are contained in the same symmetric global boundedness component, then the convex hull of their union is also upper, lower or symmetric compact.*

Proof. Suppose that A and B are subsets of the symmetric global boundedness component $\mathcal{B}^s(c)$. The convex hull of $A \cup B$ is the image of the upper, lower or symmetric compact product $A \times B \times [0,1]$ under the continuous mapping $(a, b, \alpha) \mapsto \alpha a + (1-\alpha)b$, hence is also upper, lower or symmetric compact. □

3. Compactness in weak topologies

In this section we shall further investigate the previously introduced concepts of compactness, precompactness and completeness for the special case that a locally convex cone carries the weak topology of a given duality. Similar studies using the given rather than the relative topologies have been carried out in [5], [6] and [7]. Dual pairs of cones, bilinear forms and polar topologies were introduced in Chapter II.3 of [4].

A *dual pair* $(\mathcal{P}, \mathbb{Q})$ consists of two cones $\mathcal{P}$ and $\mathbb{Q}$ together with a bilinear form $(a, x) \mapsto \langle a, x\rangle : \mathcal{P} \times \mathbb{Q} \to \overline{\mathbb{R}}$. Various locally convex cone topologies on $\mathcal{P}$ are introduced by suitable families of subsets of $\mathbb{Q}$ leading to Mackey-Arens type results for the resulting dual cone $\mathcal{P}^*$ (Theorem II.3.8 in [4]). The order on $\mathcal{P}$ is defined canonically as $a \le b$ for $a, b \in \mathcal{P}$ if $\langle a, x\rangle \le \langle b, x\rangle$ for all $x \in \mathbb{Q}$. In our context we are interested in the weak upper, lower and symmetric topologies $\sigma(\mathcal{P}, \mathbb{Q})$ on $\mathcal{P}$ which are generated by the neighborhoods v_X corresponding to finite subsets X of $\mathbb{Q}$. The upper neighborhood $v_X(a)$ of an element $a \in \mathcal{P}$ is defined as

$$v_X(a) = \{b \in \mathcal{P} \mid \langle b, x\rangle \le \langle a, x\rangle + 1 \quad \text{for all} \quad x \in X\}.$$

The collection $\mathcal{V}_\sigma$ of all these neighborhoods forms a base for the neighborhood system $\mathcal{V}_{\sigma(\mathcal{P},\mathbb{Q})}$ and generates the locally convex cone $(\mathcal{P}, \mathcal{V}_{\sigma(\mathcal{P},\mathbb{Q})})$. The symmetric $\sigma(\mathcal{P}, \mathbb{Q})$-topology is Hausdorff if and only if the order on $\mathcal{P}$ is antisymmetric.

Example 3.1. Let $\mathcal{P}$ be the cone of all $\overline{\mathbb{R}}$-valued functions on a set X (see Example 2.9(c)) and let $\mathbb{Q}$ be the cone generated by all point evaluations on $\mathcal{P}$, that is $\mathbb{Q} = \{\sum_{i=1}^n \alpha_i \delta_{x_i} \mid \alpha_i \ge 0, \quad x_i \in X\}$. We use the canonical bilinear form on $(\mathcal{P}, \mathbb{Q})$ setting $\langle f, \sum_{i=1}^n \alpha_i \delta_{x_i}\rangle = \sum_{i=1}^n \alpha_i f(x_i)$. Then the weak topology $\sigma(\mathcal{P}, \mathbb{Q})$ on $\mathcal{P}$ is indeed the topology of pointwise convergence from Example 2.9(c).

Proposition 3.2. *If a locally convex cone $\mathcal{P}$ is endowed with the weak topology of a duality $(\mathcal{P}, \mathbb{Q})$, then its given and its weak preorder as well as its given and its relative upper, lower and symmetric topologies each coincide.*

Proof. If $a \preceq b$ for $a, b \in \mathcal{P}$, then for $x \in \mathbb{Q}$ and every $\epsilon > 0$ there is $1 \le \gamma \le 1 + \epsilon$ such that $\langle a, x\rangle \le \gamma\langle b, x\rangle + \epsilon$. This yields $\langle a, x\rangle \le \langle b, x\rangle$, hence $a \le b$. For the second statement we first argue the case of the upper topologies. Clearly, every upper relative neighborhood contains an upper neighborhood in the given topology.

For the converse let $a \in \mathcal{P}$ and let $v \in \mathcal{V}_\sigma$ be a neighborhood defined by a finite set $X \subset \mathbb{Q}$. Choose $\rho \geq \max\{|\langle a,x\rangle| \,|\, x \in X \text{ such that } \langle a,x\rangle < +\infty\}$ and set $\epsilon = 1/(1+\rho)$. Then $v_\epsilon(a) \subset v(a)$. Indeed, if $b \in v_\epsilon(a)$, then $b \leq \gamma a + \epsilon v$ for some $1 \leq \gamma \leq 1+\epsilon$, that is $\langle b,x\rangle \leq \gamma\langle a,x\rangle + \epsilon$ for all $x \in X$. Thus

$$\langle b,x\rangle \leq \langle a,x\rangle + (\gamma-1)\langle a,x\rangle + \epsilon \leq \langle a,x\rangle + \epsilon(1+\rho) = \langle a,x\rangle + 1,$$

for $x \in X$, provided that $\langle a,x\rangle < +\infty$. The case $\langle a,x\rangle = +\infty$ is obvious. Thus $b \in v(a)$ and $v_\epsilon(a) \subset v(a)$ holds as claimed. For the lower neighborhoods with the same choice for ρ and ϵ let $b \in (a)v_\epsilon$. Then $a \leq \gamma b + \epsilon v$ for some $1 \leq \gamma \leq 1+\epsilon$, hence $(1/\gamma)a \leq b + \epsilon v$ and $a \leq b + \big(1-(1/\gamma)\big)a + \epsilon v$. Thus

$$\langle a,x\rangle \leq \langle b,x\rangle + \frac{\gamma-1}{\gamma}\langle a,x\rangle + \epsilon \leq \langle b,x\rangle + \epsilon(1+\rho) = \langle b,x\rangle + 1,$$

for $x \in X$, provided that $\langle a,x\rangle < +\infty$. If $\langle a,x\rangle = +\infty$, then $\langle b,x\rangle = +\infty$ as well, and we conclude that $b \in (a)v$. The case of the symmetric neighborhoods follows. □

For weak topologies the notion of compactness therefore coincides in both the given and the relative topologies. However, the uniformities induced by both sets of topologies are generally not equivalent. Thus for precompactness and completeness we shall use the notions for the relative topologies developed in the preceding sections. We have $\big(\epsilon v\big)(a) \subset v_\epsilon(a)$, $(a)\big(\epsilon v\big) \subset (a)v_\epsilon$ and $\big(\epsilon v\big)^s(a) \subset v^s_\epsilon(a)$ for all $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\epsilon > 0$. Precompactness in the given upper, lower or symmetric topology therefore implies precompactness in the corresponding relative topology.

For our upcoming statements about weak precompactness we require the following technical lemma. By $\overline{\mathbb{Z}}$ we denote the set of all integers augmented by the element $+\infty$. For $n \in \mathbb{N}$ we endow $\overline{\mathbb{Z}}^n$ with the componentwise order. For a subset A of $\overline{\mathbb{Z}}^n$ an element $a \in A$ is *maximal* if $a \leq b$ implies that $a = b$. Similarly, $a \in A$ is *minimal* if $a \geq b$ for $b \in A$ implies that $a = b$. If $A \subset \overline{\mathbb{Z}}^n$ is bounded above or below by an element of $\mathbb{Z}^n$, then a simple counting argument shows that for every $a \in A$ there is a maximal or a minimal element $b \in A$ such that $a \leq b$ or $a \geq b$, respectively. Furthermore, if B is a subset of A such that for every $a \in A$ there is $b \in B$ such that $a \leq b$ or that $b \leq A$, then B contains all maximal or all minimal elements of A, respectively.

Lemma 3.3. *If a subset A of $\overline{\mathbb{Z}}^n$ is bounded above or below by an element of $\mathbb{Z}^n$, then it contains only finitely many maximal or minimal elements.*

Proof. We shall first consider the bounded below case. Let $A \subset \overline{\mathbb{Z}}^n$ be bounded below by an element of $\mathbb{Z}^n$. We shall demonstrate by induction that A contains only finitely many minimal elements. For $n = 1$, that is $A \subset \overline{\mathbb{Z}}$, there is nothing to prove. Now suppose that our statement holds for some $n \geq 1$ and let $A \subset \overline{\mathbb{Z}}^{n+1}$. We consider the projection mapping $P : \overline{\mathbb{Z}}^{n+1} \to \overline{\mathbb{Z}}^n$ such that $(z_1,\ldots,z_{n+1}) \mapsto (z_1,\ldots,z_n)$ and $p : \overline{\mathbb{Z}}^{n+1} \to \overline{\mathbb{Z}}$ such that $(z_1,\ldots,z_{n+1}) \mapsto z_{n+1}$. We set

$$A_\infty = \{a \in A \mid p(a) = +\infty\} \qquad \text{and} \qquad A_{oo} = \{a \in A \mid p(a) < +\infty\}.$$

Then $A = A_\infty \cup A_{oo}$ and by our assumption for n the subsets $P(A_\infty)$ and $P(A_{oo})$ of $\overline{\mathbb{Z}}^n$ have only finitely many minimal elements each. They are projections under P of finite subsets B_∞ of A_∞ and B_{oo} of A_{oo}, respectively. Let $l = \min\{p(a) \mid a \in A\}$ and $m = \max\{p(b) \mid b \in B_{oo}\}$. For all integers $l \leq k \leq m-1$ set

$$A_k = \{a \in A \mid p(a) = k\}.$$

Every set $P(A_k) \subset \mathbb{Z}^n$ has a finite subset of minimal elements, each a projection under P of a finite subset B_k of A_k. Now let $B = \big(B_\infty \cup B_{oo}\big) \cup_{k=l}^{m-1} B_k$. We claim that for every $a \in A$ we have $a \geq b$ for some $b \in B$. Indeed, if $p(a) = +\infty$, we have $a \in A_\infty$ and $P(a) \geq P(b)$ for some $b \in B_\infty$. Hence $a \geq b$ since $p(a) = p(b) = +\infty$.

If $p(a) = k$ for some $l \leq k \leq m-1$, we have $a \in A_k$ and $P(a) \geq P(b)$ for some $b \in B_k$. Hence $a \geq b$ since $p(a) = p(b) = k$. Finally, if $m \leq p(a) < +\infty$, we have $a \in A_{oo}$ and $P(a) \geq P(b)$ for some $b \in B_{oo}$. Hence $a \geq b$ since $p(a) \geq p(b)$. The set $B \subset A$ is finite, contains lower bounds for all elements of A, hence includes all of its minimal elements, our claim. For the bounded above case, suppose that $A \subset \overline{\mathbb{Z}}^n$ is bounded above by an element of $\mathbb{Z}^n$. Then $-A \subset \overline{\mathbb{Z}}^n$ is bounded below, and the preceding guarantees the existence of only finitely many minimal elements of $-A$, that is maximal elements of A. □

Using this lemma we reverse the conclusions of Proposition 2.7 and Theorem 2.8 for weak topologies.

Theorem 3.4. *Suppose that $\mathcal{P}$ is endowed with the weak topology of a duality $(\mathcal{P}, \mathbb{Q})$.*

(a) *A convex subset A of $\mathcal{P}$ is upper precompact if and only if for every $v \in \mathcal{V}_\sigma$ it is v-bounded above relative to some element $b \in A$.*

(b) *A subset A of $\mathcal{P}$ is lower precompact if and only if it is bounded below.*

(c) *A subset A of $\mathcal{P}$ is symmetric precompact if and only if for every $v \in \mathcal{V}_\sigma$ it can be expressed as a finite union of disjoint relatively v-bounded subsets $A_1, \ldots, A_n$.*

Precompactness in the given and relative topologies coincide for all subsets of $\mathcal{P}$ in the lower and symmetric case and for convex subsets of $\mathcal{P}$ in the upper case.

Proof. Let $A \subset \mathcal{P}$. Proposition 2.7 and Theorem 2.8 state that upper, lower or symmetric precompactness in the relative topologies implies the respective conditions in (a), (b) and (c). In turn, we shall prove that these conditions imply precompactness in the given topologies of $\mathcal{P}$, which as earlier remarked imply precompactness in the corresponding relative topologies. Given a neighborhood v in $\sigma(\mathcal{P}, \mathbb{Q})$ we denote by X the finite subset of $\mathbb{Q}$ which defines v. Under the assumptions of the respective cases we shall establish coverings by finite sets of neighborhoods $v(a_i)$, $(a_i)v$ and $v^s(a_i)$. Unfortunately, we shall have to discuss all three cases since the arguments are sufficiently dissimilar.

For Part (a) let $A \subset \mathcal{P}$ and $b \in A$ be as stated, that is there are $\lambda, \rho \geq 0$ such that $a \leq \rho b + \lambda v$ for all $a \in A$. Let X' be the subset of all $x \in X$ such that $\langle b, x\rangle < +\infty$ and $X'' = X \setminus X'$. If $X' = \emptyset$, then $A \subset w(b)$, and we are done. Otherwise, since $\langle a, x\rangle \leq \rho\langle b, x\rangle + \lambda$ for all $x \in X$, we find $N \in \mathbb{Z}$ such that $\langle a, x\rangle \leq N$ for all $x \in X'$.

Now let $\mathfrak{T}$ be the family of all mappings t from X' into the integers up to $2N$. It corresponds to a bounded above subset of $\mathbb{Z}^n$ for some n. For every $t \in \mathfrak{T}$ let A_t be the subset of all $a \in A$ such that $(t(x)-1)/2 < \langle a, x\rangle \le t(x)/2$ for all $x \in X'$. Then $A = \bigcup_{t\in\mathfrak{T}} A_t$. According to Lemma 3.3 the subset $\mathfrak{T}'$ of $\mathfrak{T}$ consisting of those $t \in \mathfrak{T}$ such that $A_t \ne \emptyset$ admits a finite subset $\mathfrak{T}''$ of maximal elements.

Next we choose elements $a_t \in A_t$ for each $t \in \mathfrak{T}''$. For each $t \in \mathfrak{T}''$ we find $0 < \epsilon < 1$ and $b_t = (1-\epsilon)a_t + \epsilon b \in A$ (recall that A is supposed to be convex) such that $|\langle a_t, x\rangle - \langle b_t, x\rangle| \le 1/2$ for all $x \in X'$. Combining all of the preceding, we claim that $A \subset \bigcup_{t\in\mathfrak{T}''} w(b_t)$. Indeed, given $a \in A$ we have $a \in A_s$ for some $s \in \mathfrak{T}'$ and $s \le t$ for some $t \in \mathfrak{T}''$. This shows $\langle a, x\rangle \le s(x)/2 \le t(x)/2 \le \langle a_t, x\rangle + 1/2 \le \langle b_t, x\rangle + 1$ for all $x \in X'$. For $x \in X''$, on the other hand we have $\langle b_t, x\rangle = \langle b, x\rangle = +\infty$, and $\langle a, x\rangle \le \langle b_t, x\rangle + 1$ holds as well. Thus $a \in v(b_t)$, hence our claim.

For Part (b) let $A \subset \mathcal{P}$ be bounded below, that is there is $\lambda \ge 0$ such that $0 \le a + \lambda v$ for all $a \in A$. We find $N \in \mathbb{Z}$ such that $\langle a, x\rangle \ge N$ for all $x \in X$. Now let $\mathfrak{T}$ be the family of all mappings t from X into $\{z \in \overline{\mathbb{Z}} \mid z \ge N\}$. It corresponds to a bounded below subset of $\overline{\mathbb{Z}}^n$ for some n. For every $t \in \mathfrak{T}$ let A_t be the subset of all $a \in A$ such that $t(x) \le \langle a, x\rangle < t(x) + 1$ for all $x \in X$. Then $A = \bigcup_{t\in\mathfrak{T}} A_t$. According to Lemma 3.3 the subset $\mathfrak{T}'$ of $\mathfrak{T}$ consisting of those $t \in \mathfrak{T}$ such that $A_t \ne \emptyset$ admits a finite subset $\mathfrak{T}''$ of minimal elements.

Next we choose elements $a_t \in A_t$ for each $t \in \mathfrak{T}''$. We claim that $A \subset \bigcup_{t\in\mathfrak{T}''}(a_t)v$. Indeed, given $a \in A$ we have $a \in A_s$ for some $s \in \mathfrak{T}$ and $s \ge t$ for some $t \in \mathfrak{T}''$. This shows $\langle a, x\rangle \ge s(x) \ge t(x) \ge \langle a_t, x\rangle - 1$ for all $x \in X$. Thus $a \in (a_t)v$, and our claim follows.

For Part (c) first suppose that $A \subset \mathcal{P}$ itself is relatively v-bounded. Then there are $\lambda, \rho \ge 0$ such that $a \le \rho b + \lambda v$ for all $a, b \in A$, that is $\langle a, x\rangle \le \rho\langle b, x\rangle + \lambda$ for all $x \in X$. In particular, $\langle a, x\rangle = +\infty$ if and only if $\langle b, x\rangle = +\infty$, and the neighborhood v is determined on A by those $x \in X$ that take only finite values on A. We may therefore assume that there is a bounded interval I in $\mathbb{R}$ such that $\langle a, x\rangle \in I$ for all $a \in A$ and all $x \in X$. There is a finite family $\mathfrak{I}$ of disjoint intervals in $\mathbb{R}$, each of length at most 1 and such that I is contained in their union. Now let $\mathfrak{T}$ be the finite family of all mappings $t : X \to \mathfrak{I}$, and for every $t \in \mathfrak{T}$ let A_t be the subset of all $a \in A$ such that $\langle a, x\rangle \in t(x)$ for all $x \in X$. Then $A = \bigcup_{t\in\mathfrak{T}} A_t$.

We set $\mathfrak{T}' = \{t \in \mathfrak{T} \mid A_t \ne \emptyset\}$ and choose elements $a_t \in A_t$ for all $t \in \mathfrak{T}'$. We claim that $A \subset \bigcup_{t\in\mathfrak{T}'} v^s(a_t)$. Indeed if $a \in A$, then $a \in A_t$ for some $t \in \mathfrak{T}'$, that is $|\langle a, x\rangle - \langle a_t, x\rangle| \le 1$ for all $x \in X$. Hence $a \in v^s(a_t)$, our claim. Now in the general case A can be expressed as a finite union of relatively v-bounded subsets, each of which can be covered by finitely many symmetric neighborhoods $v^s(a_t)$ by the preceding. Therefore A itself has the same property. □

A meaningful result for completeness in relative weak topologies appears to be available only in the symmetric case. In preparation let $\mathbb{Q}^\star$ be the cone of all $\overline{\mathbb{R}}$-valued monotone linear functionals on $\mathbb{Q}$, that is the algebraic dual of $\mathbb{Q}$. Then $(\mathbb{Q}^\star, \mathbb{Q})$ forms a dual pair and the symmetric topology $\sigma(\mathbb{Q}^\star, \mathbb{Q})$ on $\mathbb{Q}^\star$ is Hausdorff. $\mathcal{P}$ can be canonically embedded into $\mathbb{Q}^\star$. This embedding is one-to-one if and only if the symmetric $\sigma(\mathcal{P}, \mathbb{Q})$-topology on $\mathcal{P}$ is Hausdorff. The completeness statements below refer to the relative topologies induced by $\sigma(\mathcal{P}, \mathbb{Q})$.

Theorem 3.5. *Suppose that $\mathcal{P}$ is endowed with the weak topology of a duality $(\mathcal{P},\mathbb{Q})$ and that the symmetric $\sigma(\mathcal{P},\mathbb{Q})$-topology is Hausdorff. Let $\mathbb{Q}^\star$ be the algebraic dual of $\mathbb{Q}$. A subset A of $\mathcal{P}$ is symmetric complete if and only if it is closed as a subset of $Q^\star$ in the symmetric $\sigma(\mathbb{Q}^\star,\mathbb{Q})$-topology.*

Proof. Let $A \subset \mathcal{P}$ and let $(a_i)_{i\in\mathcal{I}}$ be a symmetric Cauchy net in A. We shall verify that for every $x \in \mathbb{Q}$ the net $\big(\langle a_i, x\rangle\big)_{i\in\mathcal{I}}$ is Cauchy in the symmetric topology of the locally convex cone $\overline{\mathbb{R}}$. First, for the neighborhood $v_{\{x\}}$ and $\epsilon = 1$ according to our definition of a Cauchy net with respect to the symmetric relative topology there is $a \in A$ and $i_0 \in \mathcal{I}$ such that $a_i \in v_{\{x\}\,\epsilon}(a)$, that is

$$\langle a_i, x\rangle \le \gamma\langle a, x\rangle + 1 \qquad \text{and} \qquad \langle a, x\rangle \le \gamma'\langle a_i, x\rangle + 1$$

with some $1 \le \gamma, \gamma' \le 2$ holds for all $i \ge i_0$. If $\langle a, x\rangle = +\infty$, then $\langle a_i, x\rangle = +\infty$ as well, and the net $\big(\langle a_i, x\rangle\big)_{i\in\mathcal{I}}$ converges to $+\infty \in \overline{\mathbb{R}}$. Otherwise, we choose $\rho = 2|\langle a, x\rangle| + 1$ and infer from the above that $|\langle a_i, x\rangle| \le \rho$ holds for all $i \ge i_0$. Now, given $\epsilon > 0$ we set $\delta = \epsilon/(\rho+1)$ and using Proposition 2.1 we find $i_1 \ge i_0$ such that

$$\langle a_i, x\rangle \le \gamma\langle a_k, x\rangle + \delta \qquad \text{and} \qquad \langle a_k, x\rangle \le \gamma'\langle a_i, x\rangle + \delta$$

with some $1 \le \gamma, \gamma' \le 1+\delta$ holds for all $i, k \ge i_1$. This implies

$$\langle a_i, x\rangle \le \langle a_k, x\rangle + (\gamma-1)\langle a_k, x\rangle + \delta \le \langle a_k, x\rangle + \delta(\rho+1) \le \langle a_k, x\rangle + \epsilon,$$

hence $|\langle a_i, x\rangle - \langle a_i, x\rangle| \le \epsilon$ for all $i, k \ge i_1$. The net $\big(\langle a_i, x\rangle\big)_{i\in\mathcal{I}}$ is therefore Cauchy, hence convergent in the usual topology of $\mathbb{R}$ with limit $\mu(x) \in \mathbb{R}$. The mapping $\mu : \mathbb{Q} \to \overline{\mathbb{R}}$ is linear and monotone, therefore an element of $\mathbb{Q}^\star$ and the net $(a_i)_{i\in\mathcal{I}}$ converges to μ in the symmetric$\sigma(\mathbb{Q}^\star,\mathbb{Q})$-topology. If A is closed in $\mathbb{Q}^\star$ with respect to this topology, we infer that $\mu \in A$. Furthermore, $(a_i)_{i\in\mathcal{I}}$ converges to μ in the symmetric $\sigma(\mathcal{P},\mathbb{Q})$-topology, hence in the symmetric relative topology as well (Proposition 3.2).

For the converse, suppose that $A \subset \mathcal{P} \subset \mathbb{Q}^\star$ is symmetric complete. If $\mu \in \mathbb{Q}^\star$ is an element of the symmetric $\sigma(\mathbb{Q}^\star,\mathbb{Q})$-closure of A, then there is an net $(a_i)_{i\in\mathcal{I}}$ in A converging to μ in this topology. Consequently, $(a_i)_{i\in\mathcal{I}}$ is a symmetric Cauchy net in A. Indeed, given a finite subset X of $\mathbb{Q}$, defining a neighborhood v in $\sigma(\mathcal{P},\mathbb{Q})$ and $\epsilon > 0$, there is $i_0 \in \mathcal{I}$ such that $|\langle a_i, x\rangle - \langle a_k, x\rangle| \le \epsilon$ holds for all $i, k \ge i_0$ and all $x \in X$. Thus $a_i \in v^s_\epsilon(a_k)$. Therefore $(a_i)_{i\in\mathcal{I}}$ is convergent with limit $a \in A$ with respect to the symmetric relative topology, hence also with respect to the symmetric $\sigma(\mathcal{P},\mathbb{Q})$-topology which is the trace on $\mathcal{P}$ of the symmetric $\sigma(\mathbb{Q}^\star,\mathbb{Q})$-topology. The latter is Hausdorff which implies that $\mu = a \in A$. Thus $A \subset \mathbb{Q}^\star$ is indeed closed in the symmetric $\sigma(\mathbb{Q}^\star,\mathbb{Q})$-topology. □

Combining the results of Theorems 3.4(c) and 3.5 and using Theorem 2.3 and Proposition 3.2 we obtain a characterization of compact subsets of a locally convex cone carrying the symmetric weak topology of a duality.

Theorem 3.6. *Suppose that $\mathcal{P}$ is endowed with the weak topology of a duality $(\mathcal{P},\mathbb{Q})$ and that the symmetric $\sigma(\mathcal{P},\mathbb{Q})$-topology is Hausdorff. Let $\mathbb{Q}^\star$ be the algebraic dual of $\mathbb{Q}$. A subset A of $\mathcal{P}$ is symmetric compact if and only if it is closed as a subset of $\mathbb{Q}^\star$ in the symmetric $\sigma(\mathbb{Q}^\star,\mathbb{Q})$-topology and for every $v \in \mathcal{V}_\sigma$ it can be expressed as a finite union of disjoint relatively v-bounded subsets $A_1, \ldots, A_n$.*

4. Locally compact convex cones

In this section we shall investigate locally compact convex cones, that is locally convex cones $(\mathcal{P}, \mathcal{V})$ where every element admits a base of compact neighborhoods. We shall restrict ourselves to the symmetric relative topology and assume that this topology is Hausdorff, that is that the weak preorder on $\mathcal{P}$ is antisymmetric (Proposition I.4.8 in [11]). An element of $\mathcal{P}$ admits a base of symmetric compact neighborhoods provided there is a single symmetric compact neighborhood (Theorems 17.5 and 18.2 in [13]). A subset of a symmetric compact set is again symmetric compact if and only if it is closed (Proposition 2.10). We shall also assume that $\mathcal{P}$ is endowed with its weak preorder. As mentioned before, this assumption does not interfere with topological concepts including uniformity and boundedness, since the relative topologies induced by the weak preorder are equivalent to those using the given order (Lemma 1.6). The upper, lower and symmetric relative neighborhoods are closed in the lower, upper and symmetric relative topologies in this case (Proposition 1.7). Thus if an element $a \in \mathcal{P}$ has a symmetric compact neighborhood, we may assume that this neighborhood is of the type $v_\epsilon^s(a)$. We shall use the above assumptions throughout this section.

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *locally compact (with respect to the symmetric relative topology)* if every element of $\mathcal{P}$ admits a base of compact symmetric relative neighborhoods.

An element a of a locally convex cone $\mathcal{P}$ is *invertible* if there is $a' \in \mathcal{P}$ such that $a + a' = 0$. Obviously, every invertible element a is bounded since $0 \leq a' + \lambda v$ for $v \in v$ and $\lambda \geq 0$ implies that $a \leq \lambda v$. Moreover, if $a + a' = a + a'' = 0$, then $a' = a' + (a + a'') = (a' + a) + a'' = a''$. The inverse $(-a)$ of a is therefore unique, and we can define a negative multiple of the element a setting $(-\alpha)a = \alpha(-a)$ for $\alpha \geq 0$. The invertible elements of $\mathcal{P}$ therefore form a vector subspace $\mathcal{P}_\mathrm{i}$ of $\mathcal{P}$. The trace of the symmetric relative topology of $\mathcal{P}$ coincides with the trace of the given symmetric topology on $\mathcal{P}_\mathrm{i}$ (Proposition 1.3) which is a locally convex vector space topology.

Obviously, if the sum of two elements is invertible, then both of them are themselves invertible. Thus the non-invertible elements of $\mathcal{P}$ together with $0 \in \mathcal{P}$ form a subcone $\mathcal{P}_0$ of $\mathcal{P}$.

Proposition 4.1. *$\mathcal{P}$ is the direct sum of the vector subspace $\mathcal{P}_\mathrm{i}$ of its invertible elements and the subcone $\mathcal{P}_\mathrm{o}$ consisting of all non-invertible elements adjoined by $0 \in \mathcal{P}$. If $0 \in \mathcal{P}$ has a symmetric compact neighborhood, then $\mathcal{P}_\mathrm{i}$ is finite dimensional.*

Proof. All left is to prove the last statement. Suppose that $v_\epsilon^s(0)$ is a compact neighborhood of $0 \in \mathcal{P}$. We shall demonstrate that $v_\epsilon^s(0) \cap \mathcal{P}_\mathrm{i}$ is also compact. Thus the locally convex topological vector space $\mathcal{P}_\mathrm{i}$ admits a compact neighborhood of the origin and therefore is finite dimensional. For this we need to argue that $v_\epsilon^s(0) \cap \mathcal{P}_\mathrm{i}$ is a closed subset of $v_\epsilon^s(0)$. Indeed, let $(a_i)_{i \in \mathcal{I}}$ be a net in $v_\epsilon^s(0) \cap \mathcal{P}_\mathrm{i}$ converging to a point in the closure of $v_\epsilon^s(0) \cap \mathcal{P}_\mathrm{i}$. Clearly $a \in v_\epsilon^s(0)$ since the symmetric relative neighborhoods are closed. We shall verify that a is also invertible, hence an element of $v_\epsilon^s(0) \cap \mathcal{P}_\mathrm{i}$. We note that $a_i \in v_\epsilon^s(0)$ means that both $a_i \leq \epsilon v$ and $0 \leq \gamma a_i + \epsilon v$ holds with some $1 \leq \gamma \leq 1 + \epsilon$. The second inequality implies $0 \leq a_i + (\epsilon/\gamma)v \leq a_i + v$.

Adding $-a_i$ to both sides of these inequalities yields $0 \leq -a_i + \epsilon v$ and $-a_i \leq \epsilon v$, hence $-a_i \in v^s_\epsilon(0)$ as well. Since $v^s_\epsilon(0)$ is compact, the net $(-a_i)_{i\in\mathcal{I}}$ admits a convergent subnet $(-a_j)_{j\in\mathcal{J}}$ with limit $a' \in v^s_\epsilon(0)$. Now using the continuity of the addition in $\mathcal{P}$ we infer that

$$a + a' = \lim_{j\in\mathcal{J}} a_j + \lim_{j\in\mathcal{J}}(-a_j) = \lim_{j\in\mathcal{J}}(a_j - a_j) = 0.$$

Thus $a \in \mathcal{P}$ is also invertible and therefore an element of $v^s_\epsilon(0) \cap \mathcal{P}_{\mathfrak{i}}$. □

We proceed to investigate how compact symmetric neighborhoods transfer between elements of a locally convex cone.

Proposition 4.2. *Let a and b be elements of $\mathcal{P}$.*

(a) *If a has a symmetric compact neighborhood, then for every $\alpha > 0$ the element αa has a symmetric compact neighborhood.*

(b) *If $b \in \mathcal{P}$ is invertible, then a has a symmetric compact neighborhood if and only if $a + b$ has a symmetric compact neighborhood.*

(c) *If $\mathcal{P}$ is symmetric complete, if $b \in \mathcal{P}$ is bounded and if $a + b$ has a symmetric compact neighborhood, then a has a symmetric compact neighborhood.*

Proof. (a) Let $v^s_\epsilon(a)$ be a symmetric compact neighborhood of $a \in \mathcal{P}$. For $\alpha > 0$ the neighborhood $\alpha v^s_\epsilon(a) = (\alpha v)_\epsilon(\alpha a)$ of $\alpha a \in \mathcal{P}$ is also compact (Proposition 2.13(b) and Lemma 1.5(a)).

For Part (b) let $b \in \mathcal{P}$ be invertible and let $v^s_\epsilon(a)$ be a symmetric compact neighborhood of a. Set $\delta = \epsilon/(1+\lambda)$, where $\lambda \geq 0$ is such that $0 \leq (-b) + \lambda v$. Following Lemma 1.5(b) we have $v^s_\delta(a+b) + (-b) \subset v^s_\epsilon(a)$, hence $v^s_\delta(a+b) \subset v^s_\epsilon(a) + b$. According to Proposition 2.13(c) the set $v^s_\epsilon(a) + b$ is compact, $v^s_\delta(a+b)$ is a closed subset and therefore also compact (Proposition 2.10(c)). If on the other hand $a + b$ has a symmetric compact symmetric neighborhood, so does $a = (a+b) + (-b)$ by the preceding.

For Part (c) suppose that $\mathcal{P}$ is symmetric complete, that $b \in \mathcal{P}$ is bounded and that $v^s_\epsilon(a+b)$ is compact. Set $\delta = \epsilon/(1+\lambda)$, where $\lambda \geq 0$ is such that $0 \leq b + \lambda v$. Following Lemma 1.5(b) we have $v^s_\delta(a) + b \subset v^s_\epsilon(a+b)$. The set $v^s_\delta(a)$ is symmetric closed in $\mathcal{P}$, hence symmetric complete, and so is $v^s_\delta(a) + b$ by Proposition 2.13(b). As a symmetric complete subset of $v^s_\epsilon(a+b)$ the set $v^s_\delta(a) + b$ is also symmetric closed, hence symmetric compact. Finally, Proposition 2.14(c) yields that $v^s_\delta(a)$ is compact as claimed. □

We shall say that an element $\theta \in \mathcal{P}$ is a *zero-type element* if $\theta + \theta = \theta$, that is if θ is an idempotent element of the additive semigroup $\mathcal{P}$. Apparent examples are $0 \in \mathcal{P}$, $+\infty \in \overline{\mathbb{R}}$ and functions that take only the values 0 or $= +\infty$ in Example 2.9(c). Obviously the zero-type elements form a subcone of $\mathcal{P}$.

Lemma 4.3. *For an element $\theta \in \mathcal{P}$ the following are equivalent:*

(i) *θ is a zero-type element.*

(ii) *$\alpha\theta = \theta$ for all $\alpha > 0$.*

(iii) *$\theta + a = a$ for all $a \in \mathcal{B}^s(\theta)$.*

(iv) *Linear functionals in $\mathcal{P}^*$ take only the values 0 or $+\infty$ at θ.*

Proof. Clearly, (ii) implies (i) and (i) implies (iv). Moreover, if (iv) holds and $a \in \mathcal{B}^s(\theta)$, then for every $\mu \in \mathcal{P}^*$ we know that $\mu(a) = +\infty$ if and only if $\mu(\theta) = +\infty$. Otherwise we have $\mu(\theta) = 0$, hence $\mu(\theta + a) = \mu(a)$ in any case. Following Corollary I.4.31 in [11] this implies both $\theta + a \preceq a$ and $a \preceq \theta + a$, hence $\theta = \theta + a$ since the weak preorder is supposed to be antisymmetric. Thus (iv) implies (iii).

Now if (iii) holds and $\alpha > 1$, we set $\beta = \alpha/(\alpha - 1)$, that is $\alpha = \beta/(\beta - 1)$. Then $\beta\theta = \theta + (\beta - 1)\theta = (\beta - 1)\theta$ by (iii). (Recall that $\mathcal{B}^s(\rho a) = \mathcal{B}^s(a)$ for all $a \in \mathcal{P}$ and $\rho > 0$.) Thus $\alpha\theta = (\beta/(\beta - 1))\theta = \theta$, as claimed in (ii). If $0 < \alpha < 1$, we have $(1/\alpha)\theta = \theta$ by the above, hence $\theta = \alpha\theta$ as well. Thus (iii) implies (ii) which completes our argument. □

Proposition 4.4. *All extreme points of an upper, lower, symmetric, global or v -boundedness component are zero-type elements. A symmetric global boundedness component of $\mathcal{P}$ contains at most one zero-type element.*

Proof. Let $\mathcal{B}$ be a boundedness component and suppose that $a \in \mathcal{B}$ is not a zero-type element. By Lemma 4.3(ii) there is $\alpha > 0$ such that $\alpha a \neq a$. Both $b = \alpha a$ and $c = (1/\alpha)a$ are elements of $\mathcal{B}$ and $a = \big(1/(\alpha + 1)\big)b + \big(\alpha/(\alpha + 1)\big)c$. Hence a is not an extreme point of $\mathcal{B}$. If both θ and θ' are zero-type elements in a symmetric global boundedness component $\mathcal{B}^s(a)$, then $\mathcal{B}^s(a) = \mathcal{B}^s(\theta) = \mathcal{B}^s(\theta')$ and $\theta' = \theta + \theta' = \theta$ by 4.3(iii). □

Given a neighborhood $v \in \mathcal{V}$ and a zero-type element $\theta \in \mathcal{P}$ we denote by v°_θ the set of all $\mu \in v^\circ$ such that $\mu(\theta) = 0$. The elements of v°_θ take only finite values on $\mathcal{B}^s_v(\theta)$. The following is easily verified:

Lemma 4.5. *Let $\theta \in \mathcal{P}$ be a zero-type element, $v \in \mathcal{V}$ and $\epsilon > 0$. Then*

(a) $v^s_\epsilon(\theta) = (\epsilon v)^s(\theta) = \epsilon\, v^s(\theta)$ *and* $\delta v^s_\epsilon(\theta) = v^s_{\epsilon\delta}(\theta)$ *for all* $\delta > 0$.

(b) *If $a \in \mathcal{B}^s_v(\theta)$, then $a \in v^s_\epsilon(\theta)$ if and only if $|\mu(a)| \leq \epsilon$ for all $\mu \in v^\circ_\theta$.*

(c) *If $a \in \mathcal{B}^s_v(\theta)$ is a zero-type element, then $\mu(a) = 0$ for all $\mu \in v^\circ_\theta$, that is $a \in v^s_\delta(\theta)$ for all $\delta > 0$.*

Proof. We have $a \in v^s_\epsilon(\theta)$ if and only if both $a \leq \gamma\theta + \epsilon v$ and $\theta \leq \gamma' a + \epsilon v$ holds with $1 \leq \gamma, \gamma' \leq 1 + \epsilon$. Since $\gamma\theta = (1/\gamma)\theta = \theta$ and $(1/\gamma)v \leq v$ we realize that the above conditions are equivalent to $a \leq \theta + \epsilon v$ and $\theta \leq a + \epsilon v$, that is $a \in (\epsilon v)^s(\theta)$, or equivalently $(1/\epsilon)a \in v^s(\theta)$. Hence the first statement in (a). The second statement follows from the first one, which yields $\delta v^s_\epsilon(\theta) = \delta\epsilon\, v^s(\theta) = v^s_{\epsilon\delta}(\theta)$.

For Part (b) we recall that according to our general assumption for this section $\mathcal{P}$ carries the weak preorder, hence $a \leq b + \epsilon v$ holds for $a, b \in \mathcal{P}$ if and only if $\mu(a) \leq \mu(b) + \epsilon$ for all $\mu \in v^\circ$ (Corollary I.4.34 in [11]). If $a \in \mathcal{B}^s_v(\theta)$, then $\mu(a) = +\infty$ if and only if $\mu(\theta) = +\infty$ for all $\mu \in v^\circ$. Consequently, $a \leq \theta + \epsilon v$ and $\theta \leq a + \epsilon v$ if and only if $\mu(a) \leq \epsilon$ and $0 \leq \mu(a) + \epsilon$ holds for all $\mu \in v^\circ_\theta$.

For Part (c), if $a \in \mathcal{B}^s_v(\theta)$, then $\mu(a) < +\infty$ for all $\mu \in v^\circ_\theta$. If a is a zero-type element, then $\mu(a) = 0$ by Lemma 4.3(iii) and therefore $a \in v^s_\delta(\theta)$ for all $\delta > 0$ by Part (b). □

Lemma 4.6. *For an element $a \in \mathcal{P}$, with respect to the symmetric relative topology of $\mathcal{P}$ the following are equivalent:*

(i) *There is a net $(\alpha_i)_{i\in\mathcal{I}}$ in $(0,+\infty)$ converging to $0 \in \mathbb{R}$ such that the net $(\alpha_i a)_{i\in\mathcal{I}}$ is convergent in $\mathcal{P}$.*

(ii) *If $(\alpha_i)_{i\in\mathcal{I}}$ is a net in $(0,+\infty)$ converging to $0 \in \mathbb{R}$ and $(a_i)_{i\in\mathcal{I}}$ is a net in $\mathcal{P}$ converging to a, then the net $(\alpha_i a_i)_{i\in\mathcal{I}}$ converges to a zero-type element in $\mathcal{B}^s(a)$.*

(iii) *$\mathcal{B}^s(a)$ contains a zero-type element.*

Proof. Clearly, (ii) implies (i). If $(\alpha_i)_{i\in\mathcal{I}}$ is a net in $(0,+\infty)$ converging to $0 \in \mathbb{R}$ and $(\alpha_i a)_{i\in\mathcal{I}}$ is converges to $\theta \in \mathcal{P}$, then for every $\mu \in \mathcal{P}^*$ the net $\big(\mu(\alpha_i a)\big)_{i\in\mathcal{I}}$ converges to $\mu(\theta)$. Hence either $\mu(\theta) = 0$ or $\mu(\theta) = +\infty$. Thus θ is a zero-type element according to Lemma 4.3. For every $v \in \mathcal{V}$ there is $\alpha_i a \in v_1^s(\theta) = v^s(\theta)$ for some $i \in \mathcal{I}$. This shows $\theta \in \mathcal{B}_v^s(a)$, thus $\theta \in \mathcal{B}^s(a)$ since the former holds for all $v \in \mathcal{V}$.

Accordingly (i) implies (iii) and all left to show is that (iii) implies (ii).

For this purpose suppose that $(\alpha_i)_{i\in\mathcal{I}}$ is a net in $(0,+\infty)$ converging to $0 \in \mathbb{R}$, the net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ converges to a and θ is a zero-type element in $\mathcal{B}^s(a)$. We shall demonstrate that the net $(\alpha_i a_i)_{i\in\mathcal{I}}$ converges to θ. Indeed, given $v \in \mathcal{V}$ and $\epsilon > 0$ there is $\rho > 0$ such that $a \in v_\rho^s(\theta)$. We set $\delta = \min\{1, \epsilon/(2\rho+1)\}$. There is $i_0 \in \mathcal{I}$ such that both $0 < \alpha_i \le \delta$ and $a_i \in v_\delta^s(a)$ for all $i \ge i_0$. According to Lemma 1.5(a) this implies $\alpha_i a_i \in (\alpha_i v)_\delta^s(\alpha_i a) \subset (\delta v)_\delta^s(\alpha_i a)$.

Since $\alpha_i a \in \alpha_i v_\rho^s(\theta) = (\alpha_i v)_\rho^s(\theta) \subset (\delta v)_\rho^s(\theta)$, we use Lemma 1.4(b) and infer that $\alpha_i a_i \in (\delta v)_\sigma^s(\theta)$, where $\sigma = \delta + \rho + \delta\rho \le 1 + 2\rho$. Now Lemma 4.5(a) yields that $(\delta v)_\sigma^s(\theta) = v_{\delta\sigma}^s(\theta)$, and since $\delta\sigma \le \epsilon$, we conclude that $\alpha_i a_i \in v_\epsilon^s(\theta)$ holds for all $i \ge i_0$. This completes our argument. □

Other than the global boundedness components, the larger v-boundedness components of a locally convex cone can hold more than one zero-type element. (See Example 2.9(c).)

Proposition 4.7. *If a zero-type element $\theta \in \mathcal{P}$ has a symmetric compact neighborhood $v_\epsilon^s(\theta)$, then*

(a) *every element of $\mathcal{B}_v^s(\theta)$ has a symmetric compact neighborhood;*

(b) *$\mathcal{B}^s(a)$ contains a zero-type element for every $a \in \mathcal{B}_v^s(\theta)$.*

(c) *$a \in \mathcal{B}_v^s(\theta)$ is a zero-type element if and only if $\mu(a) = 0$ for all $\mu \in v_\theta^\circ$, that is if $a \in v_\delta^s(\theta)$ for all $\delta > 0$.*

Proof. Let $v_\epsilon^s(\theta)$ be a symmetric compact neighborhood of $\theta \in \mathcal{P}$ and let $a \in \mathcal{B}_v^s(\theta)$. There is $\rho > 0$ such that $a \in v_\rho^s(\theta)$. Using Lemmas 1.4(b) and 4.5(a) we infer that

$$v_\epsilon^s(a) \subset v_{(\epsilon+\rho+\epsilon\rho)}^s(\theta) = (1+\rho+\rho/\epsilon)v_\epsilon^s(\theta).$$

As a multiple of a symmetric compact set the latter is symmetric compact, hence also its closed subset $v_\delta^s(a)$. Part (a) of our claim follows.

For Part (b) we set $\alpha_n = \epsilon/(n\rho)$ for $n \in \mathbb{N}$ and observe that $\alpha_n a \in v_\epsilon^s(\theta)$. Since this set is symmetric compact, the sequence $\big((1/\rho n)a\big)_{n\in\mathbb{N}}$ admits a convergent subnet. Now Lemma 4.6 yields Part (b) of our claim.

Lemma 4.5(c) states the first implication in (c). For the converse suppose that $\mu(a) = 0$ for all $\mu \in v^\circ_\theta$, that is $a \in v^s_\delta(\theta)$ for all $\delta > 0$. Thus $a \in v^s_{(\epsilon/n)}(\theta)$, hence $na \in nv^s_{(\epsilon/n)}(\theta) = v^s_\epsilon(\theta)$ for all $n \in \mathbb{N}$. We shall use Lemma 4.3 in order to verify that a is a zero-type element. For $\mu \in \mathcal{P}^*$, according to Proposition 2.5 the set $\mu\big(v^s_\epsilon(\theta)\big) \cap \mathbb{R}$ is compact in the usual topology of $\mathbb{R}$. Thus we have either $\mu(a) = +\infty$ or $n\mu(a)$ is contained in the compact set $\mu\big(v^s_\epsilon(\theta)\big) \cap \mathbb{R}$ for all $n \in \mathbb{N}$. The latter implies that $\mu(a) = 0$ as required by 4.3(iv). □

We shall conclude our investigations into locally compact convex cones looking at the existence of compact bases (see for example [2]). Other than in the case of subcones of locally convex vector spaces, this concept will apply to the boundedness components rather than the whole of $\mathcal{P}$. Our main result will depend on the existence of sufficiently man zero-type elements.

Let $v \in \mathcal{V}$ and let $\mathcal{B}^s_v(a)$ be the symmetric v-boundedness component of an element $a \in \mathcal{P}$. A subset C of $\mathcal{B}^s_v(a)$ is a *base* of $\mathcal{B}^s_v(a)$ if there are $\mu, \nu \in v^\circ$ such that $\mu(a), \nu(a) < +\infty$ and $\gamma > 0$ such that

$$C = \{c \in \mathcal{B}^s_v(a) \mid \mu(c) = \nu(c) + \gamma\}$$

and if every non-zero-type element of $\mathcal{B}^s_v(a)$ is a positive multiple of an element of C. Note that the requirements for the functionals μ and ν guarantee that both take only finite values on $\mathcal{B}^s_v(a)$. Thus $\mu(\theta) = \nu(\theta) = 0$ holds for any zero-type element $\theta \in \mathcal{B}^s_v(a)$ (see Lemma 4.3(iii)) which therefore can not be contained in C. We infer that $\nu(a) \leq \mu(a)$ holds for all $a \in \mathcal{B}^s_v(a)$ and that $\nu(a) = \mu(a)$ if and only if a is a zero-type element.

We shall say that the symmetric v-boundedness component $\mathcal{B}^s_v(a)$ is *proper* if the zero-type elements form a *face* in $\mathcal{B}^s_v(a)$, that is if $\theta = b + c$ for a zero-type element $\theta \in \mathcal{B}^s_v(a)$ and $b, c \in \mathcal{B}^s_v(a)$ implies that both b and c are also zero-type elements. We shall use the same notations for the global symmetric boundedness component $\mathcal{B}^s(a)$ of a, requiring that $\mu, \nu \in v^\circ$ for some $v \in \mathcal{V}$ such that $\mu(a), \nu(a) < +\infty$ in the definition of a base C of $\mathcal{B}^s(a)$. Proposition 4.4 implies that a global symmetric boundedness component is proper if and only if it either contains no zero-type elements or exactly one which is its only extreme point. For the following recall that the symmetric v-boundedness components are both open and closed in the symmetric relative topology of $\mathcal{P}$.

Lemma 4.8. *Let $a \in \mathcal{P}$ and $v \in \mathcal{V}$. If the symmetric v-boundedness component $\mathcal{B}^s_v(a)$ has a base, then it is proper and its zero-type elements form a closed subset.*

Proof. Let $a \in \mathcal{P}$, $v \in \mathcal{V}$ and let C be a base of $\mathcal{B}^s_v(a)$. There are $\mu, \nu \in v^\circ$ such that $\mu(a), \nu(a) < +\infty$ and $\gamma > 0$ such that

$$C = \{c \in \mathcal{B}^s_v(a) \mid \mu(c) = \nu(c) + \gamma\}.$$

Then $\nu(a) \leq \mu(a)$ for all $a \in \mathcal{B}^s_v(a)$ and $A = \{a \in \mathcal{B}^s_v(a) \mid \mu(c) = \nu(c)\}$ is the set of all zero-type elements in $\mathcal{B}^s_v(a)$. The functionals μ and ν are continuous with respect to the symmetric relative topology of $\mathcal{P}$, hence A is closed in $\mathcal{P}$. If $\theta = b + c$ for a zero-type element $\theta \in \mathcal{B}^s_v(a)$ and $b, c \in \mathcal{B}^s_v(a)$, then $\mu(a) + \mu(b) = \mu(\theta) = 0$

and $\nu(a)+\nu(b)=\nu(\theta)=0$. Thus $\mu(a)\geq\nu(a)=-\nu(b)\geq-\mu(b)=\mu(a)$, that is $\mu(a)=\nu(a)$ and $\mu(b)=\nu(b)$. Hence both a and b are also zero-type elements, and $\mathcal{B}^s(\theta)$ is seen to be proper. □

Theorem 4.9. *For a zero-type element $\theta\in\mathcal{P}$ and a neighborhood $v\in\mathcal{V}$ the following are equivalent:*

(i) *θ has a symmetric compact neighborhood $v^s_\epsilon(\theta)$ and $\mathcal{B}^s_v(\theta)$ is proper.*

(ii) *$\mathcal{B}^s_v(\theta)$ has a symmetric compact base and $\mathcal{B}^s(a)$ contains a zero-type element for every $a\in\mathcal{B}^s_v(\theta)$.*

Proof. In order to verify that (i) implies (ii), let $v^s_\epsilon(\theta)$ be a symmetric compact neighborhood of $\theta\in\mathcal{P}$ and suppose that $\mathcal{B}^s_v(\theta)$ is proper. Recall that we have $v^s_\epsilon(\theta)=\epsilon v^s(\theta)\subset\mathcal{B}^s_v(\theta)$. Hence according to Proposition 2.12 the symmetric neighborhood $v^s(\theta)$ is also symmetric compact. In a first step we shall construct a symmetric compact subset A of $\mathcal{B}^s_v(\theta)$ not containing any zero-type element and such that every non-zero-type element of $\mathcal{B}^s_v(\theta)$ is a positive multiple of an element of A. For this we consider the functional p on $\mathcal{B}^s_v(\theta)$ defined as

$$p(a)=\inf\{\lambda\geq 0\mid a\in\lambda v^s(\theta)\}$$

and observe that:

(i) $p(a)<+\infty$ since $\mathcal{B}^s_v(\theta)=\bigcup_{\rho>0}v^s_\rho(\theta)=\bigcup_{\rho>0}\rho v^s(\theta)$;

(ii) $p(a+b)\leq p(a)+p(b)$ and $(\alpha a)=\alpha p(a)$ for all $a,b\in\mathcal{B}^s_v(\theta)$ and $\alpha>0$;

(iii) $v^s(\theta)=p^{-1}([0,1])$.

Indeed, $v^s(\theta)\subset p^{-1}([0,1]$ is obvious.

The reverse inclusion follows since $a\in p^{-1}([0,1])$ implies that $a\in\sigma v^s(\theta)=v^s_\sigma(\theta)$ for all $\sigma>1$. Hence $a\in v_1(\theta)=v(\theta)$ by Lemma 1.6 because $\mathcal{P}$ carries the weak preorder;

(iv) The set $p^{-1}([0,1))$ is open in the symmetric relative topology since for every $a\in\rho v^s(\theta)=v^s_\rho(\theta)$ for $\rho<1$, according to Lemma 1.4(b) there is $\delta>0$ such that $v^s_\delta(a)\subset v^s_{\rho'}(\theta)=\rho' v^s(\theta)$ for some $\rho'<1$:

(v) According to Proposition 4.7(c) $a\in\mathcal{B}^s_v(\theta)$ is a zero-type element if and only if $p(a)=0$. Using these observations we set $A=p^{-1}(\{1\})=v^s(\theta)\ \setminus\ p^{-1}([0,1))$.

Following the above, A is a closed, hence compact subset of $v^s(\theta)$ in the symmetric relative topology. For every non-zero-type element $a\in\mathcal{B}^s_v(\theta)$ we have $p(a)>0$, hence $\big(1/p(a)\big)a\in A$. Thus a is a positive multiple of an element of A. Now in a second step we shall verify that the closed convex hull B of A, which is a subset of $\mathcal{B}^s_v(\theta)$, does not contain θ. In a vector space situation we might use the Krein-Milman theorem for this conclusion. Instead we shall use a suitable embedding and proceed as follows: Let H be the vector space of real-valued functions on $v^s(\theta)$, generated by the functionals $\mu\in v^\circ_\theta$, that is the functions in $h\in H$ are of the type $h=\lambda(\mu-\nu)$, where $\mu,\nu\in v^\circ_\theta$ and $\lambda\geq 0$. These are affine functions, continuous with respect to the symmetric relative topology of the compact convex set $v^s(\theta)$ (see Proposition I.4.5 in [11]). We endow the vector space H with the maximum norm for functions, its dual space H^* with the weak* topology. Next we consider the embedding $\Phi: v^s(\theta)\to H^*$ mapping $a\in v^s(\theta)$ onto the point evaluation $\delta_a\in H^*$

such that $\delta_a(h) = h(a)$ for $h \in H$. This embedding is not necessarily one-to-one, but affine and continuous with respect to the symmetric relative topology of $v^s(\theta)$ and the weak* topology of H^*. The set $\Phi\big(v^s(\theta)\big) \subset H^*$ is therefore weak*-compact and convex. Its subset $\Phi(A)$ is also compact and does not contain $\Phi(\theta)$. Indeed, for $a \in A$, we have $a \notin (1/2)v^s(\theta)$, and there is $\mu \in v_\theta^\circ$ such that $\mu(a) \neq \mu(\theta) = 0$. Since $\mu \in H$, we conclude that $\big(\Phi(a)\big)(\mu) = \mu(a) \neq \mu(\theta) = \big(\Phi(\theta)\big)(\mu)$. Thus $\Phi(\theta) \neq \Phi(a)$.

Furthermore, according to the assumption that $\mathcal{B}_v^s(\theta)$ is proper, $\Phi(\theta)$ is an extreme point of $\Phi\big(v^s(\theta)\big)$. Indeed, suppose that

$$\Phi(\theta) = \lambda\Phi(a) + (1-\lambda)\Phi(b) = \Phi\big(\lambda a + (1-\lambda)b\big)$$

for $a, b \in v^s(\theta)$ and $0 < \lambda < 1$. Then $\mu\big(\lambda a + (1-\lambda)b\big) = \mu(\theta) = 0$ for all $\mu \in v_\theta^\circ$. Thus $\theta' = \lambda a + (1-\lambda)b$ is a zero-type element according to Proposition 4.7(c). Hence both a and b are also zero-type elements since $\mathcal{B}_v^s(\theta)$ is proper. But the latter implies that $\Phi(a) = \Phi(b) = \Phi(\theta) = 0 \in H^*$ as claimed. Now the Krein-Milman Theorem (see for example [8] or [1]) applies: The closed convex hull $\widetilde{B}$ of $\Phi(A)$ in H^* is weak*-compact and does not contain $\Phi(\theta)$. Moreover, according to the Hahn-Banach theorem, there is a weak*-continuous linear functional on H^*, that is a function $h \in H$ (the dual of H^* in the weak* topology is known to be H) and $\alpha \in \mathbb{R}$ such that $h\big(\Phi(\theta)\big) < \alpha < h(\widetilde{B})$. The function $h \in H$ is of the type $h = \lambda(\mu - \nu)$ with $\mu, \nu \in v_\theta^\circ$ and $\lambda > 0$. Since $h(\Phi(\theta)) = 0$, the above reads $0 < \alpha < h(\widetilde{B})$, that is $0 < \gamma < (\mu - \nu)(\widetilde{B})$, where $\gamma = \alpha/\lambda$. Next we set $B = \Phi^{-1}(\widetilde{B}) \subset v^s(\theta)$ and understand from the preceding that $\nu(b) + \gamma < \mu(b)$ holds for all $b \in B$. We are finally ready to identify a compact base of $\mathcal{B}_v^s(\theta)$. We set

$$C = \{c \in \mathcal{B}_v^s(\theta) \mid \mu(c) = \nu(c) + \gamma\}$$

Indeed, because every non-zero-type element of $\mathcal{B}_v^s(\theta)$ is a positive multiple of some $a \in A$, for every $c \in C$ there are $a \in A$ and $\alpha > 0$ such that $c = \alpha a$. Thus $\mu(a) = \nu(a) + \gamma/\alpha$ and also $\nu(a) + \gamma < \mu(a)$ since $A \subset B$. This yields $\gamma < \gamma/\alpha$, hence $p(c) = \alpha < 1$ and $c \in v^s(\theta)$. Thus C is symmetric compact as it is closed in $v^s(\theta)$. Moreover, for every non-zero-type element $a \in \mathcal{B}_v^s(\theta)$ we have $a = \beta b$ for some $b \in B$ and $\beta > 0$, hence $\nu(a) + \beta\gamma < \mu(a)$. We set

$$c = \frac{\gamma}{\mu(a) - \nu(a)} a$$

and realize that $c \in C$. Since a is a positive multiple of c, we conclude that C is indeed a compact base of $\mathcal{B}_v^s(\theta)$. Moreover, Proposition 4.7 yields that $\mathcal{B}^s(a)$ contains a zero-type element for every $a \in \mathcal{B}_v^s(\theta)$, the second statement in (ii).

Now for the reverse implication suppose that (ii) holds true. There are $\mu, \nu \in v_\theta^\circ$ and $\gamma > 0$ such that the set

$$C = \{c \in \mathcal{B}^s(\theta) \mid \mu(c) = \nu(c) + \gamma\}$$

is a symmetric compact base of $\mathcal{B}_v^s(\theta)$. We recall that $\nu(a) \leq \mu(a)$ holds for all $a \in \mathcal{B}_v^s(\theta)$. Lemma 4.8 yields that $\mathcal{B}^s(\theta)$ is proper, the second statement in (i). For the first statement we shall establish that the set

$$C_{[0,1]} = \{\alpha c \mid c \in C,\ \alpha \in [0,1]\} = \{a \in \mathcal{B}^s(\theta) \mid \mu(a) \leq \nu(a) + \gamma\}$$

is symmetric compact and contains a neighborhood of θ. For the latter we choose $\epsilon = \gamma/2$ and for every element $a \in v_\epsilon^s(\theta)$ observe that both $|\mu(a)| \le \epsilon$ and $|\nu(a)| \le \epsilon$ (Lemma 4.5(b)), hence $\mu(a) \le \epsilon \le \nu(a) + 2\epsilon = \nu(a) + \gamma$.

Thus $a \in C_{[0,1]}$, verifying that $v_\epsilon^s(\theta) \subset C_{[0,1]}$ as claimed. For compactness, consider a net $(a_i)_{i\in\mathcal{I}}$ in $C_{[0,1]}$, whereby $a_i = \alpha_i c_i$ with $\alpha_i \in [0,1]$ and $c_i \in C$. Since both $[0,1]$ and C are compact in their respective topologies there exists a subnet $(a_j)_{j\in\mathcal{J}}$ of $(a_i)_{i\in\mathcal{I}}$ such that both nets $(\alpha_j)_{j\in\mathcal{J}}$ and $(c_j)_{j\in\mathcal{J}}$ are convergent with limits $\alpha \in [0,1]$ and $c \in C$, respectively. If $\alpha > 0$, then the continuity of the scalar multiplication with respect to the symmetric relative topology of $\mathcal{P}$ (Proposition 1.3(c)) at the point $(\alpha, c) \in (0, +\infty) \times \mathcal{P}$ implies that $(a_j)_{j\in\mathcal{J}}$ converges to the element $\alpha c \in C_{[0,1]}$.

If $\alpha = 0$ on the other hand, we use Lemma 4.6 and argue that $(a_j)_{j\in\mathcal{J}}$ converges to the unique zero-type element θ' in $\mathcal{B}^s(c) \subset \mathcal{B}_v^s(c) = \mathcal{B}_v^s(\theta)$. Since $\theta' \in v_\epsilon^s(\theta) \subset C_{[0,1]}$ (Lemma 4.5(c)), the given net $(a_i)_{i\in\mathcal{I}}$ admits a convergent subnet with limit in $C_{[0,1]}$ in any case, demonstrating that $C_{[0,1]}$ is indeed symmetric compact. So is its closed subset $v_\epsilon^s(\theta)$. Hence (ii) implies (i). □

A similar equivalence is generally not available for v-boundedness components that do not contain a zero-type element (see Example **??**(b) below). However, we are able to recover a modified version of one of the implications using a weaker version of compactness for a base of a boundedness component. For an element a of $\mathcal{P}$ and a neighborhood $v \in \mathcal{V}$ we shall say that a base C of $\mathcal{B}_v^s(s)$ *has a symmetric compact cover* if for every non-zero-type element $b \in \mathcal{B}_v^s(a)$ there is a symmetric compact subset D of C such that b is in the topological interior (with respect to the symmetric relative topology) of $D_{(0,+\infty)} = \{\lambda d \mid d \in D,\ \lambda \in (0, +\infty)\}$. A brief argument shows that in the situation of Theorem 4.9 this notion is implied by compactness.

Lemma 4.10. *Let $a \in \mathcal{P}$ and $v \in \mathcal{V}$. Every symmetric compact base for $\mathcal{B}_v^s(a)$ has a symmetric compact cover.*

Proof. Let $a \in \mathcal{P}$, $v \in \mathcal{V}$ and let C be a symmetric compact base of $\mathcal{B}_v^s(a)$. According to Lemma 4.8 the set A of all zero-type elements of $\mathcal{B}_v^s(a)$ is closed in $\mathcal{P}$ and its complement in $\mathcal{B}_v^s(a)$ is open. (Recall that $\mathcal{B}_v^s(a)$ is both open and closed in $\mathcal{P}$.) We set $D = C$ in the above definition of a symmetric compact cover of C. Then $D_{(0,+\infty)}$ coincides with the open set $\mathcal{B}_v^s(a) \setminus A$ of all non-zero-type elements of $\mathcal{B}_v^s(a)$. Our claim follows. □

Theorem 4.11. *If for $a \in \mathcal{P}$ and $v \in \mathcal{V}$ the boundedness component $\mathcal{B}_v^s(a)$ has a base with symmetric compact cover, then $\mathcal{B}_v^s(a)$ is proper and every non-zero-type element of $\mathcal{B}_v^s(a)$ has a symmetric compact neighborhood.*

Proof. Let $a \in \mathcal{P}$, $v \in \mathcal{V}$ and let C be a base of $\mathcal{B}_v^s(a)$. There are $\mu, \nu \in v^\circ$ such that $\mu(a), \nu(a) < +\infty$ and $\gamma > 0$ such that

$$C = \{c \in \mathcal{B}_v^s(a) \mid \mu(c) = \nu(c) + \gamma\}.$$

According to Lemma 4.8 $\mathcal{B}_v^s(a)$ is proper. For our second claim, let $b \in \mathcal{B}_v^s(a)$ be a non-zero-type element. Hence b is a positive multiple of some element in C, that is $b = \beta c$ for some $\beta > 0$, and therefore $\mu(b) = \nu(b) + \beta\gamma$.

If C has a symmetric compact cover, there is a compact subset D of C such that b is in the topological interior $D^{\circ}_{(0,+\infty)}$ of $D_{(0,+\infty)}$. We choose $0 < \lambda < \beta < \rho$ and observe that the set

$$C_{(\lambda,\rho)} = \{b \in \mathcal{B}^s(a) \mid \nu(b) + \gamma\lambda < \mu(b) < \nu(b) + \gamma\rho\}$$

is open in $\mathcal{P}$ since the functionals μ and ν are continuous with respect to the symmetric relative topology. Thus $u = C_{(\lambda,\rho)} \cap D^{\circ}_{(0,+\infty)}$ is an open neighborhood of b. On the other hand u is a subset of

$$D_{[\lambda,\rho]} = \{\delta d \mid d \in D,\ \delta \in [\lambda, \rho]\}$$

which according Proposition 2.12(b) is indeed symmetric compact. This yields our claim. □

Similar results are available involving the global rather than the symmetric v-boundedness components of $\mathcal{P}$. As mentioned before, these form a partition of closed but not necessarily open subsets op $\mathcal{P}$ with respect to its symmetric relative topology. Neighborhoods in the trace topology of a global boundedness component are therefore not necessarily neighborhoods in $\mathcal{P}$. Compactness, hoverer coincides in both instances.

In order to investigate this setting, given a zero-type element $\theta \in \mathcal{P}$ we consider the locally convex cone $(\mathcal{P}_\theta, \mathcal{V})$, where $\mathcal{P}_\theta = \mathcal{B}^s$ with the slightly modified scalar multiplication defining $0 \cdot a = \theta$ for all $a \in \mathcal{P}_\theta$. Thus θ is the zero element of $\mathcal{P}_\theta$ (see Lemma 4.3). The requirements for a locally convex cone are easily checked. All elements of $\mathcal{P}_\theta$ are bounded and its symmetric relative topology coincides with the trace of the symmetric relative topology on $\mathcal{B}^s(\theta)$. The dual cone of $\mathcal{P}_\theta$ consists of all elements $\mu \in \mathcal{P}^*$ such that $\mu(\theta) = 0$. Now applying Proposition 4.7 to $(\mathcal{P}_\theta, \mathcal{V})$ yields

Proposition 4.12. *If a zero-type element $\theta \in \mathcal{P}$ has a compact neighborhood in the trace of the symmetric relative topology on $\mathcal{B}^s(\theta)$, then every element of $\mathcal{B}^s(\theta)$ has a compact neighborhood in this topology.*

Theorems 4.9 and 4.11 translate into

Theorem 4.13. *For a zero-type element $\theta \in \mathcal{P}$ the following are equivalent:*
(i) *θ has a symmetric compact neighborhood $v^s_\epsilon(\theta)$ with respect to the trace of the symmetric relative topology on $\mathcal{B}^s(a)$ and $\mathcal{B}^s(\theta)$ is proper.*
(ii) *$\mathcal{B}^s(\theta)$ has a symmetric compact base.*

Theorem 4.14. *If for $a \in \mathcal{P}$ the boundedness component $\mathcal{B}^s(a)$ has a base with symmetric compact cover, then $\mathcal{B}^s(a)$ is proper and every non-zero-type element of $\mathcal{B}^s(a)$ has a compact neighborhood with respect to the trace of the symmetric relative topology on $\mathcal{B}^s(a)$.*

Examples 4.15. (a) For $n \in \mathbb{N}$ let $\mathcal{P} = \{(x_1, x_2, \ldots, x_n) \mid x_i \in [0, +\infty]\}$ with the algebraic operations and order of $\overline{\mathbb{R}}^n$ and the neighborhood system consisting of all strictly positive multiples of the neighborhood $v = (1, 1, \ldots, 1)$. The v- and global boundedness components of $\mathcal{P}$ coincide and are as follows:

For a subset Y of $\{1, 2, \ldots, n\}$ let

$$\theta_Y = \{(z_1, z_2, \ldots,_n) \mid z_i = 0 \text{ if } i \in Y, \text{ and } z_i = +\infty, \text{ else}\}.$$

Each θ_Y is a zero-type element and the boundedness components $\mathcal{B}^s(\theta_Y)$ form a partition of $\mathcal{P}$ into both closed and open sets. The set

$$C_Y = \left\{ (c_1, c_2, \ldots, c_n) \;\middle|\; 0 \le c_i, \; \sum_{i \in Y} c_i = 1 \text{ and } c_i = +\infty \text{ for } i \notin Y \right\}$$

is a base of $\mathcal{B}^s(\theta_Y)$. It is symmetric compact since it is homeomorphic to a compact subset of $\mathbb{R}^{|Y|}$, where $|Y|$ denotes the number of elements of Y. Hence $\mathcal{P}$ is locally compact according to Theorem 4.13 and Proposition 4.12.

(b) Let $\mathcal{P}$ be the cone of all polynomials $f = a_0 + a_2 x^2$ on $\mathbb{R}$, such that $a_0, a_2 \in \mathbb{R}$ and $a_2 \ge 0$, endowed with the pointwise order for functions on $\mathbb{R}$ and the neighborhood system of all positive constants, that is $\mathcal{V} = \{\epsilon > 0\}$. Obviously, the functions in $\mathcal{P}$ are bounded below. The v-boundedness components coincide with the global ones and are given by $\mathcal{B}^s(1) = \{a_0 \in \mathbb{R}\}$ and $\mathcal{B}^s(x^2) = \{a_0 + a_2 x^2 \mid a_0 \in \mathbb{R} \text{ and } a_2 > 0\}$.

A linear functional μ on $\mathcal{P}$ is defined by two numbers $\alpha_0 \in \mathbb{R}$ and $\alpha_2 \in \overline{\mathbb{R}}$, that is $\mu = (\alpha_0, \alpha_2)$ such that $\mu(a_0 + a_2 x^2) = \alpha_0 a_0 + \alpha_2 a_2$ for $a_0 + a_2 x^2 \in \mathcal{P}$. It is an element of $\mathcal{P}^*$ if and only if both $\alpha_0, \alpha_2 \ge 0$. Clearly, $\mathcal{B}^s(1)$ is locally compact since it is isomorphic to $\mathbb{R}$. But $\mathcal{B}^s(1)$ does not have a base as it is not proper. In order to show that $\mathcal{B}^s(x^2)$, which is prober but does not contain a zero-type element, is locally compact we shall use Theorem 4.11. Consider the linear functionals $\mu = (0, 1)$ and $\nu = (1, 0)$. The set

$$C = \mu^{-1}(1) = \{a_0 + x^2 \mid a_0 \in \mathbb{R}\}$$

is a base of $\mathcal{B}^s(x^2)$. It is symmetric closed, but obviously not symmetric compact. (The sequence $(f_n)_{n \in \mathbb{N}}$ in C such that $f_n = n + x^2$ does not have a convergent subnet.) We shall show that C has a symmetric compact cover. For $\lambda > 0$ let

$$D^\lambda = C \cap \nu^{-1}[-\lambda, \lambda] = \{a_0 + x^2 \mid a_0 \in [-\lambda, \lambda]\}.$$

Since D^λ is the convex hull of the elements $-\lambda + x^2$ and $\lambda + x^2$ in $\mathbb{B}^s(x^2)$, it is symmetric compact according to Proposition 2.16. We proceed to verify that the sets D^λ form a compact cover of C. First we observe that

$$D^\lambda_{(0,+\infty)} = \{a_0 + a_2 x^2 \mid a_0 \in [-\lambda a_2, \lambda a_2]\}.$$

Next for $\sigma > 0$ and $\rho > 1$ we set

$$C^\sigma_\rho = \nu^{-1}\big((-\sigma, \sigma)\big) \cap \mu^{-1}\big((1/\rho, \rho)\big) = \{a_0 + x^2 \mid a_0 \in (-\sigma, \sigma), \; a_2 \in (1/\rho, \rho)\}.$$

This set is open since the functionals μ and ν are continuous. Moreover, we realize that $C^\sigma_\rho \subset D^{\sigma\rho}_{(0,+\infty)}$. Indeed, if $a_0 + a_2 x^2 \in C^\sigma_\rho$, then $(\sigma\rho) a_2 \ge \sigma$ and $-(\sigma\rho) a_2 \le -\sigma$.

Thus

$$-(\sigma\rho) a_2 \le -\sigma \le a_0 \le \sigma \le (\sigma\rho) a_2$$

as claimed. Now for every $f = a_0 + a_2 x^2 \in \mathbb{B}(x^2)$ there are $\sigma, \rho > 0$ such that $f \in C^\sigma_\rho$ which by the above is contained in the topological interior of $D^{\sigma\rho}_{(0,+\infty)}$. Therefore the sets D^λ form a compact cover of C, and according to Theorem 4.11 the locally convex cone $\mathcal{P}$ is indeed locally compact. With some effort this example can be generalized using cones of polynomials of higher even degrees.

(c) Let X be a compact Hausdorff space. We consider regular $[0,+\infty]$-valued Borel measures μ on X, where both σ-additivity and regularity are required with respect to the symmetric topology of $\overline{\mathbb{R}}$. This is stronger than the usual requirement of convergence with respect to the one-point compactification topology of $\mathbb{R}\cup\{\infty\}$.

In particular, σ-additivity in this sense requires that $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = +\infty$ for Borel subsets A_i of X implies that $\mu\big(\bigcup_{i=1}^{n} A_i\big) = +\infty$ for some $n \in \mathbb{N}$. Regularity means that for every Borel set Y in X we have $\mu(Y) = \lim_{O\supset Y}\mu(O)$ and for every open set O in X we have $\mu(O) = \lim_{K\subset O}\mu(K)$, where the limits are taken over the downward directed net of open sets $O \supset Y$ and the upward directed net of compact set $K \subset O$, respectively. For an open set O such that $\mu(O) = +\infty$ the latter requires that $\mu(K) = +\infty$ for some compact subset K of O. We shall first list a few properties of measures μ and ν of this type:

(i) There exists a largest open subset O_μ of X such that $\mu(O_\mu) < +\infty$. Indeed, let $O_\mu = \bigcup\{O \text{ open} \mid \mu(O) < +\infty\}$. Every compact subset K of O_μ is covered by finitely many of these open sets $O_1,\ldots,O_n$, hence $\mu(K) \le \sum_{i=1}^{n}\mu(O_i) < +\infty$. Thus $\mu(O_\mu) < +\infty$ as well by the regularity of μ.

(ii) $O_{\lambda\mu} = O_\mu$ for $\lambda > 0$ and $O_{\mu+\nu} = O_\mu \cap O_\nu$. Indeed, $(\mu+\nu)(O) < +\infty$ for an open set $O \subset X$ if and only if both $\mu(O) < +\infty$ and $\nu(O) < +\infty$, that is if and only if $O \subset O_\mu \cap O_\nu$.

(iii) $\mu(Y) < +\infty$ for a Borel set Y in X if and only if $Y \subset O_\mu$. Indeed, $Y \subset O_\mu$ implies that $\mu(Y) \le \mu(O_\mu) < +\infty$. If on the other hand $\mu(Y) < +\infty$, then there is an open set O containing Y and such that $\mu(O) < +\infty$. Hence $Y \subset O \subset O_\mu$.

(iv) μ can be expressed as a sum of a finite measure $\phi_\mu \in \mathcal{P}$ and a zero-type element $\theta_\mu \in \mathcal{P}$. Indeed, for a Borel set Y in X we set $\phi_\mu(Y) = \mu(Y \cap O_\mu)$ and $\theta_\mu(Y) = 0$ if $Y \subset O_\mu$, $\theta_\mu(Y) = +\infty$, else. The required properties are readily checked.

(v) For a continuous non-negative function f on X we have $\int_X f\,d\mu < +\infty$ if and only if $f(x) = 0$ for all $x \in X \setminus O_\mu$. Indeed, if this condition holds for f, then $\int_X f\,d\mu = \int_{O_\mu} f\,d\mu \le M\,\mu(O_\mu) < +\infty$, where M denotes the maximum value of f on X. If on the other hand $f(x_0) > 0$ for some $x_0 \notin O_\mu$, then the set $Y = \{x \in X \mid f(x) \ge f(x_0)/2\}$ has measure $+\infty$ by (iii). Hence $\int_X f\,d\mu = +\infty$.

In order to construct our example we fix a finite family $\mathfrak{O}$ of both open and closed subsets of X, closed for intersections and containing X. Note that if X is locally connected, then the family of all open and closed subsets of X is finite. Indeed, there is a finite covering of X by the open, closed and connected components of X (Corollaries 27.10 and 27.11 in [13]). Given any open and closed subset Y of X, connectedness implies that each component is either disjoint from Y or contained in it. Hence Y is the union of those components which it contains. There are only finitely many different of these unions, our claim.

Now let $\mathcal{P}$ be the cone of all regular $[0,+\infty]$-valued Borel measures μ on X (in the above sense) such that $O_\mu \in \mathfrak{O}$, and let $\mathbb{Q}$ be the cone of all non-negative continuous functions on X. Together with the bilinear form $\langle \mu, f\rangle = \int_X f\,d\mu$ for $\mu \in \mathcal{P}$ and $f \in \mathbb{Q}$ we consider the dual pair $(\mathcal{P}, \mathbb{Q})$ and the symmetric weak topology $\sigma(\mathcal{P}, \mathbb{Q})$ on $\mathcal{P}$. The relative and the given topologies on $\mathcal{P}$ coincide and the order induced by the duality is the weak preorder (Proposition 3.2). This order is antisymmetric, since $\int_X f\,d\mu = \int_X f\,d\nu$ for $\mu, \nu \in \mathcal{P}$ and all $f \in \mathbb{Q}$ implies that $\mu = \nu$ by the regularity of these measures. We shall show that $\mathcal{P}$ is locally compact. According to 3.2 we shall argue using the given rather than the relative neighborhoods in $\mathcal{P}$.

The neighborhoods v_F in $\mathcal{P}$ are defined by finite subsets F of $\mathbb{Q}$ as explained in Section 3, and the symmetric v_F-boundedness components of $\mathcal{P}$ are determined by the subsets of F in the following way: Two measures μ and ν in $\mathcal{P}$ are contained in the same symmetric v_F-boundedness component if and only if for all $f \in F$ we have $\int_X f\,d\mu = +\infty$ if and only if $\int_X f\,d\nu = +\infty$. Thus there are only finitely many of these components, each determined by a subset of F.

We shall investigate a particular neighborhood: For every $O \in \mathfrak{O}$ its characteristic function χ_O, that is $\chi_O(x) = 1$ for $x \in O$ and $\chi_O(x) = 0$ else, is continuous, hence an element of $\mathbb{Q}$. We consider the symmetric neighborhood $v_{F_\mathfrak{O}}$ in $\mathcal{P}$ defined by the finite subset $F_\mathfrak{O} = \{\chi_O \mid O \in \mathfrak{O}\}$ of $\mathbb{Q}$. The symmetric $v_{F_\mathfrak{O}}$-boundedness component of an element $\mu \in \mathcal{P}$ consists of all $\nu \in \mathcal{P}$ such that $O_\nu = O_\mu$, and each of these components contains a zero-type element, namely θ_μ as defined in (iv). In the light of Proposition 4.7(a) we will only have to verify that the symmetric $v_{F_\mathfrak{O}}$ -neighborhoods of these zero-type elements are compact in order to establish that $\mathcal{P}$ itself is locally compact: Let $\theta \in \mathcal{P}$ be a zero-type element. By the above and the definition in Section 3 we have $\nu \in v^s_{F_\mathfrak{O}}(\theta)$ if and only if for all $\chi_O \in F_\mathfrak{O}$ we have either $\int \chi_O\,d\theta = \int \chi_O\,d\nu = +\infty$ or $|\int \chi_O\,d\theta - \int \chi_O\,d\nu| \le 1$, that is if and only if

$$O_\nu = O_\theta \quad \text{and} \quad \nu(O_\theta) \le 1.$$

Equivalently, we express $\nu = \theta + \phi$ as elaborated in (iv), where $\phi \in \mathcal{P}$ is a finite regular Borel measure such that $\nu(X) \le 1$. That is $\phi \in v^s_{F_\mathfrak{O}}(0)$, and we conclude

$$v^s_{F_\mathfrak{O}}(\theta) = \theta + v^s_{F_\mathfrak{O}}(0).$$

Thus according to Proposition 2.13(c) we only need to verify that the neighborhood $v^s_{F_\mathfrak{O}}(0)$ is symmetric compact. We use Theorem 3.6 for this: For symmetric precompactness, given a neighborhood v_G in $\mathcal{P}$ defined by a finite subset G of $\mathbb{Q}$ let $M \ge 0$ be an upper bound for the functions in G. If $\phi \in v^s_{F_\mathfrak{O}}(0)$, then $\phi(X) \le 1$ and therefore $\langle \phi, g\rangle = \int_X g\,d\phi \le M$ for all $g \in G$. We infer that

$$\langle \phi, g\rangle \le \langle \psi, g\rangle + M$$

holds for all $\phi, \psi \in v^s_{F_\mathfrak{O}}(0.)$ Thus $v^s_{F_\mathfrak{O}}(0)$ is relatively v_G-bounded. All left to verify is that $v^s_{F_\mathfrak{O}}(0)$ is a closed subset of $\mathbb{Q}^\star$. Let $(\phi_i)_{i\in\mathcal{I}}$ be a net in $v^s_{F_\mathfrak{O}}(0)$ converging to a functional $q \in \mathbb{Q}^\star$ in the symmetric $\sigma(\mathbb{Q}^\star, \mathbb{Q})$-topology. This functional is linear and monotone on $\mathbb{Q}$, and we have $q(\chi_X) \le 1$. According to the classical Riesz representation theorem (see for example Ch. 21 in[12]), then q itself is a finite regular Borel measure on X, hence an element of $v^s_{F_\mathfrak{O}}(0)$. This completes our argument. The locally convex cone $\mathcal{P}$ is seen to be locally compact.

For a zero-type element $\theta \in \mathcal{P}$ its symmetric $v_{F_{\mathfrak{O}}}$-boundedness component $\mathcal{B}^s_{v_{F_{\mathfrak{O}}}}(\theta)$ is obviously proper, since $\theta = \mu + \nu$ for $\mu, \nu \in \mathcal{B}^s_{v_{F_{\mathfrak{O}}}}(\theta)$ implies that $\mu = \nu = \theta$. Hence according to Theorem 4.9 it admits a symmetric compact base. Such a base for $\mathcal{B}^s_{v_{F_{\mathfrak{O}}}}(\theta)$ is given by

$$C = \{\mu \in \mathcal{B}^s_{v_{F_{\mathfrak{O}}}}(\theta) \mid \langle \mu, \chi_{O_\theta} \rangle = 1\} = \theta + \{\mu \in \mathcal{P} \mid \mu(X) = 1\}.$$

In case that $\mathfrak{O} = \{X\}$, $\mathcal{P}$ consists of all finite regular measures on X, $\theta = 0$ is the only zero-type element in $\mathcal{P}$, $\mathcal{B}^s_{v_{\{\chi_X\}}}(0)\mathcal{P}$ is the only v_{χ_X}-boundedness component and indeed the only global boundedness component of $\mathcal{P}$. Its symmetric compact base is given by $C = \{\mu \in \mathcal{P} \mid \mu(X) = 1\}$.

(d) Consider the full locally convex cone

$$\mathcal{P} = \{(x_i)_{i \in \mathbb{N}} \mid x_i \in [0, +\infty] \quad \text{only finitely many} \quad x_i \neq 0\}$$

with the componentwise algebraic operations and order and the neighborhood system $\mathfrak{V}$ consisting of the neighborhoods $v = (z_i)_{i \in \mathbb{N}} \in \mathcal{P}$ such that $z_i \in [0, +\infty)$ and only finitely many $z_i = 0$. For the global boundedness components of $\mathcal{P}$ we realize that for $(x_i), (y_i) \in \mathcal{P}$ we have $(x_i) \in \mathcal{B}^s(y_i)$ if and only if there are disjoint subsets Y_0, Y_f and Y_∞ of $\mathbb{N}$ whose union is $\mathbb{N}$, such that Y_f is finite and such that $x_i = y_i = 0$ for all $i \in Y_0$, $x_i = y_i = +\infty$ for all $i \in Y_\infty$ and $x_i, y_i \in (0 + \infty)$ for all $i \in Y_f$. If $Y_f \neq \emptyset$, then $\mathcal{B}^s(y_i)$ does not contain a zero-type element. Endowed with the neighborhood $v = (z_i)_{i \in \mathbb{N}}$, where $z_i = 1$ for all $i \in \mathbb{N}$, the boundedness component $\mathcal{B}^s(y_i)$ is homeomorphic to $(0, +\infty)^{|Y_f|}$, where $|Y_f|$ denotes the number of elements of Y_f. Hence $\mathcal{B}^s(y_i)$ is locally compact in its trace topology. Because the global boundedness components are closed but not open subsets of $\mathcal{P}$ in this case, this observation does not imply local compactness for $\mathcal{P}$ itself.

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