Journal of Convex Analysis
Volume 31 (2024), No. 1, 111–120

Small Combination of Slices and Dentability in Ideals of Banach Spaces

Sudeshna Basu
Dept. Mathematics and Statistics, Loyola University, Baltimore, U.S.A.;
and: Dept. of Mathematics, George Washington University, Washington DC, U.S.A.
sudeshnamelody@gmail.com

Susmita Seal
Dept. of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute,
Belur Math, Howrah 711202, West Bengal, India
susmitaseal1996@gmail.com

Received: November 11, 2021
Accepted: September 3, 2022

We study small diameter properties of the closed unit ball namely Ball Small Combination of Slices Property ($BSCSP$), Ball Huskable property (BHP) and Ball Dentable Property (BDP) in a Banach space and its dual in the context of ideals in Banach spaces. The related concepts for an arbitrary closed bounded convex set in a Banach space were initiated and developed by several authors in the 1970s and 1980s. These concepts were extensively studied in the context of small combination of slices, dentability, huskability, Radon Nikodym Property and Krein Milman Property by N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer [*Some topological and geometrical structures in Banach spaces*, Memoirs of the American Mathematical Society, Volume 70, Number 378 (1987)]. We show that if a Banach space X has $BSCSP$ (respectively, BHP, BDP), then any M-ideal of X also has $BSCSP$ (respectively, BHP, BDP). We further show that if Y is an M-ideal or a strict ideal of X, then weak star versions of these properties can be lifted from Y^* to X^*. We use these results to prove the stability of these properties in $C(K)$ spaces and their dual. Lastly, we show that if Y is an almost isometric ideal of X, then $BSCSP$ (respectively BHP, BDP) can be lifted from Y to X.

Keywords: Slices, M-ideals, strict ideals, almost isometric ideals, huskable, denting, dentable, small combination of slices.

2010 Mathematics Subject Classification: 46B20, 46B28.

1. Introduction

Let X be a *real* nontrivial Banach space and X^* its dual. We will denote by B_X, S_X and $B_X(x,r)$ the closed unit ball, the unit sphere and the closed ball of radius $r>0$ and center x. We refer to the monograph [2] for notions of convexity theory that we will be using here.

Definition 1.1. (i) We say $A \subseteq B_{X^*}$ is a *norming set* for X if $\|x\| = \sup\{|x^*(x)| : x^* \in A\}$, for all $x \in X$. A closed subspace $F \subseteq X^*$ is a *norming subspace* if B_F is a norming set for X.

(ii) Let $f \in X^*$, $\alpha > 0$ and $C \subseteq X$. Then the set

$$S(C,f,\alpha) = \{x \in C : f(x) > \sup f(C) - \alpha\}$$

is called *slice determined* by f and α. We assume without loss of generality that

ISSN 0944-6532 / \$ 2.50

$\|f\| = 1$. One can analogously define w^*-slices in X^*. A convex combination of slices $S_i,\ i = 1 \dots n$ is a set $S = \sum_{i=1}^n \lambda_i S_i$ where $\lambda_i > 0\ \forall i = 1, \dots, n$ with $\sum_{i=1}^n \lambda_i = 1$.

We recall the following definitions from [3], [6] and [7]. The notions of dentability, huskability and small combination of slices in closed bounded convex sets were studied in detail in [10] and [13].

Definition 1.2. A Banach space X has
(i) *Ball Dentable Property* (BDP) if the closed unit ball has slices of arbitrarily small diameter.
(ii) *Ball Huskable Property* (BHP) if the closed unit ball has nonempty relatively weakly open subsets of arbitrarily small diameter.
(iii) *Ball Small Combination of Slice Property* ($BSCSP$) if the closed unit ball has convex combinations of slices of arbitrarily small diameter.

Remark 1.3. Analogously, we can define w^*-BDP, w^*-BHP and w^*-$BSCSP$ in a dual space by taking w^*-slices, w^*-open sets and w^*-small combination of slices respectively.

It is easy to observe the following:

$$\begin{array}{ccccc} BDP & \Longrightarrow & BHP & \Longrightarrow & BSCSP \\ \Uparrow & & \Uparrow & & \Uparrow \\ w^*BDP & \Longrightarrow & w^*BHP & \Longrightarrow & w^*BSCSP \end{array}$$

In general, none of the reverse implications hold good. For details, see [7]. All the three properties BDP, BHP and $BSCSP$ are localised (to the closed unit ball) versions of three well known geometric properties RNP, PCP and SR respectively (see [10]). We also have the following result from [7] which we will need in our subsequent discussion.

Proposition 1.4. [7] *A Banach space X has BDP (resp. BHP , $BSCSP$) if and only if X^{**} has w^*-BDP (resp. w^*-BHP, w^*-$BSCSP$).*

In this work, we study these properties in ideals of Banach spaces. The notion of ideals in Banach spaces, introduced by Godefroy, Kalton and Saphar in [11], continues to be a very important concept in geometry of Banach space. We include here some of the references relevant to our work, namely [1], [15], [16], [19], [22] and [23], and recall the following definitions and results which we need in our discussion:

Definition 1.5. Let $Y \subseteq X$ be a subspace of X. The *annihilator* of Y in the dual space X^* is the subspace of X^* defined by $Y^\perp = \{x^* \in X^* : x^*(y) = 0 \quad \forall y \in Y\}$.

Definition 1.6. [11]Let Y be a closed subspace of X. Then Y is said to be an *ideal* in X if $Y^\perp$ is the kernel of a norm one projection in X^*.

Remark 1.7. (1) Suppose $P : X^* \to X^*$ is projection with $\|P\| = 1$, $\mathrm{Ker}(P) = Y^\perp$. For $x^* \in X^*$ since $P(x^*) - x^* = 0$ on Y , as $\|P\| = 1$, we see that $P(x^*)$ is a norm preserving extension of $x^*|_Y$.

(2) If we look at X as an embedding in X^{**} via the map $J : X \to X^{**}$ defined by $J(x)(x^*) = x^*(x)$, the natural projection on X^{***} by $\Gamma \to \Gamma|_X$ is a norm one linear projection with kernel $X^\perp$. Thus X is isometric to an ideal in X^{**} via this canonical embedding J.

Definition 1.8. Let X be a Banach space. A linear projection P on X is called

(i) an *L-projection* if $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in X$.

(ii) an *M-projection* if $\|x\| = Max\{\|Px\|, \|x - Px\|\}$ for all $x \in X$.

Definition 1.9. Let X be a Banach space. A closed subspace $Y \subset X$ is called an *M-ideal* if there exists an L-projection $P : X^* \to X^*$ with $\mathrm{Ker}(P) = Y^\perp$.

We recall from Chapter I of [16] that when $Y \subset X$ is an M-ideal, elements of Y^* have unique norm-preserving extension to X^* and one has the identification, $X^* = Y^* \oplus_1 Y^\perp$. Several examples from among function spaces and spaces of operators that satisfy these geometric properties can be found in the monograph [16]. See also [19].

Definition 1.10. An ideal $Y \subseteq X$ is said to be a *strict ideal* if for a projection $P : X^* \to X^*$ with $\|P\| = 1$, $\mathrm{Ker}(P) = Y^\perp$, one also has $B_{P(X^*)}$ is w^*-dense in B_{X^*} or in other words $B_{P(X^*)}$ is a norming set for X.

In the case of an ideal also one has that Y^* embeds (though there may not be uniqueness of norm-preserving extensions) as $P(X^*)$. Thus we continue to write $X^* = Y^* \oplus Y^\perp$. In what follows, we use a result from [22] that identifies strict ideals as those for which $Y \subset X \subset Y^{**}$ under the canonical embedding of Y in Y^{**}. A prime example of a strict ideal is a Banach space X under its canonical embedding in X^{**}.

Definition 1.11. Let X be a Banach space and Y be a subspace of X. An operator $f : Y^* \to X^*$ is called a *Hahn Banach extension operator* if it satisfies the following conditions:

(i) $(fy^*)|_Y = y^* \ \forall y^* \in Y^*$

(ii) $\|fy^*\| = \|y^*\| \ \forall y^* \in Y^*$.

Definition 1.12. A subspace Y of X is called an almost isometric ideal (*ai*-ideal) in X if for every $\varepsilon > 0$ and every finite dimensional subspace $E \subset X$ there exists $T : E \to Y$ such that the following hold :

(i) $Te = e \ \forall e \in E \bigcap Y$

(ii) $\frac{1}{1+\varepsilon}\|e\| \leqslant \|Te\| \leqslant (1 + \varepsilon)\|e\| \ \forall e \in E$.

Almost Isometric ideals were first introduced in [1]. It is easy to see that the notion of *ai*-ideal is a direct generalisation of Principle of local reflexivity. For more details, see [18], [21].

Proposition 1.13. ([1]) *A subspace Y of X is ai-ideal if and only if there exists a Hahn Banach extension operator $f : Y^* \to X^*$ such that for every $\varepsilon > 0$ and every finite dimensional subspace $E \subset X$, finite dimensional subspace $F \subset Y^*$, there exists $T : E \to Y$ such that the following hold:*

(i) $Te = e\ \forall e \in E \bigcap Y$

(ii) $\frac{1}{1+\varepsilon}\|e\| \leqslant \|Te\| \leqslant (1+\varepsilon)\|e\|\ \forall e \in E$

(iii) $fy^*(e) = y^*(Te)\ \forall e \in E\ \forall y^* \in F.$

We show that if a Banach space X has $BSCSP$, (respectively, BHP, BDP) then any M-ideal of X also has $BSCSP$ (respectively, BHP, BDP). We further show that if Y is an M-ideal of X, then X^* has w^*-$BSCSP$ (respectively w^*-BHP, w^*-BDP) whenever Y^* has w^*-$BSCSP$ (respectively w^*-BHP, w^*-BDP). We use these results to prove that for a compact Hausdorff space K, which has an isolated point, X has $BSCSP$ (respectively, BHP, BDP) whenever $C(K,X)$ has $BSCSP$ (respectively, BHP, BDP) and $C(K,X)^*$ has w^*-$BSCSP$ (respectively w^*-BHP, w^*-BDP) whenever X^* has w^*-$BSCSP$ (respectively w^*-BHP, w^*-BDP). We also prove that, if Y is a strict ideal of X, then X^* has w^*-BHP whenever Y^* has w^*-BHP. Lastly, we show that if Y is an almost isometric ideal of X then Y has $BSCSP$ (respectively, BHP, BDP) implies X also has $BSCSP$ (respectively, BHP, BDP). The spaces that we will be considering have been well studied in literature. A large class of function spaces like the Bloch spaces, Lorentz and Orlicz spaces, spaces of vector valued functions and spaces of compact operators are examples of the spaces we will be considering, for details, see [16]. In some of our proofs, we use similar techniques as in [1]and [15].

2. Small diameter properties in ideals of Banach spaces

We need the following Proposition in our subsequent discussion which is an easy consequence of Proposition 2.3 [23].

Proposition 2.1. *Let Y be an M-ideal in X. Then, for all $x \in B_X$, for all $\delta > 0$, and for all $y^* \in B_{Y^*}$ there exists $z \in B_Y$ such that*

(i) $\|y + x - z\| < (1+\delta) \quad \forall y \in B_Y$ *and*

(ii) $|z(y^*) - x(y^*)| < \delta.$

Proposition 2.2. *Let $Y \subset X$ be an M-ideal. Then X has $BSCSP$ implies Y has $BSCSP$.*

Proof. We will prove this result by contradiction. Suppose Y does not have $BSCSP$. Then there exists $\varepsilon > 0$, such that every convex combination of slices of B_Y has diameter greater than ε. Since X has $BSCSP$, there exists a convex combination of slices $\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha_i)$ of B_X with diameter less than $\frac{\varepsilon}{1+\varepsilon}$. Choose α such that $0 < \alpha < \min\{\varepsilon, \frac{\alpha_1}{3}, \frac{\alpha_2}{3}, \ldots, \frac{\alpha_n}{3}\}$. Since Y is an M-ideal, there exists an L-projection $P : X^* \to X^*$ with $\mathrm{Ker}(P) = Y^\perp$. If $Px_i^* = 0$ we choose $\hat{y}_i^* \in S_{Y^*}$ arbitrarily and fix it. Now define

$$y_i^* = \begin{cases} \frac{Px_i^*}{\|Px_i^*\|} & \text{if} \quad Px_i^* \neq 0 \\ \hat{y}_i^* & \text{if} \quad Px_i^* = 0 \end{cases} \quad \text{and} \quad \beta_i = \begin{cases} \frac{\alpha - \alpha\|Px_i^*\| + \alpha^2}{\|Px_i^*\|} & \text{if} \quad Px_i^* \neq 0 \\ \alpha & \text{if} \quad Px_i^* = 0. \end{cases}$$

Therefore , $\sum_{i=1}^n \lambda_i S(B_Y, y_i^*, \beta_i)$ is a convex combination of slices of B_Y, by assumption it has diameter greater than ε.

Hence, there exist $y_1^1, y_2^1, \ldots, y_n^1 \in B_Y$ and $y_1^2, y_2^2, \ldots, y_n^2 \in B_Y$ such that

$$\|\sum_{i=1}^{n} \lambda_i(y_i^1 - y_i^2)\| > \varepsilon$$

and $$y_i^*(y_i^k) > 1 - \beta_i \quad \forall k = 1, 2,\ \forall i = 1, 2, \ldots, n.$$

Now for $Px_i^* \neq 0$, we have

$$\frac{Px_i^*}{\|Px_i^*\|}(y_i^k) > 1 - \beta_i \quad \forall k = 1, 2$$

i.e. $$Px_i^*(y_i^k) > (\|Px_i^*\| - \alpha)(1 + \alpha) \quad \forall k = 1, 2.$$

Also there exist $x_1, x_2, \ldots, x_n \in B_X$ such that

$$(x_i^* - Px_i^*)(x_i) > (\|x_i^* - Px_i^*\| - \alpha)(1 + \alpha) \quad \forall i = 1, 2, \ldots n.$$

Since Y is an M-ideal in X, by Proposition 2.1, for every $x_i, P(x_i^*)$ there exists $z_i \in B_Y$ such that

$$\|y_i^k + x_i - z_i\| < 1 + \alpha \quad \forall i = 1, 2, \ldots n,\ \forall k = 1, 2$$

and $$|Px_i^*(x_i - z_i)| < \alpha \quad \forall i = 1, 2 \ldots n.$$

Consider, $x_i^k = \dfrac{y_i^k + x_i - z_i}{1 + \alpha}$ for all $k = 1, 2$, and for all $i = 1, 2, \ldots, n$.

Thus $x_i^k \in S(B_X, x_i^*, \alpha_i)$ for all $k = 1, 2$.

Indeed,
$$\begin{aligned} x_i^*(x_i^k) &= \frac{x_i^*(y_i^k + x_i - z_i)}{1 + \alpha} \\ &= \frac{Px_i^*(y_i^k) + (x_i^* - Px_i^*)(x_i) + Px_i^*(x_i - z_i)}{1 + \alpha} \\ &> \begin{cases} \|Px_i^*\| - \alpha + \|x_i^* - Px_i^*\| - \alpha - \alpha & \text{if } Px_i^* \neq 0 \\ \|x_i^*\| - \alpha & \text{if } Px_i^* = 0 \end{cases} \\ &\geqslant \begin{cases} \|x_i^*\| - 3\alpha & \text{if } Px_i^* \neq 0 \\ \|x_i^*\| - \alpha & \text{if } Px_i^* = 0 \end{cases} \\ &= \begin{cases} 1 - 3\alpha & \text{if } Px_i^* \neq 0 \\ 1 - \alpha & \text{if } Px_i^* = 0 \end{cases} > 1 - \alpha_i. \end{aligned}$$

Now we conclude

$$\|\sum_{i=1}^{n} \lambda_i(x_i^1 - x_i^2)\| = \frac{\sum_{i=1}^{n} \lambda_i(y_i^1 - y_i^2)}{1 + \alpha} > \frac{\varepsilon}{1 + \alpha} > \frac{\varepsilon}{1 + \varepsilon},$$

a contradiction. Thus Y has $BSCSP$. □

However for BDP, we have the following.

Proposition 2.3. *If X has a proper M ideal then X fails to have BDP.*

Proof. Suppose Y is a proper M-ideal of X. Thus $X^* = Y^* \oplus_1 Y^\perp$. If possible let, X have BDP. By Proposition 1.4, X^{**} has w^*BDP. It now follows from Proposition

2.17, [7], Y^{**} has w^*BDP. Applying Proposition 1.4 again we can conclude that Y has BDP, a contradiction, since a proper M-ideal fails to have BDP (see [16], Chapter II, Theorem 4.4). $\square$

In view of Proposition 2.3 and Proposition 2.2, we have the following remark.

Remark 2.4. Putting $n = 1$ in Proposition 2.2, it follows that if X is a Banach space with BDP and Y is an M-summand of X, then Y has BDP. This also follows from Proposition 2.7, [7].

Proposition 2.5. *Let X be a Banach space and let $Y \subset X$ be an M-ideal , then X has BHP implies Y has BHP.*

Proof. We prove by contradiction. Suppose Y fails BHP. Then there exists $\varepsilon > 0$ such that every nonempty relatively weakly open subset of B_Y has diameter greater than ε. Since X has BHP, there exists a basic relatively weakly open subset $U = \{x \in B_X : |x_i^*(x - x_0)| < \gamma, i = 1, 2, \dots, n\}$ (where $x_i^* \in S_{X^*} \forall i$ and $x_0 \in B_X$) with diameter less than $\frac{\varepsilon}{1+\varepsilon}$. Choose $\beta > 0$ such that $\beta < min\{\frac{\gamma}{4}, \varepsilon\}$. Since Y is an M-ideal of X, there exists a L-projection $P : X^* \to X^*$ with $\mathrm{Ker}(P) = Y^\perp$. Also, by Proposition 2.1, there exists $y_0 \in B_Y$ such that $\|y + x_0 - y_0\| < 1 + \beta \quad \forall y \in B_Y$. Consider, $V = \{y \in B_Y : |Px_i^*(y - y_0)| < \beta, i = 1, 2, \dots, n\}$. Then V is a nonempty relatively weakly open subset of B_Y, so $\mathrm{diam}(V) > \varepsilon$. Hence there exists $y, \tilde{y} \in V$ such that $\|y - \tilde{y}\| > \varepsilon$. Now $\|y + x_0 - y_0\| < 1 + \beta$ and $\|\tilde{y} + x_0 - y_0\| < 1 + \beta$.

Consider $$x = \frac{y + x_0 - y_0}{1 + \beta} \quad \text{and} \quad \tilde{x} = \frac{\tilde{y} + x_0 - y_0}{1 + \beta}.$$

Therefore, for $i = 1, 2, \dots, n$,

$$|x_i^*(x - x_0)| = \frac{|x_i^*(y - \beta x_0 - y_0)|}{1 + \beta} \leqslant \frac{|Px_i^*(y - y_0)| + \beta|x_i^*(x_0)|}{1 + \beta} < \frac{2\beta}{1 + \beta} < 2\beta < \gamma.$$

Thus, $x \in U$ and similarly $\tilde{x} \in U$.

Now, $\|x - \tilde{x}\| = \frac{\|y - \tilde{y}\|}{1+\beta} > \frac{\varepsilon}{1+\beta} > \frac{\varepsilon}{1+\varepsilon}$, a contradiction. Thus Y has BHP. $\square$

Proposition 2.6. *If $Y \subset X$ is an M-ideal in X , then Y^* has w^*BHP implies X^* has w^*BHP.*

Proof. Let $\varepsilon > 0$. Since Y^* has w^*BHP, B_{Y^*} has a nonempty w^* open subset V of diameter less than ε. Choose y_0^* in $S_{Y^*} \bigcap V$. Then there exists

$$V_0 = \{y^* \in B_{Y^*} : |y^*(y_i) - y_0^*(y_i)| < \alpha, \ i = 1, 2, \dots, n\} \subset V$$

for some $n \in \mathbb{N}$ and $y_1, y_2, \dots, y_n \in B_Y$. Since Y is an M-ideal, $X^* = Y^* \oplus_1 Y^\perp$. For $y_0^* \in S_{Y^*}$, there exists $y_0 \in B_Y$ such that $|y_0^*(y_0)| > 1 - \varepsilon$. Choose $\gamma > 0$ such that $|y_0^*(y_0)| > 1 - \varepsilon + \gamma$. Let $U_0 = \{y^* \in B_{Y^*} : |y_0^*(y_0) - y^*(y_0)| < \gamma\}$. Then, for $y^* \in U_0$ we have $|y^*(y_0)| > |y_0^*(y_0)| - \gamma > (1 - \varepsilon + \gamma) - \gamma = 1 - \varepsilon$. Choose $0 < \delta < \min\{\alpha, \gamma\}$.

Let, $W = \{x^* \in B_{X^*} : |x^*(y_i) - y_0^*(y_i)| < \delta, i = 0, 1, 2, \dots, n\}$.

Clearly , W is a relatively w^* open subset of B_{X^*}. Then, $W \subset V_0 + \varepsilon B_{Y^\perp}$. Indeed, let $x^* \in W$.

Then there exists $y^* \in Y^*$ and $y^\perp \in Y^\perp$ such that $x^* = y^* + y^\perp$. Then

$$|x^*(y_i) - y_0^*(y_i)| < \delta \quad \forall i = 0, 1, 2, \dots, n.$$

i.e. $$|(y^* + y^\perp)(y_i) - y_0^*(y_i)| < \delta \quad \forall i = 0, 1, 2, \dots, n.$$

i.e. $$|y^*(y_i) - y_0^*(y_i)| < \delta \quad \forall i = 0, 1, 2, \dots, n.$$

Hence, $|y^*(y_i) - y_0^*(y_i)| < \delta < \alpha \quad \forall i = 0, 1, 2, \dots, n$ and $|y^*(y_0) - y_0^*(y_0)| < \delta < \gamma$. Therefore, $y^* \in V_0$ and $y^* \in U_0$, which implies $y^* \in V_0$ and $\|y^*\| > 1 - \varepsilon$.
Since, $\|x^*\| = \|y^*\| + \|y^\perp\|$, it follows that $\|y^\perp\| < \varepsilon$ i.e. $y^\perp \in \varepsilon B_{Y^\perp}$. Thus we have $x^* = y^* + y^\perp \in V_0 + \varepsilon B_{Y^\perp}$. Hence,

$$\mathrm{diam}(W) \leqslant \mathrm{diam}(V_0) + \mathrm{diam}(\varepsilon B_{Y^\perp}) \leqslant \mathrm{diam}(V) + \mathrm{diam}(\varepsilon B_{Y^\perp}) < 2\varepsilon. \qquad \square$$

Remark 2.7. Similarly, for an M- ideal Y in X, Y^* has w^*BSCSP (w^*BDP) implies X^* has w^*BSCSP (w^*BDP), see [3], [6].

We recall that for a compact Hausdorff space K, $C(K,X)$ denotes the space of continuous X-valued functions on K, equipped with the supremum norm. We recall from [20] that dispersed compact Hausdorff spaces have isolated points.

Proposition 2.8. *Let K be a compact, Hausdorff space with an isolated point. Then*

(i) *If $C(K,X)$ has $BSCSP$ (respectively BHP, BDP) then X has $BSCSP$ (respectively BHP, BDP).*

(ii) *If X^* has w^*-BHP , then $C(K,X)^*$ has w^*-BHP.*

Proof. Let $k_0 \in K$ be a isolated point. Then the map P such that $F \longmapsto \chi_{k_0} F$ is an M-projection in $C(K,X)$ whose range is isometric to X. Thus X is isometric to an M-summand and hence an M-ideal in $C(K,X)$. (i) follows from Proposition 2.2, Proposition 2.5 and Remark 2.4. (ii) follows from Proposition 2.6. $\square$

Remark 2.9. Similar results as in (ii), also hold good for w^*-BDP and w^*-$BSCSP$. See [3] and [6].

Proposition 2.10. *Let Y be a strict ideal of X. If Y^* has w^*BHP, then X^* has w^*BHP.*

Proof. Let $\varepsilon > 0$. Since Y^* has w^*BHP then there exists a relatively w^*-open subset $V = \{y^* \in B_{Y^*} : |y^*(y_i) - y_0^*(y_i)| < \alpha,\ i = 1, 2, \dots, n\}$, where $y_1, y_2, \dots, y_n \in B_Y$ of B_{Y^*}, such that $\mathrm{diam}(V) < \varepsilon$. Let

$$W = \{x^* \in B_{X^*} : |x^*(y_i) - y_0^*(y_i)| < \alpha, i = 1, 2, \dots, n\}.$$

Then W is a relatively w^*-open subset of B_{X^*}. Since Y is a strict ideal in X, we have $B_{X^*} = \overline{B_{Y^*}}^{w^*}$, hence we have $W \subset \overline{V}^{w^*}$. Indeed, let $x_0^* \in W$. Then there exists a net (x_λ^*) in B_{Y^*} that w^* converges to x_0^*. Now,

$$\lim_\lambda |x_\lambda^*(y_i) - y_0^*(y_i)| = |x_0^*(y_i) - y_0^*(y_i)| \quad \forall i = 1, 2, \dots n.$$

Since $x_0^* \in W$ so we get a λ_0 such that $|x_\lambda^*(y_i) - y_0^*(y_i)| < \alpha \ \forall \lambda \geqslant \lambda_0 \ \forall i = 1, 2, \dots n$.

Hence $x^*_\lambda \in V, \forall\lambda \geqslant \lambda_0$ and therefore, $x^*_0 \in \overline{V}^{w^*}$. Thus,

$$\operatorname{diam}(W) \leqslant \operatorname{diam}(\overline{V}^{w^*}) = \operatorname{diam}(V) < \varepsilon. \qquad \square$$

Remark 2.11. Similar results hold good for w^*BSCSP and w^*BDP. For details, see [3], [6].

However, the converse of these results are not true. We give an example.

Example 2.12. Let $X = C[0,1]$. Since X is always a strict ideal in $X^{**}, C[0,1]$ is also so in its double dual. It is well known that $X^* = L_1[0,1] \oplus_1 l_1(\Gamma)$, for some set Γ (see [17], [14]). Since $l_1(\Gamma)$ has RNP (see [8]), it has BDP. By [7, Proposition 2.4], X^* has BDP and hence it has BHP and $BSCSP$. Thus it follows from Proposition 1.4, that X^{***} has w^*-BDP and hence w^*-BHP and w^*-$BSCSP$. It is also known that every convex combination of w^*-slices of B_{X^*} has diameter two (see [5]). Hence, X^* does not have w^*-$BSCSP$ and consequently it does not have w^*-BHP and w^*-BDP.

Proposition 2.13. *If Y is an ai-ideal in X, then Y has $BSCSP$ implies X has $BSCSP$.*

Proof. We prove by contradiction. Suppose X does not have $BSCSP$. Then there exists $\varepsilon > 0$, such that every convex combination of slices of B_X has diameter greater than ϵ. Since Y has $BSCSP$, for $\varepsilon > 0$ there exists a convex combination of slices, $S = \sum_{i=1}^n \lambda_i S_i(B_Y, y_i^*, \alpha_i)$ of B_Y, where $\alpha_i > 0$ $\forall i = 1, 2, \ldots, n$, $1 \geq \lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$, $y_i^* \in S_{Y^*}$ $\forall i = 1, 2, \ldots, n$ and $\operatorname{diam}(S) < \frac{\varepsilon}{4}$. Since Y is an ai-ideal in X, there exists a Hahn Banach extension operator, $f\colon Y^* \to X^*$ satisfying the conditions of Proposition 1.13.
Consider $\bar{S} = \sum_{i=1}^n \lambda_i S_i(B_X, f y_i^*, \alpha_i)$. Since $\bar{S}$ is a convex combination of slices of B_X, $\operatorname{diam}(\bar{S}) > \varepsilon$. Hence there exists $x = \sum_{i=1}^n \lambda_i x_i$, $\tilde{x} = \sum_{i=1}^n \lambda_i \tilde{x}_i \in \bar{S}$ such that $\|x - \tilde{x}\| > \varepsilon$. Now we can choose $0 < \alpha < 1$ such that

$$\tfrac{x_i}{1+\alpha} \in S_i(B_X, f y_i^*, \alpha_i) \;\text{ and }\; \tfrac{\tilde{x_i}}{1+\alpha} \in S_i(B_X, f y_i^*, \alpha_i)\ \forall i = 1, 2, \ldots, n.$$

Put $x' = \sum_{i=1}^n \lambda_i \frac{x_i}{1+\alpha} = \frac{x}{1+\alpha}$, $\tilde{x}' = \sum_{i=1}^n \lambda_i \frac{\tilde{x_i}}{1+\alpha} = \frac{\tilde{x}}{1+\alpha} \in \bar{S}$.
Also , $\|x' - \tilde{x}'\| = \frac{\|x-\tilde{x}\|}{1+\alpha} > \frac{\varepsilon}{1+\alpha}$. Define

$$E = \operatorname{span}\{x', \tilde{x}', \frac{x_1}{1+\alpha}, \frac{x_2}{1+\alpha}, \ldots, \frac{x_n}{1+\alpha}, \frac{\tilde{x_1}}{1+\alpha}, \frac{\tilde{x_2}}{1+\alpha}, \ldots, \frac{\tilde{x_n}}{1+\alpha}\}$$

and $F = \operatorname{span}\{y_1^*, y_2^*, \ldots, y_n^*\}$. For E, F and α there exists $T : E \to Y$ satisfying the conditions in Proposition 1.13. By Proposition 1.13 (ii), $\|Tx'\| \leqslant (1+\alpha)\|x'\| \leqslant 1$. Similarly $\|T\tilde{x}'\| \leqslant 1$. Again, by Proposition 1.13 (ii),

$$\|Tx' - T\tilde{x}'\| \geqslant \frac{\|x' - \tilde{x}'\|}{1+\alpha} > \frac{1}{1+\alpha}\frac{\varepsilon}{1+\alpha} > \frac{\varepsilon}{4}.$$

Also, $\forall i = 1, 2, \ldots, n$, by Proposition 1.13 (iii),

$$y_i^*(T\frac{x_i}{1+\alpha}) = f y_i^*(\frac{x_i}{1+\alpha}) > 1 - \alpha_i,$$

which implies $Tx' \in S$. Similarly $T\tilde{x}' \in S$, a contradiction as diam $S < \frac{\varepsilon}{4}$. $\square$

Arguing similarly as above for $n = 1$, we have,

Corollary 2.14. *If Y is an ai-ideal in X then Y has BDP implies X has BDP.*

Proposition 2.15. *If Y is an ai-ideal in X then Y has BHP implies X has BHP.*

Proof. We prove by contradiction. Suppose X does not have BHP. Then there exists $\varepsilon > 0$ such that every relatively weakly open subset of B_X has diameter greater than ε. Since Y has BHP, for $\varepsilon > 0$, B_Y has a relatively weakly open subset $U = \{y \in B_Y : |y_i^*(y - y_0)| < \delta, i = 1, 2, \ldots, n\}$ where $y_0 \in B_Y$, $\delta > 0$, $y_i^* \in Y^*$ $\forall i = 1, 2, \ldots, n$ and diam $(U) < \frac{\varepsilon}{4}$. Since Y is an *ai*-ideal in X, there exists a Hahn Banach extension operator $f : Y^* \to X^*$ as in Proposition 1.13.

Consider $V = \{x \in B_X : |fy_i^*(x - y_0)| < \delta, i = 1, 2, \ldots, n\}$. V is a relatively weakly open subset of B_X, hence diam $(V) > \varepsilon$. So, there exists $x_1, x_2 \in V$ such that $\|x_1 - x_2\| > \varepsilon$. Choose $0 < \alpha < 1$ such that $\tilde{x_1} = \frac{x_1}{1+\alpha} \in V$ and $\tilde{x_2} = \frac{x_2}{1+\alpha} \in V$. Also , $\|\tilde{x_1} - \tilde{x_2}\| = \frac{\|x_1 - x_2\|}{1+\alpha} > \frac{\varepsilon}{1+\alpha}$. Put $E = \operatorname{span}\{y_0, \tilde{x_1}, \tilde{x_2}\}$ and $F = \operatorname{span}\{y_1^*, y_2^*, \ldots, y_n^*\}$. For E, F and α there exists $T : E \to Y$ as in Proposition 1.13. By Proposition 1.13(ii), $\|T\tilde{x_1}\| \leqslant (1 + \alpha)\|\tilde{x_1}\| \leqslant 1$. Similarly $\|T\tilde{x_2}\| \leqslant 1$.

Again by (ii) Proposition 1.13, we obtain

$$\|T\tilde{x_1} - T\tilde{x_2}\| \geqslant \frac{\|\tilde{x_1} - \tilde{x_2}\|}{1 + \alpha} > \frac{1}{1 + \alpha} \frac{\varepsilon}{1 + \alpha} > \frac{\varepsilon}{4}.$$

Also $\forall i = 1, 2, \ldots, n$, by Proposition 1.13 (i) and (iii),

$$|y_i^*(T\tilde{x_1} - y_0)| = |y_i^*(T\tilde{x_1} - Ty_0)| = |fy_i^*(\tilde{x_1} - y_0)| < \delta.$$

Thus $T\tilde{x_1} \in U$. Similarly $T\tilde{x_2} \in U$, a contradiction since diam $U < \frac{\varepsilon}{4}$. □

Acknowledgements The authors would like to thank the referee for the valuable comments and for meticulously going through the manuscript and correcting many typos. The second author's research is funded by the National Board for Higher Mathematics (NBHM), Department of Atomic Energy (DAE), Government of India, Ref No: 0203/11/2019-R&D-II/9249.

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