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A Uniqueness Result for a Translation Invariant Problem in the Calculus of Variations

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We present a uniqueness result of uniformly continuous solutions for a general minimization problem in the Calculus of Variations. We minimize the functional $\mathcal{I}_\lambda(u) := \int_\Omega \varphi(\nabla u) + \lambda u$ with φ a convex but not necessarily strictly convex function, Ω an open set of $\mathbb{R}^N$ with $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. The proof is based on the two following main points : the functional $\mathcal{I}_\lambda$ is invariant under translations and we assume that the function φ is not affine on any non-empty open set. This provides a shorter proof and/or an extension for some already known uniqueness results for functionals of the type $\mathcal{I}_\lambda$ that are presented in the article.

Keywords: Calculus of variations, translation invariance, non strictly-convex function, uniqueness.

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1. Introduction

1.1. Presentation of the problem

We study a uniqueness result for a non-strictly convex problem in the calculus of variations in dimension $N \geq 1$. We consider functionals of the following form:

$$\mathcal{I}_\lambda : u \to \int_\Omega \varphi(\nabla u(x)) + \lambda u(x) dx \tag{1}$$

with $\lambda \in \mathbb{R}$, Ω a bounded open set of $\mathbb{R}^N$ and $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ a measurable convex function. We consider the problem:

$$\mathcal{P}_\lambda : \text{ Minimize } \mathcal{I}_\lambda(v) \text{ on } \mathcal{E}(\Omega, \psi).$$

Here, $\psi : \mathbb{R}^N \to \mathbb{R}$ is a continuous function in $W^{1,1}_{loc}(\mathbb{R}^N)$ and $\mathcal{E}(\Omega, \psi)$ is a subset of $W^{1,1}_\psi(\Omega)$, namely the set of functions in $W^{1,1}(\Omega)$ such that the extension outside Ω by $\psi : \mathbb{R}^N \to \mathbb{R}$ is in $W^{1,1}_{loc}(\mathbb{R}^N)$. We work with the set $\mathcal{E}(\Omega, \psi)$ to cover many situations: in the following $\mathcal{E}(\Omega, \psi)$ can stand for $W^{1,1}_\psi(\Omega)$, $W^{1,p}_\psi(\Omega)$, $W^{1,p}_\psi(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, $\mathcal{C}^{0,1}_\psi(\overline{\Omega})$ or $\{v \in W^{1,1}_\psi(\Omega),\ \nabla v(x) \in K \text{ for a.e. } x \in \Omega\}$ with K a convex set.

We assume that $\mathcal{I}_\lambda$ is well defined on $\mathcal{E}(\Omega, \psi)$ with values in $\mathbb{R} \cup \{+\infty\}$ and that there exists $v \in \mathcal{E}(\Omega, \psi)$ such that $\mathcal{I}_\lambda(v) < +\infty$.

We point out that φ is not necessarily strictly convex. Hence, no general result guarantees uniqueness. Another remarkable feature of the main result Theorem 1.2 is that $\varphi(z)$ can be equal to $+\infty$ for some $z \in \mathbb{R}^N$. Hence, this framework covers

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also the case when we put a pointwise constraint on the gradient. In the situations studied in [1] and [8], $\mathcal{P}_\lambda$ is the convexification of a non-convex problem that arises in shape optimization. In these cases (see also [10]) uniqueness for $\mathcal{P}_\lambda$ can be useful to prove existence or non-existence of a minimizer for related non-convex problems.

1.2. Main result

In order to present the main result of this paper we introduce the following sets:

Definition 1.1. We say that a convex set $F \in \mathbb{R}^N$ is an *affine part* of φ if, for all $z, \xi \in F$, we have:

$$\varphi(z) < +\infty,\ \varphi(\xi) < +\infty \ \text{ and } \ \varphi\Big(\frac{z+\xi}{2}\Big) = \frac{\varphi(z)+\varphi(\xi)}{2}.$$

The dimension of F is the dimension of the affine space generated by F. □

We make the two following assumptions on the set $\mathcal{E}(\Omega,\psi)$:

($\mathcal{A}1$) The set $\mathcal{E}(\Omega,\psi)$ is invariant under translation: if $v \in \mathcal{E}(\Omega,\psi)$ then we have $\widetilde{v}(\cdot) = v(\cdot+\tau)+\alpha \in \mathcal{E}\big(\Omega_\tau, \psi(\cdot+\tau)+\alpha\big)$ with $\Omega_\tau = \Omega - \tau$.

($\mathcal{A}2$) If $u \in \mathcal{E}(\Omega,\psi)$ and $v \in \mathcal{E}(V,u)$ with V a non empty open set in Ω, then the extension of v by u is in $\mathcal{E}(\Omega,\psi)$.

We make one assumption on φ:

($\mathcal{A}3$) The dimensions of the affine parts of φ are at most $N-1$.

We now present the main theorem of this article:

Theorem 1.2. *Under those three assumptions, the problem $\mathcal{P}_\lambda$ admits at most one uniformly continuous minimizer.*

The proof of Theorem 1.2 which is relatively short and simple, unifies results that are described in Section 2. For some of them it is even an extension to higher dimensions or to a more general set Ω.

The first two assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) are relatively standard. For instance, they are satisfied when $\mathcal{E}(\Omega,\psi)$ is $W^{1,1}_\psi(\Omega)$, $W^{1,p}_\psi(\Omega)$, $W^{1,p}_\psi(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, $\mathcal{C}^{0,1}(\overline{\Omega})$ or $\{v \in W^{1,1}_\psi(\Omega),\ \nabla v(x) \in K \text{ for a.e. } x \in K\}$ with K a convex set.

However, the last assumption ($\mathcal{A}3$) on the dimensions of the affine parts of φ is necessary for the statement to be true. It is easy to find counterexamples when this condition is not satisfied. For instance if we consider the problem when $N = 1$, $\Omega = (0,1)$, $\psi(0) = 0$, $\psi(1) = 1$, $\lambda = 0$ and $\varphi(\cdot) = |\cdot|$, every increasing function that satisfies the boundary condition is a solution. Another example of non uniqueness when the last assumption is not satisfied can be found in [3, Example 6] with $\varphi(\cdot) = (|\cdot|-1)^2_+$.

In this article, we prove a uniqueness result but we do not provide any result about the existence or about the regularity of minimizers. In the cases where the existence is provided by the direct methods and the uniform continuity is standard, we obtain the following corollary of Theorem 1.2:

Corollary 1.3. *We assume that the problem $\mathcal{P}_\lambda$ admits a minimizer and every minimizer is uniformly continuous. Under the three assumptions ($\mathcal{A}1$), ($\mathcal{A}2$) and ($\mathcal{A}3$), the problem $\mathcal{P}_\lambda$ has a unique minimizer.*

That is the case for instance when Ω is smooth, ψ smooth and φ satisfies a *p-growth* condition, some examples are given in Section 2. In some problems like in [2], if a uniformly continuous minimizer exists then all the minimizers are uniformly continuous. In this case we have:

Corollary 1.4. *We assume that if the problem $\mathcal{P}_\lambda$ has a uniformly continuous minimizer then every minimizer is uniformly continuous. Under the three assumptions ($\mathcal{A}1$), ($\mathcal{A}2$) and ($\mathcal{A}3$) if a minimizer of $\mathcal{P}_\lambda$ is uniformly continuous then it is the only minimizer.*

The main idea of the proof comes from [5, Section 4]. In this paper, De Silva and Savin consider the minimization in dimension two of

$$I(u) = \int_\Omega \varphi(\nabla u)dx$$

on $\mathcal{E}(\Omega,\psi) = \mathcal{C}^{0,1}_\psi(\overline{\Omega})$ with φ a convex map with domain $K := \{z \in \mathbb{R}^2, \varphi(z) < +\infty\}$ a closed polygon. In [5, Proposition 4.5], the authors use the fact the problem is invariant under translation to compare the gradient of a minimizer at a point with the gradient of another minimizer at another point.

1.3. Plan of the paper

In the next section, we present some cases where we can apply Theorem 1.2. In Section 3, we present some classical results for functionals of the type (1). The last section is devoted to the proof of Theorem 1.2.

2. Applications

In this section we apply Theorem 1.2 to obtain uniqueness results for some functionals arising in the literature and we conclude by a subsection on a comparison principle.

2.1. Radial functions

In this subsection, we assume that $\varphi(z) = g(||z||)$ with $g : \mathbb{R} \to \mathbb{R}$ a convex function such that $g(0) < g(t)$ for all $t \neq 0$ and $||\cdot||$ a strictly convex norm.

Proposition 2.1. *For such a φ, the assumption ($\mathcal{A}3$) is satisfied.*

Proof. Since g is strictly increasing and $||\cdot||$ is strictly convex, the lower level sets of φ are strictly convex. We assume by contradiction that there exists $z \in \mathbb{R}^N$ and $\epsilon > 0$ such that $B_\epsilon(z) \subset F$ where $B_\epsilon(z) := \{y \in \mathbb{R}^N \text{such that } ||y - z|| < \epsilon\}$ and F is an affine part of φ. Without loss of generality, we can assume that $z \neq 0$. Hence, there exists $z' \in B_\epsilon(z)\backslash\{z\}$ such that $||z'|| = ||z||$. Since $z, z' \in F$, φ is affine on $[z, z']$. Since $\varphi(z) = \varphi(z')$, the segment $[z, z']$ is contained in the boundary of a lower level set of φ. That is a contradiction. Thus, the interior of F is empty. Hence, the dimension of F is at most $N - 1$. □

In the rest of the subsection, we detail some examples where the norm considered is the Euclidean norm $|\cdot|$. If the set $\mathcal{E}(\Omega,\psi)$ is a subset of uniformly continuous functions that are in $W^{1,1}(\Omega)$ then every minimizer is uniformly continuous. In this case, Theorem 1.2 becomes: $\mathcal{P}_\lambda$ admits at most one minimizer. For instance that is the case in [10, Theorem 3] where the set considered is $\mathcal{C}^{0,1}_\psi(\overline{\Omega})$. A minimization among Lipschitz continuous functions also appears in [9, Theorem 1.1] but only for the dimension two. Hence, Theorem 1.2 allows to generalize this result to higher dimensions.

This theorem can also be used to prove the uniqueness part for the least gradient problem considered in [11]:

$$\text{Minimize} \int_\Omega |\nabla u| dx \text{ on } \{u \in BV(\Omega) \cap \mathcal{C}^0(\overline{\Omega}),\, u = \psi \text{ on } \partial\Omega\}.$$

In fact, we can not directly apply Theorem 1.2 because $\mathcal{E}(\Omega, \psi) \not\subseteq W^{1,1}(\Omega)$ but [11, Theorem 5.9] provides that if Ω and ψ are smooth, then the minimizers are globally Lipschitz continuous on Ω.

In the articles [9], [10] and [11] the question of existence is much harder. Now, we look at problems where the direct method guarantees the existence of a minimizer.

If φ satisfies a *p-growth* condition: there exist $C_1, \alpha > 0$ and $p > 1$ such that

$$C_1|z|^p - \alpha \leq \varphi(z) \tag{2}$$

for every $z \in \mathbb{R}^N$ then by [4, Theorem 3.30], $\mathcal{P}_\lambda$ admits at least one minimizer on $W^{1,p}_\psi(\Omega)$. In [1], we have:

$$\varphi(z) = \begin{cases} |z| & \text{if } |z| \leq 1, \\ \frac{1}{2}(|z|^2 + 1) & \text{if } |z| > 1, \end{cases} \tag{3}$$

and in [8], up to some rescaling we have :

$$\varphi(y) = \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| \leq 1, \\ |z| - \frac{1}{2} & \text{if } 1 < |z| < 2, \\ \frac{1}{4}(|z|^2 + 2) & \text{if } 2 \leq |z|. \end{cases}$$

In both cases, φ is a radial convex function with 0 as strict minimum that satisfies the assumption $(\mathcal{A}3)$ and a *p-growth* condition. Hence, these two problems, or any problem of this type, admit at least a minimizer. It remains to show that every minimizer is continuous.

By [7, Theorem 7.8], if Ω has a Lipschitz boundary, if ψ is Lipschitz continuous and if $\varphi(z) \leq C_2|z|^p + \alpha$ with $C_2 > 0$ and p, α as in (2) then every minimizer of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$ is globally Hölder continuous. In these cases by Corollary 1.3, $\mathcal{P}_\lambda$ has a unique solution on $W^{1,2}_\psi(\Omega)$.

Remark 2.2. An interesting fact here is that we do not assume that the boundary of Ω is connected in contrast with [1] and [8]. In this way, Theorem 1.2 extends the main results of [1] and [8] with a shorter proof.

2.2. When Ω is convex and φ is superlinear

In this subsection we use some results from [2]. We assume that Ω is convex, $\lambda = 0$, ψ is continuous and φ is superlinear:

$$\lim_{|z| \to +\infty} \frac{\varphi(z)}{|z|} = +\infty.$$

Under those assumptions, [2, Theorem 1.4] ensures that $\mathcal{P}_\lambda$ has at least one continuous solution on $\overline{\Omega}$ when $\mathcal{E}(\Omega,\psi) = W^{1,1}_\psi(\Omega)$. Hence if φ satisfies the assumption ($\mathcal{A}3$) then Theorem 1.2 provides that $\mathcal{P}_\lambda$ has a unique continuous minimizer.

Moreover, if we assume that the affine parts of φ have a uniformly bounded diameter and that there exists a uniformly continuous minimizer of $\mathcal{P}_\lambda$ then by [2, Proposition 1.6], see also Lemma 3.7, every minimizer is uniformly continuous on $\overline{\Omega}$. In this case if φ satisfies the assumption ($\mathcal{A}3$) then by Corollary 1.4, $\mathcal{P}_\lambda$ has a unique minimizer and this minimizer is uniformly continuous on $\overline{\Omega}$.

2.3. Functions defined on a bounded convex set

We assume here that K is a bounded convex set of $\mathbb{R}^N$, $\varphi(z) = \widetilde{\varphi}(z) + \chi_K(z)$ with $\widetilde{\varphi} : \mathbb{R}^N \to \mathbb{R}$ a convex function that satisfies the assumption ($\mathcal{A}3$) on K and

$$\chi_K(z) = \begin{cases} 0 & \text{if } z \in K, \\ +\infty & \text{if } z \notin K. \end{cases}$$

In this case, for every minimizer u and for a.e. $x \in \Omega$, $\nabla u(x) \in K$. Hence, every minimizer is globally Lipschitz continuous on $\overline{\Omega}$. Thus, Theorem 1.2 becomes: $\mathcal{P}_\lambda$ admits at most one minimizer on $\mathcal{E}(\Omega,\psi)$.

The main source of inspiration for this paper comes from [5, Section 4]. In this article, as mentioned in the introduction, φ is defined on a closed convex polygon in $\mathbb{R}^2$ and $\varphi \equiv +\infty$ outside. The function is strictly convex inside the polygon. Hence, φ satisfies the assumption ($\mathcal{A}3$). Thus, if the minimizer exists, it is the only one. We can generalize it to higher dimensions with a polytope instead of a polygon.

This type of formulation $\varphi(z) = \widetilde{\varphi}(z) + \chi_K(z)$ can appear when we introduce a constraint on the gradients of the minimizers. For instance in [12], the authors consider the least gradient problem with $\varphi(z) = |z|$ for all $z \in \mathbb{R}^N$ but the minimization is done on $\{u \in \mathcal{C}^{0,1}(\overline{\Omega}),\ |\nabla u| \leq 1 \text{ a.e.}\}$. In this case, we can set $K = \overline{B_1(0)}$ and apply Theorem 1.2 to obtain uniqueness if a minimizer exists.

2.4. Comparison principle

If we assume that the minimizers of $\mathcal{P}_\lambda$ are uniformly continuous then we have the following comparison principle:

Proposition 2.3. *Let ψ_1 and ψ_2 be two boundary conditions on $\partial\Omega$. If assumptions ($\mathcal{A}1$), ($\mathcal{A}2$), ($\mathcal{A}3$) are satisfied and $\psi_1 \leq \psi_2$ on $\partial\Omega$ then $u_1 \leq u_2$ where u_1 is the uniformly continuous solution of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega,\psi_1)$ and u_2 is the uniformly continuous solution of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega,\psi_2)$.*

Proof. By assumptions on ψ_1 and ψ_2, Lemma 3.3 and Lemma 3.4 below imply that $\min\{u_1,u_2\}$ is a solution of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega,\psi_1)$. Hence, by Theorem 1.2 we have $\min\{u_1,u_2\} = u_1$ and thus $u_1 \leq u_2$. □

3. Preliminary results

In this section, we recall some classical and simple results that we use in the proof of Theorem 1.2. The first lemma shows that the problem is invariant under translations as in [7, Section 1.1].

Lemma 3.1. *If u is a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$ then for every $\tau \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$, $u(\cdot + \tau) + \alpha$ is a minimizer on $\mathcal{E}(\Omega_\tau, \psi + \alpha)$ where $\Omega_\tau := \Omega - \tau$.*

Proof. We consider $v \in \mathcal{E}(\Omega_\tau, \psi + \alpha)$. By assumption $(\mathcal{A}1)$, $\widetilde{v}(x) := v(x - \tau) - \alpha$ is in $\mathcal{E}(\Omega, \psi)$ and thus

$$\int_\Omega \varphi(\nabla \widetilde{v}(x)) + \lambda \widetilde{v}(x) dx \geq \int_\Omega \varphi(\nabla u(x)) + \lambda u(x) dx. \tag{4}$$

But
$$\begin{aligned}\int_{\Omega_\tau} \varphi(\nabla v(y)) + \lambda v(y) dy &= \int_\Omega \varphi(\nabla \widetilde{v}(x)) + \lambda(\widetilde{v}(x) + \alpha) dx \\ &= \int_\Omega \varphi(\nabla \widetilde{v}(x)) + \lambda \widetilde{v}(x) dx + \alpha\lambda|\Omega|.\end{aligned}$$

By the inverse change of variables applied to u and (4) we obtain

$$\int_{\Omega_\tau} \varphi(\nabla v(y)) + \lambda v(y) dy \geq \int_{\Omega_\tau} \varphi\big(\nabla(u(y+\tau) + \alpha)\big) + \lambda(u(y+\tau) + \alpha) dy.$$

Since v can be any function in $\mathcal{E}(\Omega_\tau, \psi + \alpha)$ we have the desired result. □

Remark 3.2. This lemma uses the invariance under translations of this type of functionals with λ constant. Hence, it seems difficult to generalize Theorem 1.2 to more general problems. When λ is not constant, we can use different ideas to prove uniqueness for specific functionals. For instance, in an upcoming paper, we prove uniqueness for φ as in (3) with λ having small oscillations by comparing the level sets of two minimizers. □

The following lemma establishes that a minimizer on Ω is still a minimizer on the subsets of Ω.

Lemma 3.3. *Let u be a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$. For every non empty open set $V \subset \Omega$, $u|_V$ is a minimizer on V among all the functions in $\mathcal{E}(V, u)$.*

Proof. We assume by contradiction that this is not the case. Then there exists $v \in \mathcal{E}(V, u)$ such that

$$\int_V \varphi(\nabla v(x)) + \lambda v(x) dx < \int_V \varphi(\nabla u(x)) + \lambda u(x) dx.$$

If we define $w(x) = u(x)$ if $x \notin V$ and $w(x) = v(x)$ if $x \in V$ then by assumption $(\mathcal{A}2)$, $w \in \mathcal{E}(\Omega, \psi)$ and $\mathcal{I}_\lambda(w) < \mathcal{I}_\lambda(u)$. This contradicts the fact that u is a minimizer. □

In the next lemma, we prove that the extension of a minimizer by a compatible minimizer is still a minimizer.

Lemma 3.4. *Let u be a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$ and $V \subset \Omega$ a non empty open set. If v is a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(V, u)$ then*

$$w = \begin{cases} v \ \text{ on } V, \\ u \ \text{ on } \Omega \backslash V, \end{cases}$$

is a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$.

Proof. By assumption $(\mathcal{A}2)$ on $\mathcal{E}$, $w \in \mathcal{E}(\Omega,\psi)$. We compare $\mathcal{I}_\lambda(u)$ and $\mathcal{I}_\lambda(w)$:

$$\mathcal{I}_\lambda(u) = \int_\Omega \varphi(\nabla u) + \lambda u = \int_V \varphi(\nabla u) + \lambda u + \int_{\Omega\setminus V} \varphi(\nabla u) + \lambda u$$

and $$\mathcal{I}_\lambda(w) = \int_\Omega \varphi(\nabla w) + \lambda w = \int_V \varphi(\nabla v) + \lambda v + \int_{\Omega\setminus V} \varphi(\nabla u) + \lambda u.$$

By Lemma 3.3, $\int_V \varphi(\nabla u) + \lambda u = \int_V \varphi(\nabla v) + \lambda v$. Hence, $\mathcal{I}_\lambda(u) = \mathcal{I}_\lambda(w)$. Thus, w is a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega,\psi)$. □

We introduce the notion of point of *approximate continuity* that we use in the proof of Theorem 1.2.

Definition 3.5. Let $f : \Omega \to \mathbb{R}^m$ be a measurable function, we say that x_1 is a point of *approximate continuity* if for every $\eta > 0$

$$\lim_{r\to 0} \frac{|\{x \in B_r(x_1),\ |f(x) - f(x_1)| \geq \eta\}|}{|B_r(x_1)|} = 0. \tag{5}$$

We recall a classical result that can be found in [6, Section 1.7, Theorem 3].

Proposition 3.6. *Let $f : \Omega \to \mathbb{R}^m$ be a measurable function. Hence, a.e. $x \in \Omega$ is a point of* approximate continuity *of f.*

In the last result of this section, we show that given three minimizers, φ is affine on the simplex generated by their gradients at almost every point:

Lemma 3.7. *Let u, v and w be three minimizers of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega,\psi)$. Then φ is affine on the simplex with vertices $[\nabla u(x), \nabla v(x), \nabla w(x)]$ for a.e. $x \in \Omega$.*

Proof. Since u is a solution of P_λ, we have $\mathcal{I}_\lambda(u) \leq \mathcal{I}_\lambda\left(\frac{u+v+w}{3}\right)$.

By the convexity of φ,

$$\begin{aligned}
&\mathcal{I}_\lambda\left(\frac{u+v+w}{3}\right) = \int_\Omega \varphi\left(\frac{\nabla u+\nabla v+\nabla w}{3}\right) + \lambda\frac{u+v+w}{3}dx \\
&\leq \frac{1}{3}\int_\Omega (\varphi(\nabla u) + \lambda u)dx + \frac{1}{3}\int_\Omega (\varphi(\nabla v) + \lambda v)dx + \frac{1}{3}\int_\Omega (\varphi(\nabla w) + \lambda w)dx \\
&= \frac{1}{3}\mathcal{I}_\lambda(u) + \frac{1}{3}\mathcal{I}_\lambda(v) + \frac{1}{3}\mathcal{I}_\lambda(w).
\end{aligned}$$

Since v and w are other solutions,

$$I_\lambda(u) = \frac{1}{3}\mathcal{I}_\lambda(u) + \frac{1}{3}\mathcal{I}_\lambda(v) + \frac{1}{3}\mathcal{I}_\lambda(w).$$

This implies that

$$\int_\Omega \varphi\left(\frac{\nabla u+\nabla v+\nabla w}{3}\right)dx = \int_\Omega \frac{1}{3}(\varphi(\nabla u) + \varphi(\nabla v) + \varphi(\nabla w))dx.$$

Hence, for a.e. $x \in \Omega$,

$$\varphi\left(\frac{\nabla u(x) + \nabla v(x) + \nabla w(x)}{3}\right) = \frac{1}{3}\Big(\varphi(\nabla u(x)) + \varphi(\nabla v(x)) + \varphi(\nabla w(x))\Big),$$

and therefore φ is affine on the simplex with vertices $[\nabla u(x), \nabla v(x), \nabla w(x)]$. $\square$

Remark 3.8. The fact that φ is affine on the simplex with vertices $[\nabla u(x), \nabla v(x), \nabla w(x)]$ is equivalent to $\nabla u(x), \nabla v(x), \nabla w(x) \in F$ where F is an affine part of φ.

Remark 3.9. The number of minimizers considered in Lemma 3.7 can be changed. For instance, we can prove it with $n \in \mathbb{N}$ minimizers. In this case we obtain that φ is affine on the convex hull of the gradients of the solutions and they are still in the same affine part F of φ.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into three steps. The first one is very similar to the proof of [5, Proposition 4.5].

Step 1: Let u_1, u_2 be two uniformly continuous minimizers of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$. We call ω a common modulus of continuity of u_1 and u_2, namely: $\lim_{w\to 0^+} \omega(x) = 0$, ω is increasing and $\omega(a + b) \leq \omega(a) + \omega(b)$ for every $a, b \in \mathbb{R}_+$. We assume by contradiction that there exist $x_0 \in \Omega$ and $\epsilon > 0$ such that $u_1(x_0) - u_2(x_0) = \epsilon$.

We consider $\rho > 0$ such that $\omega(\rho) < \frac{\epsilon}{6}$. For every $y \in \Omega$ we have

$$u_1(y) - u_2(y) \leq 2\omega(\mathrm{dist}(y, \partial\Omega)).$$

Hence,
$$\omega(2\rho) \leq \frac{\epsilon}{3} < \frac{\epsilon}{2} = \frac{u_1(x_0) - u_2(x_0)}{2} \leq \omega(\mathrm{dist}(x_0, \partial\Omega)).$$

Thus, $B_{2\rho}(x_0) \Subset \Omega$. For every $\tau \in \mathbb{R}^N$ such that $|\tau| < \rho$ and for every $x \in B_\rho(x_0)$ we have $u_1(x) \geq u_1(x_0) - \omega(\rho)$ and $u_2(x+\tau) \leq u_2(x_0) + 2\omega(\rho)$. Hence, we have that

$$B_\rho(x_0) \subset \{x \in \Omega \cap \Omega_\tau,\, u_2(x+\tau) + \frac{\epsilon}{2} < u_1(x)\} \tag{6}$$

and for the same reasons:

$$B_\rho(x_0) \subset \{x \in \Omega \cap \Omega_\tau,\, u_1(x+\tau) - \frac{\epsilon}{2} > u_2(x)\} \tag{7}$$

with $\Omega_\tau = \Omega - \tau$. For every $x \in \Omega \cap \Omega_\tau$, we have

$$\begin{aligned}|u_1(x) - u_2(x+\tau)| &\leq \omega(\rho) + \min\{|u_1(x) - u_2(x)|, |u_1(x+\tau) - u_2(x+\tau)|\}\\ &\leq \omega(\rho) + 2\omega\big(\mathrm{dist}(x, \partial(\Omega \cap \Omega_\tau))\big).\end{aligned}$$

Since $\omega(\rho) < \frac{\epsilon}{6}$:
$$\{x \in \Omega \cap \Omega_\tau,\, u_2(x+\tau) + \frac{\epsilon}{2} < u_1(x)\} \Subset \Omega \cap \Omega_\tau \tag{8}$$

and for the same reasons:

$$\{x \in \Omega \cap \Omega_\tau,\, u_1(x+\tau) - \frac{\epsilon}{2} > u_2(x)\} \Subset \Omega \cap \Omega_\tau. \tag{9}$$

We extend the function $v_\tau(x) := \min\{u_1(x),\, u_2(x+\tau) + \frac{\epsilon}{2}\}$ defined on $\Omega \cap \Omega_\tau$ by u_1 to Ω. By assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and the inclusion (8), this extension belongs to $\mathcal{E}(\Omega, \psi)$.

Thanks to the inclusion (9), if we extend $w_\tau(x) := \max\{u_1(x+\tau) - \frac{\epsilon}{2},\, u_2(x)\}$ by u_2, we also have that $w_\tau \in \mathcal{E}(\Omega, \psi)$. By Lemma 3.1 and Lemma 3.3, on the open set $\{x \in \Omega \cap \Omega_\tau,\, u_2(x+\tau) + \frac{\epsilon}{2} < u_1(x)\}$, u_1 and $u_2(\cdot + \tau) + \frac{\epsilon}{2}$ are solutions of the same problem. By Lemma 3.4, v_τ is a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$. With the same arguments we can prove that w_τ is also a minimizer of $\mathcal{P}_\lambda$ on $\mathcal{E}(\Omega, \psi)$.

Step 2: In this crucial step of the proof we show that there exists an $(N-1)$-dimensional set containing the essential ranges of the gradients of any minimizer.

We consider $x_1 \in B_{\frac{\varrho}{2}}(x_0)$ and $x_1' \in B_{\frac{\varrho}{2}}(x_0)$ two points of *approximate continuity* of $(\nabla u_1,\, \varphi(\nabla u_1))$ and $x_2 \in B_{\frac{\varrho}{2}}(x_0)$ a point of *approximate continuity* of $(\nabla u_2,\, \varphi(\nabla u_2))$. By Proposition 3.6 and the fact that ∇u_1, ∇u_2 and φ are measurable, that is the case for a.e. $(x_1, x_1', x_2) \in B_{\frac{\varrho}{2}}(x_0)^3$. We fix $\tau := x_2 - x_1$ and $\tau' := x_1' - x_1$. Then by definition of v_τ and $w_{\tau'}$, x_1 is a point of *approximate continuity* of $(\nabla v_\tau,\, \varphi(\nabla v_\tau))$ and $(\nabla w_{\tau'},\, \varphi(\nabla w_{\tau'}))$. We define

$$f : x \to \big(\nabla u_1(x), \nabla w_{\tau'}(x), \nabla v_\tau(x), \varphi(\nabla u_1(x)), \varphi(\nabla w_{\tau'}(x)), \varphi(\nabla v_\tau(x))\big).$$

Since x_1 is a point of *approximate continuity* of f, we have for every $n \in \mathbb{N}$:

$$\lim_{r \to 0} \frac{|\{x \in B_r(x_1),\, |f(x) - f(x_1)| > \frac{1}{n}\}|}{|B_r(x_1)|} = 0.$$

Thus, for every $n \in \mathbb{N}$, there exist $r_n > 0$ and $x \in B_{r_n}(x_1)$ such that $|f(x) - f(x_1)| \le \frac{1}{n}$. By Lemma 3.7 we can further assume that

$$\varphi\Big(\frac{\nabla u_1(x) + \nabla w_{\tau'}(x) + \nabla v_\tau(x)}{3}\Big) = \frac{\varphi(\nabla u_1(x)) + \varphi(\nabla w_{\tau'}(x)) + \varphi(\nabla v_\tau(x))}{3} < +\infty. \quad (10)$$

We introduce $K := \{z \in \mathbb{R}^N,\, \varphi(z) < +\infty\}$ the convex domain of φ and we distinguish two cases:

(a) If the triplet $(x_1,\, x_1',\, x_2)$ is such that $\frac{\nabla u_1(x_1) + \nabla w_{\tau'}(x_1) + \nabla v_\tau(x_1)}{3} \in \mathrm{Int}(K)$, then by continuity of φ on $\mathrm{Int}(K)$ we have

$$\varphi\Big(\frac{\nabla u_1(x_1) + \nabla w_{\tau'}(x_1) + \nabla v_\tau(x_1)}{3}\Big) = \frac{\varphi(\nabla u_1(x_1)) + \varphi(\nabla w_{\tau'}(x_1)) + \varphi(\nabla v_\tau(x_1))}{3}.$$

By the definition of $w_{\tau'}$, v_τ and (6)-(7), this is equivalent to $\nabla u_1(x_1)$, $\nabla u_1(x_1')$ and $\nabla u_2(x_2)$ being in the same affine part of φ.

Now, we prove that φ is affine on the convex hull of $F := \{\nabla u_1(x)$ with $x \in X_1$ and $\nabla u_2(x)$ with $x \in X_2\}$ with X_i the set of points of *approximate continuity* of $(\nabla u_i, \varphi(\nabla u_i))$. If we consider $y \in B_{\frac{\varrho}{2}}(x_0)$, a point of *approximate continuity* of $(\nabla u_i, \varphi(\nabla u_i))$, and $y' \in B_{\frac{\varrho}{2}}(x_0)$, a point of *approximate continuity* of $(\nabla u_j, \varphi(\nabla u_j))$ with $i, j \in \{1, 2\}$ we have that

$$\frac{\nabla u_1(x_1) + \nabla u_1(x_1') + \nabla u_2(x_2) + \nabla u_i(y) + \nabla u_j(y')}{5} \in \mathrm{Int}(K). \quad (11)$$

By using the functions v_{y-x_1} if $i = 2$ and w_{y-x_1} if $i = 1$, and similarly for j, we obtain the analogue of (10) for these five minimizers, see Remark 3.9. Hence, we can argue as above and rely on (11) to get that the five points $\nabla u_1(x_1)$, $\nabla u_1(x_1')$, $\nabla u_2(x_2)$, $\nabla u_i(y)$ and $\nabla u_j(y')$ belong to the same affine part of φ. Thus, φ is affine on the segment $[\nabla u_i(y), \nabla u_j(y')]$. Hence, φ is affine on the convex hull of F. Thus by assumption ($\mathcal{A}3$), there exists $\mathcal{F}$ an (N-1)-dimensional affine space such that for a.e. $x \in B_{\frac{\varrho}{2}}(x_0)$, $\nabla u_1(x)$ and $\nabla u_2(x)$ belong to $\mathcal{F}$.

(b) If there is no such triplet then for every couple (x_1, x_1') of points of *approximate continuity*, we have that $\nabla u(x_1)$, $\nabla u(x_1')$ belong to the same face of ∂K, namely the intersection of an affine hyperplane with ∂K. Otherwise the convex combination would be in $\mathrm{Int}(K)$ contradicting the fact that we are not in case **(a)**. Hence, there exists an affine hyperplane $\mathcal{F}$ such that for a.e. $x \in B_{\frac{\varrho}{2}}(x_0)$, $\nabla u_1(x)$ is in $\mathcal{F}$. The fact that we are not in case **(a)** gives that for a.e. $x \in B_{\frac{\varrho}{2}}(x_0)$, $\nabla u_2(x)$ is in $\mathcal{F}$ too.

Step 3: Since the dimension of $\mathcal{F}$ is at most N-1 there exists a normal vector η to $\mathcal{F}$. The scalar product $\langle \nabla u_i(x), \eta \rangle = C$ is constant in $B_{\frac{\varrho}{2}}(x_0)$ and is independent of $i = 1, 2$. Thus, $u_1 - u_2 = \epsilon > 0$ on $(x_0 + \mathbb{R}\eta) \cap B_{\frac{\varrho}{2}}(x_0)$. Hence, the compact set $E_\epsilon := (u_1 - u_2)^{-1}(\epsilon) \Subset \Omega$ has no extreme point in Ω. We consider $\widetilde{x}_0 \in E_\epsilon$ such that $|\widetilde{x}_0| = \max_{x \in E_\epsilon} |x|$. Since E_ϵ has no extreme point, there exist y_1 and y_2 in E_ϵ such that $\widetilde{x}_0 \in (y_1, y_2)$. By the triangle inequality, either $|y_1| > |\widetilde{x}_0|$ or $|y_2| > |\widetilde{x}_0|$. That is a contradiction. Hence, we have $u_1 = u_2$ on Ω.

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