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Approximation of the Value of the Resolvent of a Maximal Monotone Operator in a Banach Space

Takanori Ibaraki
Dept. of Mathematics Education, Yokohama National University, Yokohama, Japan
ibaraki@ynu.ac.jp

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We consider the approximation of the value of the resolvent of a maximal monotone operator in a Banach space. We prove strong convergence theorems for resolvents of type (Q) and (R) in a Banach space.

Keywords: Maximal monotone operator, resolvent, Type (Q), Type (R), Banach space.

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1. Introduction

Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator with the domain $D(A) = \{x \in H : Ax \neq \emptyset\}$. The *zero point problem* is to find $u \in H$ such that

$$0 \in Au. \tag{1}$$

This problem is connected with many problems in nonlinear analysis and optimization. A well-known method for solving (1) in a Hilbert space H is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n \tag{2}$$

for each $n \in \mathbb{N}$, where $\{r_n\} \subset]0, \infty[$ and J_r for each $r > 0$ is defined by as follows: For each $r > 0$ and $x \in H$ there exists a unique point $x_r \in D(A)$ such that

$$x \in x_r + rAx_r. \tag{3}$$

Such a point x_r is denoted by $J_r x$ and J_r is called the *resolvent* of A. This algorithm was first introduced by Martinet [19]. Later, many researchers studied this problem; see [14, 15, 16, 17, 21, 24, 25] for example. In the proximal point algorithm, we use resolvents for each step to generate an iterative sequence. However, it is not easy to calculate the exact value of the resolvent; see [14, 15]. This is known as the sub-problem of this algorithm.

On the other hand, the problem solving to the equation (3) is an important subject in the theory of monotone operators; see [3, 4, 6, 12, 26] for example. In particular, Bruck [4] proves a strong convergence theorem for a maluti-valued monotone operator in a Hilbert space. We know two operators in a Banach space which generalize the concept of the monotone operator in a Hilbert space, that is, the monotone operator and the accretive operator. Let E be a Banach space.

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An operator $T \subset E \times E$ is said to be *accretive* if for each $(x_1, y_1), (x_2, y_2) \in T$ there exists some $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j\rangle \geq 0$$

where J is the normalized duality mapping on E. In 1990, Chidume [6] proves a strong convergence theorem for a single-valued accretive operator in a Banach space.

Theorem 1.1. ([6]) *Let E be a real Banach space with uniformly convex dual space E^* and let C be a nonempty bounded closed convex subset of E. Let $T : C \to C$ be a continuous single-valued accretive operator and let $z \in C$. Let $\{\lambda_n\}$ be a sequence of $]0,1[$ such that*

$$\sum_{n=0}^{\infty} \lambda_n = \infty \quad \textit{and} \quad \sum_{n=0}^{\infty} \lambda_n \beta(\lambda_n) < \infty$$

where the function β is that appears in Lemma 2.1 in Section 2. Let $\{x_n\}$ be a sequence of C defined as follows: $x_0 \in C$

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n(z - Tx_n)$$

for each $n \in \mathbb{N} \cup \{0\}$. Then $\{x_n\}$ converges strongly to a solution z_1 of (3) for $r = 1$, that is $z = z_1 + Tz_1$.

For a study of solving the equation (3) of accretive operators in Banach space, see [5, 7, 8, 9, 10, 11, 18, 27] for example. However, there are a few studies of solving the equation (3) of monotone operators in a Banach space.

In this paper, we are interested in solving the equation (3) of monotone operators in a Banach space. We deal with two types of resolvents of maximal monotone operators in a Banach space, that is, type (Q) and type (R). We prove strong convergence theorems for finding the value of the resolvents of type (Q) and (R) in a Banach space.

2. Preliminaries

Let E be a real Banach space with its dual E^*. The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for each point $x \in E$. Let E be a smooth Banach space and consider the function $V : E \times E \to \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$. We know the following properties (see [1, 13, 16]);

(1) $(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2$ for each $x, y \in E$;

(2) $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$;

(3) $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$;

(4) if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. It is known that for each $x \in E$ there exists a unique point $z \in C$ such that

$$V(z,x) = \min_{y\in C} V(y,x).$$

Such a point z is denoted by $\Pi_C x$ and Π_C is called the generalized projection of E onto C (see [1]). Let D be a subset of E with JD is closed and convex and let Π_{JD} be the generalized projection of E^* onto JD. The map R_D of E onto D defined by

$$R_D := J^{-1}\Pi_{JD}J$$

is call the *sunny generalized nonexpansive retraction*; see [13, 17] for more details. It is well known that the generalized projection and the sunny generalized nonexpansive retraction are generalizations of the metric projection in the sense of a Hilbert space, and hence these mappings coincide in a Hilbert space; see [1, 13, 16].

Let E be a smooth, strictly convex and reflexive Banach space. An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be *monotone* if $\langle x-y, x^*-y^*\rangle \geq 0$ for any $(x,x^*),(y,y^*) \in A$. A monotone operator A is said to be *maximal* if $A = A_0$ whenever $A_0 \subset E \times E^*$ is a monotone operator such that $A \subset A_0$.

Let $r > 0$ and let $A \subset E \times E^*$ and $B \subset E^* \times E$ be maximal monotone operators. Then it is known that $R(J + rA) = E^*$ and $R(I + rBJ) = E$; see [3, 13, 23]. We also know that the closures of $D(A)$ and $D(B)$ are convex; see [22]. We can define a single-valued mapping

$$(J + rA)^{-1}J : E \to D(A)$$

by $Q_r = (J + rA)^{-1}J$ for each $r > 0$. It is called the *resolvent of type (Q)* of A for $r > 0$; see [3]. Also, we define a single-valued mapping

$$(I + rBJ)^{-1} : E \to D(BJ)$$

by $R_r = (I + rBJ)^{-1}$ for each $r > 0$. It is called the *resolvent of type (R)* of B for $r > 0$; see [13].

The following results are well-known; see [2, 16, 20, 25].

Lemma 2.1. ([20]) *Let E be a uniformly smooth Banach space. Then there exists a nondecreasing continuous function $\beta\colon [0,\infty[\to [0,\infty[$ such that we have $\beta(0) = 0$, $\beta(ct) \leq c\beta(t)$ for all $c \geq 1$ and*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + \max\{\|x\|,1\}\|y\|\beta(\|y\|)$$

for all $x, y \in E$.

Lemma 2.2. ([16]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} V(x_n,y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Theorem 2.3. ([2, 25]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{a_n\}$ a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} a_n = \infty$, $\{b_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} b_n \leq 0$, and $\{c_n\}$ a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} c_n < \infty$. Assume that $s_{n+1} \leq (1-a_n)s_n + a_n b_n + c_n$. Then $\{s_n\}$ converges to 0.*

3. Main results

In this section, we first prove a strong convergence theorem for finding the value of the resolvent of type (Q) in a Banach space.

Theorem 3.1. *Let E be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone operator such that $D(A)$ is closed and $R(A)$ is bounded. Let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let r be a positive real number. Let $\{x_n\}$ be a sequence of $D(A)$ defined as follows: $z \in E$, $x_1 \in D(A)$ and*

$$y_n^* \in Ax_n, \quad x_{n+1} = \Pi_{D(A)} J^{-1}\big((1-\alpha_n)Jx_n + \alpha_n(Jz - ry_n^*)\big)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $Q_r z$.

Proof. Put $z_r := Q_r z$ and $u_n^* := Jz - ry_n^*$ for each $n \in \mathbb{N}$. From the monotonicity of A and $(Jz - Jz_r)/r \in Az_r$, it follows that

$$\begin{aligned}
\left\langle x_n - z_r, y_n^* - \frac{Jz - Jz_r}{r} \right\rangle &\geq 0 \\
\langle x_n - z_r, ry_n^* - (Jz - Jz_r)\rangle &\geq 0 \\
\langle x_n - z_r, Jz_r\rangle \geq \langle x_n - z_r, Jz - ry_n^*\rangle &= \langle x_n - z_r, u_n^*\rangle. \qquad (4)
\end{aligned}$$

Since $R(A)$ is bounded, $\{y_n^*\}$ is bounded and hence $\{u_n^*\}$ is bounded. Therefore, $\{V(z_r, J^{-1}u_n^*)\}$ is also bounded. Put

$$N := \max\Big\{\sup_{n\in\mathbb{N}} V(z_r, J^{-1}u_n^*), V(z_r, x_1)\Big\}(<\infty).$$

It easy to see that $V(z_r, x_n) \leq N$ for each $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. Put

$$M := \max\Big\{\sup_{n\in\mathbb{N}} \|u_n^*\|, \sup_{n\in\mathbb{N}} \|x_n\|, 1\Big\}(<\infty).$$

By using Lemma 2.1, we see that, for each $n \in \mathbb{N}$,

$$\begin{aligned}
V(z_r, x_{n+1}) &\leq V\left(z_r, J^{-1}\big((1-\alpha_n)Jx_n + \alpha u_n^*\big)\right) \\
&= \|z_r\|^2 - 2\langle z_r, (1-\alpha_n)Jx_n + \alpha u_n^*\rangle + \|(1-\alpha_n)Jx_n + \alpha_n u_n^*\|^2 \\
&\leq \|z_r\|^2 - 2(1-\alpha_n)\langle z_r, Jx_n\rangle - 2\alpha_n\langle z_r, u_n^*\rangle \\
&\quad + \|(1-\alpha_n)Jx_n\|^2 + 2\langle J^{-1}\left((1-\alpha_n)Jx_n\right), \alpha_n u_n^*\rangle \\
&\quad + \max\{(1-\alpha_n)\|x_n\|, 1\}\, \alpha_n\|u_n^*\|\beta(\alpha_n\|u_n^*\|),
\end{aligned}$$

hence we obtain

$$\begin{aligned}V(z_r, x_{n+1}) &\leq (1-\alpha_n)\left\{\|z_r\|^2 - 2\langle z_r, Jx_n\rangle\right\} + \alpha_n\left\{\|z_r\|^2 - 2\langle z_r, u_n^*\rangle\right\} \\ &\quad + (1-\alpha_n)^2\|x_n\|^2 + 2\alpha_n(1-\alpha_n)\langle x_n, u_n^*\rangle + \alpha_n M^2\beta(\alpha_n M) \\ &\leq (1-\alpha_n)V(z_r, x_n) + (-\alpha_n + \alpha_n^2)\|x_n\|^2 \\ &\quad + \alpha_n\left\{\|z_r\|^2 - 2\langle z_r, u_n^*\rangle + 2(1-\alpha_n)\langle x_n, u_n^*\rangle + M^3\beta(\alpha_n)\right\} \\ &\leq (1-\alpha_n)V(z_r, x_n) + \alpha_n\big\{\|z_r\|^2 + 2\langle x_n - z_r, u_n^*\rangle \\ &\quad - 2\alpha_n\langle x_n, u_n^*\rangle + (-1+\alpha_n)\|x_n\|^2 + M^3\beta(\alpha_n)\big\}\end{aligned}$$

where β is in Lemma 2.1. It follows (4) that, for each $n \in \mathbb{N}$,

$$\begin{aligned}&\|z_r\|^2 + 2\langle x_n - z_r, u_n^*\rangle - 2\alpha_n\langle x_n, u_n^*\rangle + (-1+\alpha_n)\|x_n\|^2 \\ &\quad \leq \|z_r\|^2 + 2\langle x_n - z_r, Jz_r\rangle - 2\alpha_n\langle x_n, u_n^*\rangle + (-1+\alpha_n)\|x_n\|^2 \\ &\quad \leq -\|z_r\|^2 + 2\langle x_n, Jz_r\rangle - \|x_n\|^2 + \alpha_n\left\{\|x_n\|^2 + 2\|x_n\|\|u_n^*\|\right\} \\ &\quad \leq -V(x_n, z_r) + \alpha_n(M^2 + 2M^2) \\ &\quad \leq 0 + 3\alpha_n M^2 = 3\alpha_n M^2.\end{aligned}$$

Therefore, we have $V(z_r, x_{n+1}) \leq (1-\alpha_n)V(z_r, x_n) + \alpha_n\left\{3\alpha_n M^2 + M^3\beta(\alpha_n)\right\}$. From the properties of β and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\left\{3\alpha_n M^2 + M^3\beta(\alpha_n)\right\} = 0.$$

Using Theorem 2.3 as $s_n = V(z_r, x_n)$, $a_n := \alpha_n$, $b_n := 3\alpha_n M^2 + M^3\beta(\alpha_n)$, $c_n := 0$, we obtain $s_n \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} V(z_r, x_n) = 0$$

Thus, by Lemma 2.2, $\{\|x_n - z_r\|\}$ converges to 0. This implies that $\{x_n\}$ converges strongly to z_r. $\square$

Next, we prove a strong convergence theorem for finding the value of the resolvent of type (R) in a Banach space by using Theorem 3.1.

Theorem 3.2. *Let E be a uniformly smooth and strictly convex Banach space and let $B \subset E^* \times E$ be maximal monotone operator such that $D(B)$ is closed and $R(B)$ is bounded. Let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$ and let r is a positive real number. Let $\{x_n\}$ be a sequence of $D(BJ)$ defined as follows: $z \in E$, $x_1 \in D(BJ)$ and*

$$y_n \in BJx_n, \quad x_{n+1} = R_{D(BJ)}\big((1-\alpha_n)x_n + \alpha(z - ry_n)\big)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to $R_r z$. Moreover, if the dual norm of E^* is Fréchet differentiable, then $\{x_n\}$ converges strongly to $R_r z$.

Proof. Put $x_n^* := Jx_n$ for each $n \in \mathbb{N}$. Then, we have $y_n \in Bx_n^*$ and

$$\begin{aligned} x_{n+1}^* = Jx_{n+1} &= JR_{D(BJ)}\big((1-\alpha_n)x_n + \alpha(z - ry_n)\big) \\ &= JR_{J^{-1}D(B)}J^{-1}J\big((1-\alpha_n)J^{-1}Jx_n + \alpha(z - ry_n)\big) \\ &= \Pi_{D(B)}J\big((1-\alpha_n)J^{-1}x_n^* + \alpha(J^{-1}Jz - ry_n)\big). \end{aligned}$$

Since E is uniformly smooth and strictly convex, then E^* is smooth and uniformly convex. Using Theorem 3.1, we have

$$Jx_n = x_n^* \to (J^{-1} + rB)^{-1}J^{-1}Jz = (J^{-1} + rB)^{-1}z$$

as $n \to \infty$. It is easy to see that $JR_rz = (J^{-1} + rB)^{-1}z$. Since E^* is smooth, the duality mapping J^{-1} on E^* is norm to weak continuous. Therefore, we obtain

$$x_n = J^{-1}Jx_n \rightharpoonup J^{-1}JR_rz = R_rz.$$

as $n \to \infty$. For latter part of this theorem, suppose that the dual norm of E^* is Fréchet differentiable. Then the duality mapping J^{-1} on E^* is norm to norm continuous. Therefore, we obtain, as $n \to \infty$,

$$x_n = J^{-1}Jx_n \to J^{-1}JR_rz = R_rz. \qquad \square$$

4. Deduced results

Using Theorems 3.1 and 3.2, we obtain the following results.

Theorem 4.1. *Let E be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone operator such that $D(A) = E$ and $R(A)$ is bounded. Let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let r is a positive real number. Let $\{x_n\}$ be a sequence of $D(A)$ defined as follows: $z \in E$, $x_1 \in D(A)$ and*

$$y_n^* \in Ax_n, \quad x_{n+1} = J^{-1}\big((1-\alpha_n)Jx_n + \alpha_n(Jz - ry_n^*)\big)$$

for each $\mathbb{N}$. Then $\{x_n\}$ converges strongly to Q_rz.

Proof. If $D(A) = E$, then $\Pi_{D(A)}$ is the identity mapping. As a direct consequence of Theorem 3.1, we obtain the desired result. $\square$

Theorem 4.2. *Let E be a uniformly smooth and strictly convex Banach space and let $B \subset E^* \times E$ be maximal monotone operator such that $D(B) = E^*$ and $R(B)$ is bounded. Let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let r is a positive real number. Let $\{x_n\}$ be a sequence of $D(BJ)$ defined as follows: $z \in E$, $x_1 \in D(BJ)$ and*

$$y_n \in BJx_n, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha(z - ry_n)$$

for each $\mathbb{N}$. Then $\{x_n\}$ converges weakly to R_rz. Moreover, if the dual norm of E^* is Fréchet differentiable, then $\{x_n\}$ converges strongly to R_rz.

Proof. Proceeding as in the proof of Theorem 4.1, we obtain the desired result by Theorem 3.2. □

In the case where E is a Hilbert space, the duality mapping J of E is the identity mapping. Therefore, we obtain the following results by Theorems 3.1 and 3.2.

Theorem 4.3. *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator such that $D(A)$ is closed and $R(A)$ is bounded. Let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let r is a positive real number. Let $\{x_n\}$ be a sequence of $D(A)$ defined as follows: $z \in H$, $x_1 \in D(A)$ and*

$$y_n \in Ax_n, \quad x_{n+1} = P_{D(A)}\big((1-\alpha_n)x_n + \alpha_n(z - ry_n)\big)$$

for each $\mathbb{N}$. Then $\{x_n\}$ converges strongly to $J_r z$, where $P_{D(A)}$ is the metric projection of H onto $D(A)$.

Similarly, we obtain the following result by Theorems 4.1 and 4.2.

Theorem 4.4. *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator such that $D(A) = H$ and $R(A)$ is bounded. Let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let r is a positive real number. Let $\{x_n\}$ be a sequence of $D(A)$ defined as follows: $z \in H$, $x_1 \in D(A)$ and*

$$y_n \in Ax_n, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n(z - ry_n)$$

for each $\mathbb{N}$. Then $\{x_n\}$ converges strongly to $J_r z$.

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