

When is a Minkowski Norm Strictly Sub-Convex?

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The aim of this paper is to give two complete and simple characterizations of Minkowski norms N on an *arbitrary* topological real vector space such that the sublevel sets of N are *strictly convex*. We first show that this property is equivalent to the continuity of N together with the fact that any open chord between two points of the boundary of the sublevel set $N^{-1}([0, 1])$ lies inside that set (geometric characterization). On the other hand, we prove that this is also the same as saying that N is continuous and that for an arbitrary real number $\alpha > 1$ the function N^α is strictly convex (analytic characterization).

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1. Introduction

In this paper, we shall be concerned with functions defined on a topological real vector space which are strictly sub-convex, that is, whose sublevel sets are strictly convex. The property of being strictly sub-convex is of course more general than that of being strictly convex. For example, the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by

$f(t) := 0$ for $t \leq 0$ and $f(t) := 1$ for $t > 0$ is strictly sub-convex even though it is not strictly convex. This notion will be made more precise in Section 6.

We shall focus on a particular class of strictly sub-convex functions on an arbitrary topological real vector space which are positively homogeneous and that are called Minkowski norms (see Section 2).

Minkowski norms are the most natural generalization of the usual norms defined on real vector spaces. Indeed, they differ from norms in that they are not necessarily symmetric, and hence they give rise to pseudo-metrics instead of the classical distance functions. Moreover, Minkowski norms are the main mathematical objects behind Finsler metrics which are defined on differentiable manifolds and which are the smallest extensions of the well-known Riemannian metrics (see for example [3] and [7]).

The present work gives two complete and simple characterizations of Minkowski norms which are strictly sub-convex as we shall see with both Theorem 7.1 and Theorem 7.3 in Section 7.

Indeed, in Theorem 7.1, we prove that being strictly sub-convex for a Minkowski norm N is the same as saying that N is continuous and that it satisfies the following geometric property: any open chord between two points of the boundary of the sublevel set $N^{-1}([0, 1))$ lies inside that set.

On the other hand, from the point of view of analysis, we show in Theorem 7.3 that the strict sub-convexity of a Minkowski norm N is equivalent to the continuity of N together with the strict convexity of the function N^α for an arbitrary real number α greater than 1.

As Minkowski norms belong to the class of non-negative functions which are positively homogeneous and sub-convex, Theorem 7.1 and Theorem 7.3 will be a consequence of a more general result given by Theorem 6.33 that we will state in Section 6 and which gives a relationship between strict sub-convexity of non-negative positively homogeneous functions and strict convexity of any of their powers whose exponent is greater than one.

In order to build this bridge between geometric and analytic aspects of convexity, we shall have to give some useful properties about gauge functions in Section 3 – where their continuity is fully characterized –, convexity in Section 4 and sub-additive functions in Section 5.

2. Preliminaries

Let us first give the definition of a Minkowski norm on a real vector space.

Definition 2.1. Given a real vector space V , a function $N: V \rightarrow \mathbf{R}$ is said to be a *Minkowski norm* on V whenever it satisfies the following properties:

- (1) N is non-negative.
- (2) $N(\lambda x) = \lambda N(x)$ for any $x \in V$ and $\lambda \in (0, +\infty)$. (positive homogeneity).
- (3) $N(x + y) \leq N(x) + N(y)$ for any $x, y \in V$. (sub-additivity)
- (4) $N(x) \neq 0$ for any $x \in V \setminus \{0\}$. (point-separating)

A Minkowski norm N on a real vector space V gives rise to the norm $\|\cdot\|$ on V defined by $\|x\| := (N(x) + N(-x))/2$ (the symmetric part of N) which coincides with N in the case where this latter is symmetric (that is, if we have $N(-x) = N(x)$ for any $x \in V$).

Here are now some basic facts about Minkowski norms that may be useful in the sequel.

Proposition 2.2. *Given a Minkowski norm N on a real vector space V , we have the following properties:*

- (1) $N(0) = 0$.
- (2) $N(x) \leq 2\|x\|$ for any $x \in V$.
- (3) $|N(x) - N(y)| \leq \max\{N(x - y), N(y - x)\}$ for any $x, y \in V$.
- (4) $|N(x) - N(y)| \leq 2\|x - y\|$ for any $x, y \in V$. (Lipschitz continuity)

Proof. The proof is easy and left to the reader. □

Let now $\mathcal{T}$ be the collection of all the subsets U of V such that for any $a \in U$ there exists a real number $r > 0$ which satisfies $N^{-1}([0, r]) + a \subseteq U$.

We can then easily check that $\mathcal{T}$ is a topology on V which contains $N^{-1}([0, \varepsilon))$ for any real number $\varepsilon > 0$, and for which every translation of V is a homeomorphism. It is actually the coarsest topology on V for which N is continuous at the origin, and it is coarser than the topology on V associated with the norm $\|\cdot\|$.

We note moreover that the vector addition $V \times V \rightarrow V$ and the scalar multiplication $[0, +\infty) \times V \rightarrow V$ by non-negative real numbers are continuous functions when V is endowed with the topology $\mathcal{T}$.

Nevertheless, the topology $\mathcal{T}$ does not always give rise to a topological vector space structure on V as does the topology on V associated with the norm $\|\cdot\|$.

Indeed, in order for this to happen, it is necessary and sufficient that the antipodal map $A: V \rightarrow V$ defined by $A(x) := -x$ be continuous at the origin with respect to the topology $\mathcal{T}$, which is equivalent to having the following condition:

- (#) There exists a real number $C > 1$ such that
we have $N(-x) \leq CN(x)$ for any $x \in V$.

It is to be noticed that this amounts to saying that the topology $\mathcal{T}$ is nothing else than the topology associated with $\|\cdot\|$.

The condition (#) is in particular satisfied when V is finite dimensional.

Indeed, in that case, the sphere $S := \{x \in V \mid \|x\| = 1\}$ is compact (closed and bounded) for the norm $\|\cdot\|$, and hence the continuity of N with respect to $\|\cdot\|$ (see Point (4) in Proposition 2.2) insures the existence of a real number $C > 1$ such that we have the inclusion $N(S) \subseteq [2/C, +\infty)$ owing to the point-separating property of N . Therefore, we get $\|x\| \leq (C/2)N(x)$ for any $x \in V$ by the positive homogeneity of N , which finally implies the inequality $N(-x) \leq CN(x)$ since we have $N(-x) \leq 2\|-x\| = 2\|x\|$ (see Point (2) in Proposition 2.2).

Now, if V is infinite dimensional, the above condition (#) may be false as we can see with the following example.

Let V be the real vector space $\ell^1(\mathbf{R})$ endowed with the one-norm $\|\cdot\|_1$, and let φ be the linear form on V defined by

$$\varphi(x) := \sum_{n=0}^{+\infty} \frac{n+1}{n+2} x_n.$$

Then we have $\varphi(x) < \|x\|_1$ for any $x \in V \setminus \{0\}$ (this proves in particular that φ is continuous for $\|\cdot\|_1$), which implies that the function $N := \|\cdot\|_1 + \varphi$ is a Minkowski norm on V .

Therefore, if we assume that the above condition (#) is satisfied for N , then one obtains $\varphi(x) \leq [(C-1)/(C+1)] \|x\|_1$ for any $x \in V$, which is equivalent to saying that there exists a constant $K \in (0, 1)$ such that we have $\varphi(x) \leq K\|x\|_1$ for any $x \in V$. But an easy computation shows that we have

$$\sup\{\varphi(x) \mid x \in V \text{ and } \|x\|_1 = 1\} = 1,$$

which leads to a contradiction.

Among all the Minkowski norms on a real vector space, those which satisfy the conditions given by the following result are of particular interest for the purpose of the present paper.

Proposition 2.3. *Given a real vector space V and a Minkowski norm N on V , the following properties are equivalent:*

- (1) *For any non-collinear vectors x and y in V , we have $N(x+y) < N(x) + N(y)$.*
- (2) *For any two vectors $x \neq y$ in V satisfying $N(x) = N(y) = 1$, we have $N((x+y)/2) < 1$.*
- (3) *For any two vectors $x \neq y$ in V which satisfy $N(x) = N(y) = 1$, there exists $s \in (0, 1)$ such that we have $N((1-s)x + sy) < 1$.*
- (4) *For any two vectors $x \neq y$ in V which satisfy $N(x) = N(y) = 1$ and for every $t \in (0, 1)$, we have $N((1-t)x + ty) < 1$.*

Proof. (1) $\Leftrightarrow$ (2) This equivalence is an adaptation of [6, Theorem 11.1, page 110] which is a result given for usual norms.

(2) $\Rightarrow$ (3) This is obvious by considering $s := 1/2$.

(3) $\Rightarrow$ (4) Let $x \neq y$ be two vectors in V satisfying $N(x) = N(y) = 1$. By (3), there exists $s \in (0, 1)$ such that the point $z := (1-s)x + sy$ satisfies $N(z) < 1$. Now fix an arbitrary $t \in (0, 1)$ and consider the point $a := (1-t)x + ty$.

* If we have $t \in (0, s]$, then one can write $a = (1-\alpha)x + \alpha z$ with $\alpha := t/s \in (0, 1]$, which yields $N(a) \leq (1-\alpha)N(x) + \alpha N(z) = (1-\alpha) + \alpha N(z) < (1-\alpha) + \alpha = 1$ since we have $1-\alpha \geq 0$ and since N is positively homogeneous and sub-additive.

* If we have $t \in [s, 1)$, then the points $x' := y$ and $y' := x$ satisfy $z = (1-s')x' + s'y'$ with $s' := 1-s \in (0, 1)$, and hence we get $a = (1-t')x' + t'y'$ with $t' := 1-t \in (0, s']$, which yields $N(a) < 1$ according to the previous case.

(4) $\Rightarrow$ (2) This is obvious by considering $t := 1/2$. □

When N is a norm on a real vector space V (that is, a Minkowski norm on V which is symmetric) that satisfies the four equivalent properties in Proposition 2.3, then the normed vector space (V, N) is often called “strictly convex” in the literature as in [6, page 108] and [15, page 30].

Nevertheless, some authors prefer to say that such a norm is “rotund” (see [2], [10] and [19]), which is much better since the fourth property in Proposition 2.3 exactly means that there is no non-trivial line segment in the unit sphere of N (from an intuitive point of view, this unit sphere does not contain any “flat piece”).

We will see in Section 7 that this can be expressed by a topological property of the unit ball of N , and we will call such a norm N “strictly sub-convex”.

Moreover, we will generalize the notion of “strict sub-convexity” in Theorem 7.1 to any Minkowski norm on an arbitrary topological real vector space.

3. About positive homogeneity

In this section, we introduce a couple of notions related to affine geometry (Subsection 3.1), and then give some definitions and properties about positively homogeneous functions (Subsection 3.2) in order to characterize the continuity of a gauge function on an arbitrary topological real vector space (Subsection 3.3).

3.1. Geometric aspects of positive homogeneity

Let us begin by recalling the definition of a cone (a subset of a real vector space which is closed under scalar multiplications by positive real numbers) and some related affine notions that will be needed in the sequel.

Definition 3.1. A subset C of a real vector space V is said to be

- (1) a *ray* (with initial point at the origin) in V if there exists a non-zero vector $x \in V$ which satisfies $C = \{tx \mid t \geq 0\}$.
- (2) a *cone* (with apex at the origin) in V if it satisfies $\lambda C \subseteq C$ for any scalar $\lambda > 0$ (it is said to be *pointed* if it contains the origin, and *blunt* otherwise).

Remark 3.2. (1) A pointed cone is characterized by the fact that its intersection with any ray reduces to that ray or to the origin. Therefore, pointed cones not reduced to the origin coincide with arbitrary unions of rays.

(2) In particular, the empty set is a blunt cone, and any union or intersection of cones is also a cone. Moreover, the complement of a cone is a cone, and a product of cones is a cone too.

(3) In the case where V is a topological real vector space, the interior, the closure and the boundary of a cone in V are also cones in V .

Definition 3.3. A subset S of a real vector space is said to be *star-shaped* (about the origin) if for any $x \in S$ and $t \in [0, 1]$ we have $tx \in S$.

Definition 3.4. Given a subset S of a real vector space V , the *star-shaped hull* $\widehat{S}$ of S is the smallest star-shaped subset of V which contains S .

In other words, we have $\widehat{S} = [0, 1]S$.

Definition 3.5. The *pointed conic hull* $\text{Cone}(S)$ of a subset S of a real vector space V is the smallest cone in V which contains $S \cup \{0\}$. The *blunt conic hull* $\text{Cone}_b(S)$ of S is the smallest cone in V which contains $S \setminus \{0\}$. In other words, we have $\text{Cone}(S) = \{\lambda x \mid \lambda > 0 \text{ and } x \in S \cup \{0\}\}$ and $\text{Cone}_b(S) = \text{Cone}(S) \setminus \{0\}$.

According to this definition, one has $S \cup \{0\} \subseteq \text{Cone}(S)$ and $S \setminus \{0\} \subseteq \text{Cone}_b(S)$. It is to be noticed that for any $x \in V \setminus \{0\}$ the pointed conic hull of $\{x\}$ is nothing else than the ray in V passing through x .

Remark 3.6. (1) For any subset S of a real vector space, we of course have

$$\begin{aligned} \text{Cone}(S \setminus \{0\}) &= \text{Cone}(S) = \text{Cone}(S \cup \{0\}), \\ \text{Cone}_b(S \setminus \{0\}) &= \text{Cone}_b(S) = \text{Cone}_b(S \cup \{0\}) \quad \text{and} \quad \widehat{S} \subseteq \text{Cone}(S). \end{aligned}$$

(2) For any subsets A and B of a real vector space which satisfy $A \subseteq B$, the inclusions $\text{Cone}(A) \subseteq \text{Cone}(B)$ and $\text{Cone}_b(A) \subseteq \text{Cone}_b(B)$ are straightforward.

(3) For any subset S of a real vector space, we have $\text{Cone}(\widehat{S} \setminus \{0\}) = \text{Cone}(S)$ by Points (1) and (2) above.

(4) For any subsets A and C of a real vector space, if C is a cone, then we have $\text{Cone}(A \cap C) = \text{Cone}(A) \cap (C \cup \{0\})$ and $\text{Cone}_b(A \cap C) = \text{Cone}_b(A) \cap C$.

Definition 3.7. Given a subset S of a real vector space V , the vector subspace $\text{Vect}(S)$ of V *spanned* by S is the smallest linear subspace of V which contains S .

Therefore, one has $\text{Vect}(S \setminus \{0\}) = \text{Vect}(S)$. Moreover, for any subsets A and B of V which satisfy $A \subseteq B$, we obviously have $\text{Vect}(A) \subseteq \text{Vect}(B)$.

Definition 3.8. Given a subset S of a real vector space V , the *affine hull* $\text{Aff}(S)$ of S is the smallest affine subspace of V which contains S . In other words, the affine hull of S is equal to the set of points $x \in V$ that can be written as $x = \sum_{i=1}^n \lambda_i x_i$ for some integer $n \geq 1$, some points $x_1, \dots, x_n \in S$ and some real numbers $\lambda_1, \dots, \lambda_n$ which satisfy $\sum_{i=1}^n \lambda_i = 1$.

Therefore, for any subset S of V , we have $\text{Aff}(S) \subseteq \text{Vect}(S)$.

Moreover, for any subsets A and B of V which satisfy $A \subseteq B$, we clearly have $\text{Aff}(A) \subseteq \text{Aff}(B)$.

It is to be noticed that for any subset S of V the union of all the lines passing through two distinct points of S is exactly $\text{Aff}(S)$.

Once all these definitions have been recalled, let us now give some useful relationships between the affine operations Cone , Aff and Vect .

Proposition 3.9. *For any subset S of a real vector space V , the following three properties are equivalent:*

(1) $0 \in \text{Aff}(S)$; (2) $\text{Aff}(S) = \text{Aff}(S \cup \{0\})$; (3) $\text{Aff}(S) = \text{Vect}(S)$.

Moreover, we have $\text{Vect}(\text{Cone}(S)) = \text{Aff}(S \cup \{0\}) = \text{Vect}(S)$.

Proof. (1) $\Rightarrow$ (2) Assume that we have $0 \in \text{Aff}(S)$. This implies

$$\text{Aff}(S) \cup \{0\} \subseteq \text{Aff}(S), \text{ and hence } \text{Aff}(\text{Aff}(S) \cup \{0\}) \subseteq \text{Aff}(S).$$

But, on the other hand, we obviously have $S \subseteq S \cup \{0\} \subseteq \text{Aff}(S) \cup \{0\}$, which yields

$$\text{Aff}(S) \subseteq \text{Aff}(S \cup \{0\}) \subseteq \text{Aff}(\text{Aff}(S) \cup \{0\}).$$

Therefore, we get $\text{Aff}(S) = \text{Aff}(S \cup \{0\})$.

(2) $\Rightarrow$ (3) Assume that we have $\text{Aff}(S) = \text{Aff}(S \cup \{0\})$, and pick $x \in \text{Vect}(S)$. We have $x = \lambda_1 x_1 + \dots + \lambda_n x_n$ for some integer $n \geq 1$, some points $x_1, \dots, x_n \in S$ and some real numbers $\lambda_1, \dots, \lambda_n$. Then $\lambda := \lambda_1 + \dots + \lambda_n$ satisfies the equality $x = \lambda_1 x_1 + \dots + \lambda_n x_n + (1 - \lambda)0$, which shows that x is in $\text{Aff}(S \cup \{0\})$ since we have $\lambda_1 + \dots + \lambda_n + (1 - \lambda) = 1$.

So, we proved the inclusion $\text{Vect}(S) \subseteq \text{Aff}(S)$, and hence we get $\text{Vect}(S) = \text{Aff}(S)$ since we always have $\text{Aff}(S) \subseteq \text{Vect}(S)$.

(3) $\Rightarrow$ (1) This implication is straightforward since we have $0 \in \text{Vect}(S)$.

Let us now prove the last point in Proposition 3.9.

Since we have $0 \in S \cup \{0\} \subseteq \text{Aff}(S \cup \{0\})$, we get $\text{Aff}(S \cup \{0\}) = \text{Vect}(S \cup \{0\})$ by the previous implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), where S is replaced by $S \cup \{0\}$. Therefore, this yields $\text{Aff}(S \cup \{0\}) = \text{Vect}(S)$ since we always have $\text{Vect}(S \cup \{0\}) = \text{Vect}(S)$.

Moreover, we have $S \subseteq \text{Cone}(S) \subseteq \text{Vect}(S)$, and hence

$$\text{Vect}(S) \subseteq \text{Vect}(\text{Cone}(S)) \subseteq \text{Vect}(S),$$

which implies $\text{Vect}(S) = \text{Vect}(\text{Cone}(S))$. $\square$

Proposition 3.10. *Given a real vector space V , the following properties hold:*

- (1) For any $x \in V$, we have $\text{Aff}((0, 1)x) = \text{Aff}([0, +\infty)x) = \mathbf{R}x$.
- (2) For any subset S of V which satisfies $S \setminus \{0\} \neq \emptyset$, we have $\text{Aff}(\widehat{S}) = \text{Aff}((0, 1)S) = \text{Aff}([(0, 1)S] \setminus \{0\}) = \text{Aff}(\text{Cone}_b(S)) = \text{Aff}(\text{Cone}(S))$.

Proof. (1) Let us fix a vector $x \in V$.

First of all, since one has $(0, 1)x \subseteq [0, +\infty)x \subseteq \mathbf{R}x$, we get

$$\text{Aff}((0, 1)x) \subseteq \text{Aff}([0, +\infty)x) \subseteq \text{Aff}(\mathbf{R}x) = \mathbf{R}x.$$

On the other hand, given any $\lambda \in \mathbf{R}$, we have $\lambda x = (2 - 3\lambda) \cdot (1/3)x + (3\lambda - 1) \cdot (2/3)x$, which shows that λx lies in $\text{Aff}((0, 1)x)$ since one has $(2 - 3\lambda) + (3\lambda - 1) = 1$ and since $(1/3)x$ and $(2/3)x$ belong to $(0, 1)x$.

This proves the inclusion $\mathbf{R}x \subseteq \text{Aff}((0, 1)x)$, and hence the equalities

$$\text{Aff}((0, 1)x) = \text{Aff}([0, +\infty)x) = \mathbf{R}x.$$

(2) Let us fix a subset S of V which satisfies $S \setminus \{0\} \neq \emptyset$.

* First of all, for any $x \in S$, one has

$$[0, +\infty)x \subseteq \text{Aff}([0, +\infty)x) = \text{Aff}((0, 1)x) \subseteq \text{Aff}((0, 1)S)$$

by Point (1) above, and hence $\text{Cone}(S) = \bigcup_{x \in S} [0, +\infty)x \subseteq \text{Aff}((0, 1)S)$ since we have $S \neq \emptyset$, which yields the inclusion $\text{Aff}(\text{Cone}(S)) \subseteq \text{Aff}((0, 1)S)$.

* On the other hand, since one has $(0, 1)S \subseteq \widehat{S} \subseteq \text{Cone}(S)$ by Point (1) in Remark 3.6, we get $\text{Aff}((0, 1)S) \subseteq \text{Aff}(\widehat{S}) \subseteq \text{Aff}(\text{Cone}(S))$, and hence

$$\text{Aff}((0, 1)S) = \text{Aff}(\widehat{S}) = \text{Aff}(\text{Cone}(S))$$

owing to the previous step.

* We now notice that we obviously have $[(0, 1)S] \setminus \{0\} = (0, 1)[S \setminus \{0\}]$, which implies

$$\text{Aff}([(0, 1)S] \setminus \{0\}) = \text{Aff}((0, 1)[S \setminus \{0\}]) = \text{Aff}(\text{Cone}(S \setminus \{0\})) = \text{Aff}(\text{Cone}(S))$$

owing to Point (1) in Remark 3.6 and by replacing S by $S \setminus \{0\} \neq \emptyset$ in the equality

$$\text{Aff}((0, 1)S) = \text{Aff}(\text{Cone}(S))$$

obtained in the previous step.

* Finally, since one has the inclusions $(0, 1)[S \setminus \{0\}] \subseteq \text{Cone}_b(S) \subseteq \text{Cone}(S)$, we get

$$\text{Aff}([(0, 1)S] \setminus \{0\}) \subseteq \text{Aff}(\text{Cone}_b(S)) \subseteq \text{Aff}(\text{Cone}(S)) ,$$

and hence $\text{Aff}(\text{Cone}_b(S)) = \text{Aff}(\text{Cone}(S))$ owing to the equality

$$\text{Aff}([(0, 1)S] \setminus \{0\}) = \text{Aff}(\text{Cone}(S))$$

obtained in the previous step. $\square$

Definition 3.11. Given a point x in a real vector space V , a subset S of V is said to *absorb* x if there exists a real number $\lambda > 0$ such that we have $x \in \lambda S$.

We will say that S is *absorbing* if S absorbs any $x \in V \setminus \{0\}$.

Remark 3.12. So S absorbs $x \in V \setminus \{0\}$ if and only if one has $x \in \text{Cone}_b(S)$. Moreover, S absorbs 0 if and only if S contains 0.

Proposition 3.13. *Given a subset S of a real vector space V , the following properties hold:*

- (1) We have $\text{Aff}(\text{Cone}(S)) = \text{Vect}(S)$.
- (2) If we have $S \setminus \{0\} \neq \emptyset$, then we get $\text{Aff}(\text{Cone}_b(S)) = \text{Vect}(S)$.
- (3) If there exists $x \in V$ such that S absorbs both x and $-x$, then we have the equality $\text{Aff}(S) = \text{Vect}(S)$.
- (4) We have the equivalence $(S \text{ is absorbing}) \iff \text{Cone}(S) = V$.
Moreover, having both $\text{Cone}(S) = V$ and $S \neq \emptyset$ implies $\text{Aff}(S) = \text{Vect}(S) = V$.

Proof. (1) Since $\text{Cone}(S)$ contains the origin, we have $\text{Cone}(S) = \text{Cone}(S) \cup \{0\}$, and hence $\text{Aff}(\text{Cone}(S)) = \text{Aff}(\text{Cone}(S) \cup \{0\})$.

Now, according to Proposition 3.9 with $\text{Cone}(S)$ instead of S , one has

$$\text{Aff}(\text{Cone}(S) \cup \{0\}) = \text{Vect}(\text{Cone}(\text{Cone}(S))) = \text{Vect}(\text{Cone}(S)) ,$$

which yields $\text{Aff}(\text{Cone}(S)) = \text{Aff}(\text{Cone}(S) \cup \{0\}) = \text{Vect}(S)$ since we have the equality $\text{Vect}(\text{Cone}(S)) = \text{Vect}(S)$ owing to Proposition 3.9 once again.

(2) Assume that we have $S \setminus \{0\} \neq \emptyset$. Then one obtains $\text{Aff}(\text{Cone}_b(S)) = \text{Aff}(\text{Cone}(S))$ by Proposition 3.10, which yields the equality $\text{Aff}(\text{Cone}_b(S)) = \text{Vect}(S)$ owing to Point (1) above.

(3) Let $x \in V$ such that S absorbs both x and $-x$, which means that there exist $\lambda > 0$ and $\mu > 0$ satisfying $\lambda x \in S$ and $-\mu x \in S$. Since we have $\lambda + \mu > 0$ and $0 = \frac{\mu}{\lambda + \mu}(\lambda x) + \frac{\lambda}{\lambda + \mu}(-\mu x)$, one gets $0 \in \text{Aff}(S)$, which yields $\text{Aff}(S) = \text{Vect}(S)$ by Point (3) in Proposition 3.9.

(4) The equivalence is given by Remark 3.12.

Assume that we have $\text{Cone}(S) = V$ and $S \neq \emptyset$. If S contains the origin, then S absorbs 0 by Remark 3.12, and hence we get $\text{Aff}(S) = \text{Vect}(S)$ by Point (3) above with $x := 0$.

Otherwise, S contains a non-zero vector $x \in V$, and hence it absorbs x and $-x$ according to the equivalence previously established, which yields $\text{Aff}(S) = \text{Vect}(S)$ by Point (3) above.

Moreover, since we always have $\text{Cone}(S) \subseteq \text{Vect}(S)$, the equality $\text{Cone}(S) = V$ implies the inclusion $V \subseteq \text{Vect}(S)$, that is, $\text{Vect}(S) = V$. $\square$

3.2. Positively homogeneous functions

We shall now deal with positively homogeneous functions and give some useful properties of their sublevel sets.

Definition 3.14. Let V, W be real vector spaces and C a cone in V . Given a real number $\alpha > 0$, a map $f: C \rightarrow W$ is said to be *positively homogeneous of degree α* if $f(\lambda x) = \lambda^\alpha f(x)$ holds for any $x \in C$ and any real number $\lambda > 0$.

In the particular case where one has $\alpha = 1$, we merely say that f is *positively homogeneous*.

If we have $W := \mathbf{R}$, then the word function is preferred to that of map.

Remark 3.15. For any real vector spaces V and W , it is clear that a positively homogeneous map $f: C \rightarrow W$ of degree $\alpha > 0$ defined on a cone C in V satisfies the inclusion $C \cap \{0\} \subseteq f^{-1}(0)$.

Definition 3.16. Given a set X , a function $f: X \rightarrow \mathbf{R}$ and a number $r \in \mathbf{R}$, the *sublevel set* of f associated with r is defined by

$$S_r(f) := \{x \in X \mid f(x) \leq r\}.$$

Remark 3.17. (1) It is straightforward that the family $(S_r(f))_{r \in \mathbf{R}}$ is non-decreasing: for any $r, r' \in \mathbf{R}$ which satisfy $r' \leq r$, we have the inclusion $S_{r'}(f) \subseteq S_r(f)$.

(2) On the other hand, notice that for any $r \in \mathbf{R}$ we have $S_r(f) = \bigcap_{a > r} S_a(f)$.

From now on, we will focus on functions $f: S \rightarrow \mathbf{R}$ defined on a subset S of a real vector space.

Proposition 3.18. Let C be a cone in a real vector space and $f: C \rightarrow \mathbf{R}$ a positively homogeneous function of degree $\alpha > 0$. Then for any real number $r > 0$, we have the following properties:

- (1) $S_r(f) = r^{1/\alpha} S_1(f)$.
- (2) $S_r(f) \cup \{0\}$ is star-shaped.
- (3) If C is not empty, then neither is $S_r(f)$, and hence we have $\widehat{S_r(f)} = S_r(f) \cup \{0\}$.

Proof. Let r be a positive real number.

- (1) For any $x \in C$, we have the equivalences

$$\begin{aligned} x \in S_r(f) &\iff f(x) \leq r = (r^{1/\alpha})^\alpha \iff f(x)/(r^{1/\alpha})^\alpha \leq 1 \\ &\iff f(x/r^{1/\alpha}) \leq 1 \iff x \in r^{1/\alpha} S_1(f). \end{aligned}$$

(2) Let us consider the set $S := S_r(f) \cup \{0\}$.

For any $t \in (0, 1]$, we then have

$$tS = tS_r(f) \cup \{0\} = tr^{1/\alpha}S_1(f) \cup \{0\} = (t^\alpha r)^{1/\alpha}S_1(f) \cup \{0\} = S_{t^\alpha r}(f) \cup \{0\} \subseteq S$$

by Point (1) above and Point (1) in Remark 3.17 with $r' := t^\alpha r \leq r$.

Moreover, since one has $0S \subseteq S$, we obtain that S is star-shaped.

(3) Assume that C is not empty, pick x in C , and consider $\lambda := |f(x)| + 1 > 0$. Then we obtain $S_{f(x)}(f) \subseteq S_\lambda(f)$ by Point (1) in Remark 3.17 since we have $f(x) \leq \lambda$.

Now, we can write $S_\lambda(f) = (\lambda/r)^{1/\alpha}S_r(f)$ by Point (1) above, which proves that $S_r(f)$ is not empty since $S_{f(x)}(f)$ contains x . Therefore, the non-emptiness of $S_r(f)$ implies that $\widehat{S_r(f)}$ contains $\{0\}$, and hence contains $S_r(f) \cup \{0\}$ since we obviously have $S_r(f) \subseteq \widehat{S_r(f)}$.

Finally, the obvious inclusion $S_r(f) \subseteq S_r(f) \cup \{0\}$ yields $\widehat{S_r(f)} \subseteq S_r(f) \cup \{0\}$ since $S_r(f) \cup \{0\}$ is star-shaped by Point (2). $\square$

Proposition 3.19. *Let C be a cone in a real vector space, $f: C \rightarrow \mathbf{R}$ a non-negative function and $\alpha > 0$ a real number. Then the following properties are equivalent:*

- (1) *The function f is positively homogeneous of degree α .*
- (2) *The function $f^{1/\alpha}$ is positively homogeneous.*
- (3) *We have $S_r(f) = r^{1/\alpha}S_1(f)$ for any real number $r > 0$.*

Proof. (1) $\Leftrightarrow$ (2) This is obvious.

(1) $\Rightarrow$ (3) This is Point (1) in Proposition 3.18.

(3) $\Rightarrow$ (1) Assume that Point (3) is satisfied, and fix $x \in C$ and a real number $\lambda > 0$. Then, for any real number $r > 0$, we have the equivalences

$$\begin{aligned} \lambda^\alpha f(x) \leq r &\iff x \in S_{\frac{r}{\lambda^\alpha}}(f) = \left(\frac{r}{\lambda^\alpha}\right)^{1/\alpha} S_1(f) = \frac{1}{\lambda} [r^{1/\alpha} S_1(f)] = \frac{1}{\lambda} S_r(f) \\ &\iff f(\lambda x) \leq r. \end{aligned}$$

This yields $\lambda^\alpha f(x) \leq f(\lambda x) + \varepsilon$ and $f(\lambda x) \leq \lambda^\alpha f(x) + \varepsilon$ for any real number $\varepsilon > 0$ by choosing $r := f(\lambda x) + \varepsilon > 0$ and $r := \lambda^\alpha f(x) + \varepsilon > 0$, respectively.

Conclusion: since these two inequalities are true for an arbitrary real number $\varepsilon > 0$, we obtain $f(\lambda x) = \lambda^\alpha f(x)$. $\square$

Proposition 3.20. *Given a real vector space V and a function $f: V \rightarrow \mathbf{R}$ which is positively homogeneous, the sublevel set $S_1(f)$ is absorbing, and hence satisfies*

$$\text{Cone}(S_1(f)) = \text{Aff}(S_1(f)) = \text{Vect}(S_1(f)) = V.$$

Proof. Given $x \in V \setminus \{0\}$, the scalar $\lambda := |f(x)| + 1 > 0$ satisfies $x/\lambda \in S_1(f)$ since f is positively homogeneous. This proves that $S_1(f)$ is absorbing.

Now, since $S := S_1(f)$ contains the origin by Remark 3.15, it is not empty, and hence Point (4) in Proposition 3.13 gives the equalities to be proved. $\square$

Corollary 3.21. *Given a cone C in a real vector space V and $f: C \rightarrow \mathbf{R}$ a positively homogeneous function, the following properties hold:*

- (1) $\text{Cone}(S_1(f)) = C \cup \{0\}$.
- (2) $\text{Vect}(S_1(f)) = \text{Vect}(C)$.
- (3) *If C is not empty, then we have $\text{Aff}(S_1(f)) = \text{Vect}(C)$.*

Proof. (1) Let us consider the function $g: V \rightarrow \mathbf{R}$ defined by $g(x) := f(x)$ for $x \in C$ and $g(x) := 0$ for $x \notin C$. Since g is positively homogeneous, we can write $\text{Cone}(S_1(g)) = V$ by Proposition 3.20, and hence

$$\text{Cone}(S_1(g)) \cap (C \cup \{0\}) = V \cap (C \cup \{0\}) = C \cup \{0\},$$

which yields $\text{Cone}(S_1(f)) = C \cup \{0\}$ owing to Point (4) in Remark 3.6 with $A := S_1(g)$ and since we obviously have $S_1(g) \cap C = S_1(f)$.

(2) From Point (1) above, we immediately obtain $\text{Aff}(\text{Cone}(S_1(f))) = \text{Aff}(C \cup \{0\})$, and this yields $\text{Vect}(S_1(f)) = \text{Vect}(C)$ since one has $\text{Aff}(C \cup \{0\}) = \text{Vect}(C)$ by Proposition 3.9 and $\text{Aff}(\text{Cone}(S_1(f))) = \text{Vect}(S_1(f))$ by Point (1) in Proposition 3.13.

(3) Assume that C is not empty, and let us prove that the set $S := S_1(f)$ satisfies $\text{Aff}(S) = \text{Vect}(S)$ by considering two cases.

* First case: $S \setminus \{0\} = \emptyset$.

Then S contains the origin since it is not empty owing to Point (3) in Proposition 3.18, and hence we get $0 \in \text{Aff}(S)$, which yields $\text{Aff}(S) = \text{Vect}(S)$ by Point (3) in Proposition 3.9.

* Second case: $S \setminus \{0\} \neq \emptyset$.

Since $S \cup \{0\}$ is star-shaped by Point (2) in Proposition 3.18, we have

$$(0, 1)S \cup \{0\} = (0, 1)(S \cup \{0\}) \subseteq S \cup \{0\},$$

which implies $[(0, 1)S] \setminus \{0\} \subseteq S \setminus \{0\} \subseteq S \subseteq \widehat{S}$. Hence, the condition $S \setminus \{0\} \neq \emptyset$ yields

$$\text{Aff}(\widehat{S}) = \text{Aff}([(0, 1)S] \setminus \{0\}) \subseteq \text{Aff}(S) \subseteq \text{Aff}(\widehat{S}) = \text{Aff}(\text{Cone}(S))$$

according to Point (2) in Proposition 3.10, and we thus obtain

$$\text{Aff}(S) = \text{Aff}(\widehat{S}) = \text{Aff}(\text{Cone}(S)).$$

But, on the other hand, we have $\text{Aff}(\text{Cone}(S)) = \text{Vect}(S)$ by Point (1) in Proposition 3.13, which implies $\text{Aff}(S) = \text{Vect}(S)$, and finally we get $\text{Aff}(S) = \text{Vect}(C)$ by Point (2) above. $\square$

A particular important class of positively homogeneous functions is given by gauge functions.

In the sequel, we will denote by $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$ the real extended line.

Definition 3.22. Given a subset S of a real vector space V , the *gauge* function $p_S: V \rightarrow \overline{\mathbf{R}}$ of S is defined by $p_S(x) := \inf \{\lambda \geq 0 \mid x \in \lambda S\} \in [0, +\infty]$.

Remark 3.23. (1) If S is void, then we have $p_S = +\infty$ since the empty set does not absorb any vector in V .

- (2) If S is not empty, then we obviously have $p_S(0) = 0$.
- (3) For any $x \in V$ which satisfies $(0, +\infty)x \subseteq S$, we clearly have $p_S(x) = 0$.
- (4) Moreover, given any subsets A and B of V which satisfy $A \subseteq B$, we have $p_B \leq p_A$.

Proposition 3.24. *The gauge function p_S of a subset S of a real vector space V satisfies the following properties:*

- (1) For any $x \in V \setminus \{0\}$, we have the equivalences
 S absorbs $x \iff x \in \text{Cone}_b(S) \iff S \cap (0, +\infty)x \neq \emptyset \iff p_S(x) \in [0, +\infty)$.
 As a consequence, one has $\text{Cone}(S) = p_S^{-1}(\mathbf{R})$ whenever S is non-void.
- (2) For any $x \in V \setminus \{0\}$, we have

$$p_S(x) = \inf\{\lambda > 0 \mid x \in \lambda S\} = \frac{1}{\sup\{\mu > 0 \mid \mu x \in S\}}$$
 (with the conventions $1/0 := +\infty$ and $1/(+\infty) := 0$).
- (3) The gauge p_S is positively homogeneous (with the convention $\lambda \times (+\infty) := +\infty$ for any real number $\lambda > 0$).
- (4) We have $p_S = p_{\widehat{S}}$ (the gauge function of the star-shaped hull $\widehat{S}$ of S).
- (5) In case when S is star-shaped, we have $p_S^{-1}([0, 1)) \subseteq S \subseteq p_S^{-1}([0, 1])$.

Proof. (1) Given $x \in V \setminus \{0\}$, we have the equivalences

$$\begin{aligned} S \text{ absorbs } x &\iff x \in \text{Cone}_b(S) \text{ (see Remark 3.12)} \iff S \cap (0, +\infty)x \neq \emptyset \\ &\iff \{\lambda \geq 0 \mid x \in \lambda S\} \neq \emptyset \iff p_S(x) \in [0, +\infty). \end{aligned}$$

This proves that we have $\text{Cone}_b(S) = p_S^{-1}(\mathbf{R}) \setminus \{0\}$.

Moreover, in case when S is not empty, we have $0 \in p_S^{-1}(\mathbf{R})$ by Point (2) in Remark 3.23, and hence we get

$$\begin{aligned} \text{Cone}(S) &= \text{Cone}_b(S) \cup \{0\} = [p_S^{-1}(\mathbf{R}) \cap (V \setminus \{0\})] \cup \{0\} \\ &= [p_S^{-1}(\mathbf{R}) \cup \{0\}] \cap [(V \setminus \{0\}) \cup \{0\}] = p_S^{-1}(\mathbf{R}) \cup \{0\} = p_S^{-1}(\mathbf{R}). \end{aligned}$$

- (2) For any $x \in V \setminus \{0\}$, we have $x \notin 0S$, and hence we get the first equality.

On the other hand, the second equality is a mere consequence of classical properties about the infimum and the supremum in $\mathbf{R}$ with $\mu := 1/\lambda$.

- (3) Given $x \in V$ and a real number $t > 0$, we have

$$\begin{aligned} p_S(tx) &= \inf\{\lambda \geq 0 \mid tx \in \lambda S\} = \inf\{\lambda \geq 0 \mid x \in (\lambda/t)S\} \\ &= \inf\{t(\lambda/t) \mid \lambda \geq 0 \text{ and } x \in (\lambda/t)S\} \\ &= \inf\{t\mu \mid \mu \geq 0 \text{ and } x \in \mu S\} = tp_S(x). \end{aligned}$$

- (4) First of all, the inclusion $S \subseteq \widehat{S}$ implies $p_{\widehat{S}} \leq p_S$ by Point (4) in Remark 3.23. On the other hand, given $x \in V$ and a real number $\lambda \geq 0$ such that we have $x \in \lambda \widehat{S}$, there exists $t \in [0, 1]$ which satisfies $x \in \lambda tS$.

This yields $p_S(x) \leq \lambda t$, and hence we get $p_S(x) \leq \lambda$ since one has $t \leq 1$.

The inequality $p_S(x) \leq p_{\widehat{S}}(x)$ then follows.

(5) For any $x \in S$, one can write $1 \in \{\lambda \geq 0 \mid x \in \lambda S\}$, which yields $p_S(x) \leq 1$.

This proves the inclusion $S \subseteq p_S^{-1}([0, 1])$.

For any $x \in V$ which satisfies $p_S(x) < 1$, there exists $\lambda \in (0, 1)$ such that we have $x \in \lambda S$, and this yields $x \in S$ since S is star-shaped.

This proves the inclusion $p_S^{-1}([0, 1]) \subseteq S$. $\square$

Remark 3.25. It is to be mentioned that owing to Point (4) in Proposition 3.24 it is enough to deal with *star-shaped* subsets when considering gauge functions.

It is now time to give the relationship between non-negative positively homogeneous functions and gauge functions. For this purpose, let us denote by $j: \mathbf{R} \rightarrow \overline{\mathbf{R}}$ the canonical inclusion of $\mathbf{R}$ into $\overline{\mathbf{R}}$.

Proposition 3.26. *Let C be a pointed cone in a real vector space, S a star-shaped subset of C and p_S the gauge function of S . Then any positively homogeneous function $f: C \rightarrow \mathbf{R}$ which is non-negative satisfies the following equivalence:*

$$(p_S)|_C = j \circ f \iff f^{-1}([0, 1]) \subseteq S \subseteq f^{-1}([0, 1]) = S_1(f).$$

Proof. ($\Rightarrow$) Assume that we have $(p_S)|_C = j \circ f$.

Since one has $p_S^{-1}([0, 1]) \subseteq S \subseteq p_S^{-1}([0, 1])$ by Point (5) in Proposition 3.24, we get

$$((p_S)|_C)^{-1}([0, 1]) = p_S^{-1}([0, 1]) \cap C \subseteq S \cap C \subseteq p_S^{-1}([0, 1]) \cap C = ((p_S)|_C)^{-1}([0, 1]),$$

which yields $f^{-1}([0, 1]) \subseteq S \subseteq f^{-1}([0, 1])$ by using $S \cap C = S$.

($\Leftarrow$) Assume that we have $f^{-1}([0, 1]) \subseteq S \subseteq f^{-1}([0, 1])$.

Then Point (4) in Remark 3.23 implies $p_{f^{-1}([0, 1])} \leq p_S \leq p_{f^{-1}([0, 1])}$.

On the other hand, given $x \in C$ and a real number $\lambda > 0$, we can write the equivalences

$$\begin{aligned} x \in \lambda f^{-1}([0, 1]) &\iff x/\lambda \in f^{-1}([0, 1]) \iff f(x/\lambda) \leq 1 \\ &\iff f(x) \leq \lambda \quad (\text{since } f \text{ is positively homogeneous}). \end{aligned}$$

If we have $x \neq 0$, then Point (2) in Proposition 3.24 yields

$$p_{f^{-1}([0, 1])}(x) = \inf \{\lambda > 0 \mid x \in \lambda f^{-1}([0, 1])\} = \inf \{\lambda > 0 \mid f(x) \leq \lambda\} = f(x).$$

If we have $x = 0 \in C$, then $f^{-1}([0, 1])$ contains the origin since f satisfies $f(0) = 0$ by Remark 3.15. Therefore, since $f^{-1}([0, 1])$ is not empty, we get $p_{f^{-1}([0, 1])}(0) = 0$ by Point (2) in Remark 3.23, and hence we can write $p_{f^{-1}([0, 1])}(x) = f(x) = 0$.

This proves $(p_{f^{-1}([0, 1])})|_C = j \circ f$.

With the same reasoning, we also obtain $(p_{f^{-1}([0, 1])})|_C = j \circ f$.

Conclusion: summing up, we have proved $j \circ f \leq (p_S)|_C \leq j \circ f$. $\square$

Notice that Proposition 3.26 is a generalization of the result [1, Lemma 5.50, Point (1), page 192] in the situation where C is not equal to the whole space V .

Actually, the following result shows that Proposition 3.26 extends to blunt cones.

Corollary 3.27. *Let C be a cone in a real vector space, S a star-shaped subset of $C \cup \{0\}$ and p_S the gauge function of S . Then any non-negative function $f: C \rightarrow \mathbf{R}$ which is positively homogeneous satisfies the following equivalence:*

$$(p_S)|_C = j \circ f \iff f^{-1}([0, 1)) \subseteq S \subseteq f^{-1}([0, 1]) \cup \{0\} = S_1(f) \cup \{0\}.$$

Proof. We may assume that C is not empty since the equivalence to be proved is obvious otherwise.

Let us consider the pointed cone $D := C \cup \{0\}$ and the function $g: D \rightarrow \mathbf{R}$ defined by $g(0) := 0$ and $g(x) := f(x)$ for $x \in C$ (in case when C is pointed, this makes sense since we then have $f(0) = 0$ by Remark 3.15).

Since g is non-negative and positively homogeneous, Proposition 3.26 yields the equivalence

$$(p_S)|_D = j \circ g \iff g^{-1}([0, 1)) \subseteq S \subseteq g^{-1}([0, 1]) = S_1(g). \quad (\star)$$

($\Rightarrow$) Assume now that we have $(p_S)|_C = j \circ f$.

This first implies that S is not empty by Point (1) in Remark 3.23. Therefore, Point (2) in Remark 3.23 yields $(p_S)|_D(0) = p_S(0) = 0 = g(0) = (j \circ g)(0)$. Since one has $C = D \setminus \{0\}$, this proves $(p_S)|_D = j \circ g$, and hence we can deduce the inclusions $g^{-1}([0, 1)) \subseteq S \subseteq g^{-1}([0, 1])$ from the equivalence ($\star$).

Finally, we get $f^{-1}([0, 1)) = g^{-1}([0, 1)) \cap \{0\} \subseteq S \cap \{0\} \subseteq S$

and $S \subseteq g^{-1}([0, 1]) = f^{-1}([0, 1]) \cup \{0\}.$

($\Leftarrow$) Conversely, assume that we have $f^{-1}([0, 1)) \subseteq S \subseteq f^{-1}([0, 1]) \cup \{0\}.$

Since C is not empty, the same is true for $f^{-1}([0, 1))$, which yields $S \neq \emptyset$.

This implies that S contains the origin since it is star-shaped, and hence one obtains

$$g^{-1}([0, 1)) = f^{-1}([0, 1)) \cup \{0\} \subseteq S \cup \{0\} = S \subseteq f^{-1}([0, 1]) \cup \{0\} = g^{-1}([0, 1]).$$

Finally, this yields $(p_S)|_D = j \circ g$ owing to the equivalence ($\star$), which gives

$$(p_S)|_C = ((p_S)|_D)|_C = (j \circ g)|_C = j \circ (g|_C) = j \circ f,$$

completing the proof. $\square$

Remark 3.28. For any non-negative function $g: C \rightarrow \mathbf{R}$ which is positively homogeneous of degree $\alpha > 0$, Corollary 3.27 obviously applies to $f := g^{1/\alpha}$.

3.3. Continuity of gauge functions

The aim of this subsection is to give a characterization of the continuity of the gauge function of an *arbitrary* star-shaped subset of a *general* topological real vector space.

As we will need in the sequel some elementary but useful properties about topological real vector spaces, let us begin with the following remark.

Remark 3.29. (1) Given a neighborhood U of the origin in a topological real vector space V , the following easy-to-prove properties hold:

- (a) For any vector $x \in V$, there exists a real number $\varepsilon > 0$ such that we have the inclusion $[-\varepsilon, \varepsilon]x \subseteq U$.
- (b) The subset U of V is absorbing by Point (1)(a) above, and therefore satisfies the equality $\text{Vect}(U) = V$ by Point (4) in Proposition 3.13.
- (2) A cone C in a topological real vector space V whose interior $\overset{\circ}{C}$ contains the origin is equal to V . Indeed, C is then a neighborhood of 0 in V , and hence is absorbing by Point (1)(b), which yields $C = V$ by Point (4) in Proposition 3.13.
- (3) Given a finite-dimensional real vector space W , there exists a *unique* topological real vector space structure on W which is Hausdorff. Endowed with this structure, W is then isomorphic to the canonical topological real vector space $\mathbf{R}^n$, where n denotes the dimension of W (see [5, Chapitre I, Théorème 2, page 14]).
- (4) That said, it is to be mentioned that any finite-dimensional topological real vector space is isomorphic to the Cartesian product $\mathbf{R}^k \times \mathbf{R}^{n-k}$ for some integers $0 \leq k \leq n$, where the first factor is equipped with the usual topology and the second one with the trivial topology (see for example [16, Chapter 2, Section 7, Problem A, page 64]).

Lemma 3.30. *The gauge function p_S of a star-shaped subset S of a topological real vector space V satisfies the following properties:*

$$(1) \quad p_S^{-1}([0, 1]) \subseteq \overline{p_S^{-1}([0, 1])}; \quad (2) \quad \widehat{\overset{\circ}{p_S^{-1}([0, 1])}} \subseteq p_S^{-1}([0, 1]).$$

Proof. (1) Let $x \in V$ such that $p_S(x) \leq 1$ holds, and let U be a neighborhood of x in V . Since the map $t \mapsto tx$ from $\mathbf{R}$ to V is continuous at $t := 1$, there exists $\alpha \in (0, 1)$ satisfying $[\alpha, 1]x \subseteq U$, which in particular yields $\alpha x \in U$. But we have $p_S(\alpha x) = \alpha p_S(x) < 1$, which implies $\alpha x \in p_S^{-1}([0, 1])$.

This proves the inclusion $p_S^{-1}([0, 1]) \subseteq \overline{p_S^{-1}([0, 1])}$.

(2) Given $x \in \widehat{\overset{\circ}{p_S^{-1}([0, 1])}}$, the continuity at $t := 1$ of the map $t \mapsto tx$ from $\mathbf{R}$ to V insures the existence of a real number $\alpha > 1$ such that we have $[1, \alpha]x \subseteq p_S^{-1}([0, 1])$. In particular, we get $\alpha x \in p_S^{-1}([0, 1])$, which implies $p_S(x) \leq 1/\alpha$ by the positive homogeneity of p_S (see Point (3) in Proposition 3.24), and hence $p_S(x) < 1$.

This proves the inclusion $\widehat{\overset{\circ}{p_S^{-1}([0, 1])}} \subseteq p_S^{-1}([0, 1])$. □

We shall now use Lemma 3.30 to establish a result which gives a complete and simple characterization of the continuity of a gauge function (see Point (2)(c) below).

Proposition 3.31. *The gauge function p_S of a star-shaped subset S of a topological real vector space V satisfies the following properties:*

- (1) We have the equalities
 - (a) $\widehat{\overset{\circ}{p_S^{-1}([0, 1])}} = \overset{\circ}{S} = \widehat{\overset{\circ}{p_S^{-1}([0, 1])}}$, and (b) $\overline{p_S^{-1}([0, 1])} = \overline{S} = \overline{p_S^{-1}([0, 1])}$.
- (2) We have the equivalences
 - (a) p_S is lower semi-continuous $\iff p_S^{-1}([0, 1])$ is closed in V
 $\iff p_S^{-1}([0, 1]) = \overline{S},$

- (b) p_S is upper semi-continuous $\iff p_S^{-1}([0, 1))$ is open in V
 $\iff p_S^{-1}([0, 1)) = \overset{\circ}{S}$, and
- (c) p_S is continuous $\iff p_S^{-1}(1) = \text{bd } S$.

Proof. (1.a) As we have $p_S^{-1}([0, 1)) \subseteq S \subseteq p_S^{-1}([0, 1])$ by Point (5) in Proposition 3.24, one first obtains $\overline{p_S^{-1}([0, 1))}^{\circ} \subseteq \overset{\circ}{S} \subseteq \overline{p_S^{-1}([0, 1])}^{\circ}$.

Then, combining these two inclusions with Point (2) in Lemma 3.30 and taking the interior, we get $\overline{p_S^{-1}([0, 1))}^{\circ} = \overset{\circ}{S} = \overline{p_S^{-1}([0, 1])}^{\circ}$.

(1.b) As we have $p_S^{-1}([0, 1)) \subseteq S \subseteq p_S^{-1}([0, 1])$ by Point (5) in Proposition 3.24, one first obtains $\overline{p_S^{-1}([0, 1))} \subseteq \overline{S} \subseteq \overline{p_S^{-1}([0, 1])}$.

Then, combining these two inclusions with Point (1) in Lemma 3.30 and taking the closure, we get $\overline{p_S^{-1}([0, 1))} = \overline{S} = \overline{p_S^{-1}([0, 1])}$.

(2.a) The first implication $\implies$ is straightforward since p_S is non-negative.

The second implication $\implies$ is a consequence of the second equality in Point (1)(b).

Now, assume that the equality $p_S^{-1}([0, 1]) = \overline{S}$ holds, and pick any $\alpha \in \mathbf{R}$. We get

$$p_S^{-1}([-\infty, \alpha]) = \begin{cases} \emptyset & \text{if one has } \alpha < 0 \text{ (since } p_S \text{ is non-negative)} \\ p_S^{-1}([0, \alpha]) = \alpha p_S^{-1}([0, 1]) = \alpha \overline{S} & \text{if one has } \alpha > 0 \text{ (since } p_S \text{ is non-negative} \\ & \text{and owing to Point (3) in Proposition 3.24),} \\ \bigcap_{t>0} p_S^{-1}([-\infty, t]) = \bigcap_{t>0} t \overline{S} & \text{if one has } \alpha = 0 \text{ (by the previous line)} \end{cases}$$

which shows that $p_S^{-1}([-\infty, \alpha])$ is closed in V . This proves that p_S is lower semi-continuous.

(2.b) The first implication $\implies$ is straightforward since p_S is non-negative.

The second implication $\implies$ is a consequence of the first equality in Point (1)(a).

Now, assume that the equality $p_S^{-1}([0, 1]) = \overset{\circ}{S}$ holds, and pick any $\alpha \in \mathbf{R}$. We get

$$p_S^{-1}([-\infty, \alpha]) = \begin{cases} \emptyset & \text{if one has } \alpha \leq 0 \text{ (since } p_S \text{ is non-negative)} \\ p_S^{-1}([0, \alpha]) = \alpha p_S^{-1}([0, 1]) = \alpha \overset{\circ}{S} & \text{if one has } \alpha > 0 \text{ (since } p_S \text{ is non-negative} \\ & \text{and owing to Point (3) in Proposition 3.24),} \end{cases}$$

which shows that $p_S^{-1}([-\infty, \alpha])$ is open in V . This proves that p_S is upper semi-continuous.

(2.c) ($\Rightarrow$) Assume that p_S is continuous.

Then we have $p_S^{-1}([0, 1]) = \overline{S}$ by Point (2)(a) and $p_S^{-1}([0, 1]) = \overset{\circ}{S}$ by Point (2)(b), which yields the equalities $p_S^{-1}(1) = p_S^{-1}([0, 1]) \setminus p_S^{-1}([0, 1]) = \overline{S} \setminus \overset{\circ}{S} = \text{bd } S$.

($\Leftarrow$) Conversely, assume that we have $p_S^{-1}(1) = \text{bd } S$.

Then we get

$$p_S^{-1}([0, 1]) = p_S^{-1}([0, 1]) \setminus p_S^{-1}(1) \subseteq \overline{p_S^{-1}([0, 1])} \setminus p_S^{-1}(1) = \overline{S} \setminus \text{bd } S = \overset{\circ}{S}$$

since we proved $\overline{p_S^{-1}([0, 1])} = \overline{S}$ in Point (1)(b).

On the other hand, we have $\overset{\circ}{S} = \overline{p_S^{-1}([0, 1])} \subseteq p_S^{-1}([0, 1])$ by Point (1)(a). So we get $p_S^{-1}([0, 1]) = \overset{\circ}{S}$, which proves that p_S is upper semi-continuous by Point (2)(b).

Finally, using again the last equality, we obtain

$$p_S^{-1}([0, 1]) = p_S^{-1}([0, 1]) \cup p_S^{-1}(1) = \overset{\circ}{S} \cup \text{bd } S = \overline{S},$$

which proves that p_S is lower semi-continuous according to Point (2)(a).

Conclusion: the function p_S is continuous. $\square$

Remark 3.32. (1) Each of the inclusions $\overset{\circ}{S} \subseteq p_S^{-1}([0, 1]) \subseteq S \subseteq p_S^{-1}([0, 1]) \subseteq \overline{S}$ contained in the proof of Point (1) in Proposition 3.31 are strict in general as we can check with the star-shaped subset S of $\mathbf{R}^2$ defined by

$$S := D \cup \{(r \cos \theta, r \sin \theta) \mid r \in [0, 2] \text{ and } \theta \in \mathbf{Q}\},$$

where D denotes the open unit disk in $\mathbf{R}^2$.

(2) As a consequence of Point (2)(a) in Proposition 3.31, the lower semi-continuity of p_S yields the equality $p_S^{-1}(0) = \bigcap_{t>0} t\overline{S}$.

(3) Since we have $S \subseteq p_S^{-1}([0, 1])$ by Point (5) in Proposition 3.24 and $p_S^{-1}([0, 1]) \subseteq \overline{S}$ by the second equality in Point (1)(b) in Proposition 3.31, the equality $p_S^{-1}([0, 1]) = \overline{S}$ occurs whenever S is closed in V , and this then implies the lower semi-continuity of p_S owing to Point (2)(a) in Proposition 3.31.

(4) On the other hand, as regards the upper semi-continuity, let us notice that if S is not empty and if p_S is upper semi-continuous then S must be a neighborhood of the origin in V by Point (2)(b) in Proposition 3.31.

Nevertheless, the converse is not true as we can check by considering the star-shaped subset S of $\mathbf{R}^2$ defined by $S := D \cup ([0, +\infty) \times \{0\})$, where D denotes the open unit disk in $\mathbf{R}^2$. Indeed, in that case, the point $x := (2, 0)$ satisfies $p_S(x) = 0 \in [0, 1)$ by Point (3) in Remark 3.23 but does not belong to $\overset{\circ}{S} = D$, which shows that the condition $p_S^{-1}([0, 1]) = \overset{\circ}{S}$ in Point (2)(b) in Proposition 3.31 does not hold.

We shall see in Proposition 5.6 which extra condition is needed to fill this gap.

4. About convexity

In this section, we give some definitions and properties about convex sets and convex functions before moving towards the notion of strict convexity for subsets of an arbitrary topological real vector space.

4.1. Geometric aspects of convexity

We shall first focus on some useful properties of convexity for both sets and functions.

Definition 4.1. Given points x and y in a real vector space V , the set

$$[x, y] := \{(1 - t)x + ty \mid t \in [0, 1]\}$$

is called the (*closed*) *line segment* between x and y , whereas the set

$$]x, y[:= [x, y] \setminus \{x, y\}$$

is called the *open line segment* between x and y (the latter set is therefore empty in case when one has $x = y$).

A subset C of V is said to be *convex* if we have $[x, y] \subseteq C$ for all $x, y \in C$.

Remark 4.2. For any collection of convex subsets of V which is an upward directed set for the inclusion, its union is itself a convex subset of V .

Definition 4.3. The *convex hull* $\text{Conv}(S)$ of a subset S of a real vector space V is the smallest convex subset of V which contains S . In other words, the convex hull of S is equal to the set of points $x \in V$ which can be written in the form $x = \sum_{i=1}^n \lambda_i x_i$ for some integer $n \geq 1$, some points $x_1, \dots, x_n \in S$ and some real numbers $\lambda_1, \dots, \lambda_n \in [0, +\infty)$ which satisfy $\sum_{i=1}^n \lambda_i = 1$.

We obviously have $\text{Conv}(S) \subseteq \text{Aff}(S)$.

Remark 4.4. (1) Any convex subset C of V is star-shaped if and only if it contains the origin since the convexity of C is equivalent to saying that C is star-shaped about any of its points.

For example, any convex subset C of V for which we can find $x \in V$ such that C absorbs x and $-x$ contains the origin.

Indeed, there exist real numbers $\lambda > 0$ and $\mu > 0$ satisfying $\lambda x \in C$ and $-\mu x \in C$, and hence we get

$$0 = \frac{\mu}{\lambda + \mu}(\lambda x) + \frac{\lambda}{\lambda + \mu}(-\mu x) \in C$$

since C is convex and since we have $\frac{\mu}{\lambda + \mu} \geq 0$, $\frac{\lambda}{\lambda + \mu} \geq 0$ and $\frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} = 1$.

(2) The convex hull of a cone C in V is still a cone in V . Indeed, for any real number $\lambda > 0$, the multiplication by λ is an affine mapping from V to itself, and hence satisfies $\lambda \text{Conv}(C) \subseteq \text{Conv}(\lambda C)$, which yields $\lambda \text{Conv}(C) \subseteq \text{Conv}(C)$ since we have $\lambda C \subseteq C$.

Proposition 4.5. For any convex subset C of a real vector space, its star-shaped hull $\widehat{C}$ is also convex.

Proof. We refer to [24, Theorem 7.2.3, page 336] for a proof of a general result which implies Proposition 4.5 when we take $A := C$ and $B := \{0\}$. $\square$

Remark 4.6. The pointed conic hull $\text{Cone}(C)$ of a convex subset C of a real vector space V is convex.

Indeed, assuming that C is not empty (since this property is obvious otherwise), we know that for each real number $\lambda > 0$ the subset $\lambda \widehat{C}$ of V is convex by Proposition 4.5 and since the multiplication by λ is an affine mapping from V to itself.

Moreover, we have $\lambda\widehat{C} = [0, \lambda]C$.

Therefore, the family $(\lambda\widehat{C})_{\lambda>0}$ is non-decreasing for the inclusion, and hence its union, which is equal to $\text{Cone}(C)$, is a convex subset of V by Remark 4.2.

Proposition 4.7. *Given a cone C in a real vector space V , we have the following equivalence: C is convex $\iff C$ is stable with respect to $+$.*

Proof. We refer to [20, Theorem 2.6, page 14] for a proof of this result. $\square$

Let us now switch to functions by recalling the definition of a (strictly) convex function on a convex subset of a real vector space.

Definition 4.8. A function $f: C \rightarrow \mathbf{R}$ defined on a convex subset C of a real vector space is said to be

- (1) *convex* if we have $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$ for any points $x, y \in C$ and any number $t \in (0, 1)$;
- (2) *strictly convex* if we have $f((1-t)x + ty) < (1-t)f(x) + tf(y)$ for any distinct points $x, y \in C$ and any number $t \in (0, 1)$.

It is to be noticed that both convexity and strict convexity of functions are mere affine notions.

Remark 4.9. (1) A strictly convex function is of course convex.

(2) Given a scalar product $\langle \cdot, \cdot \rangle$ on a real vector space V and a real number $\alpha > 1$, the norm $\|\cdot\|$ associated with $\langle \cdot, \cdot \rangle$ is such that the function $f := \|\cdot\|^\alpha$ is strictly convex. Indeed, pick x and y in V which satisfy $x \neq y$, and let us fix $t \in (0, 1)$. First of all, if we have $\|x\| \neq \|y\|$, then one can write

$$f((1-t)x + ty) \leq ((1-t)\|x\| + t\|y\|)^\alpha < (1-t)f(x) + tf(y)$$

since the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by $\varphi(s) := s^\alpha$ is non-decreasing and strictly convex.

Now, assume that the equality $\|x\| = \|y\|$ holds.

If we had $\langle x, y \rangle = \|x\|\|y\|$, the vectors x and y would be collinear by the equality case in the Cauchy-Schwarz inequality, which means that there would exist $\lambda \in \mathbf{R}$ satisfying $y = \lambda x$. Therefore, we would obtain $\lambda\|x\|^2 = \langle x, y \rangle = \|x\|^2$, which yields $\lambda = 1$ since x is not equal to the zero vector (otherwise we would have $x = y = 0$ which is not possible), and hence $y = x$, which is not possible.

So we have $\langle x, y \rangle < \|x\|\|y\|$, which implies

$$\begin{aligned} \|(1-t)x + ty\|^2 &= (1-t)^2\|x\|^2 + 2t(1-t)\langle x, y \rangle + t^2\|y\|^2 \\ &= 2t(1-t)[\langle x, y \rangle - \|x\|\|y\|] + \|x\|^2 < \|x\|^2, \end{aligned}$$

and hence we get

$$f((1-t)x + ty) = (\|(1-t)x + ty\|^2)^{\alpha/2} < (\|x\|^2)^{\alpha/2} = \|x\|^\alpha = (1-t)f(x) + tf(y)$$

since the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by $\varphi(s) := s^{\alpha/2}$ is increasing.

The next property gives a way for constructing new (strictly) convex functions from old ones.

Proposition 4.10. *Let C be a subset of a real vector space, $f: C \rightarrow \mathbf{R}$ a non-negative function and $\alpha \geq 1$ a real number. Then we have the implication*

$$f \text{ is (strictly) convex} \implies f^\alpha \text{ is (strictly) convex}.$$

Proof. This is straightforward since the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by $\varphi(t) := t^\alpha$ is both convex and increasing. $\square$

Remark 4.11. Of course, the converse of the implication in Proposition 4.10 is not true as we can easily see with $\alpha := 2$ and $f: [0, +\infty) \rightarrow \mathbf{R}$ defined by $f(t) := \sqrt{t}$ (for convexity) or by $f(t) := t$ (for strict convexity).

4.2. Topological aspects of convexity

We shall now deal with the topological notion of strict convexity for subsets of a general topological real vector space. For this purpose, we will first need to introduce the relative interior, closure and boundary of an arbitrary subset of a topological real vector space. We shall use notations based on [21, Chapter 2, page 64] and [17, Chapter 2, page 80].

Definition 4.12. Let S be a subset of a topological real vector space V .

- (1) The *relative interior* $\text{rint}(S)$ of S is the interior of S with respect to the relative topology of $\text{Aff}(S)$ in V .
- (2) The *relative closure* $\text{rcl}(S)$ of S is the closure of S with respect to the relative topology of $\text{Aff}(S)$ in V .
- (3) The *relative boundary* $\text{rbd}(S)$ of S is the boundary of S with respect to the relative topology of $\text{Aff}(S)$ in V (so we have $\text{rbd}(S) = \text{rcl}(S) \setminus \text{rint}(S)$).

We obviously have $\overset{\circ}{S} \subseteq \text{rint}(S)$ and $\text{rcl}(S) \subseteq \overline{S}$, which yields $\text{rbd}(S) \subseteq \text{bd } S$.

Proposition 4.13. *Let V be a topological real vector space.*

- (1) *For any subsets A and B of V which satisfy $A \subseteq B$, we have $\text{rcl}(A) \subseteq \text{rcl}(B)$.*
- (2) *For any subset S of V , we have*
 - (a) $\text{Aff}(\text{rcl}(S)) = \text{Aff}(S)$, *and*
 - (b) $\text{rint}(S) \neq \emptyset \iff (S \neq \emptyset \text{ and } \text{Aff}(\text{rint}(S)) = \text{Aff}(S)).$

Proof. We refer to [23, Proposition 2.7, page 805] for a proof of this result. $\square$

Before we give the definition of a strictly convex set – which is a central notion in the present work – let us first recall some useful topological results about convex subsets of a general topological real vector space.

Proposition 4.14. (see [5, Chapitre II, pages 14 and 15]) *For any convex subset C of a topological real vector space, we have the following properties:*

- (1) *The closure $\overline{C}$ of C is also convex.*
- (2) *For any $x \in \overset{\circ}{C}$ and $y \in \overline{C}$, we have $]x, y[\subseteq \overset{\circ}{C}$.*

Remark 4.15. In particular, Point (2) in Proposition 4.14 implies that for any $x, y \in \overline{C}$ we have the implication

$$]x, y[\cap \overset{\circ}{C} \neq \emptyset \implies]x, y[\subseteq \overset{\circ}{C}.$$

Definition 4.16. A subset C of a topological real vector space V is said to be *strictly convex* if for any two distinct points $x, y \in \text{rcl}(C)$ we have $]x, y[\subseteq \text{rint}(C)$.

Remark 4.17. (1) This definition coincides with the usual one when V is the canonical topological real vector space $\mathbf{R}^n$ (see for example [13, page 2] and [22, page 87]) since in this case the closeness of $\text{Aff}(C)$ in V yields $\text{rcl}(C) = \overline{C}$.

(2) A strictly convex subset of V is of course convex.

(3) It is to be noticed that the property of being strictly convex for C involves the topology of V whereas the property of just being convex does not.

(4) A subset C of V is strictly convex if and only if it is convex and for any two distinct points $x, y \in \text{rbd}(C)$ we have $]x, y[\subseteq \text{rint}(C)$. Indeed, the necessary condition is an easy consequence of the very definition of the strict convexity of C combined with Point (2) above. And the sufficient condition is obtained by Point (2) in Proposition 4.14. According to the common geometric intuition, this means that C is convex and that there is no non-trivial segment in the relative boundary of C .

Proposition 4.18. *For any strictly convex subset C of a topological real vector space, we have the implication*

$$C \neq \emptyset \implies \text{rint}(C) \neq \emptyset.$$

Proof. We refer to [23, Proposition 2.9, page 807] for a proof of this result. $\square$

Remark 4.19. When dealing with a single strictly convex subset C of a general topological real vector space V , we will always assume in the hypotheses that C has a non-empty interior in V in order to insure $\text{Aff}(C) = V$, and this makes sense by Proposition 4.18 and Point (2)(b) in Proposition 4.13.

Proposition 4.20. *Let C_1 and C_2 be subsets of a topological real vector space which satisfy $\text{rint}(C_1) \subseteq C_2 \subseteq \text{rcl}(C_1)$. Then we have the implication*

$$C_1 \text{ is strictly convex} \implies C_2 \text{ is strictly convex}.$$

Proof. Assume that C_1 is strictly convex.

* In case when C_1 is empty, the sets $\text{rint}(C_1)$ and $\text{rcl}(C_1)$ are empty too, which implies that C_2 is empty, and hence strictly convex.

* In case when C_1 is not empty, we have $\text{rint}(C_1) \neq \emptyset$ according to Proposition 4.18.

Therefore, Points (2)(b) and (2)(a) in Proposition 4.13 yield

$$\text{Aff}(\text{rint}(C_1)) = \text{Aff}(C_1) = \text{Aff}(\text{rcl}(C_1)),$$

and hence we get $\text{Aff}(C_2) = \text{Aff}(C_1)$ owing to the hypothesis $\text{rint}(C_1) \subseteq C_2 \subseteq \text{rcl}(C_1)$.

As a consequence, one obtains $\text{rint}(C_1) \subseteq \text{rint}(C_2)$.

Now, any two points $x \neq y$ in $\text{rcl}(C_2) \subseteq \text{rcl}(\text{rcl}(C_1)) = \text{rcl}(C_1)$ satisfy $]x, y[\subseteq \text{rint}(C_1)$ since C_1 is strictly convex, and hence we get $]x, y[\subseteq \text{rint}(C_2)$ from the obvious inclusion $\text{rint}(C_1) \subseteq \text{rint}(C_2)$.

This proves that C_2 is strictly convex. $\square$

Proposition 4.21. *Let C be a strictly convex cone in a topological real vector space V whose interior is not empty and which is not equal to the whole space V . Then V is one-dimensional and $C \cup \{0\}$ is a ray.*

Proof. * Assume first that C is pointed.

Then the boundary $\text{bd } C$ of C is also a cone by Point (3) in Remark 3.2.

Since C is not equal to the whole space V , we have $0 \in C \setminus \overset{\circ}{C} \subseteq \text{bd } C$ by Point (2) in Remark 3.29, and hence the boundary $\text{bd } C$ is reduced to the origin (indeed, if there were a point $x \neq 0$ in $\text{bd } C$, then the ray $[0, +\infty)x$ would lie in the pointed cone $\text{bd } C$, and this is not possible since C is strictly convex).

Hence, we get $C \subseteq \overline{C} = \overset{\circ}{C} \cup \text{bd } C = \overset{\circ}{C} \cup \{0\} \subseteq C$, that is, $C = \overline{C} = \overset{\circ}{C} \cup \{0\}$.

We thus have $C \setminus \{0\} = \overline{C} \cap (V \setminus \{0\})$, which proves that $C \setminus \{0\}$ is closed in $V \setminus \{0\}$.

On the other hand, one has $C \setminus \{0\} = \overset{\circ}{C} \subseteq V \setminus \{0\}$, which shows that $C \setminus \{0\}$ is open in $V \setminus \{0\}$. Since $C \setminus \{0\} = \overset{\circ}{C}$ is not empty and different from $V \setminus \{0\}$ by hypothesis, this implies that $V \setminus \{0\}$ is not connected. Therefore, V is one-dimensional since any topological real vector space of dimension greater than one is arcwise connected.

Finally, since V is the union of two distinct rays, the pointed cone C is equal to one of these rays by Point (1) in Remark 3.2 since we have $C \neq V$ and $C \neq \{0\}$.

* Assume now that C is blunt.

Let us consider the pointed cone $D := C \cup \{0\}$ whose interior contains $\overset{\circ}{C} \neq \emptyset$. For any $x \in C$, the sequence $(x/n)_{n \geq 1}$ is in the cone C and converges to 0, which yields $0 \in \overline{C}$. This proves the inclusion $D \subseteq \overline{C}$, which implies that D is strictly convex according to Proposition 4.20 with $C_1 := C$ and $C_2 := D$.

Therefore, the previous point implies that the pointed cone D is equal to either V or a ray. But the first case is not possible since we would get $C = D \setminus \{0\} = V \setminus \{0\}$, which is not a convex set (notice that we have $\emptyset \neq \overset{\circ}{C} \subseteq C \subseteq V \setminus \{0\}$, and hence V is not reduced to $\{0\}$). $\square$

Let us now end this section by recalling some useful results about the continuity of convex functions that we will need in the sequel.

Proposition 4.22. *For any closed convex subset S of a topological real vector space which contains the origin, the gauge function p_S of S is lower semi-continuous.*

Proof. This is a straightforward consequence of Point (3) in Remark 3.32. $\square$

Remark 4.23. This result can be found for example in Point (a) in [17, Proposition 3.4.5, page 132].

Theorem 4.24. (see [5, Chapitre II, Proposition 21, page 20]) *Given a non-empty open convex subset C of a topological real vector space V , a convex function $f: C \rightarrow \mathbf{R}$ is continuous if and only if there exists a non-empty open subset U of V which satisfies $U \subseteq C$ and such that f is bounded from above on U .*

Corollary 4.25. (see [5, Chapitre II, Corollaire, page 20]) *Given an open convex subset C of the canonical topological real vector space $\mathbf{R}^n$, any convex function from C to $\mathbf{R}$ is continuous.*

5. About sub-additive functions

In this section, we give some relationships between the notions of sub-additivity and convexity for positively homogeneous functions.

Let us begin with the definition of sub-additivity for an arbitrary function defined on a subset of a real vector space.

Definition 5.1. Given a subset S of a real vector space which is stable with respect to $+$, a function $f: S \rightarrow \mathbf{R}$ is said to be *sub-additive* if we have $f(x+y) \leq f(x)+f(y)$ for any $x, y \in S$.

Remark 5.2. For example, we may take a convex cone for S according to Proposition 4.7.

Proposition 5.3. (see [20, Theorem 4.7, page 30]) *Let C be a convex cone in a real vector space and $f: C \rightarrow \mathbf{R}$ a positively homogeneous function. Then we have the equivalence: f is convex $\iff f$ is sub-additive.*

Corollary 5.4. *For any convex subset S of V which contains the origin, its pointed conic hull $C := \text{Cone}(S)$ and its gauge function p_S satisfy the following properties:*

- (1) *The set C is convex and we have $C = p_S^{-1}(\mathbf{R})$.*
- (2) *The function $h: C \rightarrow \mathbf{R}$ defined by $h(x) := p_S(x)$ is convex.*

Proof. (1) Since S is convex, the same holds for C according to Remark 4.6. On the other hand, we have $C = p_S^{-1}(\mathbf{R})$ by Point (1) in Proposition 3.24.

(2) Let us now notice that h is positively homogeneous by Point (3) in Proposition 3.24.

Next, fix $x, y \in C$, and let λ and μ be positive real numbers satisfying $h(x) < \lambda$ and $h(y) < \mu$. This yields $h(x/\lambda) < 1$ and $h(y/\mu) < 1$, which implies that x/λ and y/μ are in S since we have $h^{-1}([0, 1)) = p_S^{-1}([0, 1)) \subseteq S$ by Point (5) in Proposition 3.24 knowing that S is star-shaped by Point (1) in Remark 4.4.

Now, we will mimic a standard calculation used for example in [17, Lemma 3.4.2, page 130]. Namely, the convexity of S yields

$$\frac{x+y}{\lambda+\mu} = \left(\frac{\lambda}{\lambda+\mu}\right)\frac{x}{\lambda} + \left(\frac{\mu}{\lambda+\mu}\right)\frac{y}{\mu} \in S,$$

and hence we get $h((x+y)/(\lambda+\mu)) = p_S((x+y)/(\lambda+\mu)) \leq 1$ owing to the obvious inclusion $S \subseteq \text{Cone}(S) = C$ and since one has $S \subseteq p_S^{-1}([0, 1])$ by Point (5) in Proposition 3.24.

So, we get $h(x + y) \leq \lambda + \mu$, which implies $h(x + y) \leq h(x) + h(y)$ since λ and μ are arbitrary.

This proves that h is sub-additive, and hence convex by Proposition 5.3. $\square$

Remark 5.5. It is to be noticed that the function h in Corollary 5.4 may be convex even though S is not convex.

Indeed, the subset $S := (-1, 1)^2 \cup \{(1, 1), (1, -1)\}$ of $\mathbf{R}^2$ is not convex and we obviously have $C := \text{Cone}(S) = \mathbf{R}^2$. On the other hand, we can easily check that the subset $T = [-1, 1]^2$ of $\mathbf{R}^2$ satisfies $\text{Cone}(T) = C = \mathbf{R}^2$ and that h can be written $h(x, y) := p_S(x, y) = p_T(x, y)$ for any $(x, y) \in \mathbf{R}^2$. Therefore, since T is star-shaped and convex, the function h is convex by Point (2) in Corollary 5.4.

Proposition 5.6. *Let S be a non-empty convex subset of a topological real vector space V and p_S the gauge function of S . Then we have the equivalence*

$$p_S \text{ is continuous} \quad \Longleftrightarrow \quad S \text{ is a neighborhood of the origin in } V.$$

Proof. $(\Rightarrow)$ Assume that p_S is continuous.

Since it is upper semi-continuous, S must be a neighborhood of the origin in V by Point (2)(b) in Proposition 3.31.

$(\Leftarrow)$ Assume that S is a neighborhood of the origin in V .

Then we have $C := \text{Cone}(S) = V$ by Point (1)(b) in Remark 3.29, and hence Corollary 5.4 implies that p_S is real-valued and that the function $h: C \rightarrow \mathbf{R}$ defined by $h(x) := p_S(x)$ is convex.

On the other hand, the non-empty open set $\overset{\circ}{S}$ lies in $p_S^{-1}([0, 1])$ by Point (1)(a) in Proposition 3.31, which proves that h is bounded from above on S . Therefore, h is continuous owing to Theorem 4.24, and hence the same holds for $p_S = j \circ h$, where j denotes the canonical inclusion of $\mathbf{R}$ into $\overline{\mathbf{R}}$. $\square$

Remark 5.7. It is to be mentioned that Proposition 5.6 is a generalization to arbitrary topological vector spaces of a result given in Point (c) in [17, Proposition 3.4.5, page 132] for normed vector spaces.

6. About sub-convex functions

In this section, we deal with sub-convex functions, and then define the key notion of *strict* sub-convexity, which we shall link up with strict convexity for positively homogeneous functions.

6.1. Geometric aspects of sub-convexity

We first introduce (strictly) quasi-convex functions, whose class is wider than that of (strictly) convex functions, but whose properties are nevertheless very close to (strict) convexity (we may refer to [12] and [14, Part I, page 3] for an overview).

It is to be mentioned that the notion of quasi-convex functions has been extensively developed by Crouzeix and Volle (see for example [8], [9] and [18]).

Definition 6.1. A function $f: C \rightarrow \mathbf{R}$ defined on a convex subset C of a real vector space is said to be

- (1) *quasi-convex* if we have $f((1-t)x + ty) \leq \max\{f(x), f(y)\}$ for any points $x, y \in C$ and any number $t \in (0, 1)$;
- (2) *strictly quasi-convex* if we have $f((1-t)x + ty) < \max\{f(x), f(y)\}$ for any two distinct points $x, y \in C$ and any number $t \in (0, 1)$.

It is to be noticed that both quasi-convexity and strict quasi-convexity of functions are mere affine notions.

Remark 6.2. (1) A strictly quasi-convex function is of course quasi-convex.

(2) A convex function is of course quasi-convex, but the converse is obviously not true as we can check with the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) := \sqrt{|x|}$.

(3) On the other hand, a strictly convex function is strictly quasi-convex, but the converse is clearly false as we can see with the absolute value function on the real line.

(4) Given a convex cone C in a real vector space V and a positively homogeneous function $f: C \rightarrow \mathbf{R}$ which is strictly quasi-convex, we have $f^{-1}(0) \subseteq \{0\}$. Indeed, any vector $x \neq 0$ in V belongs to the open line segment $]x/2, 2x[$, which yields

$$f(x) < \max\{f(x/2), f(2x)\} = \max\{f(x)/2, 2f(x)\}$$

owing to the strict quasi-convexity and the positive homogeneity of f , and hence we cannot have $f(x) = 0$.

Proposition 6.3. Let C be a convex subset of a real vector space, $f: C \rightarrow \mathbf{R}$ a non-negative function and $\alpha > 0$ a real number. Then we have the equivalence

$$f \text{ is (strictly) quasi-convex} \iff f^\alpha \text{ is (strictly) quasi-convex}.$$

Proof. This is straightforward since the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by the formula $\varphi(t) := t^\alpha$ is increasing. $\square$

Proposition 6.4. Given a convex subset C of a real vector space, any non-negative function $f: C \rightarrow \mathbf{R}$ such that f^α is (strictly) convex for some real number $\alpha > 0$ is (strictly) quasi-convex.

Proof. Assume that f^α is (strictly) convex for some real number $\alpha > 0$. According to Points (2) and (3) in Remark 6.2, the function f^α is (strictly) quasi-convex, and hence f is (strictly) quasi-convex by Proposition 6.3. $\square$

We shall now consider the class of sub-convex functions and give its relationships with both convexity and quasi-convexity.

Definition 6.5. Given a subset C of a real vector space, a function $f: C \rightarrow \mathbf{R}$ is said to be *sub-convex* if the sublevel set $S_r(f)$ is convex for any $r \in \mathbf{R}$.

Remark 6.6. (1) The domain C of such a function f is actually convex since the non-decreasing family $(S_r(f))_{r \in \mathbf{R}}$ covers C . Therefore, when dealing with sub-convexity, we will always consider functions defined on convex domains.

(2) If C is a cone and f is non-negative and positively homogeneous, then Point (1) in Proposition 3.18 with $\alpha := 1$ and Point (2) in Remark 3.17 with $r := 0$ show that the sub-convexity of f is equivalent to the convexity of $S_1(f)$ since both the image of a convex set by a homothety and the intersection of a family of convex sets are convex sets.

Proposition 6.7. *A function $f: C \rightarrow \mathbf{R}$ defined on a convex subset C of a real vector space is sub-convex if and only if $f^{-1}((-\infty, r))$ is convex for any $r \in \mathbf{R}$.*

Proof. We refer to [14, Lemma 1.27, page 32] for a proof of this result. $\square$

Proposition 6.8. *For any convex subset C of a real vector space and any function $f: C \rightarrow \mathbf{R}$, we have the following implication:*

$$f \text{ is convex} \implies f \text{ is sub-convex}.$$

Proof. Since the preimage of a convex set by a convex function is itself convex, the implication is proved. $\square$

The converse of the implication in Proposition 6.8 is of course not true as one can check by considering the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(t) := 0$ for $t < 0$ and $f(t) := -t$ for $t \geq 0$.

Let us now turn our attention to the relationship between sub-convexity and quasi-convexity.

Proposition 6.9. *For any convex subset C of a real vector space and any function $f: C \rightarrow \mathbf{R}$, the following equivalence holds:*

$$f \text{ is sub-convex} \iff f \text{ is quasi-convex}.$$

Proof. We refer to [11, Point 50, page 69 (reformatted by Border)] for a proof of this result. $\square$

Remark 6.10. This equivalence shows that there is no need to be cautious when mixing sub-convexity and quasi-convexity, and this is actually what many authors do in the literature (see for example [14, Chapter 1]). On the contrary, this is no longer the case when dealing with strict quasi-convexity and strict sub-convexity as we shall see in the next subsection.

The next property gives a way for constructing new sub-convex functions from old ones.

Proposition 6.11. *Let C be a convex subset of a real vector space, $f: C \rightarrow \mathbf{R}$ a sub-convex function, D a subset of $\mathbf{R}$ which contains $f(C)$ and $\varphi: D \rightarrow \mathbf{R}$ a non-decreasing function. Then the function $g: C \rightarrow \mathbf{R}$ defined by $g(x) := \varphi[f(x)]$ is sub-convex.*

Proof. For any $x, y \in C$ and $t \in [0, 1]$, we have $f((1-t)x + ty) \leq \max\{f(x), f(y)\}$ since the sub-convexity of f is equivalent to its quasi-convexity by Proposition 6.9, which yields

$$g((1-t)x + ty) \leq \varphi(\max\{f(x), f(y)\}) = \max\{g(x), g(y)\}.$$

This proves that g is quasi-convex, and hence sub-convex by Proposition 6.9. $\square$

If we pick a real number $\alpha > 0$ and make use of Proposition 6.11 with the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by $\varphi(t) := t^\alpha$, then we obtain the following result.

Corollary 6.12. *Let C be a convex subset of a real vector space, $f: C \rightarrow \mathbf{R}$ a non-negative function and $\alpha > 0$ a real number. Then we have the equivalence*

$$f^\alpha \text{ is sub-convex} \iff f \text{ is sub-convex}.$$

Let us now improve Proposition 6.8 for non-negative and positively homogeneous functions.

Proposition 6.13. *Let C be a convex cone in a real vector space and $f: C \rightarrow \mathbf{R}$ a non-negative function which is positively homogeneous. Then we have the equivalence*

$$f \text{ is sub-convex} \iff f \text{ is convex}.$$

Proof. First of all, we may assume that C is not empty since in this case the equivalence to be proved is trivial. As we only have to prove the implication $\implies$ according to Proposition 6.8, let us assume that f is sub-convex.

So, $S_1(f)$ is convex, and hence the set $S := \widehat{S_1(f)}$ is also convex by Proposition 4.5. Moreover, according to Point (3) in Proposition 3.18 with $r := 1 > 0$, we have $S = S_1(f) \cup \{0\}$. Now, since S is a star-shaped subset of $C \cup \{0\}$ which contains $f^{-1}([0, 1))$, we have $(p_S)|_C = j \circ f$ by Corollary 3.27, where j denotes the canonical inclusion of $\mathbf{R}$ into $\overline{\mathbf{R}}$.

On the other hand, we have $\text{Cone}(S) = \text{Cone}(S_1(f)) = C \cup \{0\}$ by Point (1) in Remark 3.6 and Point (1) in Corollary 3.21.

Finally, since the set S is non-empty, star-shaped and convex, this implies that the function $h: C \cup \{0\} \rightarrow \mathbf{R}$ defined by $h(x) := p_S(x)$ is convex by Corollary 5.4, and hence $f = h|_C$ is convex too. $\square$

Remark 6.14. (1) According to Point (2) in Remark 6.6, it is to be noticed that Proposition 6.13 corresponds to the result given in [17, Lemma 3.4.2, page 130].

(2) On the other hand, Proposition 6.13 has also been proved in [4, Theorem 3, page 208] in the particular case where V is equal to $\mathbf{R}^n$ and where f is positive outside the origin.

(3) Proposition 6.13 is useful to avoid long computations in differential calculus. For example, if we consider the convex cone $C := (0, +\infty) \times \mathbf{R}$ in $\mathbf{R}^2$, the function $f: C \rightarrow \mathbf{R}$ defined by $f(x, y) := (x^2 + y^2)/(2x)$ is convex since it is non-negative, positively homogeneous and sub-convex (indeed, $S_0(f)$ is empty and $S_1(f)$ is the closed disk in $\mathbf{R}^2$ about $(1, 0)$ with radius 1 less the origin).

Corollary 6.15. *For any convex cone C in a real vector space and any non-negative function $f: C \rightarrow \mathbf{R}$ which is positively homogeneous of degree $\alpha \geq 1$, we have the equivalence:*

$$f \text{ is sub-convex} \iff f \text{ is convex}.$$

Proof. As we only have to prove the implication $\implies$ according to Proposition 6.8, let us assume that f is sub-convex. Therefore, the function $f^{1/\alpha}$ is sub-convex by Corollary 6.12, and hence Proposition 6.13 implies that it is convex since it is positively homogeneous.

Finally, using Proposition 4.10, we get that $f = (f^{1/\alpha})^\alpha$ is convex. $\square$

Remark 6.16. It is to be noticed that the implication $\implies$ is no longer true if we have $\alpha < 1$ since the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(t) := \sqrt{|t|}$ is not convex even though it is sub-convex and positively homogeneous of degree $1/2$.

6.2. Topological aspects of sub-convexity

We shall now deal with the topological notion of strict sub-convexity for functions defined on a general topological real vector space, and show how it is related to continuity and strict convexity when these functions have the extra property of being positively homogeneous.

Definition 6.17. Given a subset C of a topological real vector space, a function $f: C \rightarrow \mathbf{R}$ is said to be *strictly sub-convex* if the sublevel set $S_r(f)$ is strictly convex for any $r \in \mathbf{R}$.

Remark 6.18. (1) According to Point (2) in Remark 4.17, any strictly sub-convex function $f: C \rightarrow \mathbf{R}$ defined on a subset C of a topological real vector space is sub-convex, and hence its domain C is necessarily convex by Point (1) in Remark 6.6.

(2) A function defined on an interval of $\mathbf{R}$ is of course strictly sub-convex if and only if it is sub-convex.

(3) A norm $\|\cdot\|$ on a real vector space V is strictly sub-convex with respect to the topology associated with $\|\cdot\|$ if and only if the normed vector space $(V, \|\cdot\|)$ is “strictly convex” in the sense given in [6, page 108] and [15, page 30]. This is a mere consequence of the equivalence (1) $\iff$ (4) in Proposition 2.3 since in $(V, \|\cdot\|)$ the topological boundary of the unit closed ball is exactly the unit sphere.

(4) It is to be noticed that Proposition 6.7 does not hold when sub-convexity is replaced by strict sub-convexity. In other words, the strict sub-convexity of a function $f: C \rightarrow \mathbf{R}$ defined on a subset C of a topological real vector space cannot be characterized by saying that $f^{-1}((-\infty, r))$ is strictly convex for any $r \in \mathbf{R}$.

* Indeed, if we consider the set $C := \mathbf{R}^2$ and the function $f: C \rightarrow \mathbf{R}$ defined by

$$f(x, y) := \max\{0, \sqrt{2(x^2 + y^2)} - 2y\},$$

the sublevel set $S_0(f) = \{(x, y) \in \mathbf{R}^2 \mid y \geq |x|\}$ is not strictly convex, which shows that f is not strictly sub-convex. Nevertheless, we have

$$f^{-1}((-\infty, r)) = \{(x, y) \in \mathbf{R}^2 \mid (y+r)^2 - x^2 > r^2/2\} \cap [\mathbf{R} \times (-r/2, +\infty)]$$

for any $r \in (0, +\infty)$ and $f^{-1}((-\infty, r)) = \emptyset$ for any $r \in (-\infty, 0]$, which shows that the set $f^{-1}((-\infty, r))$ is strictly convex.

* On the other hand, if we consider the set $C := \mathbf{R}^2$ and the function $f: C \rightarrow \mathbf{R}$ defined by

$$f(x, y) := \min\{0, \sqrt{2(x^2 + y^2)} - 2y\},$$

we get $S_r(f) = \mathbf{R}^2$ for any $r \in (0, +\infty)$ and

$$S_r(f) = \{(x, y) \in \mathbf{R}^2 \mid (y+r)^2 - x^2 \geq r^2/2\} \cap [\mathbf{R} \times [-r/2, +\infty)]$$

for any $r \in (-\infty, 0]$, which shows that f is strictly sub-convex.

Nevertheless, the set $f^{-1}((-\infty, 0)) = \{(x, y) \in \mathbf{R}^2 \mid y > |x|\}$ is not strictly convex.

Let us now give the relationship between strict sub-convexity and strict quasi-convexity.

Proposition 6.19. *For any strictly convex subset C of a topological real vector space V and any continuous function $f: C \rightarrow \mathbf{R}$, we have the following implication:*

$$f \text{ is strictly quasi-convex} \implies f \text{ is strictly sub-convex}.$$

Proof. Assume that f is strictly quasi-convex, fix a number $r \in \mathbf{R}$, and consider two points $x \neq y$ in $\text{rcl}(S_r(f)) \subseteq \text{rcl}(C)$ (see Point (1) in Proposition 4.13).

Then we have $]x, y[\subseteq \text{rint}(C)$ since C is strictly convex.

Moreover, the quasi-convexity of f insures that $S_r(f)$ is convex by Proposition 6.9, which implies that $\text{rcl}(S_r(f))$ is also convex owing to Point (1) in Proposition 4.14, and hence we obtain the inclusion $]x, y[\subseteq \text{rcl}(S_r(f))$.

On the other hand, we compute

$$\begin{aligned} \text{rcl}(S_r(f)) \cap \text{rint}(C) &= \overline{S_r(f)} \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) \\ &= \overline{S_r(f)} \cap C \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) \quad (\text{since one has } \text{rint}(C) \subseteq C) \\ &= S_r(f) \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) \quad (\text{since } S_r(f) \text{ is closed in } C \text{ owing to} \\ &\quad \text{the lower semi-continuity of } f) \\ &= S_r(f) \cap \text{rint}(C) \quad (\text{since the inclusion } S_r(f) \subseteq \text{Aff}(S_r(f)) \text{ holds}), \end{aligned}$$

which proves $]x, y[\subseteq S_r(f) \cap \text{rint}(C)$.

Now, given an arbitrary $\alpha \in (0, 1)$, fix $s \in (0, \alpha)$ and $t \in (\alpha, 1)$, and define the distinct points $x' := (1 - s)x + sy$ and $y' := (1 - t)x + ty$ in V . Then x' and y' are in $]x, y[$, and hence in $S_r(f) \cap \text{rint}(C)$ as shown above.

Moreover, the point $z := (1 - \alpha)x + \alpha y$ belongs to the open line segment $]x', y'[,$ and hence the strict quasi-convexity of f yields $f(z) < \max\{f(x'), f(y')\} \leq r$, which implies $z \in f^{-1}((-\infty, r))$.

Finally, since $\text{rint}(C)$ is open in $\text{Aff}(C)$ and since $f^{-1}((-\infty, r))$ is open in C by the upper semi-continuity of f , there exist open sets U and W in V such that one can write $\text{rint}(C) = U \cap \text{Aff}(C)$ and $f^{-1}((-\infty, r)) = W \cap C$, which yields

$$\begin{aligned} z \in f^{-1}((-\infty, r)) \cap]x, y[&\subseteq f^{-1}((-\infty, r)) \cap \text{rint}(C) \\ &\quad (\text{since we proved }]x, y[\subseteq \text{rint}(C) \text{ above}) \\ &\subseteq f^{-1}((-\infty, r)) \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) \\ &\quad (\text{since we have } f^{-1}((-\infty, r)) \subseteq S_r(f) \subseteq \text{Aff}(S_r(f))) \\ &= (W \cap C) \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) = W \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) \\ &\quad (\text{since the inclusion } \text{rint}(C) \subseteq C \text{ holds}) \\ &= W \cap \text{Aff}(S_r(f)) \cap (U \cap \text{Aff}(C)) \\ &= \Omega := (U \cap W) \cap \text{Aff}(S_r(f)) \\ &\quad (\text{since we have } \text{Aff}(S_r(f)) \subseteq \text{Aff}(C) \text{ from } S_r(f) \subseteq C). \end{aligned}$$

But, using again the second line of the above computations, we have

$$\Omega = f^{-1}((-\infty, r)) \cap \text{Aff}(S_r(f)) \cap \text{rint}(C) \subseteq f^{-1}((-\infty, r)) \subseteq S_r(f),$$

and this yields the inclusion $\Omega \subseteq \text{rint}(S_r(f))$ since Ω is an open set in $\text{Aff}(S_r(f))$, which finally leads to $z \in \text{rint}(S_r(f))$.

Conclusion: since the points $x \neq y$ in $\text{rcl}(S_r(f))$ and $\alpha \in (0, 1)$ have been chosen arbitrarily, the sublevel set $S_r(f)$ is therefore strictly convex. This proves that f is strictly sub-convex since $r \in \mathbf{R}$ is arbitrary. $\square$

Remark 6.20. (1) Let us notice that Proposition 6.19 does not hold if C is not strictly convex as we can see with the non-strictly convex subset $C := [-1, 1] \times [-1, 1]$ of $\mathbf{R}^2$ and the strictly convex function $f: C \rightarrow \mathbf{R}$ defined by $f(x, y) := x^2 + y^2$ for in this case f is continuous and strictly quasi-convex (see Point (3) in Remark 6.2) but the sublevel set $S_2(f) = C$ is not strictly convex.

(2) Moreover, Proposition 6.19 is false if f is not continuous.

Indeed, let us consider the convex subset $C := \{u \in \mathbf{R}^2 \mid \|u\| \leq \sqrt{2}\}$ of $\mathbf{R}^2$, where $\|\cdot\|$ stands for the canonical Euclidean norm on $\mathbf{R}^2$.

Then the function $f: C \rightarrow \mathbf{R}$ defined by $f(u) := \|u\|^2$ for $u \in [-1, 1] \times [-1, 1]$ and $f(u) := \|u\|^2 + 2$ for $u \in C \setminus ([-1, 1] \times [-1, 1])$ is obviously not upper semi-continuous.

Now, given two points v and w in C , there are three cases to be considered.

* If v and w both belong to $[-1, 1] \times [-1, 1]$ or to one of the four connected components of $C \setminus ([-1, 1] \times [-1, 1])$, we have $f((1-t)v + tw) < \max\{f(v), f(w)\}$ for all $t \in (0, 1)$ since f is strictly convex on each of these five convex sets – and hence strictly quasi-convex owing to Point (3) in Remark 6.2.

* If v is in $[-1, 1] \times [-1, 1]$ and w belongs to one of the four connected components of $C \setminus ([-1, 1] \times [-1, 1])$, we have $f((1-t)v + tw) \leq 2 < f(w)$ for all $t \in [0, 1]$, and this yields $f((1-t)v + tw) < \max\{f(v), f(w)\}$ since the first inequality with $t := 0$ implies $f(v) \leq f(w)$.

* If v and w belong to different connected components of $C \setminus ([-1, 1] \times [-1, 1])$, we have $f(u) \leq \|u\|^2 + 2 < \max\{f(v), f(w)\}$ for all $u \in]v, w[$ since the second inequality is given by the strict quasi-convexity of the strictly convex function $\|\cdot\|^2 + 2$ (see Point (3) in Remark 6.2).

Summing up, this proves that f is strictly quasi-convex.

Nevertheless, the function f is not strictly sub-convex since the sublevel set $S_2(f)$ is not strictly convex.

It should also be noticed that the function f is lower semi-continuous (indeed, given any real number α , the set $\{u \in C \mid f(u) \leq \alpha\}$ is obviously closed in C).

(3) On the other hand, the converse of the implication in Proposition 6.19 is not true – even though the function is continuous – as we can see with the zero function defined on the real line.

(4) Replacing “strictly sub-convex” by “strictly convex” in Proposition 6.19 is not possible as one can check with the absolute value function defined on the real line.

Owing to the first part of Point (3) in Remark 6.2 and Proposition 6.19, we get the next result.

Corollary 6.21. *For any strictly convex subset C of a topological real vector space and any continuous function $f: C \rightarrow \mathbf{R}$, we have the following implication:*

$$f \text{ is strictly convex} \implies f \text{ is strictly sub-convex}.$$

Remark 6.22. (1) Point (1) in Remark 6.20 shows that we cannot drop the strict convexity of C in the hypotheses of Corollary 6.21.

(2) Moreover, Corollary 6.21 is false if f is not continuous as we can see by considering the strictly convex subset $C := \mathbf{R}$ of the real vector space $\mathbf{R}$ endowed with the trivial topology and the strictly convex function $f: C \rightarrow \mathbf{R}$ defined by $f(x) := x^2$ since in this case f is not continuous and the sublevel set $S_1(f) = [-1, 1]$ is not strictly convex.

(3) The converse of the implication in Corollary 6.21 is not true since otherwise this would lead to a contradiction by combining Proposition 6.19 and the second part of Point (3) in Remark 6.2.

The next property gives a way for constructing new strictly sub-convex functions from old ones.

Proposition 6.23. *Let C be a strictly convex subset of a topological real vector space, $f: C \rightarrow \mathbf{R}$ a strictly sub-convex function, D a subset of $\mathbf{R}$ which contains $f(C)$ and $\varphi: D \rightarrow \mathbf{R}$ a non-decreasing function which is lower semi-continuous. Then the function $g: C \rightarrow \mathbf{R}$ defined by $g(x) := \varphi[f(x)]$ is strictly sub-convex.*

Proof. Let us fix $r \in \mathbf{R}$.

If we have $S_r(\varphi) = D$, then $S_r(g)$ is reduced to C , which is strictly convex.

Otherwise, there exists $t_0 \in D$ which satisfies $\varphi(t_0) > r$, and hence we get

$$S_r(\varphi) \subseteq \mathbf{R} \setminus [t_0, +\infty) = (-\infty, t_0) \subseteq (-\infty, t_0]$$

since φ is non-decreasing. So, since $S_r(\varphi)$ is bounded from above, let us consider its supremum $a \in \mathbf{R}$.

We first have $S_r(\varphi) \subseteq (-\infty, a]$, which yields $S_r(g) \subseteq S_a(f)$ by using the obvious equality $S_r(g) = f^{-1}(S_r(\varphi))$.

Now, in order to prove the reverse inclusion, there are two cases to be considered depending on whether a belongs to D or not.

* Assume that we have $a \in D$.

Then, since one has $a \in \overline{S_r(\varphi)}^{\mathbf{R}}$, we obtain $a \in \overline{S_r(\varphi)}^D$, which writes $a \in S_r(\varphi)$ since $S_r(\varphi)$ is closed in D by the lower semi-continuity of φ . So we have $\varphi(a) \leq r$, which implies $g(x) = \varphi[f(x)] \leq \varphi(a) \leq r$ for any $x \in S_a(f)$ since φ is non-decreasing, and this proves $S_a(f) \subseteq S_r(g)$.

* Assume that we have $a \notin D$.

Then, for each $x \in S_a(f)$, one gets $f(x) < a$, and hence there exists $b \in S_r(\varphi)$ satisfying $f(x) \leq b \leq a$ by the very definition of a , and this yields $g(x) = \varphi[f(x)] \leq \varphi(b) \leq r$.

This proves the inclusion $S_a(f) \subseteq S_r(g)$.

Conclusion: in both cases, we proved that $S_r(g)$ is equal to the strictly convex set $S_a(f)$. The function g is therefore strictly sub-convex since $r \in \mathbf{R}$ is arbitrary. $\square$

Remark 6.24. (1) It is to be noticed that the strict convexity of C cannot be dropped in the hypotheses of Proposition 6.23.

Indeed, let us consider the open convex subset $C := (-1, 1) \times (-1, 1)$ of $\mathbf{R}^2$, the smooth function $f: C \rightarrow \mathbf{R}$ defined by $f(x, y) := 1/[(1 - x^2)(1 - y^2)]$, the subset $D := [0, +\infty)$ of $\mathbf{R}$ and the function $\varphi: D \rightarrow \mathbf{R}$ defined by $\varphi(t) := t/(t + 1)$.

Next, let us fix any real number $r > 1$, define $\lambda := 1/r \in (0, 1)$ and $a := \sqrt{1 - \lambda} > 0$, and introduce the function $\sigma: [-a, a] \rightarrow \mathbf{R}$ defined by $\sigma(x) := \sqrt{1 - \lambda/(1 - x^2)}$. Then, for any $x \in (-a, a)$, we compute $\sigma'(x) = -\lambda x/[(1 - x^2)^2 \sigma(x)]$, which shows that the derivative of σ is decreasing on $(-a, a)$ (use the fact that σ is a symmetric function which is positive and decreasing on $(0, a)$), and hence that σ is strictly concave. Therefore, the sublevel set $S_r(f) = \{(x, y) \mid |y| \leq \sigma(x)\}$ is strictly convex. On the other hand, for any $r \in (-\infty, 1)$, the sublevel set $S_r(f) = \emptyset$ is also strictly convex.

Finally, since $S_1(f) = \{(0, 0)\}$ is strictly convex too, the function f is strictly sub-convex. Nevertheless, even though φ is non-decreasing and lower semi-continuous (since it is continuous), the function $g: C \rightarrow \mathbf{R}$ defined by $g(x, y) := \varphi[f(x, y)]$ is *not* strictly sub-convex. This is because we have $\varphi(D) \subseteq (-\infty, 1]$, and hence the sublevel set $S_1(g)$ is equal to $f^{-1}(D) = C$, which is not strictly convex.

(2) Moreover, Proposition 6.23 is false if φ is not semi-continuous as we can check by considering the second function f that we used in Point (4) in Remark 6.18 and the non-decreasing function $\varphi := -\mathbf{1}_{(-\infty, 0)}$ defined on $\mathbf{R}$.

Indeed, we then get that φ is not semi-continuous at the origin and $g := \varphi \circ f$ is not strictly sub-convex since $S_{-1}(g) = f^{-1}((-\infty, 0)) = \{(x, y) \in \mathbf{R}^2 \mid y > |x|\}$ is not strictly convex.

From now on, let us focus on functions which are non-negative and positively homogeneous.

Proposition 6.25. *For any convex cone C in $\mathbf{R}^n$ and any non-negative function $f: C \rightarrow \mathbf{R}$ which is positively homogeneous of degree $\alpha > 0$, we have the implication*

$$f \text{ is sub-convex} \implies f \text{ is continuous on } \text{rint}(C).$$

Proof. Since $f^{1/\alpha}$ is sub-convex by Corollary 6.12, it is convex by Corollary 6.15 (indeed, the function $f^{1/\alpha}$ is positively homogeneous of degree one). Now, using Corollary 4.25 in $\text{Aff}(C) \subseteq \mathbf{R}^n$, we get that $f^{1/\alpha}$ is continuous on $\text{rint}(C)$, and the same holds for f . $\square$

Remark 6.26. (1) Proposition 6.25 may be false if $\text{rint}(C)$ is replaced by C .

Indeed, if we consider the convex cone $C := ((0, +\infty) \times \mathbf{R}) \cup \{(0, 0)\}$ in $\mathbf{R}^2$ and the function $f: C \rightarrow \mathbf{R}$ defined by $f(0, 0) := 0$ and $f(x, y) := (x^2 + y^2)/(2x)$ in case when one has $x > 0$, then f is non-negative, positively homogeneous and sub-convex (and even strictly sub-convex since we have $S_0(f) = \{0\}$ and since $S_1(f)$ is the closed disk in $\mathbf{R}^2$ about $(1, 0)$ of radius 1), but it is not continuous at $(0, 0)$ since we have $f(1/n, 1/\sqrt{n}) \rightarrow 1/2 \neq f(0, 0)$ and $(1/n, 1/\sqrt{n}) \rightarrow (0, 0)$ as $n \rightarrow +\infty$.

(2) On the other hand, Proposition 6.25 is no longer true if $\mathbf{R}^n$ is replaced by an arbitrary topological real vector space.

Indeed, if we consider the vector space $V := C^0(\mathbf{R}, \mathbf{R}) \cap \mathcal{L}^2(\mathbf{R}, \mathbf{R}) \subseteq \mathbf{R}^{\mathbf{R}}$ endowed with the topology $\mathcal{T}$ of pointwise convergence (this is nothing else than the product topology) and the function $f: V \rightarrow \mathbf{R}$ defined by $f(u) := \|u\|_2^2$, then f is non-negative, positively homogeneous of degree $\alpha := 2$ and sub-convex (it is even strictly convex since for any $u \in V$ its Hessian at u with respect to the norm $\|\cdot\|_2$ on V is equal to $2\langle \cdot, \cdot \rangle$, which is positive definite).

Nevertheless, f is not continuous with respect to $\mathcal{T}$ for it is not even upper semi-continuous at the origin with respect to $\mathcal{T}$ (since the sequence $(u_n)_{n \geq 1}$ in V defined in [23, Example 1.1, page 797] converges to zero with respect to $\mathcal{T}$ and satisfies $f(u_n) \geq 4 > 0 = f(0)$ for any integer $n \geq 1$).

We shall now prove that the situation described in Point (2) in Remark 6.26 does not appear when sub-convexity is replaced by *strict* sub-convexity.

In order to do this, the following easy-to-prove result will be useful.

Lemma 6.27. *Let C be a convex subset of a topological real vector space, $f: C \rightarrow \mathbf{R}$ a non-negative function and $\alpha > 0$ a real number. Then we have the equivalence*

$$f^\alpha \text{ is strictly sub-convex} \quad \Longleftrightarrow \quad f \text{ is strictly sub-convex}.$$

Proposition 6.28. *Given a convex cone C in a topological real vector space and a non-negative function $f: C \rightarrow \mathbf{R}$ which is positively homogeneous of degree $\alpha > 0$, we have the implication*

$$f \text{ is strictly sub-convex} \quad \implies \quad f \text{ is continuous on } \text{rint}(C).$$

Proof. Assume that $\text{rint}(C)$ is not empty (since otherwise there is nothing to prove) and that f is strictly sub-convex. First, notice that we have $\text{Aff}(C) = \text{Aff}(\text{rint}(C))$ by Point (2)(b) in Proposition 4.13 with $S := C$. Now, in case when C is not equal to $\{0\}$, we have $\text{Cone}_b(C) = C \setminus \{0\}$ and $\text{Cone}(C) = C \cup \{0\}$ by Point (4) in Remark 3.6 with $A := V$, which yields the two equalities $\text{Aff}(C \setminus \{0\}) = \text{Vect}(C)$ and $\text{Aff}(C \cup \{0\}) = \text{Vect}(C)$ according to Points (1) and (2) in Proposition 3.13 with $S := C$.

Therefore, the inclusions $C \setminus \{0\} \subseteq C \subseteq C \cup \{0\}$ imply $\text{Aff}(C) = W := \text{Vect}(C)$, and this latter equality still holds in case when we have $C = \{0\}$. Since f is sub-convex by Point (1) in Remark 6.18, the positively homogeneous function $g := f^{1/\alpha}$ is also sub-convex by Corollary 6.12, and hence it is convex owing to Proposition 6.13.

On the other hand, since C is not empty, we have $\text{Aff}(S_1(g)) = W$ by Point (3) in Corollary 3.21, and this implies that the interior of $S_1(g)$ in W is nothing else than $U := \text{rint}(S_1(g))$. Now, according to Lemma 6.27, the function g is strictly sub-convex, and hence the sublevel set $S_1(g)$ is strictly convex.

Therefore, since $S_1(g)$ is not empty by Point (3) in Proposition 3.18, we get $U \neq \emptyset$ by Proposition 4.18 applied with the strictly convex subset $S_1(g)$ of W .

Finally, since g is bounded from above (by the constant 1) on the non-empty open subset U of the open convex subset $\text{rint}(C)$ of W , it is continuous on $\text{rint}(C)$ by Theorem 4.24, and this proves that f is continuous on $\text{rint}(C)$ too. $\square$

Remark 6.29. It is to be noticed that Proposition 6.28 is no longer true if $\text{rint}(C)$ is replaced by C as Point (1) in Remark 6.26 shows.

A straightforward consequence of Proposition 6.28 when C is the whole space is the following result.

Corollary 6.30. *Given a topological real vector space V and a non-negative function $f: V \rightarrow \mathbf{R}$ which is positively homogeneous of degree $\alpha > 0$, we have the implication*

$$f \text{ is strictly sub-convex} \implies f \text{ is continuous}.$$

Let us now give the relationship between strict sub-convexity and strict convexity for positively homogeneous functions.

Proposition 6.31. *Given a convex cone C in a topological real vector space and a non-negative positively homogeneous function $f: C \rightarrow \mathbf{R}$, we have the implication*

$$\begin{aligned} & f \text{ is strictly sub-convex and satisfies } f^{-1}(0) \subseteq \{0\} \\ \implies & f^\alpha \text{ is strictly convex for any real number } \alpha > 1. \end{aligned}$$

Proof. Assume that C is not empty (since otherwise the implication to be proved is trivial) and that f is strictly sub-convex and satisfies $f^{-1}(0) \subseteq \{0\}$, fix a real number $\alpha > 1$, and pick two distinct points $x, y \in C$.

There are two cases to be considered depending on whether $f(x)$ and $f(y)$ are equal or not.

* Case when we have $f(x) \neq f(y)$.

According to Point (1) in Remark 6.18 and Corollary 6.15, the function f is convex, which yields $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$ for any $t \in (0, 1)$, and hence we obtain $f^\alpha((1-t)x + ty) \leq [(1-t)f(x) + tf(y)]^\alpha < (1-t)f^\alpha(x) + tf^\alpha(y)$ since the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by $\varphi(s) := s^\alpha$ is non-decreasing and strictly convex.

* Case when we have $f(x) = f(y)$.

The property $f^{-1}(0) \subseteq \{0\}$ together with $x \neq y$ imply $r := f(x) > 0$, and hence we can write $x, y \in S_r(f) \subseteq \text{rel}(S_r(f))$, which yields $(1-t)x + ty \in \text{rint}(S_r(f))$ for any $t \in (0, 1)$ since f is strictly sub-convex.

Now, we have $\widehat{S_1(f)} = S_1(f) \cup \{0\}$ by Point (3) in Proposition 3.18, and hence $(p_{\widehat{S_1(f)}})|_C = j \circ f$ owing to Corollary 3.27 with $S := \widehat{S_1(f)}$, which immediately yields $(p_{S_1(f)})|_C = j \circ f$ since one has $p_{\widehat{S_1(f)}} = p_{S_1(f)}$ according to Point (4) in Proposition 3.24 with $S := S_1(f)$ and $V := \text{Vect}(C)$. Therefore, we get the equalities $p_{S_1(f)}^{-1}([0, 1]) \cap C = S_1(f)$ and $p_{S_1(f)}^{-1}([0, 1]) \cap C = f^{-1}([0, 1])$. As a consequence, the first equality implies $S_1(f) \subseteq p_{S_1(f)}^{-1}([0, 1])$, and hence we obtain the inclusion

$$\text{Aff}(S_1(f)) \subseteq \text{Aff}(p_{S_1(f)}^{-1}([0, 1])),$$

which leads to $\text{Vect}(C) \subseteq \text{Aff}(p_{S_1(f)}^{-1}([0, 1]))$ since one has $\text{Aff}(S_1(f)) = \text{Vect}(C)$ by Point (3) in Corollary 3.21.

On the other hand, using Point (1) in Proposition 3.24 with $S := S_1(f)$, Point (2) in Remark 3.6 and Proposition 3.9 with $S := C$, one can write

$$\begin{aligned} p_{S_1(f)}^{-1}([0, 1]) &\subseteq p_{S_1(f)}^{-1}([0, +\infty)) \subseteq \text{Cone}_b(S_1(f)) \cup \{0\} = \text{Cone}(S_1(f)) \subseteq \text{Cone}(C) \\ &\subseteq \text{Vect}(\text{Cone}(C)) = \text{Vect}(C) \end{aligned}$$

since we have $S_1(f) \subseteq C$, and this yields

$$\text{Aff}(p_{S_1(f)}^{-1}([0, 1])) \subseteq \text{Aff}(\text{Vect}(C)) \subseteq \text{Vect}(\text{Vect}(C)) = \text{Vect}(C) .$$

Therefore, we have obtained $\text{Aff}(p_{S_1(f)}^{-1}([0, 1])) = \text{Vect}(C)$.

The inclusion $S_1(f) \subseteq p_{S_1(f)}^{-1}([0, 1])$ then implies $\text{rint}(S_1(f)) \subseteq \text{rint}(p_{S_1(f)}^{-1}([0, 1]))$ (notice that both interiors are computed in $\text{Vect}(C)$), and hence

$$\text{rint}(S_1(f)) \subseteq \text{rint}(p_{S_1(f)}^{-1}([0, 1])) \cap C$$

since we have $\text{rint}(S_1(f)) \subseteq S_1(f) \subseteq C$. Now, according to Point (2) in Lemma 3.30 with $S := \widehat{S_1(f)}$ and $V := \text{Vect}(C)$, we can write $\text{rint}(p_{S_1(f)}^{-1}([0, 1])) \subseteq p_{S_1(f)}^{-1}([0, 1])$ since one has $p_{\widehat{S_1(f)}} = p_{S_1(f)}$, which implies

$$\text{rint}(S_1(f)) \subseteq p_{S_1(f)}^{-1}([0, 1]) \cap C = f^{-1}([0, 1]) .$$

The positive homogeneity of f and Point (3) in Proposition 3.19 then yield

$$\text{rint}(S_r(f)) = r \cdot \text{rint}(S_1(f)) \subseteq r \cdot f^{-1}([0, 1]) = f^{-1}([0, r])$$

given that the homothety with ratio r is a topological vector space automorphism which preserves $\text{Vect}(C)$.

Finally, one obtains $f^\alpha((1-t)x + ty) < r^\alpha = f^\alpha(x) = (1-t)f^\alpha(x) + tf^\alpha(y)$ for any $t \in (0, 1)$ since the function $\varphi: [0, +\infty) \rightarrow \mathbf{R}$ defined by $\varphi(s) := s^\alpha$ is increasing. $\square$

Remark 6.32. (1) Proposition 6.31 is not true without the assumption $f^{-1}(0) \subseteq \{0\}$ as we can easily check when considering the zero function defined on the real line.

(2) It is to be noticed that the converse of the implication in Proposition 6.31 is not true. Indeed, let us consider the vector space $V := C^0(\mathbf{R}, \mathbf{R}) \cap \mathcal{L}^2(\mathbf{R}, \mathbf{R}) \subseteq \mathbf{R}^{\mathbf{R}}$ endowed with its natural scalar product $\langle \cdot, \cdot \rangle$ whose associated norm is denoted by $\|\cdot\|_2$. Then the function $f: V \rightarrow \mathbf{R}$ defined by $f(u) := \|u\|_2$ is non-negative and positively homogeneous.

Moreover, f^α is strictly convex for any real number $\alpha > 1$ according to Point (2) in Remark 4.9.

Nevertheless, f is not strictly sub-convex when V is endowed with the topology $\mathcal{T}$ of pointwise convergence (this is nothing else than the product topology) since if it were the case the function f would be continuous with respect to $\mathcal{T}$ by Corollary 6.30.

But f is not even upper semi-continuous at the origin with respect to $\mathcal{T}$ since the sequence $(u_n)_{n \geq 1}$ in V defined in [23, Example 1.1, page 797] converges to zero with respect to $\mathcal{T}$ and satisfies $f(u_n) \geq 4 > 0 = f(0)$ for any integer $n \geq 1$.

Finally, if we sum up all the previous results, we eventually get the following useful property.

Theorem 6.33. *Let V be a topological real vector space and $f: V \rightarrow \mathbf{R}$ a non-negative function which is positively homogeneous. Then for any real number $\alpha > 1$, the following conditions are equivalent:*

- (1) *f is continuous and f^α is strictly convex.*
- (2) *f is continuous and strictly quasi-convex.*
- (3) *f is strictly sub-convex and we have $f^{-1}(0) = \{0\}$.*

Proof. (1) $\Rightarrow$ (2) Assume that Point (1) is satisfied. Then Point (2) is a mere consequence of Proposition 6.4 with $C := V$.

(2) $\Rightarrow$ (3) Assume that Point (2) is satisfied. Then the function f is strictly sub-convex by Proposition 6.19 with $C := V$.

On the other hand, Remark 3.15 with $C := V$ and Point (4) in Remark 6.2 yield the equality $f^{-1}(0) = \{0\}$.

(3) $\Rightarrow$ (1) Assume that Point (3) is satisfied. Then Corollary 6.30 implies that the function f is continuous since it is positively homogeneous of degree one.

Moreover, Proposition 6.31 with $C := V$ shows that f^α is strictly convex. $\square$

Remark 6.34. (1) It is to be pointed out that if Point (1) in Theorem 6.33 is satisfied for some real number $\alpha > 1$ (for instance $\alpha := 2$), then it is satisfied for any real number $\alpha > 1$.

(2) In the particular case where V is equal to $\mathbf{R}^n$ equipped with the usual topology, the continuity in Point (1) in Theorem 6.33 can be dropped since the strict convexity of f^α implies its convexity by Point (1) in Remark 4.9, and hence that f is continuous according to Corollary 4.25.

(3) On the other hand, when V is as in the previous point, the continuity in Point (2) in Theorem 6.33 is useless since the strict quasi-convexity of f implies its quasi-convexity by Point (1) in Remark 6.2, and hence its sub-convexity by Proposition 6.9. Therefore, the positive homogeneity of f and Proposition 6.13 insure that f is convex, and hence continuous according to Corollary 4.25.

Now, a natural question is to know whether Theorem 6.33 holds in the more general framework when the function f is defined on a non-empty convex cone C in V and when the condition $f^{-1}(0) = \{0\}$ in Point (3) is replaced by $f^{-1}(0) \subseteq \{0\}$. Unfortunately, this is actually not always true as we can check with the open convex cone $C := \{(x, y) \in \mathbf{R}^2 \mid y > |x|\}$ in $V := \mathbf{R}^2$ and the function $f: C \rightarrow \mathbf{R}$ defined by $f(x, y) := \sqrt{x^2 + y^2}$ which satisfies $f^{-1}(0) = \emptyset \subseteq \{0\}$. Indeed, f^2 is strictly convex since its Hessian is positive definite, and hence f is strictly quasi-convex by Proposition 6.4. Moreover, f is non-negative, positively homogeneous and continuous.

Nevertheless, since the boundary of $S_1(f) = C \cap \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ in $\mathbf{R}^2$ contains the line segment $[(0, 0), (1, 1)]$, the function f is not strictly sub-convex. This shows that even if the condition $f^{-1}(0) = \{0\}$ in Point (3) in Theorem 6.33 is replaced by $f^{-1}(0) \subseteq \{0\}$, the implication Point (2) $\implies$ Point (3) is not satisfied by f .

However, in the case where the convexity of C is replaced by the *strict* convexity, Theorem 6.33 still holds.

Corollary 6.35. *Let C be a non-empty strictly convex cone in a topological real vector space V and $f: C \rightarrow \mathbf{R}$ a non-negative function which is positively homogeneous. Then for any real number $\alpha > 1$, the following conditions are equivalent:*

- (1) *f is continuous and f^α is strictly convex.*
- (2) *f is continuous and strictly quasi-convex.*
- (3) *f is strictly sub-convex and we have $f^{-1}(0) \subseteq \{0\}$.*

Proof. Let us consider the vector subspace $W := \text{Vect}(C)$ of V . Owing to Proposition 4.18, $\text{rint}(C)$ is not empty, and hence Proposition 4.21 implies that either C is equal to W or $C \cup \{0\}$ is a ray and W is one-dimensional.

* In case when C is equal to W , we may apply Theorem 6.33 with W instead of V , and then obtain the equivalence of Points (1), (2) and (3) since the inclusion $\{0\} \subseteq f^{-1}(0)$ already holds by Remark 3.15 with $C := V$.

* Now, in case when $C \cup \{0\}$ is a ray and W is one-dimensional, the situation is the same as if we had $W = \mathbf{R}$ together with $C \cup \{0\} = [0, +\infty)$ and if the function $f: C \rightarrow \mathbf{R}$ were defined by $f(t) := at$, where a is a non-negative real constant.

On the other hand, since the open set $\text{rint}(C)$ in W is not empty and since we have $\text{rint}(C) \subseteq C \neq W$, the topology on W is not trivial, and hence it is equal to the usual topology on $\mathbf{R}$ according to Point (4) in Remark 3.29. Moreover, if a is positive, then Points (1), (2) and (3) are all satisfied by f , and if we have $a = 0$ then none of these points are satisfied by f . This proves that Points (1), (2) and (3) are equivalent. $\square$

7. Application to Minkowski norms

So far, we have been dealing with positively homogeneous functions, and we obtained accurate relationships between strict convexity and strict sub-convexity, that is, between geometric and topological aspects of these functions.

We shall now apply all these properties to the particular case of Minkowski norms in order to characterize those ones which are strictly convex and to give a generalization of a result that Carothers proved for normed vector spaces (see [6, Theorem 11.1, page 110]).

The first characterization of strict convexity for Minkowski norms on arbitrary topological real vector spaces is stated as follows.

Theorem 7.1. *Given a topological real vector space V and a Minkowski norm N on V , the following properties are equivalent:*

- (1) *N is strictly sub-convex.*
- (2) *N is continuous, and for any two vectors $x \neq y$ in V satisfying $N(x) = N(y) = 1$, we have the inequality $N((x+y)/2) < 1$.*

Proof. First of all, $S := N^{-1}([0, 1]) = S_1(N)$ is a star-shaped subset of V by Point (2) in Proposition 3.18 with $C := V$ and $f := N$ since $S_1(N)$ contains the origin, and hence the gauge function p_S of S satisfies $p_S(x) = N(x)$ for any $x \in V$ owing to Proposition 3.26 with $C := V$.

Moreover, notice that we have $\text{Aff}(S) = V$ by Proposition 3.20 with $f := N$, which yields $\text{rint}(S) = \mathring{S}$, $\text{rl}(S) = \overline{S}$ and $\text{rb}(S) = \overline{S} \setminus \mathring{S} = \text{bd } S$.

(1) $\Rightarrow$ (2) Assume that Point (1) is satisfied, and let $x \neq y$ be two vectors in V such that we have $N(x) = N(y) = 1$. Then Corollary 6.30 with $\alpha := 1$ insures that N is continuous, which implies that we have both $\overset{\circ}{S} = N^{-1}([0, 1))$ and $\text{bd } S = N^{-1}(1)$ by Points (2)(b) and (2)(c) in Proposition 3.31.

Since we have $x, y \in N^{-1}(1) = \text{bd } S \subseteq \overline{\overset{\circ}{S}} = \text{rcl}(S)$ and since S is strictly convex, the open line segment $]x, y[$ lies in $\text{rint}(S) = \overset{\circ}{S} = N^{-1}([0, 1))$. Hence, we get in particular $N((x + y)/2) < 1$ since the condition $x \neq y$ implies $(x + y)/2 \in]x, y[$.

(2) $\Rightarrow$ (1) Suppose that Point (2) is true, and let us first prove that S is strictly convex. Since N is continuous, we have both $\overset{\circ}{S} = N^{-1}([0, 1))$ and $\text{bd } S = N^{-1}(1)$ by Points (2)(b) and (2)(c) in Proposition 3.31. Then, given $x, y \in \text{bd } S$, one has $N(x) = N(y) = 1$, which leads to $N((x + y)/2) < 1$. So this inequality is equivalent to $(x + y)/2 \in N^{-1}([0, 1)) = \overset{\circ}{S}$, and hence we get $]x, y[\cap \overset{\circ}{S} \neq \emptyset$.

On the other hand, since N is positively homogeneous and sub-additive, it is convex by Proposition 5.3 with $C := V$, which implies that S is a convex subset of V as the pre-image by N of the convex subset $[0, 1]$ of $\mathbf{R}$. Therefore, using the implication in Remark 4.15, we get $]x, y[\subseteq \overset{\circ}{S} = \text{rint}(S)$. So, owing to Point (4) in Remark 4.17, we have proved that S is strictly convex.

Now, for any $r \in (0, +\infty)$, we have $S_r(N) = rS$ by Point (3) in Proposition 3.19 with $C := V$, $f := N$ and $\alpha := 1$, which shows that $S_r(N)$ is also strictly convex since the homothety of V with ratio r is an affine homeomorphism. Moreover, since we have $N^{-1}(0) = \{0\}$, the sublevel set $S_0(N)$ reduces to $\{0\}$, and hence is strictly convex. Finally, for any $r \in (-\infty, 0)$, the sublevel set $S_r(N)$ is empty, and hence is strictly convex too. This proves that N is strictly sub-convex. $\square$

Remark 7.2. (1) When the Minkowski norm N in Theorem 7.1 is reduced to a norm and when V is endowed with the topology associated with N , then we obtain a generalization of the characterization of “strictly convex” normed vector spaces (see [6, page 108] and [15, page 30] for the definition) given by Point (i) in [6, Theorem 11.1, page 110]. This is indeed a consequence of Point (3) in Remark 6.18.

(2) It is to be noticed that if we drop continuity in Point (2) in Theorem 7.1, the implication (2) $\Rightarrow$ (1) is no longer true.

Indeed, if we consider the vector space $V := C^0(\mathbf{R}, \mathbf{R}) \cap \mathcal{L}^2(\mathbf{R}, \mathbf{R}) \subseteq \mathbf{R}^{\mathbf{R}}$ endowed with the topology $\mathcal{T}$ of pointwise convergence (this is nothing else than the product topology) and the function $f: V \rightarrow \mathbf{R}$ defined by $f(u) := \|u\|_2^2$, then f is not continuous with respect to $\mathcal{T}$ for it is not even upper semi-continuous at the origin with respect to $\mathcal{T}$ (since the sequence $(u_n)_{n \geq 1}$ in V defined in [23, Example 1.1, page 797] converges to zero with respect to $\mathcal{T}$ and satisfies $f(u_n) \geq 4 > 0 = f(0)$ for any integer $n \geq 1$).

On the other hand, f is strictly convex since for any $u \in V$ its Hessian at u with respect to the norm $\|\cdot\|_2$ on V is equal to $2\langle \cdot, \cdot \rangle$, which is positive definite.

Now, let us consider the Minkowski norm N on V defined by $N(u) := \|u\|_2$.

Then, for any two vectors $u \neq v$ in V which satisfy $N(u) = N(v) = 1$, we get

$$f((u + v)/2) < [f(u) + f(v)]/2 = 1$$

by the strict convexity of f , and hence $N((u + v)/2) < 1$ holds since we have $f = N^2$.

However, if N were strictly sub-convex, then it would be continuous by Proposition 6.28 with $C := V$, $f := N$ and $\alpha := 1$, and hence the function $N^2 = \|\cdot\|_2^2$ would be continuous too, which is not the case as shown above.

(3) Even though all the norms on a real vector space are not strictly sub-convex, it is well known that any norm which gives rise to a separable Banach space is actually equivalent to a smooth and strictly sub-convex norm (see for example [15, page 33]). Therefore, since there are many separable Banach spaces (especially in functional analysis), this shows that there is a lot of Banach spaces whose norm is strictly sub-convex. This is a good reason for which strictly sub-convex norms are worth being studied, not to mention the fact that it is more convenient to deal with such norms.

The second characterization of strict convexity for Minkowski norms on arbitrary topological real vector spaces is given by the following result.

Theorem 7.3. *Let V be a topological real vector space and $N: V \rightarrow \mathbf{R}$ a function. Then for any real number $\alpha > 1$, the following conditions are equivalent:*

- (1) *N is strictly sub-convex and is a Minkowski norm on V .*
- (2) *N is non-negative, positively homogeneous, continuous and the function N^α is strictly convex.*

Of course, once N satisfies Point (2), it automatically satisfies both Points (2) and (3) in Theorem 6.33.

Proof. Let us fix a real number $\alpha > 1$.

(1) $\Rightarrow$ (2) Assume that Point (1) is satisfied. Since we have $N^{-1}(0) = \{0\}$, the function N is continuous and N^α is strictly convex owing to the implication (3) $\Rightarrow$ (1) in Theorem 6.33.

(2) $\Rightarrow$ (1) Assume that Point (2) is satisfied. Then N is sub-convex by Point (1) in Remark 6.18 with $C := V$, and hence it is convex owing to Proposition 6.13 with $C := V$. Therefore, Proposition 5.3 with $C := V$ implies that N is sub-additive.

On the other hand, the strict sub-convexity of N and the equality $N^{-1}(0) = \{0\}$ are a consequence of the implication (1) $\Rightarrow$ (3) in Theorem 6.33. $\square$

Remark 7.4. (1) When the function N in Theorem 7.3 is reduced to a norm and when V is endowed with the topology associated with N , then we obtain a generalization of the characterization of “strictly convex” normed vector spaces (see [6, page 108] and [15, page 30] for the definition) given by Point (ii) in [6, Theorem 11.1, page 110]. This is indeed a consequence of Point (3) in Remark 6.18.

(2) It is to be noticed that a function $N: V \rightarrow \mathbf{R}$ which satisfies the equivalent conditions in Theorem 7.3 is a continuous Minkowski norm on V , which shows that the topology $\mathcal{T}$ defined by N (see Section 1) is coarser than the vector space topology on V .

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