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Lifting Approach to Integral Representations of Rich Multimeasures with Values in Banach Spaces I

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Let (Ω, Σ, μ) be a complete probability space and M be a μ-continuous multimeasure of σ-finite variation with values in the family of non-empty closed convex subsets of a Banach space X. I prove that if M is rich in countably additive selections possessing strongly measurable and Pettis integrable densities, then there exists an Effros measurable multifunction that is a Pettis integrable density of M. The above assumptions are in particular satisfied in case of X with RNP. In particular, if X has RNP and Γ is a multifunction that is Pettis integrable in the family of non-empty closed convex and bounded subsets of X, then there exists an Effros measurable multifunction that is scalarly equivalent to Γ. The paper is a continuation of a previous paper of the author [*Multimeasures with values in conjugate Banach spaces and the Weak Radon-Nikodým Property*, J. Convex Analysis 28/3 (2021) 879–902], where it has been proven that if X^* has WRNP, then a multimeasure as above but with values in X^* can be represented as a Pettis integral of a multifunction with closed bounded and convex values that is Effros measurable with respect to weak* open sets.

Keywords: Measurable multimeasures, rich multimeasures, Radon-Nikodym property, integral representations, lifting.

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1. Introduction

The current work concerns multimeasures of σ-finite variation with values in the family of non-empty closed convex subsets of a Banach space X. The multimeasures under consideration are assumed to be rich in countably additive selections possessing strongly measurable and Pettis integrable density. In particular, X may have RNP. I prove that if a multimeasure M is absolutely continuous with respect to a finite measure, then it can be represented as a Pettis integral of a multifunction Γ that is Effros measurable with respect to norm open sets (Theorem 3.8). In particular, if Γ is Pettis integrable and its integral is rich in strongly measurable desities, then Γ is scalarly equivalent to a measurable multifunction. The measurability of multifunctions with respect to norm open sets is usually called measurability and is widely assumed in several investigations of multifunctions. Its utility follows from the fact that in case of a separable range space X it is equivalent with the Castaing representation of the multifunction via a sequence of selections. If the values of a multimeasure are non-separable, such a representation by a countable collection of selections is impossible.

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Earlier treatments of the existence problem of the Radon-Nikodỳm derivative can be found in [2] and still earlier in [4, 5, 6, 14, 22].

I proved recently in [20, Theorem 5.10] that X^* has the WRNP if and only if for every complete probability space (Ω, Σ, μ) and every μ-continuous multimeasure $M\colon \Sigma \to cw^*k(X^*)$, $[cb(X^*)]$ (non-empty weak*-compact [closed] and convex subsets of X^*) of σ-finite variation there exists a Pettis integrable multifunction $\Gamma\colon \Omega \to cw^*k(X^*)$, $[cb(X^*)]$ such that $M(E) = (Pettis)\int_E \Gamma\, d\mu$. I present a more precise formulation of that result in Section 4.

A presentation of the general theory of Pettis integration of vector functions can be found in [16], [17] and [23]. For the general theory of Pettis integrable multifunctions I recommend [1, 10, 15, 18, 19].

2. Preliminaries

Throughout (Ω, Σ, μ) is a complete probability space. $\overline{\int}$ and $\underline{\int}$ denote respectively the upper and lower integral. ρ is an arbitrary lifting on $L_\infty(\mu)$ and on (Ω, Σ, μ) (formally these are two different liftings but each of them uniquely determines the other). The family of all liftings on $L_\infty(\mu)$ is denoted by $\Lambda(\mu)$. X is a Banach space with its dual X^* and the closed unit ball of X is denoted by B_X. If $A \subset X^*$, then $\overline{A}^*$ is the weak*-closure of A and $\overline{A}$ is its norm closure. If $A \subset X$, then $\overline{A}^{**}$ is the weak*-closure of A in X^{**}. If $A \subset X$ is nonempty, then we write $|A| := \sup\{\|x\| : x \in A\}$.

$c(X)$ denotes the collection of all non-empty closed convex subsets of X, $cb(X)$ is the collection of all bounded elements of $c(X)$, $cwk(X)$ denotes the family of all weakly compact elements of $cb(X)$ and $cw^*k(X^*)$ denotes the family of all non-empty convex and weak*-compact subsets of X^*. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle : \ x \in C\}$, for each $x^* \in X^*$.

A function $f : \Omega \to \mathbb{R}$ is *quasi-integrable* (cf. [21]), if the integral $\int_\Omega f\, d\mu$ exists. A map $\Gamma : \Omega \to c(X)$ is called a *multifunction*. Γ is said to be *scalarly measurable* (*scalarly integrable, scalarly quasi-integrable*) if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable (integrable, quasi-integrable).

A multifunction $\Gamma : \Omega \to c(X^*)$ is said to be *weak***-scalarly measurable* (*weak***-scalarly integrable, weak***-scalarly quasi-integrable*) if for every $x \in X$, the map $s(x, \Gamma(\cdot))$ is measurable (integrable, quasi-integrable).

Multifunctions $\Delta, \Gamma : \Omega \to cb(X)[cb(X^*)]$ are *scalarly (weak** *scalarly) equivalent*, if for each $x^* \in X^*$ $[x \in X]$ we have $s(x^*, \Delta) = s(x^*, \Gamma)$ $[s(x, \Delta) = s(x, \Gamma)]$ a.e. (the exceptional sets depend on functionals). We write then $\Delta \overset{s}{\approx} \Gamma$ $[\Delta \overset{w^*s}{\approx} \Gamma]$. We write $\Delta \subset \Gamma$, if $\Delta(\omega) \subset \Gamma(\omega)$ for every $\omega \in \Omega$.

A multifunction $\Gamma : \Omega \to c(X^*)$ $[\Gamma : \Omega \to c(X)]$ is *weak***-scalarly bounded* [*scalarly bounded*] if there is a constant $M > 0$ such that for every $y \in X$ $[y \in X^*]$

$$|s(y, \Gamma)| \le M\|y\| \qquad a.e.$$

A multifunction $\Gamma : \Omega \to c(X^*)$ is called *weak*-open measurable* (Effros measurable with respect to weak*-open sets), if for every weak*-open $U \subset X^*$, we have

$$\{\omega \in \Omega \colon U \cap \Gamma(\omega) \neq \emptyset\} \in \Sigma\,. \tag{1}$$

Following the generally accepted terminology $\Gamma : \Omega \to c(X)$ is called *measurable* if (1) is fulfilled for norm open sets $U \subset X$. One can easily see that measurability of Γ yields its scalar measurability.

A function $\gamma : \Omega \to X$ is called a *selection of* Γ, if for almost every $\omega \in \Omega$, one has $\gamma(\omega) \in \Gamma(\omega)$. $\gamma : \Omega \to X$ ($\gamma : \Omega \to X^{**}$) is called a *quasi-selection (see [18]) of* Γ (a *weak** quasi-selection of* Γ), if γ is scalarly measurable (weak* scalarly measurable) and for each $x^* \in X^*$ one has $\langle x^*, \gamma(\omega)\rangle \in \langle x^*, \Gamma(\omega)\rangle$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x^*). The collection of all quasi-selections of Γ (weak** quasi-selections of Γ) will be denoted by $\mathcal{QS}_\Gamma$ ($\mathcal{W}^{**}\mathcal{QS}_\Gamma$). A scalarly measurable $\gamma\colon \Omega \to X$ (a weak* scalarly measurable $\gamma\colon \Omega \to X^{**}$) is a (weak**) quasi-selection of $\Gamma : \Omega \to c(X)$ if and only if for each $x^* \in X^*$

$$\langle x^*, \gamma\rangle \le s(x^*, \Gamma) \quad a.e.$$

A function $\gamma : \Omega \to X^*$ is said to be a *weak*-quasi-selection* of a multifunction $\Gamma : \Omega \to c(X^*)$ if γ is weak*-scalarly measurable and we have $\langle x, \gamma(\omega)\rangle \in \langle x, \Gamma(\omega)\rangle$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x). The collection of all weak*-quasi-selections of Γ will be denoted by $\mathcal{W}^*\mathcal{QS}_\Gamma$. A weak*-scalarly measurable $\gamma\colon\Omega \to X^*$ is a weak*-quasi-selection of $\Gamma\colon\Omega \to c(X^*)$ if and only if for each $x \in X$

$$\langle x, \gamma\rangle \le s(x, \Gamma) \quad a.e. \tag{2}$$

One can easily check that if $\gamma : \Omega \to X$, then $\gamma \in \mathcal{QS}_\Gamma$ if and only if $\gamma \in \mathcal{W}^*\mathcal{QS}_{\Gamma^{**}}$, where $\Gamma^{**}(\omega) := \overline{\Gamma(\omega)}^{**} \subset X^{**}$.

If $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$ ($\gamma \in \mathcal{QS}_\Gamma$) is Pettis integrable, then we will write it as $\gamma \in \mathcal{W}^*\mathcal{QS}^P_\Gamma$ ($\gamma \in \mathcal{QS}^P_\Gamma$).

If $\gamma : \Omega \to X^*$ is a weak*-scalarly measurable and weak*-scalarly bounded function, then $\rho(\gamma) : \Omega \to X^*$ is defined by $\langle x, \rho(\gamma)(\omega)\rangle := \rho(\langle x, \gamma\rangle)(\omega)$ (see [13, ch. IV.5]).

If $\gamma : \Omega \to X$ is scalarly bounded and scalarly measurable, then $\rho(\gamma) : \Omega \to X^{**}$ is defined by $\langle x^*, \rho(\gamma)(\omega)\rangle := \rho(\langle x^*, \gamma\rangle)(\omega)$.

A map $M : \Sigma \to c(X)$ is *additive*, if $M(A \cup B) = M(A) \oplus M(B) := \overline{M(A) + M(B)}$ for each pair of disjoint elements of Σ. An additive map $M : \Sigma \to c(X)$ is called a *multimeasure* if $s(x^*, M(\cdot)) : \Sigma \to (-\infty, +\infty]$ is a measure, for every $x^* \in X^*$. $M : \Sigma \to c(X^*)$ is called a *weak*-multimeasure* if $s(x, M(\cdot)) : \Sigma \to (-\infty, +\infty]$ is a measure, for every $x \in X$. One can easily see that if M is a weak*-multimeasure (multimeasure), and if for $y \in X(X^*)$ the equality $s(y, M) = s(y, M)^+ - s(y, M)^-$ is the Jordan decomposition of $s(y, M)$, then

$$s(y, M(\Omega))^- < \infty. \tag{3}$$

If $M : \Sigma \to c(X)$, then $M^{**} : \Sigma \to c(X^{**})$ is defined by $M^{**}(E) = \overline{M(E)}^{**}$, for every $E \in \Sigma$. If M is a multimeasure, then M^{**} is a weak*-multimeasure. If $M : \Sigma \to c(X^*)$ is a weak*-multimeasure, then $M^* : \Sigma \to c(X^*)$ defined by $M^*(E) = \overline{M(E)}^{*}$ is also a weak*-multimeasure.

If M is a point map, then we talk about weak*-measure and measure, respectively.

If $m : \Sigma \to X^*$ is a weak*-measure such that $m(A) \in M(A)$, for every $A \in \Sigma$, then m is called a *weak**-*selection* of M. $\mathcal{W}^*\mathcal{S}(M)$ will denote the set of all weak*-countably additive selections of M. Similarly, a vector measure $m : \Sigma \to X$ such that $m(A) \in M(A)$, for every $A \in \Sigma$, is called a *selection* of M. $\mathcal{S}(M)$ will denote the set of all countably additive selections of M.

If M is a weak*-multimeasure or a multimeasure, then M is called *absolutely continuous with respect to* μ (we write then $M \ll \mu$), if $\mu(E) = 0$ yields $M(E) = \{0\}$. M is *dominated by* μ if there exists $\alpha > 0$ such that $|M(E)| \le \alpha\mu(E)$, for every $E \in \Sigma$.

If M is a weak*-multimeasure or a multimeasure, then the *variation of* M is defined by

$$|M|(E) := \sup_{\mathcal{P}} \Big\{ \sum_{A \in \mathcal{P}} |M(A)| : \mathcal{P} \text{ is a finite measurable partition of } E \Big\}.$$

One can easily see that $|M|$ is a measure (cf. [19, page 863]) and, $M \ll \mu$ if and only if $|M| \ll \mu$ (that is $\mu(E) = 0 \Rightarrow |M|(E) = 0$). If $M \ll \mu$ is of σ-finite variation, then there exists a partition $(\Omega_n)_n$ of Ω such that $|M(E)| \le n\mu(E)$ (equivalently $|M|(E) \le n\mu(E)$) for every $E \in \Omega_n \cap \Sigma$, $n \in \mathbb{N}$ (this is a consequence of the classical Radon-Nikodým theorem).

Definition 2.1. Denote by $\mathcal{C}$ a non-empty subfamily of $c(X^*)$. A weak*-scalarly quasi-integrable multifunction $\varGamma : \Omega \to c(X^*)$ is *Gelfand integrable* in $\mathcal{C}$, if for each set $A \in \Sigma$ there exists a set $M_\varGamma^G(A) \in \mathcal{C}$ such that

$$s(x, M_\varGamma^G(A)) = \int_A s(x, \varGamma)\, d\mu\,, \text{ for every } x \in X.$$

$M_\varGamma^G$ is a weak*-multimeasure and $M_\varGamma^G(A)$ is called the Gelfand integral of $\varGamma$ on A: $(G)\int_E \varGamma\, d\mu := M_\varGamma^G(A)$. One should notice that the Gelfand integral is uniquely determined in the family $cw^*k(X^*)$ but not in $cb(X^*)$ (see [20, Remark 3.12]). Throughout this paper $\mathcal{C} := cw^*k(X^*)$ or $\mathcal{C} = cb(X^*)$. To distinguish the two Gelfand integrals the first one will be denoted by $M_\varGamma^{*G}$ and the second by $M_\varGamma^G$. $\square$

Theorem 2.2. ([20, Theorem 3.8]) *Every weak**-*scalarly integrable* $\varGamma : \Omega \to cb(X^*)$ *is Gelfand integrable in* $cw^*k(X^*)$. *Moreover,* $\gamma \in \mathcal{W}^*\mathcal{QS}_\varGamma$ *if and only if we have* $(G)\int_\bullet \gamma \in \mathcal{W}^*\mathcal{S}(M_\varGamma^{*G})$.

Definition 2.3. Denote by $\mathcal{D}$ a non-empty subfamily of $c(X)$. A scalarly quasi-integrable multifunction $\varGamma\colon \Omega \to c(X)$ is *Pettis integrable* in $\mathcal{D}$, if for each $A \in \Sigma$ there exists a set $M_\varGamma(A) \in \mathcal{D}$ such that

$$s(x^*, M_\varGamma(A)) = \int_A s(x^*, \varGamma)\, d\mu \quad \text{for every } x^* \in X^*.$$

$M_\varGamma : \Sigma \to cb(X)$ is a multimeasure and $M_\varGamma(A)$ called the *Pettis integral of* $\varGamma$ over A. We write $(P)\int_A \varGamma\, d\mu := M_\varGamma(A)$. $\square$

A multimeasure $M : \Sigma \to c(X)$ is called *rich* if $M(A) = \overline{\{m(A), m \in S(M)\}}$. By a result of Costé, quoted in [11, Theorem 7.9], this is verified for multimeasures which take as their values weakly compact convex subsets of a Banach space X, and $cb(X)$-valued, when X is a Banach space possessing RNP ([5, Théorème 1]).

A μ-continuous multimeasure $M : \Sigma \to c(X)$ is called *μ-rich in Pettis integrable (and strongly measurable densities)* if M is rich and each $m \in S(M)$ has a (strongly measurable) density that is Pettis integrable with respect to μ. We will use the phrase "*μ-rich in (strong) densities*".

The following well known property of lifting will be applied several times in this paper:

Proposition 2.4. (cf. [13]) *Let ρ be a lifting on the complete space (Ω, Σ, μ) and $\{A_t : t \in T\} \subset \Sigma$ be a collection of sets such that $A_t \subset \rho(A_t)$, for each $t \in T$. Then $\bigcup_{t\in T} A_t \in \Sigma$ and $\bigcup_{t\in T} A_t \subset \rho(\bigcup_{t\in T} A_t)$.*
If $\{f_t : t \in T\}$ is a uniformly bounded family of real-valued functions such that $f_t \le \rho(f_t)$, for each $t \in T$, then $\sup_{t\in T} f_t$ is a measurable function and we have $\sup_{t\in T} f_t \le \rho(\sup_{t\in T} f_t)$ everywhere.

3. Radon-Nikodým property

The following result generalizes an old result from [4].

Theorem 3.1. [20, Theorem 3.6] *If $M : \Sigma \to cw^*k(X^*)$ is a weak*-multimeasure, then*

$$M(E) = \{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\} \quad \textit{for each } E \in \Sigma\,.$$

The subsequent auxiliary lemma is essential for the further investigation.

Lemma 3.2. [8] *If $f : \Omega \to X$ is a bounded strongly measurable function, then $\rho(f)$ is a strongly measurable function with values in X^{**} but almost all values of $\rho(f)$ are in X. The range of $\rho(f)$ is weak*-separable.*

Proof. Since f is strongly measurable it is almost everywhere a uniform limit of a sequence of countably valued functions f_n , $n \in \mathbb{N}$. Consequently, $\rho(f) = \lim_n \rho(f_n)$ pointwise. If $f_n = \sum_{k=1}^{\infty} x_{nk}\chi_{E_{nk}}$, where for each n the sets E_{nk}, $k = 1, 2, \dots$ are pairwise disjoint, then $f'_n := \sum_{k=1}^{\infty} x_{nk}\chi_{\rho(E_{nk})}$ is strongly measurable and $f_n = f'_n$ a.e. Consequently $\rho(f) = \lim_n f'_n$ a.e. is strongly measurable and takes almost all valued in X.

Suppose the set $\rho(f)(\Omega)$ is not weak*-separable. Let $\Omega_0 \in \Sigma$ be such that $\mu(\Omega_0) = 1$, the set $\rho(f)(\Omega_0)$ is weak*-separable but $\rho(f)(\Omega_0^c)$ is not. Let $Z := \overline{span}^* \rho(f)(\Omega_0) \subset X^{**}$. Z is weak*-separable. Let $\omega_0 \notin \Omega_0$ be such that $\rho(f)(\omega_0) \notin Z$. According to the Hahn-Banach separation theorem there exists a weak*-closed hyperplane $H := \{x^{**} \in X^{**} : x_0^*(x^{**}) = a\}$ with $\langle x_0^*, \rho(f)(\omega)\rangle \le a$ for every $\omega \in \Omega_0$ and $\langle x_0^*, \rho(f)(\omega_0)\rangle > a$. But then, it follows from the properties of lifting that $\langle x_0, \rho(f)(\omega_0)\rangle \le a$, what gives a contradiction. Thus, the range of a strongly measurable lifted function has always weak*-separable range. □

Remark 3.3. If f is not strongly measurable, then the thesis of Lemma 3.2 may fail (see [8, Theorem 4.1]). In a more general setting the problem was undertaken by Edgar and Talagrand in [9]. In particular, they proved that if X has WRNP and the Pettis integral Property but is without RNP, then for a non-atomic complete probability space (Ω, Σ, μ) always there exists a scalarly measurable $f : \Omega \to X$ such that $\rho(f)(\omega) \notin X$ on a set of positive measure.

The subsequent result is a composition of [20, Proposition 4.3, Proposition 4.8].

Proposition 3.4. *If $\Gamma : \Omega \to cb(X^*)$ is a weak*-scalarly integrable and weak*-scalarly bounded multifunction and $M : \Sigma \to cb(X^*)$ is its Gelfand integral, then the multifunctions $\rho\Gamma, \rho\Gamma_M : \Omega \to cw^*k(X^*)$ and $\Gamma_\rho, (\Gamma_M)_\rho : \Omega \to cb(X^*)$ are defined for every $\omega \in \Omega$ by the formulae*

$$\Gamma_\rho(\omega) := \overline{conv}\,\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\} = \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\}}\,;$$

$$(\Gamma_M)_\rho(\omega) := \overline{\left\{\rho(\gamma)(\omega) : (G)\!\int_\bullet \rho(\gamma)\,d\mu \in \mathcal{W}^*\mathcal{S}(M)\right\}}, \quad and$$

$$\rho\Gamma(\omega) := \overline{conv}^{\,*}\,\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\} = \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\}}^{\,*}\,;$$

$$\rho\Gamma_M(\omega) := \overline{\left\{\rho(\gamma)(\omega) : (G)\!\int_\bullet \rho(\gamma)\,d\mu \in \mathcal{W}^*\mathcal{S}(M)\right\}}^{\,*} = \overline{(\Gamma_M)_\rho(\omega)}^{\,*}.$$

Here Γ_ρ, $(\Gamma_{M^*})_\rho$, $\rho\Gamma$ and $\rho(\Gamma_{M^*})$ are Gelfand integrable densities of M such that $(\Gamma_{M^*})_\rho = (\Gamma^*)_\rho = \Gamma_\rho \overset{w^*s}{\approx} \rho\Gamma = \rho(\Gamma_{M^*}) = \rho(\Gamma^*)$ (see Theorem 2.2) and

$$s(x,\Gamma_\rho(\omega)) \le s(x,\rho\Gamma(\omega)) = \rho(s(x,\Gamma))(\omega) \quad \textit{for every } x\in X \textit{ and every } \omega\in\Omega. \qquad (4)$$

Moreover, the multifunctions are $\rho\Gamma$, $\rho(\Gamma_M)$, $(\Gamma_M)_\rho$, Γ_ρ are weak-open measurable.*

Remark 3.5. The proof of weak*-open measurability of $\rho(\Gamma_M)$ and $(\Gamma_M)_\rho$ is similar to that of $\rho\Gamma$ and Γ_ρ (see [20, Proposition 4.8]).

The situation becomes more complicated when we investigate $c(X)$-valued multifunctions and X is not necessarily a conjugate Banach space. If a multimeasure M is defined on Σ, then we denote by $\mathcal{D}_M^P$ the collection of all Pettis integrable densities of selections of M and by $\mathcal{D}_M^{mP}$ the collection of all strongly measurable members of $\mathcal{D}_M^P$. It is clear that if Γ is a Pettis integrable density of M, then $\mathcal{D}_M^P \subset \mathcal{Q}S_\Gamma$.

Definition 3.6. Let $M : \Omega \to cb(X)$ be a multimeasure dominated by μ. If we have $\mathcal{D}_M^P \neq \emptyset$, then we define $\Gamma_{M\rho} : \Omega \to cb(X^{**})$ by

$$\Gamma_{M\rho}(\omega) := \overline{conv}\,\{\rho(\gamma)(\omega)\colon \gamma \in \mathcal{D}_M^P\} = \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{D}_M^P\}} \subset (\Gamma_{M^{**}})_\rho(\omega)$$

and $\Gamma_{M\rho}^X : \Omega \to cb(X) \cup \emptyset$ by

$$\Gamma_{M\rho}^X(\omega) := \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{D}_M^P\} \cap X} \subset \Gamma_{M\rho}(\omega) \cap X\,. \qquad \square$$

If X has RNP or M is rich in strong densities, then each element of $\mathcal{D}_M^P$ is scalarly equivalent to an element of $\mathcal{D}_M^{mP}$ and so everywhere in the above definition $\mathcal{D}_M^P$ may be replaced by $\mathcal{D}_M^{mP}$. In particular it follows immediately from Lemma 3.2 that we have $\mu\{\omega \in \Omega : \Gamma_{M\rho}^X(\omega) = \emptyset\} = 0$ in such a case.

Proposition 3.7. *If $M : \Sigma \to cb(X)$ is a multimeasure dominated by μ, then $\Gamma_{M\rho}$ is weak* scalarly measurable and for each $x^* \in X^*$*

$$s(x^*,\Gamma_{M\rho}) \le \rho[s(x^*,\Gamma_{M\rho})] \qquad (5)$$

everywhere. Moreover, the function $|\Gamma_{M\rho}|$ is measurable.

If M is also μ-rich in strong densities (e.g. X has RNP), then $\Gamma^X_{M\rho}$, $\Gamma_{M\rho}$, $\rho(\Gamma_{M^{**}})$, $(\Gamma_{M^{**}})_\rho$ are weak* scalarly equivalent and

$$s(x^*,\Gamma^X_{M\rho}) \le s(x^*,\Gamma_{M\rho}) \le \rho[s(x^*,\Gamma^X_{M\rho})] = \rho[s(x^*,\Gamma_{M\rho})] \tag{6}$$

everywhere on the set $\{\omega\in\Omega : \Gamma^X_{M\rho}(\omega)\neq\emptyset\}$. Moreover, the function $|\Gamma^X_{M\rho}|$ is measurable.

Proof. Weak* scalar measurability of $\Gamma_{M\rho}$, (5) and measurability of $|\Gamma^X_{M\rho}|$ are consequence of Proposition 2.4 and of the following relations:

$$\begin{aligned} s(x^*,\Gamma_{M\rho}(\omega)) &= \sup\left\{\langle x^*,\rho(\gamma)\rangle(\omega) : \gamma\in\mathcal{D}^P_M\right\} \\ &\overset{Prop.2.4}{\le} \rho\big(\sup\left\{\langle x^*,\rho(\gamma)\rangle(\omega) : \gamma\in\mathcal{D}^P_M\right\}\big) = \rho[s(x^*,\Gamma_{M\rho})](\omega)\,. \end{aligned}$$

If M is μ-rich in strong densities, then

$$\begin{aligned} s(x^*,M(E)) &= \sup\left\{\Big\langle x^*,(P)\int_E\gamma\,d\mu\Big\rangle\colon \gamma\in\mathcal{D}^{mP}_M\right\} \\ &= \sup\left\{\int_E\langle x^*,\rho(\gamma)\rangle\chi_X(\rho(\gamma))\,d\mu\colon \gamma\in\mathcal{D}^{mP}_M\right\} \le \underline{\int}_E s(x^*,\Gamma^X_{M\rho})\,d\mu \\ &\le \overline{\int}_E s(x^*,\Gamma^X_{M\rho})\,d\mu \le \int_E s(x^*,\Gamma_{M\rho})\,d\mu \le \int_E s(x^*,(\Gamma_{M^{**}})_\rho)\,d\mu \\ &\le \int_E s(x^*,\rho(\Gamma_{M^{**}}))\,d\mu = s(x^*,M^{**}(E)) = s(x^*,M(E))\,. \end{aligned}$$

It follows that $\Gamma^X_{M\rho}\,,\Gamma_{M\rho}\,,(\Gamma_{M^{**}})_\rho\,,\rho(\Gamma_{M^{**}})$ are weak* scalarly equivalent. The equivalence and (5) imply (6). Measurability of $|\Gamma_{M\rho}|$ and $|\Gamma^X_{M\rho}|$ is a consequence of Proposition 2.4. □

The subsequent result is an essential extension of [20, Theorem 5.10] (compare with Theorem 4.1) but under different assumptions. The range space X is not necessarily a conjugate space but it is assumed that X has RNP. It is also a generalization of [2, Theorem 4.1], proved there without the measurability of the density and under a stronger assumption that M is a strong multimeasure (cf. [12, Def. 8.4.4.1]). It gives also some kind of a generalization of Castaing representation of measurable multifunctions (see [3]) in case of a non separable X and for very special multifunctions.

Theorem 3.8. *Let $M:\Sigma\to c(X)$ be a μ-continuous multimeasure of σ-finite variation. If M is μ-rich in strong densities (e.g. X has RNP and $M:\Sigma\to cb(X)$) and $\rho\in\Lambda(\mu)$, then there exists a measurable multifunction $\Gamma:\Omega\to cb(X)$ such that*

$$M(E) = \overline{\left\{(P)\int_E\gamma\,d\mu\colon \gamma\in\mathcal{D}^{mP}_M\right\}} = (P)\int_E\Gamma\,d\mu \quad \textit{for every } E\in\Sigma \tag{7}$$

and
$$\{\omega\in\Omega\colon \Gamma(\omega)\cap U\neq\emptyset\}\subset\rho\{\omega\in\Omega\colon \Gamma(\omega)\cap U\neq\emptyset\} \tag{8}$$

for every norm open $U\subset X$.

If $M:\Sigma\to cb(X)$ and $c_0\not\subseteq X$ isomorphically, then $\mathcal{QS}_\Gamma\subset\mathcal{D}^P_\Gamma$. If $M:\Sigma\to cwk(X)$, then always $\mathcal{QS}_\Gamma\subset\mathcal{D}^P_\Gamma$. In each case each quasi-selection of Γ is scalarly equivalent to a strongly measurable selection of Γ.

If M is dominated by μ, then one may set $\Gamma = X\cap\Gamma_{M\rho}$ or $\Gamma=\Gamma^X_{M\rho}$. In particular there exists a collection (in general uncountable) $\{\gamma_t: t\in T\}\subset\mathcal{S}^{mP}_\Gamma$ such that

$$\Gamma(\omega)=\overline{\{\gamma_t(\omega): t\in T\}}\quad \textit{for every } \omega\in\Omega.$$

In each case the inequality $s(x^*,\Gamma)\le\rho(s(x^*,\Gamma))$ holds true everywhere and for every $x^*\in X^*$.

Proof. Without loss of generality one may assume that μ is atomless. Let us notice at the beginning that the first equality in (7) is a direct consequence of the riches of M. To continue we first assume that M is dominated by μ. We will prove the assertion for $\Gamma^X_{M\rho}$.

Step 1. Let

$$\mathfrak{F}:=\left\{G:\Omega\to cb(X)\cup\emptyset\;\middle|\;\begin{array}{l}\Gamma^X_{M\rho}\supset G\overset{s}{\equiv}\Gamma^X_{M\rho}, \text{ (8) is satisfied with } \Gamma \text{ replaced}\\ \text{by } G \text{ and } M(E)=(P)\displaystyle\int_E G\,d\mu \text{ for every } E\in\Sigma\end{array}\right\}.$$

Let us notice that due to Lemma 3.2 the equality $\Gamma^X_{M\rho}=\emptyset$ can happen only on a set of measure zero. To continue one needs to know that $\mathfrak{F}\neq\emptyset$.

Since M is μ-rich in strong densities, each $m\in\mathcal{S}(M)$ generates a strongly measurable function that is a Pettis integrable density of m. If $\gamma\in\mathcal{D}^{mP}_M$, then also $\rho(\gamma):\Omega\to X^{**}$ is strongly measurable and almost all values of $\rho(\gamma)$ are in X (see Lemma 3.2).

Let $\mathcal{X}$ be the linear subspace of X^{**}-valued functions spanned by $\{\rho(\gamma)\colon \gamma\in\mathcal{D}^{mP}_M\}$ and let $\mathcal{H}=\{\rho(\beta):\beta\in\mathbb{B}\}$ be a Hamel basis of $\mathcal{X}$. Since each $\rho(\beta)$ is strongly measurable with almost all values in X there exist pairwise disjoint and norm compact sets $L_{n\rho(\beta)}\subset X$, $n\in\mathbb{N}$, and pairwise disjoint sets $A_{n\rho(\beta)}:=\rho(\beta)^{-1}(L_{n\rho(\beta)})\in\Sigma$, $n\in\mathbb{N}$ such that $\mu(\Omega\setminus\bigcup_n A_{n\rho(\beta)})=0$. Then $K_{n\rho(\beta)}:=\overline{\rho(\beta)(A_{n\rho(\beta)})}$ is a norm compact subset of X and $A_{n\rho(\beta)}=\rho(\beta)^{-1}(K_{n\rho(\beta)})$. The sets $L_{n\rho(\beta)}$ and $K_{n\rho(\beta)}$ are not uniquely determined so for each $\beta\in\mathbb{B}$ we fix one possible sequence of sets $K_{n\rho(\beta)}$, $n\in\mathbb{N}$, and set

$$A_{\rho(\beta)}:=\bigcup_n\rho[\rho(\beta)^{-1}(K_{n\rho(\beta)})]=\bigcup_n\rho(A_{n\rho(\beta)})\quad\text{and}\quad K_{\rho(\beta)}:=\bigcup_n K_{n\rho(\beta)}\,.$$

Clearly $\mu(\Omega\setminus A_{\rho(\beta)})=0$. According to [13, page 52] we have for every $\beta\in\mathbb{B}$ $\rho(\beta)^{-1}(K_{n\rho(\beta)})\supset\rho\left(\rho(\beta)^{-1}(K_{n\rho(\beta)})\right)=\rho(A_{n\rho(\beta)})$, and so

$$A_{\rho(\beta)}\subset\bigcup_n\rho(\beta)^{-1}(K_{n\rho(\beta)})=\rho(\beta)^{-1}(K_{\rho(\beta)}).$$

Take now an arbitrary $\mathcal{H}\not\ni\rho(\gamma)\in\mathcal{D}^{mP}_M$. Then $\rho(\gamma)=\sum_{i=1}^m a_i\rho(\beta_i)$, where $\beta_i\in\mathbb{B}$ and the representation is unique. We define the sets $K_{n_1n_2\dots n_m\rho(\gamma)}$ by setting

$$K_{n_1n_2\dots n_m\rho(\gamma)}:=\bigcap_{i=1}^m K_{n_i\rho(\beta_i)},\quad A_{n_1n_2\dots n_m\rho(\gamma)}:=\bigcap_{i=1}^m\rho(\beta_i)^{-1}(K_{n_i\rho(\beta_i)})=\bigcap_{i=1}^m A_{n_i\rho(\beta_i)}.$$

Then we set
$$A_{\rho(\gamma)} := \bigcup_{(n_1,\ldots,n_m)\in\mathbb{N}^m} \rho\left(A_{n_1 n_2\ldots n_m \rho(\gamma)}\right) = \bigcap_{i=1}^{m} A_{\rho(\beta_i)}\,. \tag{9}$$

We define a new multifunction $\Delta : \Omega \to cb(X)\cup\emptyset$ by the formula

$$\Delta(\omega) := \overline{\left\{\rho(\gamma)(\omega)\colon \gamma\in\mathcal{D}_M^{mP}\ \&\ \omega\in A_{\rho(\gamma)}\right\}} \subset \Gamma^X_{M\rho}(\omega) \subset X\,.$$

Claim. *For each $\omega\in\Omega$ the set $D(\omega) := \left\{\rho(\gamma)(\omega) : \gamma\in\mathcal{D}_M^{mP}\ \&\ \omega\in A_{\rho(\gamma)}\right\}$ is convex.*

Proof. We have $\mu\{\omega : D(\omega)=\emptyset\}=0$, so assume that $D(\omega)\neq\emptyset$ and let $\omega\in\bigcap_{j=1}^m A_{\rho(\gamma_j)}$ be fixed with $\gamma_j\in\mathcal{D}_M^{mP}$, $j\le m$. If $x:=\sum_{j=1}^m a_j\rho(\gamma_j)(\omega)$, where a_j's are positive and $\sum_{j=1}^m a_j = 1$, then $\rho(\gamma_j)=\sum_{i=1}^{p_j} b_{ji}\rho(\beta_{ji})$ with $\beta_{ji}\in\mathbb{B}$ and the representation is unique. In particular $\omega\in\bigcap_{i=1}^{p_j} A_{\rho(\beta_i)}$, for each $j\le m$. Consequently, we have

$$x=\sum_{j=1}^m a_j\rho(\gamma_j)(\omega)=\sum_{j=1}^m a_j\sum_{i=1}^{p_j} b_{ji}\rho(\beta_{ji})(\omega)=\sum_{j=1}^m\sum_{i=1}^{p_j} a_j b_{ji}\rho(\beta_{ji})(\omega)=\rho(\gamma)(\omega)$$

where $\rho(\gamma):=\sum_{j=1}^m a_j\rho(\gamma_j)\in\mathcal{D}_M^{mP}$. From (9) we get $\omega\in\bigcap_{j=1}^m\bigcap_{j=1}^{p_j} A_{\rho(\beta_{ij})}=A_{\rho(\gamma)}$, what yields $x\in D(\omega)$. Thus, $D(\omega)$ is convex. $\square$

If $\emptyset\neq U\subset X$ is norm open, then $\Delta(\omega)\cap U\neq\emptyset \Leftrightarrow \exists\,\gamma\in\mathcal{D}_M^{mP}\ \omega\in\rho(\gamma)^{-1}(U)\cap A_{\rho(\gamma)}$.

Thus,
$$\{\omega\in\Omega : \Delta(\omega)\cap U\neq\emptyset\} = \bigcup_{\gamma\in\mathcal{D}_M^{mP}} \rho(\gamma)^{-1}(U)\cap A_{\rho(\gamma)}\,. \tag{10}$$

Since each $\rho(\gamma):\Omega\to X^{**}$ is strongly measurable, the sets $\rho(\gamma)^{-1}(U)$ are measurable. We now fix $\gamma\in\mathcal{D}_M^{mP}$ with its unique representation $\rho(\gamma)=\sum_{i=1}^m a_i\rho(\beta_i)$, where $\beta_i\in\mathbb{B}$.

Since the sets $K_{n\rho(\beta_i)}=\overline{\rho(\beta_i)(A_{n\rho(\beta_i)})}$ are norm compact, the weak topology coincides with the norm topology on each $K_{n\rho(\beta_i)}$. That means that for each norm open U and each $\beta_i\in\mathcal{D}_M^{mP}$ there exists a weakly open $U_{n\beta_i}$ such that $K_{n\rho(\beta_i)}\cap U = K_{n\rho(\beta_i)}\cap U_{n\beta_i}$. Then,

$$\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\cap\rho(\gamma)^{-1}(U)=\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\cap\rho(\gamma)^{-1}(U_{n\beta_i})\,.$$

But $\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\supset\rho\left[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\right]$ and so

$$\rho\left[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\right]\cap\rho(\gamma)^{-1}(U)=\rho\left[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)}))\right]\cap\rho(\gamma)^{-1}(U_{n\beta_i})\,.$$

Hence,
$$\rho(\gamma)^{-1}(U)\cap A_{\rho(\beta_i)}=\bigcup_n\rho[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})]\cap\rho(\gamma)^{-1}(U_{n\beta_i})\,.$$

But according to [13, p. 52] we have $\rho(\gamma)^{-1}(U_{n\beta_i})\subset\rho[\rho(\gamma)^{-1}(U_{n\beta_i})]$. Hence

$$\begin{aligned}\rho(\gamma)^{-1}(U)\cap A_{\rho(\beta_i)}&=\bigcup_n\rho\left[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\right]\cap\rho(\gamma)^{-1}(U_{n\beta_i})\\&\subset\bigcup_n\rho\left[\rho\left[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\right]\cap\rho(\gamma)^{-1}(U_{n\beta_i})\right]\\&\subset\rho\Big[\bigcup_n\rho\left[\rho(\gamma)^{-1}(K_{n\rho(\beta_i)})\right]\cap\rho(\gamma)^{-1}(U_{n\beta_i})\Big]=\rho[\rho(\gamma)^{-1}(U)\cap A_{\rho(\beta_i)}]\,.\end{aligned}$$

Hence

$$A_{\rho(\gamma)} \cap \rho(\gamma)^{-1}(U) = \bigcap_{i=1}^{m} A_{\rho(\beta_i)} \cap \rho(\gamma)^{-1}(U) \subset \bigcap_{i=1}^{m} \rho[A_{\rho(\beta_i)} \cap \rho(\gamma)^{-1}(U)]$$
$$= \rho\left[A_{\rho(\gamma)} \cap \rho(\gamma)^{-1}(U)\right].$$

It follows from the basic properties of lifting that the right hand side of (10) is a measurable set. In view of Proposition 2.4 the multifunction $\varDelta$ is measurable and

$$\{\omega \in \varOmega \colon \varDelta(\omega) \cap U \neq \emptyset\} \subset \rho[\{\omega \in \varOmega \colon \varDelta(\omega) \cap U \neq \emptyset\}].$$

It remains to prove that $\varDelta \stackrel{s}{\equiv} \varGamma^X_{M\rho}$.

If $\gamma \in \mathcal{D}^{mP}_M$, then $\rho(\gamma)(\omega) \in \varDelta(\omega)$ for almost all $\omega \in \varOmega$. Hence

$$(P)\!\int_E \rho(\gamma)\,d\mu = (G)\!\int_E \rho(\gamma)\,d\mu \in (G)\!\int_E \varDelta\,d\mu \quad \text{for every } E \in \varSigma.$$

Consequently, since M is μ-rich in strong densities,

$$M(E) = \overline{\left\{(P)\!\int_E \rho(\gamma)\,d\mu \colon \gamma \in \mathcal{D}^{mP}_M\right\}} \subset (G)\!\int_E \varDelta\,d\mu \in cw^*k(X^{**}). \tag{11}$$

If $x^* \in X^*$ and $E \in \varSigma$, then Propositions 3.7 and 3.4 and equation (11) yield

$$\int_E s(x^*, \varGamma^X_{M\rho})\,d\mu = s(x^*, M(E)) \stackrel{(11)}{\leq} \int_E s(x^*, \varDelta)\,d\mu \leq \int_E s(x^*, \varGamma^X_{M\rho})\,d\mu. \tag{12}$$

It follows that $\varGamma^X_{M\rho} \stackrel{s}{\equiv} \varDelta$ and $M(E) = (P)\!\int_E \varDelta\,d\mu$ for every $E \in \varSigma$ and consequently $\mathfrak{F} \neq \emptyset$.

Step 2. Let $H(\omega) := \bigcup\{G(\omega) : G \in \mathfrak{F}\}$. In view of Proposition 2.4 that H is a measurable multifunction with convex values and $\varDelta(\omega) \subset H(\omega) \subset \varGamma^X_{M\rho}(\omega)$ for every $\omega \in \varOmega$. It follows that H satisfies (11) and (12) if $\varDelta$ is replaced by H.

But if $U \subset X$ is norm open, then $B \cap U \neq \emptyset$ if and only if $\overline{B} \cap U \neq \emptyset$, where B is an arbitrary set. Thus, the multifunction $\overline{H}(\omega) := \overline{H(\omega)}$ is the largest element of $\mathfrak{F}$.

I am going to prove that $\overline{H}(\omega) = \varGamma^X_{M\rho}(\omega)$. So suppose that there is $\omega_0 \in \varOmega$ and $x_0 \in \varGamma^X_{M\rho}(\omega_0) \setminus \overline{H}(\omega_0)$. Define $P : \varOmega \to bd(X)$ by

$$P(\omega) := \begin{cases} \overline{H}(\omega), & if\ \omega \neq \omega_0; \\ conv\{\{x_0\} \cup \overline{H}(\omega_0)\}, & if\ \omega = \omega_0. \end{cases}$$

It is obvious that $\{\omega \in \varOmega \colon U \cap P(\omega) \neq \emptyset\} \triangle \{\omega \in \varOmega \colon U \cap \overline{H}(\omega) \neq \emptyset\} \subset \{\omega_0\}$ and so $\{\omega \in \varOmega \colon U \cap P(\omega) \neq \emptyset\} \in \varSigma$. Since $\overline{H} \in \mathfrak{F}$ and μ is atomless, we obtain the inclusion

$$\{\omega \in \varOmega \colon U \cap P(\omega) \neq \emptyset\} \subset \rho\{\omega \in \varOmega \colon U \cap P(\omega) \neq \emptyset\},$$

what proves that $P \in \mathfrak{F}$ and is larger than $\overline{H}$. It follows that $\overline{H} = \varGamma^X_{M\rho}$. This ends the proof in case of dominated M.

Step 3. Replacing in Step 2 of the above proof the multifunction $\varGamma^X_{M\rho}$ by $\varGamma_{M\rho} \cap X$ one proves that also that multifunction is a measurable and Pettis integrable density of M.

Step 4. Let now M be arbitrary. Then one can decompose $\varOmega$ into countably many pairwise disjoint sets $\varOmega_n$ in such a way that $|M|(E \cap \varOmega_n) \le n\mu(E \cap \varOmega_n)$. According to the first part of the proof, for each $n \in \mathbb{N}$ there exists a measurable multifunction $\varGamma_n : \varOmega_n \to cb(X)$ that is Pettis integrable in $cb(X)$ and

$$M(E \cap \varOmega_n) = (P)\int_{E\cap\varOmega_n} \varGamma_n \, d\mu \quad \text{for every } E \in \varSigma.$$

Setting $\varGamma(\omega) := \sum_n \varGamma_n(\omega)\chi_{\varOmega_n}(\omega)$, we get a Pettis integrable multifunction. Indeed, if $x^* \in X^*$ and $|s(x^*, M)|(\varOmega)) < \infty$, then

$$\int_\varOmega |s(x^*, \varGamma)| \, d\mu = \sum_n \int_{\varOmega_n} |s(x^*, \varGamma)| \, d\mu = \sum_n |s(x^*, M)|(\varOmega_n) = |s(x^*, M)|(\varOmega)) < \infty$$

and so $s(x^*, \varGamma)$ is integrable. Consequently, if $E \in \varSigma$, then

$$\int_E s(x^*, \varGamma) \, d\mu = \sum_n \int_E s(x^*, \varGamma_n) \, d\mu = \sum_n s(x^*, M(E \cap \varOmega_n)) = s(x^*, M(E)) \quad (13)$$

If $|s(x^*, M)|(\varOmega)) = \infty$, then let $F := \{\omega \in \varOmega : s(x^*, \varGamma(\omega)) < 0\}$, then we have (13) for $E = \varOmega \setminus F$ because $s(x^*, \varGamma) \ge 0$ on F^c. But according to (3), we have also

$$\int\limits_F s(x^*, \varGamma)^- \, d\mu = \sum_{n=1}^{\infty} \int\limits_{F\cap\varOmega_n} s(x^*, \varGamma)^- \, d\mu = \sum_{n=1}^{\infty} s(x^*, M(F \cap \varOmega_n)) = s(x^*, M(F)) \neq \pm\infty.$$

This yields scalar quasi-integrability of $\varGamma$ and the equality $M(E) = (P)\int_E \varGamma \, d\mu$ for every $E \in \varSigma$.

Step 5. Assume now that $\gamma \in \mathcal{QS}_\varGamma$ and $M : \varSigma \to cb(X)$. Then γ is scalarly equivalent to a strongly measurable and scalarly integrable selection $\widetilde{\gamma}$ of $\varGamma$. Since $c_0 \nsubseteq X$, $\widetilde{\gamma}$ is Pettis integrable, hence also γ. If $M : \varSigma \to cwk(X)$, then we apply [18, Corollary 1.5]. $\square$

Remark 3.9. Unfortunately, even if $M : \varSigma \to ck(X)$ the density with weakly compact values may not exist (see [5]).

The shape of Pettis integrable densities is simpler in case of conjugate spaces.

Theorem 3.10. *Assume that $M : \varSigma \to c(X^*)$ $[cw^*k(X^*)]$ is a μ-continuous multimeasure of σ-finite variation and $\rho \in \varLambda(\mu)$. If M is μ-rich in strong densities (e.g. X^* has RNP and $M : \varSigma \to cb(X^*)$), then there exists a measurable Pettis integrable multifunction $\varGamma : \varOmega \to cb(X^*)$ $[cw^*k(X^*)]$ such that*

$$M(E) = \overline{\left\{(P)\int_E \gamma \, d\mu \colon \gamma \in \mathcal{D}^{mP}_M\right\}} = (P)\int_E \varGamma \, d\mu \quad \textit{for every } E \in \varSigma \qquad (14)$$

and $$\{\omega \in \Omega\colon \Gamma(\omega) \cap U \neq \emptyset\} \subset \rho\{\omega \in \Omega\colon \Gamma(\omega) \cap U \neq \emptyset\} \tag{15}$$

for every norm open $U \subset X$.

If M is dominated by μ, then one may choose $\Gamma = (\Gamma_{M^*})_\rho$ in case of $c(X^*)$ and $\Gamma = \rho(\Gamma_M)$ in case of $cw^*k(X^*)$. If $M : \Sigma \to cb(X^*)$ and $c_0 \nsubseteq X^*$ isomorphically, then each weak quasi-selection of Γ is weak*-scalarly equivalent to a strongly measurable and Pettis integrable selection of Γ. If $M : \Sigma \to cwk(X^*)$, then each weak* quasi-selection of Γ is Pettis integrable and scalarly equivalent to a strongly measurable and Pettis integrable selection of Γ.*

Proof. If one replaces X by X^* in Theorem 3.8 and $\Gamma^X_{M\rho}$ by $\Gamma_{M\rho}$, then we obtain (14) with $\Gamma : \Omega \to cb(X^*)$. It remains to show that Γ may take its values in $cw^*k(X^*)$.

We assume that μ is atomless and dominated by μ. Let

$$\mathfrak{G} := \left\{ G\colon \Omega \to cb(X^*) \cup \emptyset \;\middle|\; \begin{array}{l} \rho(\Gamma_M) \supset G \overset{w^*s}{\equiv} \rho(\Gamma_M)\,,\ (15) \text{ is satisfied with } \Gamma \text{ re-} \\ \text{placed by } G \text{ and } M(E) = (P)\displaystyle\int_E G\,d\mu \text{ for every } E \in \Sigma \end{array} \right\}.$$

We already know that $(\Gamma_M)_\rho \in \mathfrak{G}$. Let $L(\omega) := \bigcup\{G(\omega) : G \in \mathfrak{G}\}$. I am going to prove that $\overline{L(\omega)} = \rho(\Gamma_M)(\omega)$. The proof is similar to that in Theorem 3.8. So suppose that there is $\omega_0 \in \Omega$ and $x_0 \in \rho(\Gamma_M)(\omega_0) \setminus \overline{L(\omega_0)}$. Define $P\colon \Omega \to bd(X)$ by

$$P(\omega) := \begin{cases} \overline{L(\omega)}, & if\ \omega \neq \omega_0; \\ conv\{\{x_0\} \cup \overline{L(\omega_0)}\}, & if\ \omega = \omega_0. \end{cases}$$

It is obvious that $\{\omega \in \Omega\colon U \cap P(\omega) \neq \emptyset\} \triangle \{\omega \in \Omega\colon U \cap \overline{L(\omega)} \neq \emptyset\} \subset \{\omega_0\}$ and so $\{\omega \in \Omega\colon U \cap P(\omega) \neq \emptyset\} \in \Sigma$. Since $\overline{L} \in \mathfrak{G}$ and μ is atomless, we obtain the inclusion

$$\{\omega \in \Omega\colon U \cap P(\omega) \neq \emptyset\} \subset \rho\{\omega \in \Omega\colon U \cap P(\omega) \neq \emptyset\},$$

what proves that $P \in \mathfrak{G}$ and is larger than $\overline{L}$. It follows that $\overline{L} = \rho(\Gamma_M)$.

It is a consequence of the construction of $\rho(\Gamma_M)$ that each weak* quasi-selection of Γ is weak*-equivalent to a strongly measurable and scalarly integrable selection of Γ. If $c_0 \nsubseteq X^*$, then the selection is Pettis integrable (cf. [7, II.3.Theorem 7]).

If $M : \Sigma \to cwk(X^*)$, then please read the proof of Theorem 4.1. $\square$

The following result is an immediate consequence of Theorem 3.8.

Theorem 3.11. *Assume that $\Gamma : \Omega \to c(X)$ is Pettis integrable in $c(X)$ and its integral is μ-rich in strong densities. Then there exists a measurable multifunction $\widetilde{\Gamma} : \Omega \to cb(X)$ that is scalarly equivalent to Γ and*

$$\{\omega \in \Omega\colon X \cap \widetilde{\Gamma}(\omega) \cap U \neq \emptyset\} \subset \rho\{\omega \in \Omega\colon X \cap \widetilde{\Gamma}(\omega) \cap U \neq \emptyset\}$$

for every norm-open $U \subset X$.

If Γ is scalarly bounded and M is its Pettis integral, then one may set $\widetilde{\Gamma} = X \cap \Gamma_{M\rho}$ or $\widetilde{\Gamma} = \Gamma^X_{M\rho}$. In particular there exists a collection $\{\gamma_t : t \in T\} \subset \mathcal{S}^P_{\widetilde{\Gamma}}$ (in general uncountable) such that

$$\widetilde{\Gamma}(\omega) = \overline{\{\gamma_t(\omega) : t \in T\}} \quad \textit{for every } \omega \in \Omega.$$

In each case the inequality $s(x^*, \widetilde{\Gamma}) \leq \rho(s(x^*, \widetilde{\Gamma}))$ holds true everywhere and for every $x^* \in X^*$.

If Γ is Pettis integrable in $cwk(X)$, then every quasi-selection of $\widetilde{\Gamma}$ is Pettis integrable and scalarly equivalent to a strongly measurable and Pettis integrable selection of $\widetilde{\Gamma}$.

Remark 3.12. Let X be a separable Banach space with RNP and $\Gamma : \Omega \to cb(X)$ be a scalarly bounded multifunction that is measurable and Pettis integrable in $cb(X)$. What is the relation between Γ and $X \cap \Gamma_{M\rho}$ or $\Gamma^X_{M\rho}$? One may expect that the multifunctions coincide a.e. Indeed, this is true.

Let $\Gamma(\omega) = \overline{conv}\{f_i(\omega) : i \in \mathbb{N}, f_i \in \mathcal{S}(\Gamma)\}$ and

$$\widehat{\Gamma}(\omega) := \overline{conv}\{\rho(f_i)(\omega) : \rho(f_i)(\omega) = f_i(\omega) \in X \text{ and } f_i \in \mathcal{S}(\Gamma),\ i \in \mathbb{N}\}.$$

It follows from Lemma 3.2 that there exists $N_1 \in \Sigma$ of measure zero such that $\Gamma(\omega) = \widehat{\Gamma}(\omega)$ for every $\omega \notin N_1$.

Since $\Gamma^X_{M\rho}(\omega) := \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{D}^{mP}_M\} \cap X}$, we have $\Gamma^X_{M\rho}(\omega) \supset \widehat{\Gamma}(\omega) = \Gamma(\omega)$ for every $\omega \notin N_1$. In virtue of Theorem 3.8 the multifunction $\Gamma^X_{M\rho}$ is measurable and so there exists a collection $\{\rho(g_i) : g_i \in \mathcal{D}^{mP}_M,\ i \in \mathbb{N}\}$ such that

$$\Gamma^X_{M\rho}(\omega) = \overline{conv}\{\rho(g_i)(\omega) \in X : g_i \in \mathcal{D}^{mP}_M,\ i \in \mathbb{N}\},$$

for every $\omega \in \Omega$. If $N_2 := \{\omega : \exists\ i \in \mathbb{N}\ g_i(\omega) \neq \rho(g_i)(\omega)\}$, then $\mu(N_2) = 0$. If $\omega \notin N_1 \cup N_2$, then

$$\begin{aligned} \Gamma(\omega) \subset \Gamma^X_{M\rho}(\omega) &\subset \overline{conv}(\{\rho(g_i)(\omega) : i \in \mathbb{N}\} \cup \{\rho(f_i)(\omega) : i \in \mathbb{N}\}) \\ &= \overline{conv}(\{g_i(\omega) : i \in \mathbb{N}\} \cup \{f_i(\omega) : i \in \mathbb{N}\}) \\ &\subset \overline{\{f(\omega) : f \in \mathcal{S}(\Gamma)\}} = \Gamma(\omega) \end{aligned}$$

That completes the proof. The generalization to multifunctions with Pettis integral of σ-finite variation is obvious.

4. Weak Radon-Nikodým property

The subsequent result is a generalization of [20, Theorem 5.10].

Theorem 4.1. *If X^* has the WRNP and $M : \Sigma \to cw^*k(X^*)$ $[c(X^*)]$ is a μ-continuous weak*-multimeasure of σ-finite variation, then there exist Pettis integrable multifunctions $\Gamma : \Omega \to cw^*k(X^*)$ $[cb(X^*)]$ that are weak*-open measurable and*

$$s(x^{**}, M(E)) = \int_E s(x^{**}, \Gamma)\, d\mu \quad \textit{for every } E \in \Sigma \textit{ and } x^{**} \in X^{**}. \tag{16}$$

Moreover, each weak quasi-selection of Γ is weak*-scalarly equivalent to a Pettis integrable selection of Γ. If M is dominated by μ, then one may choose Γ satisfying the equality $\Gamma = \rho(\Gamma_M)$ in case of $cw^*k(X^*)$ and $\Gamma = (\Gamma_{M^*})_\rho$ in case of $c(X^*)$.*

If $M : \Sigma \to cwk(X^*)$, then every weak quasi-selection of Γ is Pettis integrable.*

Proof. The weak*-open measurability of Γ is a consequence of weak*-open measurability of $\rho(\Gamma_m)$ and $(\Gamma_{M^*})_\rho$. Weak*-scalar equivalence of weak* quasi-selections of Γ to Pettis integrable selections of Γ is a direct consequence of WRNP of X^*. Only the last assertion needs a proof.

If Γ is Pettis integrable in $cwk(X^*)$, it is determined by a WCG subspace Y of X^* (see [18, Theorem 2.5]). Without loss of generality, we may assume that $Y = \overline{span} \bigcup M(\Sigma) := \overline{span} \bigcup_{E\in\Sigma} M(E)$. If f is a weak* quasi-selection of Γ, then $-s(-x^{**}, \Gamma(\omega)) \leq \langle x^{**}, f(\omega)\rangle \leq s(x^{**}, \Gamma(\omega))$ μ-a.e. and for all $x^{**} \in X^{**}$ (the exceptional sets depend on x^*). In particular, if $x^{**} \in Y^\perp = \{y \in X^{**} : y|_Y = 0\}$, then $\langle x^{**}, f(\omega)\rangle = s(x^{**}, \Gamma) = -s(-x^{**}, \Gamma) = 0$ μ-a.e. Thus, f is determined by Y.

We will prove yet that f is scalarly measurable. So let us fix $x_0^{**} \in B(X^{**})$ and assume that

$$B(X) \ni x_\alpha \overset{\tau(X^{**},X^*)}{\longrightarrow} x_0^{**}.$$

Then, there exists $\{x_n : n \in \mathbb{N}\} \subset \{x_\alpha : \alpha\}$ uniformly convergent to x_0^{**} on $\bigcup M(\Sigma)$. Since $l_1 \not\subseteq X$, we may assume that the sequence $\{x_n : n \in \mathbb{N}\}$ is weakly Cauchy and so we may assume that $x_n \to y_0^{**}$ in $\sigma(X^{**}, X^*)$. Clearly, $y_0^{**} = x_0^{**}$ on $\bigcup M(\Sigma)$, what means that $x_0^{**} - y_0^{**} \in Y^\perp$. Consequently, $\langle x_0^{**}, f\rangle = \langle y_0^{**}, f\rangle$ a.e. and since $\langle y_0^{**}, f\rangle$ is measurable, also $\langle x_0^{**}, f\rangle$ is measurable.

According to [18, Corollary 1.5] f is Pettis integrable. □

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