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Extensions of Continuous Linear Functionals and Smoothness of Banach Spaces

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We introduce a notion called strong extension property which is stronger than the well known Hahn-Banach extension property introduced by R. R. Phelps in 1960 and weaker than the uniform extension property introduced by V. Zizler in 1972. Characterizations of the strong extension property in terms of strong Chebyshevness and strong rotundity are presented in this article. We observe that the strong extension and uniform extension properties coincide when the extensions are considered on finite dimensional subspaces. We also obtain stability results pertaining to extension, strong extension and uniform extension properties. Further, for a subspace Y of a Banach space X, we consider different notions of smoothness of Y in X. These smoothness notions are characterized in terms of extension, strong extension and uniform extension properties.

Keywords: Extension property, strong extension property, strongly Chebyshev, smooth, Fréchet smooth.

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1. Introduction

Let X be a real Banach space and X^* its dual. The closed unit ball and the unit sphere of X are denoted by B_X and S_X respectively. Let Y be a closed subspace of X. For any $f \in X^*$, let $f|_Y$ denote the *restriction* of f to Y and $\|f|_Y\|$ denote the *norm* of f on Y.

The *annihilator* $Y^\perp$ of Y is defined by $Y^\perp = \{f \in X^* : f(y) = 0 \text{ for all } y \in Y\}$.

The *diameter* of a non-empty bounded subset A of X is denoted by $diam(A)$.

For any $x \in X$ and $\delta \geq 0$, we define $P_Y(x, \delta) = \{y \in Y : \|x - y\| \leq d(x, Y) + \delta\}$ where $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$. We write $P_Y(x, 0)$ as $P_Y(x)$ which is the set of all the best approximations of x in Y. We say that Y is *proximinal* (respectively *Chebyshev*) at $x \in X$ if $P_Y(x)$ is non-empty (respectively a singleton). For $x \in X$, a sequence (x_n) in Y is called a *minimizing sequence* for x in Y, if $\|x_n - x\| \to d(x, Y)$.

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We say that Y is *strongly Chebyshev* at $x \in X$, if every minimizing sequence for x in Y converges. If Y is proximinal at every element of a non-empty subset A of X, then we say that Y is proximinal on A. Similarly, we define Chebyshev on A and strongly Chebyshev on A. It is easy to observe that Y is strongly Chebyshev at $x \in X$ if and only if $diam(P_Y(x, \frac{1}{n})) \to 0$. We say that Y is *uniformly strongly Chebyshev* on a non-empty subset A of X if for every $\epsilon > 0$, there exists $\delta > 0$ such that $diam(P_Y(x, \delta)) \leq \epsilon$ for every $x \in A$.

The following definition is due to Phelps [19].

Definition 1.1. The closed subspace Y is said to have the *extension property* in X, if for every $f \in X^*$ there exists a unique $g \in X^*$ such that $g|_Y = f|_Y$ and $\|g\| = \|f|_Y\|$.

The following well known result which relates the extension property in X and the rotundity of X^* is due to Taylor [22] and Foguel [12].

Theorem 1.2. *Every closed subspace of X has the extension property in X if and only if X^* is rotund.*

Phelps [19] characterized the extension property in terms of Chebyshevness as follows.

Theorem 1.3. *The closed subspace Y has the extension property in X if and only if $Y^\perp$ is Chebyshev on X^*.*

In 1972, Zizler introduced a property called uniform extension property in [24] and obtained results which are analogous to Theorems 1.2 and 1.3. Zizler, in fact, characterized the uniform extension property in terms of uniform strong Chebyshevness (see [24, Proposition 4] and Theorem 2.6 of this article) and uniform rotundity (see [24, Proposition 9] and Theorem 5.10 of this article). It is natural to ask whether there exists a property which is stronger than the extension property and which can be characterized in terms of both strong rotundity as well as strong Chebyshevness. One of the main aims of this article is to answer this question.

In this paper, we introduce a property called strong extension property which is stronger than the extension property and weaker than the uniform extension property. We characterize the strong extension property in terms of strong Chebyshevness and strong rotundity which answers the above question. Since the extensions of continuous linear functionals are closely associated with the smoothness aspects, for a given subspace, we consider three different notions of smoothness of this subspace with respect to the whole space as follows.

Definition 1.4. (1) A point $x \in S_X$ is called a *smooth point* of B_X if there is a unique $x^* \in S_{X^*}$ such that $x^*(x) = 1$.

(2) A point $x \in S_X$ is called a *Fréchet smooth point* of B_X if (x_n^*) converges whenever (x_n^*) is a sequence in S_{X^*} satisfying $x_n^*(x) \to 1$.

Definition 1.5. Let Y be a closed subspace of X.

(1) The subspace Y is said to be *smooth* (respectively *Fréchet smooth*) in X if every $y \in S_Y$ is a smooth (respectively Fréchet smooth) point of B_X.

(2) The subspace Y is said to be *uniformly Fréchet smooth* in X if $x_n^* - y_n^* \to 0$ whenever (y_n) is a sequence in S_Y and (x_n^*), (y_n^*) are sequences in S_{X^*} such that $x_n^*(y_n) \to 1$ and $y_n^*(y_n) \to 1$.

The smoothness notions defined in Definition 1.5 are characterized in terms of extension, strong extension and uniform extension properties in this article.

The paper is organized as follows. Section 2 is mainly devoted to the uniform extension property. We reformulate the definition of the uniform extension property in a way that leads to the strong extension property. Based on this reformulation, we present a strengthened version of [24, Proposition 4] which deals with the equivalence of uniform extension property and uniform strong Chebyshevness. Some consequences of this strengthened version are obtained. One of these consequences states that every M-ideal has the uniform extension property.

We introduce the strong extension property in Section 3 and characterize it in terms of strong Chebyshevness and strong rotundity. Sufficient conditions which lead to the strong extension property are also provided. We prove that the strong extension and uniform extension properties coincide when the extensions are considered on finite dimensional subspaces. Further, we show that the extension and strong extension properties coincide when the extensions are considered on finite co-dimensional subspaces. It is observed that the strong and uniform extension properties are transitive in nature. We also present stability results pertaining to extension, strong extension and uniform extension properties.

In Section 4, the smoothness properties defined for subspaces in Definition 1.5 are characterized in terms of Chebyshevness of hyperplanes. Some of these characterizations are needed in Section 5. It is observed that the smoothness property defined for subspaces is preserved under l_p direct sums of closed subspaces, where $1 < p < \infty$. Stability results associated with the smoothness properties considered in Definition 1.5 are also presented in Section 4.

The smoothness, Fréchet smoothness and uniform Fréchet smoothness defined in Definition 1.5 are characterized in terms of extension property, strong extension property and uniform extension property respectively in Section 5.

2. Uniform extension property

In the entire paper, unless specified otherwise, we assume Y to be a closed subspace of a real Banach space X. For the notions which are not defined in this paper, we refer to [17].

By strengthening the extension property, Zizler [24] introduced the following definition.

Definition 2.1. We say that the subspace Y has the *uniform extension property* in X, if $f_n - g_n \to 0$ whenever $(f_n), (g_n)$ are any two sequences in X^* such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $\|f_n|_Y\| = 1$ and $f_n - g_n \in Y^\perp$.

In this section, we first reformulate Definition 2.1 and then primarily study the uniform extension property. We also state two results which deal with the extension property. Some of the results stated in this section are needed for the subsequent sections.

For $f \in X^*$ and $\delta \geq 0$, define

$$E_Y(f,\delta) = \{g \in X^* : g = f \text{ on } Y,\ \|g\| \leq \|f|_Y\| + \delta\}.$$

For $\delta = 0$, we write $E_Y(f,0)$ as $E_Y(f)$ which is the set of all the Hahn-Banach extensions of $f|_Y$ to the whole of X.

It is easy to check that for any $f \in X^*$ and $\delta \geq 0$, the set $E_Y(f,\delta)$ is non-empty, closed, bounded and convex. Further, if $f \in X^*$ and $\|f\| = \|f|_Y\|$, then $f \in E_Y(f,\delta)$ for every $\delta \geq 0$. From the definition of $E_Y(f)$, it is clear that Y has the extension property in X if and only if $E_Y(f)$ is a singleton for every $f \in X^*$. The reformulation of the definition of the uniform extension property in terms of the set $E_Y(f,\delta)$ is observed in Theorem 2.6.

The properties of the set $E_Y(f,\delta)$ which are presented in the next result are easy to verify.

Proposition 2.2. *Let $\delta \geq 0$ and $f, g \in X^*$. Then*

(1) *there exists $h \in Y^\perp$ such that $f + h \in E_Y(f+h)$;*

(2) *$E_Y(f,\delta) = E_Y(f+h,\delta)$ for any $h \in Y^\perp$;*

(3) *$E_Y(f,\delta) \cap E_Y(g,\delta)$ is non-empty if and only if $f - g \in Y^\perp$;*

(4) *$E_Y(\lambda f,\delta) = \lambda E_Y(f,\frac{\delta}{\lambda})$ for any $\lambda > 0$.*

Let $D(Y) = \{x \in S_X : d(x,Y) = 1\}$ and $Y^\# = \{f \in X^* : \|f\| = \|f|_Y\| = 1\}$. Since for any $f \in X^*$, $d(f, Y^\perp) = \|f|_Y\|$, we have $D(Y^\perp) = Y^\#$.

The following result is useful for verifying the extension property.

Proposition 2.3. *If $E_Y(f)$ is a singleton for every $f \in Y^\#$, then Y has the extension property in X.*

Proof. Let $f \in X^*$. If $f \in Y^\perp$, then $E_Y(f) = \{0\}$. Suppose $f \in X^* \backslash Y^\perp$ and $g_1, g_2 \in E_Y(f)$. Then $g_1 = g_2 = f$ on Y and $\|f|_Y\| = \|g_1\| = \|g_2\|$. Denote $\|f|_Y\|$ by α. Note that $\alpha > 0$ and $\frac{g_1}{\alpha} \in Y^\#$. By Proposition 2.2, $\frac{g_1}{\alpha}, \frac{g_2}{\alpha} \in E_Y(\frac{g_1}{\alpha})$. Hence, by the assumption, $g_1 = g_2$. □

The set $\{x \in X : x^*(x) = 0\}$, for any $x^* \in X^* \setminus \{0\}$, is called a hyperplane of X.

The following result shows that if every hyperplane of X containing Y has the extension property in X, then Y has the extension property in X. In fact, it proves more.

Proposition 2.4. *Let $k \in \mathbb{N}$ be such that $1 \leq k < dim(X)$ and Y be a closed subspace of X with $codim(Y) \geq k$. If every closed subspace Z of X, satisfying $Y \subseteq Z$ and $codim(Z) = k$, has the extension property in X, then Y has the extension property in X.*

Proof. Let $k \in \mathbb{N}$ and $1 \leq k < dim(X)$. If $codim(Y) = k$, then there is nothing to prove. Assume $codim(Y) > k$. Suppose Y does not have the extension property in X. Then, by Proposition 2.3, there exists $f \in Y^\#$ and $g_1, g_2 \in E_Y(f)$ such that $g_1 \neq g_2$. Define $H = ker(g_1 - g_2)$. Note that $Y \subseteq H$, $g_1 = g_2$ on H and $\|g_1\| = \|g_2\| = 1$. Now, choose a closed subspace Z of H containing Y such that

$codim(Z) = k$. Since $1 = \|g_2|_Y\| \leq \|g_2|_Z\| \leq \|g_2\| = 1$, we have $\|g_2|_Z\| = 1$. Thus, $g_1, g_2 \in E_Z(g_2)$, which is a contradiction. Hence, Y has the extension property in X. $\square$

In general, the converse of Proposition 2.4 need not be true. To see this, we assume $X = (\mathbb{R}^3, \|\cdot\|_\infty)$, $Y = span\{(0,0,1)\}$ and $H = \{x \in X : x^*(x) = 0\}$ where $x^* = (1,1,0) \in X^*$. The hyperplane H of X contains Y. It is easy to see from Theorem 1.3 that Y has the extension property in X but H does not have the extension property in X.

The following lemma which relates the extensions of continuous linear functionals and the best approximations plays a crucial role in this and the subsequent sections.

Lemma 2.5. *Let $\delta \geq 0$.*

(1) *If $f \in X^*$, then $E_Y(f,\delta) = f - P_{Y^\perp}(f,\delta)$.*

(2) *If Y is reflexive and $x \in X$, then $E_{Y^\perp}(x,\delta) = x - P_Y(x,\delta)$.*

Proof. (1): Let $\delta \geq 0$ and $f \in X^*$. Then

$$\begin{aligned} E_Y(f,\delta) &= \{g \in X^* : f - g \in Y^\perp, \|g\| \leq \|f|_Y\| + \delta\} \\ &= \{f - h \in X^* : h \in Y^\perp, \|f-h\| \leq \|f|_Y\| + \delta\} \\ &= f - \{h \in Y^\perp : \|f-h\| \leq d(f,Y^\perp) + \delta\} = f - P_{Y^\perp}(f,\delta). \end{aligned}$$

(2): Let Y be a reflexive subspace. Then (2) follows from (1) and the fact that $Q(Y) = Y^{\perp\perp}$ where Q is the canonical embedding. $\square$

For $0 \leq \rho \leq \beta$, define $D_\rho^\beta(Y^\perp) = \{f \in X^* : \rho \leq d(f,Y^\perp) \leq \beta\}$.

The following theorem is a reformulated and strengthened version of [24, Proposition 4].

Theorem 2.6. *The following statements are equivalent.*

(1) *Y has the uniform extension property in X.*

(2) *For every $\epsilon > 0$ there exists $\delta > 0$ such that $diam(E_Y(f,\delta)) \leq \epsilon$ for every $f \in Y^\#$.*

(3) *$Y^\perp$ is uniformly strongly Chebyshev on $D(Y^\perp)$.*

(4) *$Y^\perp$ is uniformly strongly Chebyshev on $D_0^\beta(Y^\perp)$ for every $\beta > 0$.*

(5) *$Y^\perp$ is uniformly strongly Chebyshev on M, whenever M is a bounded subset of X^*.*

Proof. (1) $\Rightarrow$ (2): Suppose that (2) does not hold. Then there exist $\epsilon > 0$, a sequence (f_n) in $Y^\#$ and two sequences $(g_n), (h_n) \in X^*$ such that $g_n, h_n \in E_Y(f_n, \frac{1}{n})$ and $\|g_n - h_n\| > \epsilon$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, we have $g_n = f_n = h_n$ on Y and

$$1 = \|f_n|_Y\| = \|g_n|_Y\| \leq \|g_n\| \leq \|f_n|_Y\| + \frac{1}{n} = 1 + \frac{1}{n}.$$

Therefore $\|g_n\| \to 1$. Similarly, we get $\|h_n\| \to 1$. Further, we have $\|g_n|_Y\| = 1$ and $g_n - h_n \in Y^\perp$ for all $n \in \mathbb{N}$. By (1), we get $\|g_n - h_n\| \to 0$ which is a contradiction.
(2) $\Rightarrow$ (1): First, observe from (2) and Proposition 2.3 that Y has the extension property in X.

Let (g_n) and (h_n) be two sequences in X^* such that $\|g_n\| \to 1, \|h_n\| \to 1, \|g_n|_Y\| = 1$ and $g_n - h_n \in Y^\perp$. Let $\epsilon > 0$. By (2), there exists $\delta > 0$ such that $diam(E_Y(f, \delta)) \leq \epsilon$ for every $f \in Y^\#$. Since for each $n \in \mathbb{N}$, $E_Y(g_n)$ is a singleton, denote $E_Y(g_n)$ by f_n. Then $\|g_n|_Y\| = \|f_n\| = 1$ and $g_n = f_n$ on Y for all $n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $\|g_n\| \leq 1 + \delta$ and $\|h_n\| \leq 1 + \delta$ for every $n \geq n_0$. Since $\|f_n|_Y\| = 1$ for all $n \in \mathbb{N}$, for every $n \geq n_0$, $\|g_n\| \leq \|f_n|_Y\| + \delta$, $\|h_n\| \leq \|f_n|_Y\| + \delta$ and $g_n = h_n = f_n$ on Y. This implies that $g_n, h_n \in E_Y(f_n, \delta)$ for every $n \geq n_0$. By our assumption $\|g_n - h_n\| \leq \epsilon$ for every $n \geq n_0$. Hence, $g_n - h_n \to 0$.

(2) $\Leftrightarrow$ (3): These implications follow from Lemma 2.5.

(3) $\Rightarrow$ (4): We first show that $Y^\perp$ is Chebyshev on X^*. Let $x^* \in X^* \setminus Y^\perp$. Since $Y^\perp$ is proximinal on X^*, choose $y^* \in P_{Y^\perp}(x^*)$. Let $d = d(x^*, Y^\perp)$ and $x_0^* = \frac{x^* - y^*}{d}$. Then $x_0^* \in D(Y^\perp)$ and $P_{Y^\perp}(x_0^*) = \frac{1}{d} P_{Y^\perp}(x^* - y^*) = \frac{1}{d}[P_{Y^\perp}(x^*) - y^*]$. Hence, $P_{Y^\perp}(x^*)$ is a singleton which establishes that $Y^\perp$ is Chebyshev on X^*. Let $\beta > 0$. We now show that $Y^\perp$ is uniformly strongly Chebyshev on $D_0^\beta(Y^\perp)$. Let $\epsilon > 0$. Choose $\rho = \min\{\frac{\beta}{2}, \frac{\epsilon}{4}\}$ and $0 < \delta_1 < \frac{\epsilon}{4}$. Let x^* be any element of $D_0^\rho(Y^\perp)$. If $y_1^*, y_2^* \in P_{Y^\perp}(x^*, \delta_1)$, then

$$\|y_1^* - y_2^*\| \leq \|y_1^* - x^*\| + \|x^* - y_2\| \leq 2d(x^*, Y^\perp) + 2\delta_1 \leq 2(\rho + \delta_1) < \epsilon.$$

Thus, $diam(P_{Y^\perp}(x^*, \delta_1)) \leq \epsilon$ for all $x^* \in D_0^\rho(Y^\perp)$. It follows from [10, Theorem 2.2] that $Y^\perp$ is uniformly strongly Chebyshev on $D_\rho^\beta(Y^\perp)$. Hence, there exists $\delta_2 > 0$ such that $diam(P_{Y^\perp}(x^*, \delta_2)) < \epsilon$ for all $x^* \in D_\rho^\beta(Y^\perp)$. Thus, $Y^\perp$ is uniformly strongly Chebyshev on $D_0^\beta(Y^\perp)$.

(4) $\Rightarrow$ (5): If M is a bounded subset of X^*, then $M \subset D_0^\beta(Y^\perp)$ for some $\beta > 0$. Hence, (5) follows from (4).

(5) $\Rightarrow$ (3): This implication is obvious. $\square$

We remark that the implications (1) $\Leftrightarrow$ (5) of Theorem 2.6 are established in [24, Proposition 4]. We now present some applications of Theorem 2.6.

The following result is an immediate consequence of Lemma 2.5 and Theorem 2.6.

Corollary 2.7. *The following statements are equivalent.*

(1) *Y has the uniform extension property in X.*

(2) *For every $\epsilon > 0$ and $\beta > 0$ there exists $\delta > 0$ such that $diam(E_Y(f, \delta)) \leq \epsilon$ for every $f \in X^*$ satisfying $\|f|_Y\| \leq \beta$.*

(3) *For every $\epsilon > 0$ and every bounded subset M of X^* there exists $\delta > 0$ such that $diam(E_Y(f, \delta)) \leq \epsilon$ for every $f \in M$.*

It is known that every M-ideal has the extension property (see [13, Proposition 1.12]). As a consequence of Theorem 2.6, we observe in the following corollary that every M-ideal has, in fact, the uniform extension property. We refer to [13] for the definitions and properties of M-ideal, L-summand and $1\frac{1}{2}$-ball property.

The following result will be used in Section 5.

Corollary 2.8. *If Y is an M-ideal in X, then it has the uniform extension property in X.*

Proof. Let Y be an M-ideal in X. Then $Y^{\perp}$ is an L-summand in X^*. It follows from [13, page 96] that $Y^{\perp}$ is Chebyshev on X^* and it has the $1\frac{1}{2}$-ball property. By [10, Proposition 2.11], $Y^{\perp}$ is uniformly strongly Chebyshev on the entire X^*. Hence, by Theorem 2.6, Y has the uniform extension property in X. $\square$

An inspection of the proof of Corollary 2.8 shows that if Y is an L-summand in X, then it is uniformly strongly Chebyshev on X.

Let $CB(X)$ denote the class of all non-empty closed and bounded subsets of X. Note that for a given closed subspace Y of X and $\delta > 0$, $E_Y : X^* \to CB(X^*)$, $E_Y(\cdot, \delta) : X^* \to CB(X^*)$ and $P_Y : X \to CB(X)$. Observe that E_Y is a single-valued map from X^* to X^* if and only if Y has the extension property in X. The following example illustrates that if Y has the extension property in X, then the map E_Y need not be continuous on X^*.

Example 2.9. It is known [3] that there exists a closed subspace Z of a rotund reflexive space X such that the single-valued map P_Z is discontinuous on X. Let Y be the closed subspace of X^* such that $Z = Y^{\perp}$. Then, by Lemma 2.5, E_Y is a single-valued map from X^* to X^* but it is not continuous. $\square$

The proof of the following result follows from Lemma 2.5, Theorem 2.6 and [10, Remark 2.3]. However, we present it here for reader's convenience.

Corollary 2.10. *If Y has the uniform extension property in X, then E_Y is uniformly continuous on $D_0^{\beta}(Y^{\perp})$ for all $\beta > 0$. In particular, E_Y is uniformly continuous on bounded subsets of X^*.*

Proof. Let $\beta > 0$. It follows from Theorem 2.6 that $Y^{\perp}$ is uniformly strongly Chebyshev on $D_0^{\beta}(Y^{\perp})$. Now assume $\epsilon > 0$. Then there exists $\delta > 0$ such that $diam(P_{Y^{\perp}}(x^*, \delta)) \leq \epsilon$ for all $x^* \in D_0^{\beta}(Y^{\perp})$. Let $x^*, y^* \in D_0^{\beta}(Y^{\perp})$ be such that we have $\|x^* - y^*\| \leq \frac{\delta}{3}$. Then $P_{Y^{\perp}}(y^*) \in P_{Y^{\perp}}(x^*, \delta)$ which implies immediately that $\|P_{Y^{\perp}}(x^*) - P_{Y^{\perp}}(y^*)\| \leq \epsilon$. Since the map $P_{Y^{\perp}}$ is uniformly continuous on $D_0^{\beta}(Y^{\perp})$, by Lemma 2.5, E_Y is uniformly continuous on $D_0^{\beta}(Y^{\perp})$. $\square$

It follows from Corollary 2.10 that the space Y defined in Example 2.9 does not have the uniform extension property in X. In this case, since E_Y is a single-valued map from X^* to X^*, Y has the extension property in X.

For $A, B \in CB(X)$, the Hausdorff distance $\mathcal{H}(A, B)$ between A and B is defined to be $\max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$.

Example 2.9 shows that, in case $\delta = 0$, the map $E_Y(\cdot, \delta) : X^* \to (CB(X^*), \mathcal{H})$ need not be continuous even if Y has the extension property in X. The following example illustrates that the map $E_Y : X^* \to (CB(X^*), \mathcal{H})$ may fail to be continuous even if X is finite dimensional.

Example 2.11. It is shown in [8] that there exists a one dimensional subspace Z of a three dimensional space X such that the set-valued map $P_Z : X \to (CB(X), \mathcal{H})$ is not continuous. Let Y be the subspace of X^* such that $Z = Y^{\perp}$. By Lemma 2.5, $E_Y(\cdot, 0)$ is not continuous. $\square$

If $\delta > 0$, without any additional assumption either on X or Y, the map $E_Y(\cdot, \delta)$ has the continuity property as stated in the following result.

Theorem 2.12. *If $\delta > 0$, then $E_Y(\cdot, \delta) : X^* \to (CB(X^*), \mathcal{H})$ is uniformly continuous on bounded subsets of X^*.*

Proof. The proof is derived from [15, Theorem 2.10] and Lemma 2.5 as follows. For $F \in CB(X^*)$ and $f \in X^*$, let

$$r(f, F) = \sup\{\|f - g\| : g \in F\} \quad \text{and} \quad rad_{Y^\perp}(F) = \inf\{r(g, F) : g \in Y^\perp\}.$$

Let $\delta > 0$. Define $\delta - cent_{Y^\perp}(F) = \{g \in Y^\perp : r(g, F) \leq rad_{Y^\perp}(F) + \delta\}$. Observe that, for any $f \in X^*$, $\delta - cent_{Y^\perp}(\{f\}) = P_{Y^\perp}(f, \delta)$. It follows from [15, Theorem 2.10] that the map $P_{Y^\perp}(\cdot, \delta)$ is uniformly continuous on bounded subsets of X^*. Hence, by Lemma 2.5, the map $E_Y(\cdot, \delta)$ is uniformly continuous on bounded subsets of X^*. □

The subspace Y considered in Example 2.11 is of dimension 2 and the corresponding map E_Y is discontinuous. The following result shows that if the subspace Y is of dimension one, then the map E_Y is uniformly continuous.

Theorem 2.13. *If Y is a one dimensional subspace of the space X, then the map $E_Y : X^* \to (CB(X^*), \mathcal{H})$ is uniformly continuous.*

Proof. Let $Y = span\{x\}$ for some $x \in S_X$ and $J_{X^*}(x) = \{g \in S_{X^*} : g(x) = 1\}$. Let $f \in X^*$. Observe that

$$f - P_{Y^\perp}(f) = f - \big(f - f(x)J_{X^*}(x)\big) = f(x)J_{X^*}(x).$$

It follows from the above observation and Lemma 2.5 that $E_Y(f) = f(x)J_{X^*}(x)$. Now, it is easy to verify that E_Y is uniformly continuous on X^*. □

The following theorem deals with uniform versions of Theorems 1.2 and 1.3 of [19].

Theorem 2.14. *Consider the following statements.*

(1) *$Y^\perp$ has the uniform extension property in X^*.*

(2) *Y is uniformly strongly Chebyshev on $D(Y)$.*

Then (1) $\Rightarrow$ (2). *Further, if X is reflexive, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): We first prove that Y is proximinal on X. Let $x \in X$ and (w_n) be a minimizing sequence for x in Y. Since $d(x, Y) = d(Q(x), Y^{\perp\perp})$ (see [4, Corollary 4.10] also [14, Lemma 2.1]), $(Q(w_n))$ is a minimizing sequence for $Q(x)$ in $Y^{\perp\perp}$. It follows from (1) and Theorem 2.6 that $Y^{\perp\perp}$ is strongly Chebyshev in X^{**}. Hence, $Q(w_n)$ converges to $Q(y)$ for some $y \in Y$. Therefore $Q(y) \in P_{Y^{\perp\perp}}(Q(x))$. Since $d(x, Y) = d(Q(x), Y^{\perp\perp})$, $y \in P_Y(x)$. Thus, Y is proximinal on X. Let (x_n) be a sequence in $D(Y)$. Suppose that (y_n) and (z_n) are sequences in Y such that $\|y_n - x_n\| \to 1$ and $\|z_n - x_n\| \to 1$. It is enough to show that $\|y_n - z_n\| \to 0$.

Observe that $d(x_n, Y) = d(Q(x_n), Y^{\perp\perp}) = 1$. By (1) and Theorem 2.6, $Y^{\perp\perp}$ is uniformly strongly Chebyshev on $D(Y^{\perp\perp})$. Therefore $\|Q(y_n) - Q(z_n)\| \to 0$. Thus, $\|y_n - z_n\| \to 0$.

Let X be reflexive and Y be uniformly strongly Chebyshev on $D(Y)$. Then $Y^{\perp\perp}$ is uniformly strongly Chebyshev on $D(Y^{\perp\perp})$. Therefore, by Theorem 2.6, $Y^\perp$ has the uniform extension property in X^* which proves the implication (2) $\Rightarrow$ (1). □

The following result which will be used in the next section follows from [24, Proposition 4] and [24, Proposition 7].

Theorem 2.15. *The following statements are equivalent.*

(1) *X^* is uniformly rotund in every direction.*

(2) *Every hyperplane of X has the uniform extension property in X.*

Proof. It follows from [24, Proposition 7] that X^* is uniformly rotund in every direction if and only if every one dimensional subspace of X^* is uniformly strongly Chebyshev on bounded subsets of X^*. The result now follows from [24, Proposition 4] or Theorem 2.6. □

3. Strong extension property

In this section, we introduce and study the strong extension property.

Definition 3.1. The subspace Y is said to have the strong extension property in X if for every $f \in X^*$ and $\epsilon > 0$ there exists $\delta > 0$ such that $diam(E_Y(f, \delta)) \leq \epsilon$.

It is clear that if Y has the strong extension property in X, then it has the extension property in X. We observe from Corollary 2.7 that if Y has the uniform extension property in X, then it has the strong extension property in X. We will present some examples to illustrate that the converses of the above two statements are not true. Before presenting the examples, we first characterize the strong extension property and provide different sufficient conditions which lead to the strong extension property.

Theorem 3.2. *The following statements are equivalent.*

(1) *Y has the strong extension property in X.*

(2) *For every $f \in Y^\#$ and $\epsilon > 0$ there exists $\delta > 0$ such that $diam(E_Y(f, \delta)) \leq \epsilon$.*

(3) *If $f \in Y^\#$ and (g_n) is a sequence in X^* such that $\|g_n\| \to \|f|_Y\|$ and $g_n - f \in Y^\perp$, then (g_n) converges to f.*

(4) *$Y^\perp$ is strongly Chebyshev on X^*.*

Proof. (1) ⇒ (2) ⇔ (3): These implications follow from Definition 3.1.
(2) ⇒ (1): Let $\epsilon > 0$. If $f \in Y^\perp$, then $diam(E_Y(f, \delta)) \leq 2\delta$ for all $\delta \geq 0$. Let $f \in X^* \backslash Y^\perp$. By the Hahn-Banach theorem, there exists $h \in X^*$ such that $h = f$ on Y and $\|h\| = \|f|_Y\|$. Denote $\|f|_Y\|$ by α. Since $\frac{h}{\alpha} \in Y^\#$, by (2), there exists $\delta > 0$ such that $diam(E_Y(\frac{h}{\alpha}, \delta)) \leq \frac{\epsilon}{\alpha}$. By Proposition 2.2, we have $E_Y(\frac{h}{\alpha}, \delta) = \frac{1}{\alpha} E_Y(h, \alpha\delta) = \frac{1}{\alpha} E_Y(f, \alpha\delta)$. Hence, $diam(E_Y(f, \alpha\delta)) \leq \epsilon$ which proves that Y has the strong extension property in X.
(1) ⇔ (4): These implications follow from Lemma 2.5. □

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let Y be a w^*-closed subspace of X^*. Then the following statements are equivalent.*

(1) *${}^\perp Y$ has the strong extension property in X, where*
$${}^\perp Y = \{x \in X : f(x) = 0 \text{ for every} f \in Y\}.$$

(2) *Y is strongly Chebyshev on X^*.*

We now establish a result which is analogous to Theorem 2.14.

Theorem 3.4. *Consider the following statements.*

(1) *$Y^\perp$ has the strong extension property in X^*.*

(2) *Y is strongly Chebyshev on X.*

Then (1) $\Rightarrow$ (2). *Further, if X is reflexive, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): Let $x \in X \backslash Y$ and (x_n) be a minimizing sequence for x in Y. Since $d(x, Y) = d(Q(x), Y^{\perp\perp})$, $Q(x_n)$ is a minimizing sequence for $Q(x)$ in $Y^{\perp\perp}$. It follows from (1) and Theorem 3.2 that $Y^{\perp\perp}$ is strongly Chebyshev on X^{**}. Hence, $Q(x_n)$ converges to $Q(y)$ for some $y \in Y$. Therefore $x_n \to y$ which shows that Y is strongly Chebyshev on X.
If X is reflexive, then by (2), $Y^{\perp\perp}$ is strongly Chebyshev on X^{**}. Therefore by Theorem 3.2, $Y^\perp$ has the strong extension property in X^*. □

The implication (2) $\Rightarrow$ (1) of Theorem 3.4 need not be true even if Y is reflexive. This can be verified as follows. It is shown in [19, page 252] that there exists a finite dimensional subspace Y of c_0 such that Y is Chebyshev on c_0, but $Y^\perp$ does not have the extension property in $(c_0)^*$. Since Y is finite dimensional and Chebyshev on c_0, it is strongly Chebyshev on c_0.

Next we present different sufficient conditions which lead to the space Y to have the strong extension property in X. For specifying the sufficient conditions, we need the following definitions.

1. We say that the space X has the *Kadets-Klee property* (in short, *KK-property*) if for any sequence on S_X, the norm and weak convergence coincide.

2. The dual space X^* is said to have the *Namioka-Phelps property*, if weak* and norm topologies coincide on S_{X^*}.

3. The space X is said to be *compactly locally uniformly rotund* [23] if (x_n) has a convergent subsequence whenever $x \in S_X$ and (x_n) is a sequence in X with $\|x_n\| \to 1$ and $\|x_n + x\| \to 2$.

Theorem 3.5. *Let Y have the extension property in X. Then Y has the strong extension property in X if any one of the following statements holds.*

(1) *Y has finite co-dimension.*

(2) *X^* has the Namioka-Phelps property.*

(3) *X^* is reflexive and has the KK-property.*

(4) *X^* is compactly locally uniformly rotund.*

Proof. (1): Let $f \in Y^\#$ and (g_n) be a sequence in X^* such that $\|g_n\| \to \|f|_Y\|$ and $g_n - f \in Y^\perp$. By Theorem 3.2, it is enough to show that (g_n) converges to f. Since $g_n - f \in Y^\perp$ and $Y^\perp$ is finite dimensional, there exists a subsequence (g_{n_k}) of (g_n) such that $g_{n_k} \to f + g$ for some $g \in Y^\perp$. Observe that $\|f + g\| = \|f|_Y\|$ which implies that $f + g \in E_Y(f)$. Since Y has the extension property in X, $E_Y(f) = \{f\}$. Thus, $g_n \to f$. Hence, Y has the strong extension property in X.

(2): Let $f \in Y^\#$ and (g_n) be a sequence in X^* such that $\|g_n\| \to \|f|_Y\|$ and $g_n - f \in Y^\perp$. Since (g_n) is a bounded sequence, there exist a subnet (g_{n_α}) of (g_n)

and $g \in X^*$ such that $g_{n_\alpha} \xrightarrow{w^*} g$. Observe that for every $y \in Y$, $g_{n_\alpha}(y) \to g(y)$. This shows that $f = g$ on Y. Since

$$\|f|_Y\| = \|g|_Y\| \leq \|g\| \leq \liminf \|g_{n_\alpha}\| = \|f|_Y\|,$$

it follows that $\|g\| = \|f|_Y\|$. Hence, $g \in E_Y(f)$. Note that $\|g_{n_\alpha}\| \to \|g\|$. Hence, it follows from the Namioka-Phelps property of X^* that $g_{n_\alpha} \to g$. Since Y has the extension property in X, $g = f$. Thus, $g_n \to f$. Hence, by Theorem 3.2, Y has the strong extension property in X.

(3): Since X^* is reflexive, every bounded sequence of X^* has a weakly convergent subsequence. Hence, the proof of (3) follows by replacing w^*-convergent subnet by weakly convergent subsequence and repeating the argument used in the proof of (2).

(4): Let $f \in Y^\#$ and (g_n) be a sequence in X^* such that $\|g_n\| \to \|f|_Y\|$ and $g_n - f \in Y^\perp$. Since,

$$1 = \|f|_Y\| = \left\|\frac{g_n + f}{2}|_Y\right\| \leq \left\|\frac{g_n + f}{2}\right\| \leq \frac{\|g_n\| + \|f\|}{2},$$

we conclude that $\left\|\frac{g_n+f}{2}\right\| \to 1$. Since X^* is compactly locally uniformly rotund, there exists a subsequence (g_{n_k}) of (g_n) such that $g_{n_k} \to g$ for some $g \in X^*$. Note that $\|g\| = \|f|_Y\|$ and $g = f$ on Y. Since $E_Y(f) = \{f\}$, it follows that $g = f$. Thus, $g_n \to f$. Hence, by Theorem 3.2, Y has the strong extension property in X. □

From Theorem 3.5, we conclude that the extension and strong extension properties coincide for finite co-dimensional subspaces.

Corollary 3.6. *Let $k \in \mathbb{N}$ be such that $1 \leq k < dim(X)$. Then the following statements are equivalent.*

(1) *X^* is rotund.*

(2) *Every closed subspace Y of X with $codim(Y) = k$ has the extension property in X.*

(3) *Every closed subspace Y of X with $codim(Y) = k$ has the strong extension property in X.*

Proof. If (3) holds, then it follows from Theorem 3.2 that every k-dimensional subspace of X^* is Chebyshev on X^*. This proves the implication (3) ⇒ (1). The implications (1) ⇒ (2) and (2) ⇒ (3) follow from Theorem 1.2 and Theorem 3.5 respectively. □

We remark that the equivalence of (1) and (2) of Corollary 3.6 for $k = 1$ is observed by Phelps [19, page 240].

The following corollary is an immediate consequence of Theorem 3.5 and [7, Theorem 5.3].

Corollary 3.7. *Let X be an L_1-predual space and Y be of finite co-dimension. Then the following statements are equivalent.*

(1) *Y has the extension property in X.*

(2) *$Y^{\perp\perp}$ has the extension property in X^{**}.*

(3) *$Y^{\perp\perp}$ has the strong extension property in X^{**}.*

The next result will be used in this section as well as in Section 5. Moreover, it also leads to an interesting observation which will be discussed in Remark 5.14.

Corollary 3.8. *If X^* is locally uniformly rotund, then every closed subspace of X has the strong extension property in X.*

Proof. Let Y be a closed subspace of X. Since X^* is rotund and compactly locally uniformly rotund, by Theorem 1.2 and Theorem 3.5, Y has the strong extension property in X. □

We now characterize the strong rotundity of X^* in terms of the strong extension property in X.

Theorem 3.9. *The following statements are equivalent.*

(1) *X^* is strongly rotund.*

(2) *X is reflexive and every closed subspace of X has the strong extension property in X.*

Proof. (1) $\Rightarrow$ (2): This implication follows from Theorem 1.2 and Theorem 3.5.
(2) $\Rightarrow$ (1): It follows from (2) and Theorem 3.2 that every closed subspace of X^* is strongly Chebyshev on X^*. Hence, by [20, Theorem 35], X^* is strongly rotund. □

We now present an example of Y which has the strong extension property but it does not have the uniform extension property.

Example 3.10. Consider X such that X^* is locally uniformly rotund but X^* is not uniformly rotund in every direction (see [21, Example 4]). By Theorem 2.15, there exists a hyperplane Y of X such that Y does not have the uniform extension property in X. Observe that by Corollary 3.8, Y has the strong extension property in X. □

The following two examples illustrate that the strong extension property is stronger than the extension property.

Example 3.11. Let $X = (l_1, \|\cdot\|_1)$, $x_0 = (\frac{1}{2^n}) \in S_X$ and $Y = span\{x_0\}$. Let $f \in Y^{\#}$. Then $f(x_0) = \pm 1$. First, we show that $E_Y(f)$ is a singleton. Assume $f(x_0) = 1$. Then $f = (1, 1, \dots)$ which is an element of $(l_\infty, \|\cdot\|_\infty)$. Thus we have $E_Y(f) = \{f\}$. Similarly, if $f(x_0) = -1$, we can show that $E_Y(f)$ is a singleton. By Proposition 2.3, Y has the extension property in X. We now show that Y does not have the strong extension property in X. Let $f = (1, 1, \dots) \in Y^{\#}$. Define $g_n = \frac{-1}{2^n} e_1 + e_{n+1}$ and $h_n = f - g_n$ for all $n \in \mathbb{N}$. Observe that $h_n \in X^*$ for all $n \in \mathbb{N}$. Note that $\|h_n\|_\infty = 1 + \frac{1}{2^n} = \|f|_Y\| + \frac{1}{2^n}$. Let $y \in Y$. Then $y = (\frac{\alpha}{2^n})$ for some $\alpha \in \mathbb{R}$ and

$$h_n(y) = \alpha(\frac{1}{2} + \frac{1}{2^{n+1}} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \dots) = \sum_{n=1}^{\infty} \frac{\alpha}{2^n} = f(y).$$

Thus, $h_n \in E_Y(f, \frac{1}{2^n})$ and $\|h_n - h_{n+1}\|_\infty = 1$ for all $n \in \mathbb{N}$. Therefore (h_n) does not converge. Hence, Y does not have the strong extension property in X. □

Example 3.12. As observed in Example 2.9, there exists a closed subspace Z of a rotund reflexive space X such that the map P_Z is discontinuous. Observe that Z is

Chebyshev on X. It follows from [2, Theorem 2.5] that Z is not strongly Chebyshev on X. By [19, Theorem 1.2] and Theorem 3.4, $Z^\perp$ has the extension property in X^* but it does not have the strong extension property in X^*. □

We see from Example 3.11 that the extension and strong extension properties do not coincide even when the extensions are considered on finite dimensional subspaces. We now establish that the strong extension property and the uniform extension property do coincide if we consider the extensions on finite dimensional subspaces.

We need the following lemma.

Lemma 3.13. *If Y has the strong extension property in X, then the maps $P_{Y^\perp}$ and E_Y are continuous on X^*.*

Proof. Suppose Y has the strong extension property in X. Then, by Theorem 3.2, $Y^\perp$ is strongly Chebyshev on X^*. The proof of the continuity of $P_{Y^\perp}$ on X^* follows from [2, Theorem 2.5]. By Lemma 2.5, E_Y is continuous on X^*. □

Theorem 3.14. *Let Y be a finite dimensional subspace of X. If Y has the strong extension property in X, then it has the uniform extension property in X.*

Proof. Suppose Y is finite dimensional and has the strong extension property in X. We first show that $Y^\#$ is compact. By Lemma 3.13, it follows that $P_{Y^\perp}$ is continuous on X^*. Since $Y^\perp$ is finite co-dimensional and Chebyshev on X^*, by [5, Theorem 8],

$$P_{Y^\perp}^{-1}(0) \cap \{x^* \in X^* : \|x^* - y^*\| \leq r\}$$

is compact for all $y^* \in X^*$ and $r > 0$. Observe that $Y^\# = P_{Y^\perp}^{-1}(0) \cap S_{X^*}$. Hence, $Y^\#$ is compact. Let (f_n) be a sequence in $Y^\#$. Suppose that (g_n) and (h_n) are any two sequences X^* such that $g_n, h_n \in E_Y(f_n, \frac{1}{n})$ for all $n \in \mathbb{N}$. By Theorem 2.6, it is enough to show that $g_n - h_n \to 0$. Since $Y^\#$ is compact, there exists a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f_0$ for some $f_0 \in Y^\#$. Since

$$\|f_0|_Y\| \leq \|f_0 - f_{n_k} + g_{n_k}\| \leq \|f_0 - f_{n_k}\| + \|g_{n_k}\| \leq \|f_0 - f_{n_k}\| + \|f_{n_k}|_Y\| + \frac{1}{n_k},$$

it follows that $\|f_0 - f_{n_k} + g_{n_k}\| \to \|f_0|_Y\|$. Similarly, $\|f_0 - f_{n_k} + h_{n_k}\| \to \|f_0|_Y\|$. Note that $f_0 - f_{n_k} + g_{n_k} = f_0 - f_{n_k} + h_{n_k} = f_0$ on Y. Since Y has the strong extension property, $g_{n_k} - h_{n_k} \to 0$. Therefore $g_n - h_n \to 0$. □

The following corollary follows immediately from Theorem 3.5 and Theorem 3.14.

Corollary 3.15. *If X is finite dimensional, then the following statements are equivalent.*

(1) *Y has the extension property in X.*

(2) *Y has the strong extension property in X.*

(3) *Y has the uniform extension property in X.*

We remark that Corollary 3.15 can also be obtained from a remark made in [24, page 381].

We now discuss some stability results for the extension, strong extension and uniform extension properties. The following stability result for the extension property is obtained in [6, Theorem 3.1].

Theorem 3.16. *Let Y be a closed subspace of X with dim$(Y) \geq 2$. Then the following statements are equivalent.*

(1) *Y has the extension property in X.*

(2) *Y/Z has the extension property in X/Z whenever Z is a proper closed subspace of Y.*

(3) *Y/H has the extension property in X/H whenever H is a hyperplane of Y.*

We first present a stability result for the extension property which strengthens Theorem 3.16. We then present similar stability results for the strong extension and uniform extension properties.

We need the following two preliminary results.

Lemma 3.17. *Let Z and Y be any two closed subspaces of X such that $Z \subset Y \subseteq X$, and $q : X \to X/Z$ be the canonical map. Then*

(1) *the map $T : (Y/Z)^{\perp} \to Y^{\perp}$, defined by $T(f) = f \circ q$ for all $f \in (Y/Z)^{\perp}$, is an onto isometric isomorphism;*

(2) *$d(f, (Y/Z)^{\perp}) = d(f \circ q, Y^{\perp})$ for any $f \in (X/Z)^*$.*

Proof. (1) Let B_X° and $B_{X/Z}^{\circ}$ denote the open unit balls of X and X/Z respectively. It is known that $q(B_X^{\circ}) = B_{X/Z}^{\circ}$ (see [17, Lemma 1.7.11]). Observe that T is linear and for any $h \in (X/Z)^*$ we have,

$$\|h \circ q\| = \sup\{|(h \circ q)(x)| : x \in B_X^{\circ}\} = \sup\{|h(x + Z)| : x + Z \in B_{X/Z}^{\circ}\} = \|h\|.$$

Therefore T is an isometry. If $g \in Y^{\perp}$, then $Z \subset ker(g)$. By [17, Theorem 1.7.13], there exists a unique $f \in (X/Z)^*$ such that $g = f \circ q$. Observe that $f \in (Y/Z)^{\perp}$. Hence, T is an onto isometric isomorphism.

(2): Let $f \in (X/Z)^*$. From (1), we have

$$\begin{aligned} d(f \circ q, Y^{\perp}) &= \inf\{\|f \circ q - g \circ q\| : g \in (Y/Z)^{\perp}\} = \inf\{\|f - g\| : g \in (Y/Z)^{\perp}\} \\ &= d(f, (Y/Z)^{\perp}). \end{aligned}$$

□

The isometric isomorphism given in the following result will be used in this section and the subsequent sections.

Lemma 3.18. [17] *Let Y be a proper closed subspace of X and $q : X \to X/Y$ be the canonical map. Then the map $S : (X/Y)^* \to Y^{\perp}$ defined by $S(f) = f \circ q$ for all $f \in (X/Y)^*$, is an onto isometric isomorphism.*

The statement (2) of the following result can be added to the equivalent statements given in Theorem 3.16.

Theorem 3.19. *Let Y be a closed subspace of X with dim$(Y) \geq 2$. Then the following statements are equivalent.*

(1) *Y has the extension property in X.*

(2) *Y/F has the extension property in X/F whenever F is a one dimensional subspace of Y.*

Proof. (1) $\Rightarrow$ (2): This implication follows from Theorem 3.16.

(2) $\Rightarrow$ (1): Let $f \in Y^{\#}$ and $g \in E_Y(f)$. Since $dim(Y) \geq 2$, there exists a non-zero element $y \in ker(f) \cap Y$. Let $F = span\{y\}$ and $q : X \to X/F$ be the canonical map. By Lemma 3.18, the map $S : (X/F)^* \to F^{\perp}$, defined by $S(\tilde{f}) = \tilde{f} \circ q$ for any $\tilde{f} \in (X/F)^*$, is an onto isometric isomorphism. Since $g, f \in F^{\perp}$ and S is onto, there exist $\tilde{g}, \tilde{f} \in (X/F)^*$ such that $\tilde{g} \circ q = g$ and $\tilde{f} \circ q = f$. By Lemma 3.17, we have

$$1 = \|f|_Y\| = d(\tilde{f} \circ q, Y^{\perp}) = d(\tilde{f}, (Y/F)^{\perp}) = \|\tilde{f}|_{Y/F}\|.$$

Since $\tilde{f} = \tilde{g}$ on Y/F and $\|\tilde{f}|_{X/F}\| = \|\tilde{g}|_{X/F}\| = 1$, we have $\tilde{g}, \tilde{f} \in E_{(Y/F)}(\tilde{f})$. By (2), $\tilde{g} = \tilde{f}$ which implies that $g = f$. □

Since the proofs of the stability results for the strong extension property and the uniform extension property are similar, we first present a stability result for the uniform extension property.

Theorem 3.20. *Let Y be a closed subspace of X with $dim(Y) \geq 2$. Then the following statements are equivalent.*

(1) *Y has the uniform extension property in X.*

(2) *For every $n \in \mathbb{N}$, let Z_n be any proper closed subspace of Y, $h_n \in (Y/Z_n)^{\#}$ and $f_n, g_n \in (X/Z_n)^*$. If $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in (Y/Z_n)^{\perp}$ and $g_n - h_n \in (Y/Z_n)^{\perp}$, then $\|f_n - g_n\| \to 0$.*

(3) *For every $n \in \mathbb{N}$, let H_n be any hyperplane of the subspace Y, $h_n \in (Y/H_n)^{\#}$ and $f_n, g_n \in (X/H_n)^*$. If $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in (Y/H_n)^{\perp}$ and $g_n - h_n \in (Y/H_n)^{\perp}$, then $\|f_n - g_n\| \to 0$.*

(4) *For every $n \in \mathbb{N}$, let F_n be any one dimensional subspace of Y, $h_n \in (Y/F_n)^{\#}$ and $f_n, g_n \in (X/F_n)^*$. If $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in (Y/F_n)^{\perp}$ and $g_n - h_n \in (Y/F_n)^{\perp}$, then $\|f_n - g_n\| \to 0$.*

Proof. (1) $\Rightarrow$ (2): For each $n \in \mathbb{N}$, let Z_n be a proper closed subspace of Y and (h_n) be a sequence in $D((Y/Z_n)^{\perp})$. Let $(\tilde{f_n})$ and $(\tilde{g_n})$ be sequences in $(Y/Z_n)^{\perp}$ such that $\|\tilde{f_n} - h_n\| \to 1$ and $\|\tilde{g_n} - h_n\| \to 1$. To prove (2), it is enough to show that $\|\tilde{f_n} - \tilde{g_n}\| \to 0$. For each $n \in \mathbb{N}$, let $q_n : X \to X/Z_n$ be the canonical map. By Lemma 3.17, we have $(h_n \circ q_n) \in D(Y^{\perp})$ and $(\tilde{f_n} \circ q_n)$, $(\tilde{g_n} \circ q_n) \in Y^{\perp}$ for all $n \in \mathbb{N}$. Further, $\|\tilde{f_n} \circ q_n - h_n \circ q_n\| \to 1$ and $\|\tilde{g_n} \circ q_n - h_n \circ q_n\| \to 1$. By (1) and Theorem 2.6, $Y^{\perp}$ is uniformly strongly Chebyshev on $D(Y^{\perp})$. Therefore, it follows that $\|\tilde{f_n} \circ q_n - \tilde{g_n} \circ q_n\| \to 0$. Hence, by Lemma 3.17, $\|\tilde{f_n} - \tilde{g_n}\| \to 0$.

(2) $\Rightarrow$ (3): This implication is obvious.

(3) $\Rightarrow$ (1): Let $h_n \in Y^{\#}$ and $f_n, g_n \in X^*$ be such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in Y^{\perp}$ and $g_n - h_n \in Y^{\perp}$. By (1) $\Leftrightarrow$ (2) of Theorem 2.6, it is enough to show that $\|f_n - g_n\| \to 0$. Define $H_n = ker(h_n) \cap Y$. Since $\|h_n|_Y\| = 1$, H_n is a hyperplane of Y. Let $q_n : X \to X/H_n$ be the canonical map. By Lemma 3.18, for every $n \in \mathbb{N}$ there exists an onto isometric isomorphism S_n from $(X/H_n)^*$ to $H_n^{\perp}$, defined by $S_n(\tilde{f}) = \tilde{f} \circ q_n$ for any $\tilde{f} \in (X/H_n)^*$. Since $f_n, g_n, h_n \in H_n^{\perp}$ and S_n is onto, there exist $\tilde{f_n}, \tilde{g_n}, \tilde{h_n} \in (X/H_n)^*$ such that $\tilde{f_n} \circ q_n = f_n$, $\tilde{g_n} \circ q_n = g_n$ and $\tilde{h_n} \circ q_n = h_n$. By Lemma 3.17, we have

$$1 = \|h_n|_Y\| = d(\tilde{h_n} \circ q_n, Y^{\perp}) = d(\tilde{h_n}, (Y/H_n)^{\perp}) = \|\tilde{h_n}|_{Y/H_n}\|.$$

Therefore, using the isometry of S_n, we obtain that $\tilde{h_n} \in (Y/H_n)^{\#}$, $\|\tilde{f_n}\| \to 1$ and $\|\tilde{g_n}\| \to 1$. Observe that $\tilde{f_n} - \tilde{h_n}, \tilde{g_n} - \tilde{h_n} \in (Y/H_n)^{\perp}$. By (3), $\|\tilde{f_n} - \tilde{g_n}\| \to 0$. Hence, by Lemma 3.18, $\|f_n - g_n\| \to 0$.

(2) $\Rightarrow$ (4): This implication is obvious.

(4) $\Rightarrow$ (1): Take some $h_n \in Y^{\#}$ and $f_n, g_n \in X^*$ such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in Y^{\perp}$ and $g_n - h_n \in Y^{\perp}$. By Theorem 2.6, it is sufficient to show that $\|f_n - g_n\| \to 0$. Since $dim(Y) \geq 2$, for every $n \in \mathbb{N}$, choose $y_n \in ker(h_n) \cap Y$ such that $y_n \neq 0$. Define $F_n = span\{y_n\}$. By replacing H_n by F_n in the proof of (3) $\Rightarrow$ (1) and repeating the argument involved in that proof, we get the proof of (4) $\Rightarrow$ (1). □

A stability result for the strong extension property is presented below.

Theorem 3.21. *Let Y be a closed subspace of X with $dim(Y) \geq 2$. Then the following statements are equivalent.*

(1) *Y has the strong extension property in X.*

(2) *Y/Z has the strong extension property in X/Z whenever Z is a proper closed subspace of Y.*

(3) *Y/H has the strong extension property in X/H whenever H is a hyperplane of Y.*

(4) *Y/F has the strong extension property in X/F whenever F is a one dimensional subspace of Y.*

Proof. (1) $\Rightarrow$ (2): Let Z be a proper closed subspace of Y. By Theorem 3.2, it is enough to show that $(Y/Z)^{\perp}$ is strongly Chebyshev on $(X/Z)^*$. Let $f \in (X/Z)^*$ and (h_n) be a minimizing sequence for f in $(Y/Z)^{\perp}$. By Lemma 3.17, we have $d(f, (Y/Z)^{\perp}) = d(f \circ q, Y^{\perp})$ and $\|h_n \circ q - f \circ q\| = \|h_n - f\|$ where $q : X \to X/Z$ is the canonical map. Thus, $(h_n \circ q)$ is a minimizing sequence for $f \circ q$ in $Y^{\perp}$. Since, by (1) and Theorem 3.2, $Y^{\perp}$ is strongly Chebyshev on X^*, $(h_n \circ q)$ converges in $Y^{\perp}$. It follows from Lemma 3.17 that (h_n) converges in $(Y/Z)^{\perp}$. This establishes that $(Y/Z)^{\perp}$ is strongly Chebyshev on $(X/Z)^*$.

The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are obvious.

The proofs of (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are similar to those of (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) of Theorem 3.20, respectively. We outline only the proof of (3) $\Rightarrow$ (1) here.

(3) $\Rightarrow$ (1): Let $h \in Y^{\#}$ and $f_n, g_n \in X^*$ be such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h \in Y^{\perp}$ and $g_n - h \in Y^{\perp}$. By (1) $\Leftrightarrow$ (2) of Theorem 3.2, it is enough to show that $\|f_n - g_n\| \to 0$. Define $H = ker(h) \cap Y$. In the proof of (3) $\Rightarrow$ (1) of Theorem 3.20, if we take $h_n = h$ and $H_n = H$ for all $n \in \mathbb{N}$ and proceed with that proof, we obtain $\|g_n - f_n\| \to 0$. □

We now discuss the transitive properties of the strong extension and uniform extension properties. In the rest of this section, we assume that Z is a closed subspace of Y. If Z has the extension property in Y and Y has the extension property in X, then it is easy to verify that Z has the extension property in X. We establish similar transitive properties for the strong extension and uniform extension properties.

For $f \in X^*$ and $\delta \geq 0$, let $E_Z^Y(f|_Y, \delta) := \{g \in Y^* : g = f \text{ on } Z,\ \|g|_Y\| \leq \|f|_Z\| + \delta\}$.

Theorem 3.22. *Let Y and Z be two closed subspaces of X such that $Z \subseteq Y \subseteq X$.*

(1) *If Z has the strong extension property in Y and Y has the strong extension property in X, then Z has the strong extension property in X.*

(2) *If Z has the uniform extension property in Y and Y has the uniform extension property in X, then Z has the uniform extension property in X.*

Proof. (1): Let $f \in Z^{\#}$ and $\epsilon > 0$. Since the extension property is transitive and $\|f|_Z\| = \|f|_Y\| = \|f\| = 1$, we have $f|_Y = E_Z^Y(f|_Y)$ and $f = E_Y^X(f) = E_Z^X(f)$. Since Y has the strong extension property in X, by Theorem 3.2, $Y^{\perp}$ is strongly Chebyshev on X^*. So, there exists $\delta > 0$ such that $\delta < \epsilon$ and $diam(P_{Y^{\perp}}(f, \delta)) \leq \frac{\epsilon}{3}$. It follows from the strong extension property of Z in Y that there exists $\delta_1 > 0$ such that $\delta_1 < \frac{\delta}{3}$ and $diam(E_Z^Y(f|_Y, \delta_1)) \leq \frac{\delta}{3}$. We now show that $diam(E_Z^X(f, \delta_1)) \leq \epsilon$. Let $g_1, g_2 \in E_Z^X(f, \delta_1)$. Then $g_1 = g_2 = f$ on Z, $\|g_1\| \leq 1 + \delta_1$ and $\|g_2\| \leq 1 + \delta_1$. Note that $g_1|_Y = g_2|_Y = f|_Y$ on Z and

$$\|g_1|_Y\| \leq \|g_1\| \leq 1 + \delta_1.$$

Similarly, $\|g_2|_Y\| \leq 1 + \delta_1$. Therefore, $g_1|_Y, g_2|_Y, f|_Y \in E_Z^Y(f|_Y, \delta_1)$. Since we have $diam(E_Z^Y(f|_Y, \delta_1)) \leq \frac{\delta}{3}$, it follows that $\|g_1|_Y - f|_Y\| \leq \frac{\delta}{3}$ and $\|g_2|_Y - f|_Y\| \leq \frac{\delta}{3}$. Observe that $H = Y^{\perp} + f$ is Chebyshev on X^*. Let $h_1 = P_H(g_1)$ and $h_2 = P_H(g_2)$. Note that

$$\|h_1 - g_1\| = d(g_1, H) = d(g_1, Y^{\perp} + f) = d(g_1 - f, Y^{\perp}) = \|(g_1 - f)|_Y\| \leq \frac{\delta}{3}$$

and $$\|h_1\| \leq \|h_1 - g_1\| + \|g_1\| \leq \frac{\delta}{3} + 1 + \delta_1 < 1 + \delta = d(f, Y^{\perp}) + \delta.$$

Similarly, we have $\|h_2 - g_2\| \leq \frac{\delta}{3}$ and $\|h_2\| < d(f, Y^{\perp}) + \delta$. In consequence we obtain $f - h_1, f - h_2 \in P_{Y^{\perp}}(f, \delta)$. Since $diam(P_{Y^{\perp}}(f, \delta)) \leq \frac{\epsilon}{3}$, we conclude that $\|h_1 - h_2\| \leq \frac{\epsilon}{3}$. Now,

$$\|g_1 - g_2\| \leq \|g_1 - h_1\| + \|h_1 - h_2\| + \|h_2 - g_2\| \leq \frac{\delta}{3} + \frac{\epsilon}{3} + \frac{\delta}{3} < \epsilon.$$

Therefore, $diam(E_Z^X(f, \delta_1)) \leq \epsilon$. By Theorem 3.2, Z has the strong extension property in X.

For the proof of (2), we use Theorem 2.6 and replicate the argument involved in the proof of (1). □

In [7, Theorem 5.1], it is proved that Y has the extension property in $span\{Y \cup \{x\}\}$ for all $x \in X$ if and only if Y has the extension property in X. The following example illustrates that, in the statement of the above result, the extension property cannot be replaced by the strong extension property.

Example 3.23. Consider Y, such that Y has the extension property in X but Y does not have the strong extension property in X (see, Examples 3.11 and 3.12). Clearly, Y has the extension property in $span\{Y, x\}$ for all $x \in X$. By Theorem 3.5, Y has the strong extension property in $span\{Y, x\}$ for all $x \in X$ but, as stated above, Y does not have the strong extension property in X. □

Remark 3.24. In [18], Oya introduced a property called property$-SU$ which is stronger than the extension property. We observe from [7, Proposition 6.5] and Corollary 3.15 that the property-SU is different from the strong extension property.

4. Characterizations of smoothness

In Definition 1.5, we defined the smoothness, Fréchet smoothness and uniform Fréchet smoothness of Y in X. We relate these smoothness aspects with the extension, strong extension and uniform extension properties in Section 5. In this section, we mainly present several characterizations of these smoothness aspects. Some of these characterizations will be used in Section 5.

We first make some basic observations. The proof of the next result which is due to Smulian follows from the proof of [11, Lemma 8.4].

Lemma 4.1. *The norm of X is uniformly Fréchet differentiable on S_X if and only if $x_n^* - y_n^* \to 0$ whenever (x_n) is a sequence in S_X and (x_n^*), (y_n^*) are sequences in S_{X^*} such that $x_n^*(x_n) \to 1$ and $y_n^*(x_n) \to 1$.*

If we take $Y = X$ in Definition 1.5, then by [17, Corollary 5.4.3, Theorem 5.6.11] and Lemma 4.1, Definition 1.5 leads to the usual definitions of smoothness, Fréchet smoothness and uniform Fréchet smoothness of X respectively. That is, X is smooth (respectively Fréchet smooth, uniformly Fréchet smooth) in X if and only if X is smooth (respectively Fréchet smooth, uniformly Fréchet smooth). We write FS to denote Fréchet smooth and UFS to denote uniformly Fréchet smooth. From Definition 1.5, it is easy to observe the following implications:

(1) Y is UFS in $X \implies Y$ is FS in $X \implies Y$ is smooth in X.

(2) Y is smooth (resp. FS, UFS) in $X \implies Y$ is smooth (resp. FS, UFS).

(3) X is smooth (resp. FS, UFS) $\implies Y$ is smooth (resp. FS, UFS) in X.

To illustrate that none of the above implications can be reversed, we present two examples in this section and the rest in Section 5 (see Example 5.18).

Example 4.2. (a) Let $X = (\mathbb{R}^2, \|\cdot\|_2) \oplus_\infty \mathbb{R}$ and $Y = (\mathbb{R}^2, \|\cdot\|_2)$. It is easy to see that every point of S_Y is a smooth point of B_X. Hence, Y is smooth in X. Since X is finite dimensional, by Definition 1.5, Y is UFS in X. Observe that X is not smooth.
(b) Let $X = (\mathbb{R}^2, \|\cdot\|_2) \oplus_1 \mathbb{R}$ and $Y = (\mathbb{R}^2, \|\cdot\|_2)$. Clearly, Y is UFS but Y is not smooth in X. $\square$

For $x^* \in S_{X^*}$, $x \in S_X$, and $0 \leq \delta \leq 1$, we define $S(X, x^*, \delta) = \{x \in B_X : x^*(x) \geq 1-\delta\}$ and $S(X^*, x, \delta) = \{x^* \in B_{X^*} : x^*(x) \geq 1 - \delta\}$.

The next result which will be used latter, is a generalization of a result due to Smulian [17, Theorem 5.4.19].

Proposition 4.3. *The following statements are equivalent.*

(1) *Y is smooth in X.*

(2) *$S(X^*, y, 0)$ is a singleton for every $y \in S_Y$.*

(3) *Whenever $y \in S_Y$ and (x_n^*) is a sequence in S_{X^*} such that $x_n^*(y) \to 1$, then (x_n^*) is w^*-convergent.*

Proof. (1) $\Leftrightarrow$ (2): These implications follow from Definition 1.5.
(2) $\Leftrightarrow$ (3): The proofs are similar to those of $(a) \Leftrightarrow (b)$ of [17, Theorem 5.4.19]. $\square$

For a given $x \in X \setminus \{0\}$, the set $\{x^* \in X^* : x^*(x) = 0\}$ is called the w^*-closed hyperplane of X^* generated by x. If $x_0 \in X$ and $H = \{x \in X : x^*(x) = c\}$ for some $x^* \in X^* \setminus \{0\}$ and $c \in \mathbb{R}$, then it is known [11, page 53] that

$$d(x_0, H) = \frac{|x^*(x_0) - c|}{\|x^*\|}$$

which is also known as Ascoli's formula.

In the following result, we characterize the smoothness of Y in X in terms of the Chebyshevness of the w^*-closed hyperplanes of X^*.

Proposition 4.4. *The following statements are equivalent.*

(1) *Y is smooth in X.*

(2) *Every w^*-closed hyperplane, generated by an element of S_Y, is Chebyshev on X^*.*

Proof. (1) $\Rightarrow$ (2): Let $y \in S_Y$ and $H = \{x^* \in X^* : x^*(y) = 0\}$. It is clear that H is proximinal on X^*. Let $y^* \in X^* \setminus H$ and $x_1^*, x_2^* \in P_H(y^*)$. By Ascoli's formula, we have $d(y^*, H) = |y^*(y)|$. Without loss of generality, we assume that $d(y^*, H) = y^*(y)$. Note that

$$\frac{y^* - x_1^*}{d(y^*, H)}, \frac{y^* - x_2^*}{d(y^*, H)} \in S_{X^*} \quad \text{and} \quad \frac{y^* - x_1^*}{d(y^*, H)}(y) = \frac{y^* - x_2^*}{d(y^*, H)}(y) = 1.$$

By (1), we have $x_1^* = x_2^*$. Hence, H is Chebyshev on X^*.
(2) $\Rightarrow$ (1): Suppose Y is not smooth in X. Then there exist $y \in S_Y$ and two distinct elements $x_1^*, x_2^* \in S_{X^*}$ such that $x_1^*(y) = x_2^*(y) = 1$. Let $H = \{x^* \in X^* : x^*(y) = 0\}$. By (2), H is Chebyshev on X^*. It follows from Ascoli's formula that $d(x_1^*, H) = 1$. Observe that $0, x_1^* - x_2^* \in P_H(x_1^*)$ which is a contradiction. $\square$

We now prove that the smoothness property defined for subspaces is preserved under l_p direct sum, where $1 < p < \infty$.

Theorem 4.5. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of Banach spaces and Y_i be any closed subspace of X_i for each $i \in \mathbb{N}$. For $1 < p < \infty$, let $Y = \oplus_p Y_i$ and $X = \oplus_p X_i$. Then the following statements are equivalent.*

(1) *Y is smooth in X.*

(2) *For every $i \in \mathbb{N}$, Y_i is smooth in X_i.*

Proof. (1) $\Rightarrow$ (2): This follows from Definition 1.5.
(2) $\Rightarrow$ (1): Let $x = (x_i)_{i \in \mathbb{N}} \in S_Y$ and x^*, y^* be elements in S_{X^*} such that we have $x^*(x) = y^*(x) = 1$. Note that x^* and y^* can be identified as $(x_i^*)_{i \in \mathbb{N}} \in \oplus_q X_i^*$ and $(y_i^*)_{i \in \mathbb{N}} \in \oplus_q X_i^*$ respectively, where $q = \frac{p}{p-1}$. By the Hölder inequality, we have

$$1 = \sum_{i \in \mathbb{N}} x_i^*(x_i) \leq \sum_{i \in \mathbb{N}} \|x_i^*\| \|x_i\| \leq \|(\|x_i\|)\|_p \|(\|x_i^*\|)\|_q = 1.$$

Therefore, $\|x_i^*\|^q = \|x_i\|^p$ and $x_i^*(x_i) = \|x_i^*\| \|x_i\|$ for all $i \in \mathbb{N}$.

Similarly, we have $\|y_i^*\|^q = \|x_i\|^p$ and $y_i^*(x_i) = \|y_i^*\|\|x_i\|$ for all $i \in \mathbb{N}$. Thus, $\|x_i^*\| = \|y_i^*\|$ for all $i \in \mathbb{N}$. If $x_i = 0$ for some $i \in \mathbb{N}$, then $x_i^* = y_i^*$. Assume that $x_i \neq 0$ for some $i \in \mathbb{N}$. Since Y_i is smooth in X_i, $\frac{x_i^*}{\|x_i^*\|} = \frac{y_i^*}{\|y_i^*\|}$. So $x_i^* = y_i^*$ for all $i \in \mathbb{N}$. Thus, $x^* = y^*$ which implies that Y is smooth in X. □

We present the characterizations of uniform Fréchet smoothness and Fréchet smoothness of Y in X in terms of slices and Chebyshevness. These characterizations will be used in Section 5. We first present the characterization of uniform Fréchet smoothness of Y in X.

Theorem 4.6. *The following statements are equivalent.*

(1) *Y is UFS in X.*

(2) *$diam(S(X^*, y, \frac{1}{n})) \to 0$ uniformly for all $y \in S_Y$.*

(3) *For a given $\epsilon > 0$ there exists $\delta > 0$ such that $diam(P_{H_y}(x^*, \delta)) \leq \epsilon$ whenever H_y is a w^*-closed hyperplane of X^* generated by any $y \in S_Y$ and $x^* \in D(H_y)$.*

Proof. (1) $\Rightarrow$ (3): Suppose (3) does not hold. Then there exist $\epsilon > 0$, w^*-closed hyperplane H_n generated by some $y_n \in S_Y$ and $x_n^* \in D(H_n)$ such that $diam(P_{H_n}(x_n^*, \frac{1}{n})) > \epsilon$ for all $n \in \mathbb{N}$. This implies that there exist $y_n^*, z_n^* \in P_{H_n}(x_n^*, \frac{1}{n})$ such that $\|y_n^* - z_n^*\| > \epsilon$ for all $n \in \mathbb{N}$. Observe that $\|x_n^* - y_n^*\| \to 1$ and $\|x_n^* - z_n^*\| \to 1$. By Ascoli's formula, $d(x_n^*, H_n) = |x_n^*(y_n)| = 1$. Assume that for a subsequence (y_{n_k}) of (y_n), we have $x_{n_k}^*(y_{n_k}) = 1$ for every $k \in \mathbb{N}$. Thus,

$$\frac{x_{n_k}^* - y_{n_k}^*}{\|x_{n_k}^* - y_{n_k}^*\|}(y_{n_k}) \to 1 \quad \text{and} \quad \frac{x_{n_k}^* - z_{n_k}^*}{\|x_{n_k}^* - z_{n_k}^*\|}(y_{n_k}) \to 1.$$

By (1),

$$\frac{x_{n_k}^* - y_{n_k}^*}{\|x_{n_k}^* - y_{n_k}^*\|} - \frac{x_{n_k}^* - z_{n_k}^*}{\|x_{n_k}^* - z_{n_k}^*\|} \to 0.$$

Hence, $\|y_{n_k}^* - z_{n_k}^*\| \to 0$. This leads to a contradiction.

(3) $\Rightarrow$ (2): Let $\epsilon > 0$. By (3), there exists $\delta > 0$ such that we have $\delta < \frac{\epsilon}{3}$ and $diam(P_{H_y}(x^*, \delta)) \leq \frac{\epsilon}{3}$ whenever H_y is a w^*-closed hyperplane of X^* generated by any $y \in S_Y$ and $x^* \in D(H_y)$. Choose $\delta_1 = \frac{\delta}{2}$. It is enough to show that $diam(S(X^*, y, \delta_1)) < \epsilon$ for all $y \in S_Y$. Let $y \in S_Y$ and $x^*, y^* \in S(X^*, y, \delta_1)$. Define $H_1 = \{f \in X^* : f(y) = 1\}$. Observe that H_1 is Chebyshev on X^*. Let $\tilde{x^*} = P_{H_1}(x^*)$ and $\tilde{y^*} = P_{H_1}(y^*)$. By Ascoli's formula,

$$\|\tilde{x^*} - x^*\| = d(x^*, H_1) = 1 - x^*(y) \leq \delta_1 < \frac{\epsilon}{3}.$$

Similarly, we have $\|\tilde{y^*} - y^*\| < \frac{\epsilon}{3}$. By the Hahn-Banach theorem, there exists $z^* \in S_{X^*}$ such that $z^*(y) = \|y\| = 1$. Define $H_y = \{f \in X^* : f(y) = 0\}$. Now,

$$\|z^* - \tilde{x^*} - z^*\| = \|\tilde{x^*}\| \leq \|\tilde{x^*} - x^*\| + \|x^*\| \leq 1 + \delta_1 = d(z^*, H_y) + \delta_1.$$

Similarly, $\|z^* - \tilde{y^*} - z^*\| \leq d(z^*, H_y) + \delta_1$. Therefore, $z^* - \tilde{x^*}, z^* - \tilde{y^*} \in P_{H_y}(z^*, \delta)$ and $z^* \in D(H_y)$. Since $diam(P_{H_y}(z^*, \delta)) \leq \frac{\epsilon}{3}$, it follows that $\|\tilde{y^*} - \tilde{x^*}\| \leq \frac{\epsilon}{3}$. Thus,

$$\|x^* - y^*\| \leq \|x^* - \tilde{x^*}\| + \|\tilde{x^*} - \tilde{y^*}\| + \|\tilde{y^*} - y^*\| < \epsilon.$$

This proves that $diam(S(X^*, y, \delta_1)) < \epsilon$.

(2) $\Rightarrow$ (1): This is easy to verify. □

Corollary 4.7. *Consider the following statements.*

(1) $Y^\perp$ *is UFS in* X^*.

(2) $diam(S(X^{**}, x^*, \frac{1}{n})) \to 0$ *uniformly for all* $x^* \in S_{Y^\perp}$.

(3) $diam(S(X, x^*, \frac{1}{n})) \to 0$ *uniformly for all* $x^* \in S_{Y^\perp}$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Further, if* Y *is reflexive, then* (3) $\Rightarrow$ (2).

Proof. (1) $\Leftrightarrow$ (2): These implications follow from Theorem 4.6.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (2): Let Y be a reflexive subspace. By (3), we observe that X/Y is reflexive. Let $x^* \in S_{Y^\perp}$. Because of (3), $S(X, x^*, 0)$ is non-empty. Since $Y^\perp$ is isometrically isomorphic to $(X/Y)^*$, by the James theorem, X/Y is reflexive. By [17, Corollary 1.11.19], X is reflexive. Hence, (3) $\Rightarrow$ (2). □

We now present characterizations of Fréchet smoothness of Y in X in terms of slices and strong Chebyshevness.

Theorem 4.8. *The following statements are equivalent.*

(1) Y *is FS in* X.

(2) $diam(S(X^*, y, \frac{1}{n})) \to 0$ *for all* $y \in S_Y$.

(3) *If* $y \in S_Y$ *and* H *is the* w^**-closed hyperplane generated by* y*, then* H *is strongly Chebyshev on* $D(H)$.

(4) *If* $y \in S_Y$ *and* H *is the* w^**-closed hyperplane generated by* y*, then* H *is strongly Chebyshev on* X^*.

Proof. (1) $\Rightarrow$ (3): Suppose (3) does not hold. Then there exist $\epsilon > 0$, a w^*-closed hyperplane H generated by some $y \in S_Y$ and $x^* \in D(H)$ with $diam(P_H(x^*, \frac{1}{n})) > \epsilon$ for all $n \in \mathbb{N}$. If we take $y_n = y$, $x_n^* = x^*$ and $H_n = H$ for all $n \in \mathbb{N}$ in the proof of (1) $\Rightarrow$ (3) of Theorem 4.6 and repeat the argument involved in that proof, we get a contradiction.

(3) $\Rightarrow$ (4): Let $y \in S_Y$ and $H = \{f \in X^* : f(y) = 0\}$. Let $x^* \in X^* \setminus H$. Since H is proximinal on X^*, choose $y^* \in P_H(x^*)$ and define $x_0^* = \frac{x^*-y^*}{d}$, where $d = d(x^*, H)$. Note that $x_0^* \in D(H)$. For any $\delta > 0$, observe that $P_H(x_0^*, \delta) = \frac{1}{d}(P_H(x^*, d\delta) - y^*)$. The proof of (3) $\Rightarrow$ (4) follows from this observation.

(4) $\Rightarrow$ (2): Let $\epsilon > 0$ and $y \in S_Y$. Define $H_y = \{f \in X^* : f(y) = 0\}$. By the Hahn-Banach theorem, there exists $z^* \in S_{X^*}$ such that $z^*(y) = \|y\| = 1$. Note that $z^* \in D(H_y)$. Hence, by (4), there exists $\delta > 0$ such that $\delta < \frac{\epsilon}{3}$ and $diam(P_{H_y}(z^*, \delta)) \leq \frac{\epsilon}{3}$. Choose $\delta_1 = \frac{\delta}{2}$. If we fix y in the proof of (3) $\Rightarrow$ (2) of Theorem 4.6 and repeat the argument involved in that proof, we obtain $diam(S(X^*, y, \delta_1)) < \epsilon$. This proves the implication.

(2) $\Rightarrow$ (1): This implication follows from Definition 1.5. □

Corollary 4.9. *Consider the following statements.*

(1) $Y^\perp$ *is FS in* X^*.

(2) $diam(S(X^{**}, x^*, \frac{1}{n})) \to 0$ *for all* $x^* \in S_{Y^\perp}$.

(3) $diam(S(X, x^*, \frac{1}{n})) \to 0$ *for all* $x^* \in S_{Y^\perp}$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Further, if* Y *is reflexive, then* (3) $\Rightarrow$ (2).

Proof. (1) $\Leftrightarrow$ (2): These implications follow from Theorem 4.8.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (2): Let Y be a reflexive subspace and $x^* \in S_{Y^\perp}$. We see from the proof of the implication (3) $\Rightarrow$ (2) of Corollary 4.7 that X is reflexive. In consequence we obtain $diam(S(X^{**}, x^*, \frac{1}{n})) \to 0$ for all $x^* \in S_{Y^\perp}$. □

The next three results show that the implications

$$Y \text{ is UFS in } X \implies Y \text{ is FS in } X \implies Y \text{ is smooth in } X$$

can be reversed under certain additional assumptions on Y and X.

Proposition 4.10. *Suppose X^* has the Namioka-Phelps property. If Y is smooth in X, then Y is FS in X.*

Proof. Let $y \in S_Y$ and (x_n^*) be a sequence in S_{X^*} such that $x_n^*(y) \to 1$. Since Y is smooth in X, by Proposition 4.3, we have $x_n^* \xrightarrow{w^*} x^*$ for some $x^* \in B_{X^*}$. Observe that $x^* \in S_{X^*}$. It follows from the Namioka-Phelps property of X^* that $x_n^* \to x^*$. Hence, Y is FS in X. □

Proposition 4.11. *Let Y be a finite dimensional subspace of X. If Y is FS in X, then Y is UFS in X.*

Proof. Let (x_n) be a sequence in S_Y and $(x_n^*), (y_n^*)$ be sequences in S_{X^*} such that $x_n^*(x_n) \to 1$ and $y_n^*(x_n) \to 1$. Since Y is finite dimensional, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$ for some $x_0 \in S_Y$. Note that

$$0 \leq |x_{n_k}^*(x_{n_k} - x_0)| \leq \|x_{n_k} - x_0\|.$$

Thus, $x_{n_k}^*(x_0) \to 1$. Similarly, we obtain that $y_{n_k}^*(x_0) \to 1$. Since Y is FS in X, $x_{n_k}^* - y_{n_k}^* \to 0$. Therefore, $x_n^* - y_n^* \to 0$. Hence, Y is UFS in X. □

The following corollary is a consequence of Proposition 4.10 and Proposition 4.11.

Corollary 4.12. *If X is a finite dimensional space, then the following statements are equivalent.*

(1) *Y is smooth in X.*

(2) *Y is FS in X.*

(3) *Y is UFS in X.*

We now present different stability results associated with the smoothness properties.

Theorem 4.13. *Let Y be any closed subspace of X with $dim(Y) \geq 3$. Then the following statements are equivalent.*

(1) *Y is smooth in X.*

(2) *Y/Z is smooth in X/Z, whenever Z is proper proximinal subspace of Y.*

(3) *Y/F is smooth in X/F, whenever F is a one dimensional subspace of Y.*

Proof. (1) $\Rightarrow$ (2): Let Z be a proper proximinal subspace of Y and $y + Z \in S_{Y/Z}$. Suppose that f and g are elements of $S_{(X/Z)^*}$ such that $f(y+Z) = g(y+Z) = 1$. Since

Z is proximinal at y, there exists $z_0 \in Z$ such that $\|z_0 - y\| = d(y, Z) = \|y + Z\| = 1$. Let q be the canonical map from X to X/Z. Then $(f \circ q)(y - z_0) = (g \circ q)(y - z_0) = 1$ and hence $g \circ q, f \circ q \in S_{X^*}$. By (1), we have $f \circ q = g \circ q$ which implies that $f = g$. Hence, Y/Z is smooth in X/Z.

(2) $\Rightarrow$ (3): This implication is obvious.

(3) $\Rightarrow$ (1): Suppose statement (1) does not hold. Then there exist $y \in S_Y$ and two distinct elements $f, g \in S_{X^*}$ with $f(y) = g(y) = 1$. Clearly, $codim(ker(f) \cap ker(g)) = 2$. Since $dim(Y) \geq 3$, there exists a non-zero element $y_0 \in ker(f) \cap \ker(g) \cap Y$. Define $F = span\{y_0\}$. Let $q : X \to X/F$ be the canonical map. By Lemma 3.18, the map S, defined by $S(\tilde{h}) = \tilde{h} \circ q$ for any $\tilde{h} \in (X/F)^*$, is an onto isometric isomorphism from $(X/F)^*$ to $F^\perp$. Since $f, g \in F^\perp$ and S is onto, there exist $\tilde{f}, \tilde{g} \in (X/F)^*$ such that $\tilde{f} \circ q = f$ and $\tilde{g} \circ q = g$. Therefore, using the isometry of S, we obtain that $\tilde{f}, \tilde{g} \in S_{(X/F)^*}$ and

$$1 = f(y) = (\tilde{f} \circ q)(y) = \tilde{f}(y + F) \leq \|y + F\| \leq \|y\| = 1.$$

Thus, $y + F \in S_{Y/F}$ and $\tilde{f}(y + F) = 1$. Similarly, $\tilde{g}(y + F) = 1$. By (3), it follows that $\tilde{f} = \tilde{g}$. Thus, $f = g$, which is a contradiction. □

We remark that in Theorem 4.13, the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) are true even if $dim(Y) \geq 2$. However, the implication (3) $\Rightarrow$ (1) need not be true when $dim(Y) = 2$.

Example 4.14. Let $X = (\mathbb{R}^2, \|\cdot\|_1) \oplus_2 \mathbb{R}$ and $Y = (\mathbb{R}^2, \|\cdot\|_1)$. Clearly Y is not smooth and hence Y is not smooth in X. Let F be any one dimensional subspace of Y. Then $F = span\{(a, b, 0)\}$ for some $(a, b, 0) \in Y$. Clearly $F^\perp = span\{(-b, a, 0), (0, 0, 1)\}$. By Lemma 3.18, $(X/F)^*$ and $F^\perp$ are isometrically isomorphic. We now show that $F^\perp$ and $(\mathbb{R}^2, \|\cdot\|_2)$ are isometrically isomorphic. Consider the map $T : F^\perp \to (\mathbb{R}^2, \|\cdot\|_2)$ defined by $T((-\alpha b, \alpha a, \beta)) = (\mu\alpha, \beta)$ for any $\alpha, \beta \in \mathbb{R}$, where $\mu = \max\{|a|, |b|\}$. Then, it is easy to verify that T is linear. Observe that for any $\alpha, \beta \in \mathbb{R}$,

$$\|(-\alpha b, \alpha a, \beta)\|_{X^*} = ((\max\{|\alpha a|, |\alpha b|\})^2 + \beta^2)^{\frac{1}{2}} = \|(\mu\alpha, \beta)\|_2 = \|T((-\alpha b, \alpha a, \beta))\|_2.$$

Therefore, T is an isometry. To show that T is onto, let $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then, $(-\frac{\alpha_1}{\mu} b, \frac{\alpha_1}{\mu} a, \alpha_2) \in F^\perp$ and $T((-\frac{\alpha_1}{\mu} b, \frac{\alpha_1}{\mu} a, \alpha_2)) = (\alpha_1, \alpha_2)$. This shows that T is onto. Hence, T is an onto isometric isomorphism. Therefore $(X/F)^*$ is rotund which implies that X/F is smooth. Hence, Y/F is smooth in X/F. □

Theorem 4.15. *Let Z and Y be any two closed subspaces of X with $Z \subset Y \subseteq X$.*

(1) *If Z is proximinal on Y and Y is FS in X, then Y/Z is FS in X/Z.*

(2) *If Y is UFS in X, then Y/Z is UFS in X/Z.*

Proof. (1) Let $y + Z \in S_{Y/Z}$ and (f_n) be a sequence in $S_{(X/Z)^*}$ with $f_n(y + Z) \to 1$. Let q be the canonical map from X to X/Z. Since Z is proximinal at y, there exists $z_0 \in Z$ such that $\|y - z_0\| = d(y, Z) = \|y + Z\| = 1$. Consider the map $S : (X/Z)^* \to Z^\perp$ defined by $S(g) = g \circ q$ for all $g \in (X/Z)^*$. By Lemma 3.18, S is an onto isometric isomorphism. Therefore $f_n \circ q \in S_{X^*}$ and $(f_n \circ q)(y - z_0) \to 1$.

Since Y is FS in X, $(f_n \circ q)$ converges. Since S is an isometric isomorphism, (f_n) converges. Hence, Y/Z is FS in X/Z.

(2): Let $y_n + Z \in S_{Y/Z}$ for all $n \in \mathbb{N}$. Consider two sequences (f_n) and (g_n) in $S_{(X/Z)^*}$ satisfying $f_n(y_n + Z) \to 1$ and $g_n(y_n + Z) \to 1$. Let q be the canonical map from X to X/Z. Since Y is UFS in X, Y is UFS. Therefore by [17, Theorem 5.5.13], Y is reflexive which implies that Z is proximinal on X. Thus, there exists a sequence (z_n) in Z such that $\|y_n - z_n\| = d(y_n, Z) = \|y_n + Z\| = 1$ for all $n \in \mathbb{N}$. Let S be the map defined as in (1). Then, $(f_n \circ q)$ and $(g_n \circ q)$ are sequences in S_{X^*} satisfying $(f_n \circ q)(y_n - z_n) \to 1$ and $(g_n \circ q)(y_n - z_n) \to 1$. Since Y is UFS in X, $\|f_n \circ q - g_n \circ q\| \to 0$. Since S is an isometric isomorphism, $\|f_n - g_n\| \to 0$. Hence, Y/Z is UFS in X/Z. □

The following example shows that the converses of the statements of Theorem 4.15 need not be true.

Example 4.16. Let $X = (\mathbb{R}^2, \|\cdot\|_2) \oplus_1 \mathbb{R}$, $Y = \{(0, x_2, x_3) \in X : x_2, x_3 \in \mathbb{R}\}$ and $Z = \{(0, 0, x_3) \in X : x_3 \in \mathbb{R}\}$. Observe that Z is proximinal on X and X/Z is isometrically isomorphic to $(\mathbb{R}^2, \|\cdot\|_2)$. Since X/Z is UFS, Y/Z is UFS in X/Z. For $y = (0, 0, 1) \in S_Y$, consider, $x^* = (0, -1, 1)$ and $y^* = (0, 1, 1)$. Note that $x^*, y^* \in S_{X^*}$ and $x^*(y) = y^*(y) = 1$. Hence, Y is not smooth in X. □

The following example shows that, in general, the smoothness (respectively Fréchet smoothness) of Y in X need not imply the smoothness (respectively Fréchet smoothness) of Y/Z in X/Z.

Example 4.17. Let X be a smooth (respectively FS) Banach space with non-rotund dual space [9, Theorems 5.2, 5.4, Chapter VII]. Then there exist distinct elements $x_1^*, x_2^* \in S_{X^*}$ such that $\left\|\frac{x_1^*+x_2^*}{2}\right\| = 1$. Let $Y = ker(x_1^*) \cap ker(x_2^*)$. Clearly, Y is a subspace of X such that $Y^\perp$ is not rotund. This implies that X/Y is not smooth. Now, consider the spaces $X_1 = X \oplus_2 X$, $X_2 = Y \oplus_2 X$ and $X_3 = Y \oplus_2 Y$. Observe that X_2 is smooth (respectively FS) in X_1 but X_2/X_3 is not smooth. Hence, X_2/X_3 is not smooth in X_1/X_3. □

We will see in Section 5 (see Example 5.19) that, in general, the Fréchet smoothness of Y in X and the Fréchet smoothness of Y/Z in X/Z need not imply the proximinality of Z on Y.

5. Smoothness and extension properties

In this section, we mainly characterize the smoothness, Fréchet smoothness and uniform Fréchet smoothness of Y in X in terms of extension property, strong extension property and uniform extension property respectively.

We first relate the extension property and the smoothness of Y in X.

Proposition 5.1. *The following statements are equivalent.*

(1) *Y is smooth in X.*

(2) *If $x \in S_Y$ and $f, g \in S_{X^*}$ satisfy $f(x) = g(x) = 1$, then $E_Y(f) = E_Y(g) = \{f\}$.*

Proof. (1) ⇒ (2): Let $x \in S_Y$ and $f, g \in S_{X^*}$ satisfying $f(x) = g(x) = 1$. Then, by (1), $f = g$ which implies that $E_Y(f) = E_Y(g)$. It is clear that $f \in E_Y(f)$. Suppose $h \in E_Y(f)$. Then $\|h\| = \|f|_Y\| = 1$ and $h(x) = f(x) = 1$. By (1), we see that $h = f$ on X. Hence, $E_Y(f) = E_Y(g) = \{f\}$.

(2) ⇒ (1): Let $x \in S_Y$ and $f, g \in S_{X^*}$ satisfying $f(x) = g(x) = 1$. Clearly, $f \in E_Y(f)$ and $g \in E_Y(g)$. It follows from (2) that $f = g$ on X. Hence, Y is smooth in X. □

Theorem 5.2. *Consider the following statements.*

(1) *Y has the extension property in X and Y is smooth.*

(2) *Y is smooth in X.*

Then (1) ⇒ (2)*. Further, if Y is reflexive, then* (2) ⇒ (1)*.*

Proof. (1) ⇒ (2): Let $y \in S_Y$. Suppose that f and g are elements in S_{X^*} such that $f(y) = g(y) = 1$. Then, $f|_Y(y) = g|_Y(y) = 1$ and $f|_Y, g|_Y \in S_{Y^*}$. Using the smoothness of Y, we obtain $f = g$ on Y. Hence, $f, g \in E_Y(f)$. Since Y has the extension property in X, $f = g$ on X. Hence, Y is smooth in X.

Let Y be a reflexive subspace. By (2), it is clear that Y is smooth. Let $f \in Y^\#$ and $g_1, g_2 \in E_Y(f)$. Then, $g_1, g_2 \in S_{X^*}$. Since Y is reflexive, there exists $y \in S_Y$ such that $f(y) = \|f|_Y\| = 1$. Note that $g_1(y) = g_2(y) = 1$. Since Y is smooth in X, we have $g_1 = g_2$. By Proposition 2.3, Y has the extension property in X. □

We illustrate below that the implication (2) ⇒ (1) of Theorem 5.2 need not be true and the extension property of Y in X need not imply the smoothness of Y in X.

Example 5.3. (1) Let X be a Fréchet smooth space with non-rotund dual [9, Theorems 5.2, 5.4, Chapter VII]. By Theorem 1.2, there exists a closed subspace Y of X such that Y does not have the extension property in X. However, Y is FS in X. Hence, Y is smooth in X.

(2) Let $X = (l_\infty, \|\cdot\|_\infty)$ and $Y = c_0$. Since c_0 is an M-ideal in l_∞ (see, [13, page 4]), by Corollary 2.8, Y has the uniform extension property in X. Since Y is not smooth, it is not smooth in X. □

We now see various consequences of Theorem 5.2.

In case Y is reflexive, statement (1) of the following result can be added to the equivalent statements given in Theorem 4.13.

Corollary 5.4. *Let Y be a closed subspace of X with $dim(Y) \geq 3$. Consider the following statements.*

(1) *Y/H is smooth in X/H, whenever H is a closed subspace of Y of co-dimension two.*

(2) *Y is smooth in X.*

Then (1) ⇒ (2)*. Further, if Y is reflexive, then* (2) ⇒ (1)*.*

Proof. (1) ⇒ (2): We first prove that Y^* is rotund. Let Z be any two dimensional subspace of Y^*. Suppose that $Z = span\{y_1^*, y_2^*\}$ for some $y_1^*, y_2^* \in S_{Y^*}$. Define $H = ker(y_1^*) \cap ker(y_2^*)$. Clearly, H is a closed subspace of Y of co-dimension two and $H_{Y^*}^{\perp} = Z$. Since $(Y/H)^*$ is isometrically isomorphic to Z, by (1), Z is rotund. By [17, Proposition 5.1.21], Y^* is rotund.

Now, because of Theorem 5.2, it is sufficient to prove that Y has the extension property in X. By (1) and Theorem 5.2, Y/F has the extension property in X/F whenever F is a closed subspace of Y of co-dimension two. Let $f \in Y^{\#}$ and $g \in E_Y(f)$. Define $H = ker(f) \cap Y$. Clearly, H is closed subspace of Y of co-dimension one. Since $dim(Y) \geq 3$, choose a closed subspace of F of H of co-dimension one. Note that $f, g \in F^{\perp}$. If we consider the canonical map $q : X \to X/F$ and proceed with the argument given in the proof of (2) $\Rightarrow$ (1) of Theorem 3.19, we get $f = g$.

Let Y be reflexive. The implication (2) $\Rightarrow$ (1) follows from Theorem 4.13. $\square$

We will see in Example 5.20, that (2) $\Rightarrow$ (1) of Corollary 5.4 need not be true.

Definition 5.5. [16]The subspace Y is said to be *totally smooth* if every closed subspace of Y has the extension property in X.

Example 5.3 illustrates that the smoothness of Y in X need not imply the total smoothness of Y. In the next result, we see that these two notions coincide when Y is reflexive.

Corollary 5.6. *Consider the following statements.*

(1) *Y is totally smooth in X.*

(2) *F has the extension property in X, whenever F is a one dimensional subspace of Y.*

(3) *Y is smooth in X.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *Further, if Y is reflexive, then* (3) $\Rightarrow$ (1).

The space X is said to be rotund with respect to Y [1] if S_X contains no segment parallel to Y. Uniform rotundity of X with respect to Y [1] is defined similarly.

In light of the classical duality between smoothness and rotundity [17, Propositions 5.4.5, 5.4.6, 5.4.7], it is natural to ask whether Y is smooth in X whenever X^* is rotund with respect to $Y^{\perp}$ and whether these two notions coincide when either Y or X is reflexive. The following examples address these questions.

Example 5.7. (a) Let $X = (\mathbb{R}^2, \|\cdot\|_2) \oplus_\infty \mathbb{R}$ and $Y = \{(0, x_2, x_3) \in X : x_2, x_3 \in \mathbb{R}\}$. Observe that $Y^{\perp}$ is Chebyshev on X^*. By [1, Lemma 1.4], X^* is rotund with respect to $Y^{\perp}$. But Y is not smooth in X, because, Y is not smooth.

(b) Let $X = (\mathbb{R}^3, \|\cdot\|_\infty)$ and $Y = \{(0, 0, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\}$. Observe that $(0, 0, 1) \in S_Y$ is a smooth point of B_X. Therefore, Y is smooth in X. But $Y^{\perp}$ is not rotund. Hence, X^* is not rotund with respect to $Y^{\perp}$. $\square$

Example 5.7, in fact, reveals more. Since the spaces involved in Example 5.7 are finite dimensional, we conclude that the uniform rotundity of X^* with respect to $Y^{\perp}$ does not imply that Y is smooth in X. Moreover, the uniform Fréchet smoothness of Y in X does not imply that X^* is rotund with respect to $Y^{\perp}$. However, we note that, by [1, Corollary 1.3], Theorem 1.3 and Theorem 5.2, if X^* is rotund with respect to $Y^{\perp}$ and Y is smooth, then Y is smooth in X.

We now present the characterizations of Fréchet smoothness and uniform Fréchet smoothness of Y in X in terms of strong extension property and uniform extension property of Y respectively. We first present the characterization for uniform Fréchet smoothness.

Theorem 5.8. *The following statements are equivalent.*

(1) *Y is UFS in X.*

(2) *Y has the uniform extension property in X and Y is UFS.*

Proof. (1) $\Rightarrow$ (2): By (1), Y is UFS and hence Y is reflexive. Let $h_n \in Y^{\#}$ and $f_n, g_n \in X^*$ be such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in Y^{\perp}$ and $g_n - h_n \in Y^{\perp}$. By (1) $\Leftrightarrow$ (2) of Theorem 2.6, it is enough to show that $\|f_n - g_n\| \to 0$. Since Y is reflexive, there exists $y_n \in S_Y$ such that $h_n(y_n) = \|h_n|_Y\| = 1$. Thus, $\frac{f_n}{\|f_n\|}(y_n) \to 1$ and $\frac{g_n}{\|g_n\|}(y_n) \to 1$. Since Y is UFS in X, $\frac{f_n}{\|f_n\|} - \frac{g_n}{\|g_n\|} \to 0$. Hence, $\|f_n - g_n\| \to 0$.

(2) $\Rightarrow$ (1): Because of Theorem 4.6 it suffices to show that $diam(S(X^*, y_n, \frac{1}{n})) \to 0$, for any sequence (y_n) in S_Y. Let $x_n^*, y_n^* \in S(X^*, y_n, \frac{1}{n})$ for all $n \in \mathbb{N}$. By the Hahn-Banach theorem, there exists $z_n^* \in S_{X^*}$ such that $z_n^*(y_n) = \|y_n\| = 1$. Observe that $z_n^* \in D(Y^{\perp})$ for all $n \in \mathbb{N}$. By (2) and Theorem 2.6, $Y^{\perp}$ is uniformly strongly Chebyshev on $D(Y^{\perp})$. Define $H_n = Y^{\perp} + z_n^*$ for all $n \in \mathbb{N}$. Clearly, H_n is Chebyshev on X^*. Let $\tilde{x_n^*} = P_{H_n}(x_n^*)$ and $\tilde{y_n^*} = P_{H_n}(y_n^*)$ for all $n \in \mathbb{N}$. Since Y is UFS and $x_n^*|_Y, y_n^*|_Y, z_n^*|_Y \in S(Y^*, y_n, \frac{1}{n})$, we immediately obtain $\|x_n^*|_Y - z_n^*|_Y\| \to 0$ and $\|y_n^*|_Y - z_n^*|_Y\| \to 0$. Note that $\|(x_n^* - z_n^*)|_Y\| = d(x_n^* - z_n^*, Y^{\perp}) = d(x_n^*, H_n) = \|\tilde{x_n^*} - x_n^*\|$ and $d(0, H_n) = 1$. Since $\|\tilde{x_n^*} - x_n^*\| \to 0$ and

$$1 = d(0, H_n) \leq \|\tilde{x_n^*}\| \leq \|\tilde{x_n^*} - x_n^*\| + \|x_n^*\|,$$

we have $\|\tilde{x_n^*}\| \to 1$. Similarly, $\|\tilde{y_n^*} - y_n^*\| \to 0$ and $\|\tilde{y_n^*}\| \to 1$. Therefore we obtain $\|(z_n^* - \tilde{x_n^*}) - z_n^*\| \to 1$ and $\|(z_n^* - \tilde{y_n^*}) - z_n^*\| \to 1$. Since $Y^{\perp}$ is uniformly strongly Chebyshev on $D(Y^{\perp})$, $\|\tilde{x_n^*} - \tilde{y_n^*}\| \to 0$. Now,

$$\|x_n^* - y_n^*\| \leq \|x_n^* - \tilde{x_n^*}\| + \|\tilde{x_n^*} - \tilde{y_n^*}\| + \|\tilde{y_n^*} - y_n^*\|,$$

which implies that $\|x_n^* - y_n^*\| \to 0$. Hence, $diam(S(X^*, y_n, \frac{1}{n})) \to 0$. □

In view of Example 5.3, we conclude that the uniform extension property of Y in X is not sufficient for the uniform Fréchet smoothness of Y in X.

It follows from [24, Proposition 3], Theorem 2.6 and Theorem 5.8 that if X^* is uniformly rotund with respect to $Y^{\perp}$ and Y is UFS, then Y is UFS in X. Note that Example 5.7 shows that the uniform Fréchet smoothness of Y in X neither implies nor is implied by the uniform rotundity of X^* with respect to $Y^{\perp}$.

We define a property called uniform equi-extension property as follows. Let $\mathcal{F}$ be a collection of closed subspaces of X. We say that $\mathcal{F}$ has the uniform equi-extension property in X if for $\epsilon > 0$, there exists $\delta > 0$ such that $diam(E_F(f, \delta)) < \epsilon$ for every $F \in \mathcal{F}$ and $f \in F^{\#}$. It is clear from Theorem 2.6 that if $\mathcal{F}$ has the uniform equi-extension property in X, then F has the uniform extension property in X whenever $F \in \mathcal{F}$.

In the next two results, we relate the uniform Fréchet smoothness and the uniform equi-extension property.

Theorem 5.9. *Let Y be a closed subspace of X with $dim(Y) \geq 2$. Then the following statements are equivalent.*

(1) *Y is UFS in X.*

(2) *Whenever Z_n are proper closed subspaces of Y, $y_n \in S_{Z_n}$ and $f_n, g_n \in S_{X^*}$ such that $f_n(y_n) \to 1$ and $g_n(y_n) \to 1$, then $\|f_n - g_n\| \to 0$.*

(3) *Whenever H_n are hyperplanes of Y, $y_n \in S_{H_n}$ and $f_n, g_n \in S_{X^*}$ such that $f_n(y_n) \to 1$ and $g_n(y_n) \to 1$, then $\|f_n - g_n\| \to 0$.*

(4) *Whenever Z_n are proper closed subspaces of Y, $h_n \in Z_n^{\#}$ and $f_n, g_n \in X^*$ such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in Z_n^{\perp}$ and $g_n - h_n \in Z_n^{\perp}$, then $\|f_n - g_n\| \to 0$.*

(5) *$\mathcal{F}$ has the uniform equi-extension property in X, where $\mathcal{F}$ is the collection of all proper closed subspaces of Y.*

(6) *$\mathcal{F}$ has the uniform equi-extension property in X, where $\mathcal{F}$ is the collection of all one dimensional subspaces of Y.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): These implications follow from Definition 1.5.

(3) $\Rightarrow$ (1): Let $y_n \in S_Y$ and $f_n, g_n \in S_{X^*}$ such that $f_n(y_n) \to 1$ and $g_n(y_n) \to 1$. For every $n \in \mathbb{N}$, choose $w_n \in S_Y \backslash span\{y_n\}$. By the Hahn-Banach theorem, there exists $x_n^* \in S_{X^*}$ such that $span\{y_n\} \subseteq ker(x_n^*)$ and $x_n^*(w_n) = d(w_n, span\{y_n\})$. Define $H_n = ker(x_n^*) \cap Y$. Clearly H_n is a hyperplane of Y. By (3), we obtain that $\|f_n - g_n\| \to 0$.

(1) $\Rightarrow$ (4): Let Z_n be proper closed subspaces of Y, $h_n \in Z_n^{\#}$ and $f_n, g_n \in X^*$ be such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - h_n \in Z_n^{\perp}$ and $g_n - h_n \in Z_n^{\perp}$. Note that, by (1), Y is reflexive. Therefore, there exists $y_n \in S_{Z_n}$ such that $h_n(y_n) = \|h_n|_{Z_n}\| = 1$. Thus, $\frac{f_n}{\|f_n\|}(y_n) \to 1$ and $\frac{g_n}{\|g_n\|}(y_n) \to 1$. Since Y is UFS in X, $\frac{f_n}{\|f_n\|} - \frac{g_n}{\|g_n\|} \to 0$. Hence, $\|f_n - g_n\| \to 0$.

(4) $\Leftrightarrow$ (5): These implications are easy to verify.

(5) $\Rightarrow$ (6): This implication is obvious.

(6) $\Rightarrow$ (1): Let $\epsilon > 0$. By (6), there exists $\delta > 0$ such that $diam(E_F(h, \delta)) < \epsilon$ for every one dimensional subspace F of Y and $h \in F^{\#}$. It follows from Lemma 2.5 that $diam(P_{F^\perp}(h, \delta)) < \epsilon$ for every one dimensional subspace F of Y and $h \in D(F^\perp)$. Thus, by Theorem 4.6, Y is UFS in X. $\square$

The following theorem is a consequence of Theorem 5.9 and [24, Proposition 10].

Theorem 5.10. *The following statements are equivalent.*

(1) *X is UFS.*

(2) *$\mathcal{F}$ has the uniform equi-extension property in X, where $\mathcal{F}$ is the collection of all proper closed subspaces of X.*

(3) *$\mathcal{F}$ has the uniform equi-extension property in X, where $\mathcal{F}$ is the collection of all hyperplanes of X.*

(4) *$\mathcal{F}$ has the uniform equi-extension property in X, where $\mathcal{F}$ is the collection of all one dimensional subspaces of X.*

(5) *X^* is uniformly rotund.*

We remark that the equivalence (1) $\Leftrightarrow$ (4) of Theorem 5.10 is also obtained in [24, Proposition 9]. We now characterize the Fréchet smoothness in terms of strong extension property.

Theorem 5.11. *Consider the following statements.*

(1) *Y has the strong extension property in X and Y is FS.*

(2) *Y is FS in X.*

Then (1) $\Rightarrow$ (2). *Further, if Y is reflexive, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): Let $y \in S_Y$. By Theorem 4.8, it is enough to show that $diam(S(X^*, y, \frac{1}{n})) \to 0$. Let (x_n^*) and (y_n^*) be sequences in X^* with the property $x_n^*, y_n^* \in S(X^*, y, \frac{1}{n})$ for each $n \in \mathbb{N}$. By the Hahn-Banach theorem there exists $z^* \in S_{X^*}$ such that $z^*(y) = \|y\|$. By (1) and Theorem 3.2, $Y^\perp$ is strongly Chebyshev on X^*. Let $H = Y^\perp + z^*$. If we take $y_n = y$, $z_n^* = z^*$ and $H_n = H$ for all $n \in \mathbb{N}$ in the proof of (2) $\Rightarrow$ (1) of Theorem 5.8 and repeat the argument involved in that proof, we obtain $\|x_n^* - y_n^*\| \to 0$. This proves that $diam(S(X^*, y, \frac{1}{n})) \to 0$.

Let Y be a reflexive subspace. By (2), Y is FS. Let $f \in Y^\#$ and (g_n) be a sequence in X^* such that $\|g_n\| \to 1$ and $g_n - f \in Y^\perp$. By Theorem 3.2, it is enough to show that $g_n \to f$. Since Y is reflexive, there exists $y \in S_Y$ such that $f(y) = \|f|_Y\| = 1$. Therefore, $\frac{g_n}{\|g_n\|}(y) \to 1$. Since Y is FS in X, observe that $\frac{g_n}{\|g_n\|} \to f$. Hence, $g_n \to f$. □

The following corollary is a consequence of Theorem 5.11.

Corollary 5.12. *Consider the following statements.*

(1) *Z has the strong extension property in X, whenever Z is a closed subspace of Y.*

(2) *F has the strong extension property in X, whenever F is a one dimensional subspace of Y.*

(3) *Y is FS in X.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *Further, if Y is reflexive, then* (3) $\Rightarrow$ (1).

In view of Example 5.3, we conclude that the implication (2) $\Rightarrow$ (1) of Theorem 5.11 as well as Corollary 5.12 need not be true and the strong extension property of Y in X is not sufficient for the Fréchet smoothness of Y in X.

The following corollary characterizes the Fréchet smoothness of X^* in terms of the strong extension property in X^*.

Corollary 5.13. *The following statements are equivalent.*

(1) *X^* is FS.*

(1) *Every closed subspace of X^* has the strong extension property in X^*.*

Proof. (1) $\Rightarrow$ (2): Let Y be any closed subspace of X^*. Since X^* is FS, by [17, Theorem 5.6.9], X is reflexive. By Corollary 5.12, Y has the strong extension property in X^*.

(2) $\Rightarrow$ (1): This implication follows from Corollary 5.12. □

Remark 5.14. Corollary 3.8 and Corollary 5.13 lead to the following observation. Consider a non-reflexive space X such that X^* is locally uniformly rotund [21, Example 6]. It follows from Corollary 3.8 that every closed subspace of X has the strong extension property in X. Using the preceding fact and Corollary 5.13, we observe that X cannot be a dual space.

The following corollary is a consequence of Theorem 3.9, Corollary 5.13 and [17, Theorem 5.6.9].

Corollary 5.15. *The following statements are equivalent.*

(1) *Every closed subspace of X and X^* has the strong extension property in X and X^* respectively.*

(2) *X and X^* are strongly rotund.*

The following example illustrates that, in Corollary 5.13, X^* cannot be replaced by X.

Example 5.16. Let X be a Fréchet smooth space whose dual is not rotund [9, Theorems 5.2, 5.4, Chapter VII]. By Theorem 1.2, there exists a subspace Y of X such that Y does not have the extension property in X. □

It is easy to see that Y is smooth in X if and only if Y is smooth in $span\{Y, x\}$ for every $x \in X$. In the following example, we observe that similar results do not hold for Fréchet smoothness and uniformly Fréchet smoothness.

Example 5.17. Let $X = (l_1, \|\cdot\|_1)$ and $x_0 = (\frac{1}{2^n})$. Define $Y = span\{x_0\}$. It is observed in Example 3.11 that Y has the extension property in X but Y does not have the strong extension property in X. Therefore, by Theorem 5.2 and Theorem 5.11, Y is smooth in X but Y is not FS in X. Moreover, by Corollary 4.12, Y is UFS in $\{Y, x\}$ for all $x \in X$.

We now present some examples to illustrate that some of the implications mentioned in the beginning of Section 4 cannot be reversed.

Example 5.18. (1) Let Y be a smooth (respectively FS, UFS) space and Z be any arbitrary Banach space. Consider $X = Y \oplus_\infty Z$. Then Y is an M-summand in X which implies that Y is an M-ideal [13, page 2]. By Corollary 2.8, Y has the uniform extension property in X. By Theorem 5.2 (respectively Theorem 5.11, Theorem 5.8), Y is smooth (respectively FS, UFS) in X. But X is not smooth.

(2) Let Y be a smooth (respectively FS, UFS) reflexive space and Z be any arbitrary Banach space. Consider $X = Y \oplus_1 Z$. By Lemma 3.18, it is clear that X^* is isometrically isomorphic to $Y^\perp \oplus_\infty Y^*$. Thus, $Y^\perp$ is not Chebyshev on X^*. It follows from Theorem 1.3 that Y does not have the extension property in X. By Theorem 5.2, Y is not smooth in X. Hence, Y is smooth (respectively FS, UFS) but Y is not smooth in X.

(3) Let Y be smooth but not FS (see [9, Theorem 1.9, Chapter III]) and Z be any arbitrary Banach space. Consider $X = Y \oplus_\infty Z$. By (1), Y is smooth in X but Y is not FS in X. Similarly, if we consider Y to be FS but not UFS and $X = Y \oplus_\infty Z$ for some arbitrary Banach space Z, then Y is FS in X but Y is not UFS in X. □

As noted at the end of Section 4, the following example illustrates that, in general, the Fréchet smoothness of Y in X and the Fréchet smoothness of Y/Z in X/Z need not imply the proximinality of Z in Y.

Example 5.19. Let X^* be a non-reflexive locally uniformly rotund space [21, Example 6]. Consider a non-reflexive closed subspace Y of X which is of co-dimension one. Let Z be a closed subspace of Y such that Z is of co-dimension one in Y and it is not proximinal on Y. By [9, Proposition 1.5, Chapter II], X is FS and hence Y is FS in X. By Corollary 3.8 and Theorem 3.21, it follows that Y/Z has the strong extension property in X/Z. Hence, by Theorem 5.11, Y/Z is FS in X/Z. □

As mentioned immediately after Corollary 5.4, the following example illustrates that, in general, the smoothness of Y in X need not imply the smoothness of Y/H in X/H, where H is a closed subspace of Y of co-dimension two.

Example 5.20. Let Y be a smooth space whose dual is not rotund [9, Theorems 5.2, 5.4, Chapter VII] and Z be any arbitrary Banach space. Consider $X = Y \oplus_\infty Z$. Observe that Y is smooth in X (see, Example 5.18). Since Y^* is not rotund, it follows from the proof of (1) ⇒ (2) of Corollary 5.4, there exists a closed subspace H of Y of co-dimension two such that Y/H is not smooth in X/H. □

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References

[1] D. Amir, Z. Ziegler: *Relative Chebyshev centers in normed linear spaces I*, J. Approx. Theory 29/3 (1980) 235–252.

[2] P. Bandyopadhyay, Y. Li, B. L. Lin, D. Narayana: *Proximinality in Banach spaces*, J. Math. Anal. Appl. 341/1 (2008) 309–317.

[3] A. L. Brown: *A rotund reflexive space having a subspace of codimension two with a discontinuous metric projection*, Michigan Math. J. 21 (1974) 145–151.

[4] A. K. Chakrabarty, P. Shunmugaraj, C. Zălinescu: *Continuity properties for the subdifferential and ϵ-subdifferential of a convex function and its conjugate*, J. Convex Analysis 14/3 (2007) 479–514.

[5] E. W. Cheney, D. E. Wulbert: *The existence and unicity of best approximations*, Math. Scand. 24 (1969) 113–140.

[6] S. Daptari, T. Paul: *Uniqueness of Hahn-Banach extensions and some of its variants*, Adv. Oper. Theory 7/3 (2022), art. no. 36.

[7] S. Daptari, T. Paul, T. S. S. R. K. Rao: *Stability of unique Hahn-Banach extensions and associated linear projections*, Linear Multilinear Algebra 70/21 (2022) 6067–6083.

[8] F. Deutsch, P. Kenderov: *Continuous selections and approximate selection for set-valued mappings and applications to metric projections*, SIAM J. Math. Anal. 14/1 (1983) 185–194.

[9] R. Deville, G. Godefroy, V. Zizler: *Smoothness and Renormings in Banach Spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman, Harlow (1993).

[10] S. Dutta, P. Shunmugaraj, V. Thota: *Uniform strong proximinality and continuity of metric projection*, J. Convex Analysis 24/4 (2017) 1263–1279.

[11] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, V. Zizler: *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books in Mathematics, Springer, New York (2001).

[12] S. R. Foguel: *On a theorem by A. E. Taylor*, Proc. Amer. Math. Soc. 9 (1958) 325.

[13] P. Harmand, D. Werner, W. Werner: *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics 1547, Springer, Berlin (1993).

[14] C. R. Jayanarayanan: *Characterization of strong ball proximinality in L_1-predual spaces*, J. Convex Analysis 26/2 (2019) 537–542.

[15] S. Lalithambigai, T. Paul, P. Shunmugaraj, V. Thota: *Chebyshev centers and some geometric properties of Banach spaces*, J. Math. Anal. Appl. 449/1 (2017) 926–938.

[16] C. J. Liao, N. C. Wong: *Smoothly embedded subspaces of a Banach space*, Taiwanese J. Math. 14/4 (2010) 1629–1634.

[17] R. E. Megginson: *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics 183, Springer, New York (1998).

[18] E. F. Oya: *Strong uniqueness of the extension of linear continuous functionals according to the Hahn-Banach theorem*, Math. Notes 43/1-2 (1988) 134–139.

[19] R. R. Phelps: *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. 95 (1960) 238–255.

[20] P. Shunmugaraj: *Convergence of slices, geometric aspects in Banach spaces and proximinality*, in: *Nonlinear Analysis*, Trends in Mathematics, Birkhäuser/Springer, New Delhi (2014) 61–107.

[21] M. A. Smith: *Some examples concerning rotundity in Banach spaces*, Math. Ann. 233/2 (1978) 155–161.

[22] A. E. Taylor: *The extension of linear functionals*, Duke Math. J. 5 (1939) 538–547.

[23] L. P. Vlasov: *Approximative properties of sets in normed linear spaces*, Russian Math. Surveys 28/6 (1973) 1–66.

[24] V. Zizler: *Uniform extension of linear functionals*, Časopis Pěst. Mat. 97 (1972) 379–385.