Journal of Convex Analysis
Volume 31 (2024), No. 1, 227–242

A-Numerical Radius of Semi-Hilbert Space Operators

Messaoud Guesba
Faculty of Exact Sciences, Department of Mathematics, El Oued University, Algeria
guesba-messaoud@univ-eloued.dz, guesbamessaoud2@gmail.com

Pintu Bhunia
Department of Mathematics, Indian Institute of Science, Bengaluru, Karnataka, India
pintubhunia5206@gmail.com

Kallol Paul
Department of Mathematics, Jadavpur University, Kolkata, West Bengal, India
kalloldada@gmail.com, kallol.paul@jadavpuruniversity.in

Received: May 13, 2022
Accepted: May 24, 2023

Let $\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ be a 2×2 diagonal operator matrix whose each diagonal entry is a positive bounded linear operator A acting on a complex Hilbert space $\mathcal{H}$. Let T, S and R be bounded linear operators on $\mathcal{H}$ admitting A-adjoints, where T and R are A-positive. By considering an $\mathbf{A}$-positive 2×2 operator matrix $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$, we develop several upper bounds for the A-numerical radius of S. Applying these upper bounds we obtain new A-numerical radius bounds for the product and the sum of arbitrary operators which admit A-adjoints. Related other inequalities are also derived.

Keywords: A-numerical radius, positive operator, seminorm, semi-inner product.

2020 Mathematics Subject Classification: 47A05, 47A12, 47A30, 47B15.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$ and associated norm $\|\cdot\|$. Let the symbol I stand for the identity operator on $\mathcal{H}$. Let $\mathbb{N}$ be the set of all natural numbers.

For every operator $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand respectively for the null space, the range and the closure of the range of T.

For $T \in \mathcal{B}(\mathcal{H})$, its *adjoint* is denoted by T^*. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. We will write $T \geq 0$ if T is a positive operator.

Moreover, the square root of every positive operator T will be denoted by $T^{1/2}$. Let $\mathcal{B}(\mathcal{H})^+$ be the cone of positive (semi-definite) operators, i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) : \langle Tx, x \rangle \geq 0, \forall\ x \in \mathcal{H}\}.$$

ISSN 0944-6532 / $ 2.50

Any positive operator $A \in \mathcal{B}(\mathcal{H})^+$ defines a semi-inner product $\langle .,. \rangle_A$, induced by A and is given by

$$\langle x, y \rangle_A = \langle Ax, y \rangle, \ \forall x, y \in \mathcal{H}.$$

The seminorm $\|\cdot\|_A$ induced by $\langle .,. \rangle_A$ is given by

$$\|x\|_A = \sqrt{\langle x, x \rangle_A} = \left\| A^{1/2} x \right\|, \ \forall \ x \in \mathcal{H}.$$

Observe that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. Then $\|.\|_A$ is a norm on $\mathcal{H}$ if and only if A is an injective operator and the semi-normed space $(\mathcal{H}, \|.\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. Given $T \in \mathcal{B}(\mathcal{H})$ if there exists $c > 0$ satisfying $\|Tx\|_A \leq c \|x\|_A, \forall x \in \overline{\mathcal{R}(A)}$ then

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)} \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x \in \overline{\mathcal{R}(A)} \\ \|x\|_A = 1}} \|Tx\|_A < \infty.$$

From now on, we reserve the symbol A for a non-zero positive operator in $\mathcal{B}(\mathcal{H})$ and we denote

$$\mathcal{B}^A(\mathcal{H}) = \left\{ T \in \mathcal{B}(\mathcal{H}) : \|T\|_A < \infty \right\}.$$

It can be seen that $\mathcal{B}^A(\mathcal{H})$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$ and $\|T\|_A = 0$ if and only if $T^*AT = 0$. Moreover, for $T \in \mathcal{B}^A(\mathcal{H})$ we have

$$\|T\|_A = \sup \left\{ |\langle Tx, y \rangle_A| : \ x, y \in \overline{\mathcal{R}(A)} \text{ and } \|x\|_A = \|y\|_A = 1 \right\}.$$

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an *A-adjoint of T* if for every $x, y \in \mathcal{H}$

$$\langle Tx, y \rangle_A = \langle x, Sy \rangle_A,$$

i.e., $AS = T^*A$. T is called *A-selfadjoint* if $AT = T^*A$, and it is called *A-positive* if AT is positive. For A-positive operator T, we write $T \geq_A 0$. The existence of an A-adjoint operator is not guaranteed. The set of all operators which admit A-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. By the Douglas theorem [18], we get

$$\begin{aligned} \mathcal{B}_A(\mathcal{H}) &= \left\{ T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subset \mathcal{R}(A) \right\} \\ &= \left\{ T \in \mathcal{B}(\mathcal{H}) : \exists \ c > 0 \text{ such that } \|ATx\| \leq c \|Ax\|, \forall \ x \in \mathcal{H} \right\}. \end{aligned}$$

If $T \in \mathcal{B}_A(\mathcal{H})$ then T admits A-adjoint operators. Moreover, there exists a distinguished A-adjoint operator of T, namely the reduced solution of the equation $AX = T^*A$, i.e., $T^{\sharp_A} = A^\dagger T^* A$, where $A^\dagger$ is the *Moore-Penrose inverse* of A. The A-adjoint operator $T^{\sharp_A}$ satisfies

$$AT^{\sharp_A} = T^*A, \ \mathcal{R}\left(T^{\sharp_A}\right) \subset \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}\left(T^{\sharp_A}\right) = \mathcal{N}(T^*A).$$

Again, by applying Douglas' theorem, we can see that

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \left\{ T \in \mathcal{B}(\mathcal{H}) : \exists \ c > 0; \|Tx\|_A \leq c \|x\|_A, \forall \ x \in \mathcal{H} \right\}.$$

Operators in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are called *A-bounded* operators. Moreover if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, then

$$\|T\|_A := \sup_{\substack{x \in \mathcal{H} \\ Ax \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x \in \mathcal{H} \\ \|x\|_A = 1}} \|Tx\|_A.$$

In addition, if T is A-bounded, then $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$ and

$$\|Tx\|_A \leq \|T\|_A \|x\|_A, \forall\, x \in \mathcal{H}.$$

Note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$ (see [4]). Moreover, the following inclusions

$$\mathcal{B}_A(\mathcal{H}) \subset \mathcal{B}_{A^{1/2}}(\mathcal{H}) \subset \mathcal{B}^A(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}),$$

hold with equality if A is injective and has a closed range. For more details see, e.g., [5, 6, 24, 30, 31]. Observe that if T is A-selfadjoint then it does not necessarily imply that $T = T^{\sharp_A}$. For example, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})^+$ and $T = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ then T is A-selfadjoint, but $T^{\sharp_A} = A^{\dagger}T^*A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq T$. In fact, $T = T^{\sharp_A}$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subset \overline{\mathcal{R}(A)}$.

Now, we note some properties of $T^{\sharp_A}$ and its relationship with the seminorm $\|.\|_A$. Let $T \in \mathcal{B}_A(\mathcal{H})$ and let $P_{\overline{\mathcal{R}(A)}}$ stand for the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Then the following results hold, proof of which can be found in [4].

(1) If $AT = TA$ then $T^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}T^*$.

(2) $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$, $\left(T^{\sharp_A}\right)^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $\left(\left(T^{\sharp_A}\right)^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$.

(3) $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are A-selfadjoint and A-positive.

(4) If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$.

(5) $\|T\|_A = \left\|T^{\sharp_A}\right\|_A = \left\|T^{\sharp_A}T\right\|_A^{\frac{1}{2}} = \left\|TT^{\sharp_A}\right\|_A^{\frac{1}{2}}$.

Recall that the *numerical radius* of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\omega(T) = \sup\left\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\right\}.$$

This concept is useful in studying linear operators and has attracted the attention of many authors in the last few decades (e.g., see [12, 16, 20]). It is well known that $\omega(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$. Moreover, for all $T \in \mathcal{B}(\mathcal{H})$ we have

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1}$$

Some interesting numerical radius inequalities improving inequalities (1) have been obtained by several mathematicians (see, e.g., [1,2,8,13,14,19–21,33]). The concept of the classical numerical radius was generalized to the A-numerical radius as follows:

$$\omega_A(T) = \sup\left\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\right\}.$$

It follows that $\omega_A(T) = \omega_A\left(T^{\sharp_A}\right)$ for any $T \in \mathcal{B}_A(\mathcal{H})$. A fundamental inequality for the A-numerical radius is the power inequality (see [27, 28]) which says that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$\omega_A(T^n) \leq \omega_A^n(T), \ \ \forall\, n \in \mathbb{N}.$$

It may happen that $\omega_A(T) = +\infty$ for some $T \in \mathcal{B}(\mathcal{H})$. Indeed, one can take the following operators $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Further, A-numerical radius is a seminorm on $\mathcal{B}_A(\mathcal{H})$ and that for every $T \in \mathcal{B}_A(\mathcal{H})$,

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A. \tag{2}$$

Moreover, it is known that if T is A-selfadjoint, then $\omega_A(T) = \|T\|_A$. For proofs and more facts about A-numerical radius of operators, we refer the readers to [23,30,34]. We consider the 2×2 operator diagonal matrix

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Clearly, $\mathbf{A} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})^+$. So, $\mathbf{A}$ induces the following semi-inner product

$$\langle x, y\rangle_{\mathbf{A}} = \langle \mathbf{A}x, y\rangle = \langle x_1, y_1\rangle_A + \langle x_2, y_2\rangle_A,$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$. Note that if T_{ij} are operators in $\mathcal{B}_A(\mathcal{H})$ for all $i, j \in \{1, 2\}$, then it was shown in [9] that$(T_{ij})_{2\times 2} \in \mathcal{B}_{\mathbf{A}}(\mathcal{H} \oplus \mathcal{H})$ and

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}^{\sharp_{\mathbf{A}}} = \begin{pmatrix} T_{11}^{\sharp_A} & T_{21}^{\sharp_A} \\ T_{12}^{\sharp_A} & T_{22}^{\sharp_A} \end{pmatrix}.$$

Recently, several inequalities for the $\mathbf{A}$-numerical radius of 2×2 operator matrices have been established by many authors (see [15,29,32]). In 2020, Bhunia et al. [15] studied $\mathbf{A}$-numerical radius inequalities of 2×2 operator matrices by taking the positive operator diagonal matrix $\mathbf{A} = \operatorname{diag}(A, A)$. In this direction some authors have studied many generalizations and refinements of A-numerical radius, for more details one can refer to [10, 14, 23]. This motivates us to further study on this topic. In this article, we develop several new inequalities involving seminorm and A-numerical radius of operators in $\mathcal{B}_A(\mathcal{H})$. In particular, we obtain some upper bounds for the A-numerical radius of $S \in \mathcal{B}_A(\mathcal{H})$ in terms of seminorm of A-positive operators T and R assuming that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive. Several applications of these bounds are also given. Applying these bounds we obtain upper bounds for the A-numerical radius of the product and sum of operators in $\mathcal{B}_A(\mathcal{H})$.

2. Main results

We begin this section with the following lemmas. The first lemma can be found in [15,22].

Lemma 2.1. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then the following assertions hold:*

(i) $\omega_{\mathbf{A}}\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} = \max\{\omega_A(T), \omega_A(S)\}$.

(ii) $\omega_{\mathbf{A}}\begin{pmatrix} T & S \\ S & T \end{pmatrix} = \max\{\omega_A(T+S), \omega_A(T-S)\}$.

In particular, $\omega_{\mathbf{A}}\begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} = \omega_A(S)$.

(iii) $\left\|\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}\right\|_{\mathbf{A}} = \max\{\|T\|_A, \|S\|_A\}$.

An operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be A-*unitary* if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. It is easy to see that $U^{\sharp_A}U = U^{\sharp_A\sharp_A}U^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}$ if U is A-unitary. Now, we prove the second lemma.

Lemma 2.2. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

(i) $\left\|\begin{pmatrix} T & S \\ S & T \end{pmatrix}\right\|_{\mathbf{A}} = \max\{\|T+S\|_A, \|T-S\|_A\}$.

(ii) $\max\left\{\|T+S\|_A^2, \|T-S\|_A^2\right\} \leq \|T\|_A^2 + \|S\|_A^2 + \|T\|_A\|S\|_A + \omega_A\left(S^{\sharp_A}T\right)$.

Proof. (i) Let $U=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$ and $W = \begin{pmatrix} T & S \\ S & T \end{pmatrix}$. A simple calculation shows

$$U^{\sharp_{\mathbf{A}}}WU = \begin{pmatrix} P_{\overline{\mathcal{R}(A)}} & 0 \\ 0 & P_{\overline{\mathcal{R}(A)}} \end{pmatrix}\begin{pmatrix} T-S & 0 \\ 0 & T+S \end{pmatrix}.$$

Now, using $\left\|U^{\sharp_{\mathbf{A}}}WU\right\|_{\mathbf{A}} = \|W\|_{\mathbf{A}}$ and Lemma 2.1(iii), we get

$$\left\|\begin{pmatrix} T & S \\ S & T \end{pmatrix}\right\|_{\mathbf{A}} = \left\|\begin{pmatrix} T-S & 0 \\ 0 & T+S \end{pmatrix}\right\|_{\mathbf{A}} = \max\{\|T+S\|_A, \|T-S\|_A\}.$$

(ii) The proof follows easily from [11, Th. 5] by using (i) and Lemma 2.1. □

We are now in a position to present our first theorem, which gives a necessary condition for the equality $\|T+S\|_A = \|T\|_A + \|S\|_A$.

Theorem 2.3. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. If $\|T+S\|_A = \|T\|_A + \|S\|_A$, then*

$$\omega_A\left(S^{\sharp_A}T\right) = \|T\|_A\|S\|_A.$$

Proof. If $\|T+S\|_A = \|T\|_A + \|S\|_A$, then by using Lemma 2.2 (ii), we get,

$$\|T\|_A^2 + \|S\|_A^2 + 2\|T\|_A\|S\|_A \leq \|T\|_A^2 + \|S\|_A^2 + \|T\|_A\|S\|_A + \omega_A\left(S^{\sharp_A}T\right).$$

This implies that $\|T\|_A\|S\|_A \leq \omega_A\left(S^{\sharp_A}T\right) \leq \left\|S^{\sharp_A}T\right\|_A \leq \left\|S^{\sharp_A}\right\|_A\|T\|_A$.

Consequently, we obtain that $\omega_A\left(S^{\sharp_A}T\right) = \|T\|_A\|S\|_A$. □

Observe that the necessary condition in Theorem 2.3 is not sufficient. To see this, consider $T = I$ and $S = -I$. In order to prove our next result, we need the following three well-known lemmas.

Lemma 2.4. [25, p. 26] *Let $a, b \geq 0$ and $0 \leq \alpha \leq 1$. Then, for all $r \geq 1$,*

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq \left(\alpha a^{r} + (1-\alpha) b^{r}\right)^{\frac{1}{r}}.$$

Lemma 2.5. *Let $T, S, R \in \mathcal{B}_A(\mathcal{H})$ be such that T and R are A-positive operators. Then, $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive in $\mathcal{B}_{\mathbf{A}}(\mathcal{H} \oplus \mathcal{H})$ if and only if*

$$\left|\langle Sx, y\rangle_A\right|^2 \leq \langle Tx, x\rangle_A \langle Ry, y\rangle_A \quad \textit{for all } x, y \in \mathcal{H}.$$

Proof. First we see that for any $x, y \in \mathcal{H}$ and for every $X \in \mathcal{B}_A(\mathcal{H})$ with $X \geq_A 0$,

$$|\langle Xx, y\rangle_A|^2 \leq \langle Xx, x\rangle_A \langle Xy, y\rangle_A. \tag{3}$$

As $AX \geq 0$ we obtain

$$|\langle Xx, y\rangle_A|^2 = |\langle AXx, y\rangle|^2 \leq \langle AXx, x\rangle \langle AXy, y\rangle = \langle Xx, y\rangle_A \langle Xy, y\rangle_A.$$

Now, by using the inequality (3) we get

$$\begin{aligned} |\langle Sx, y\rangle_A|^2 &= \left|\left\langle \begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle_{\mathbf{A}}\right|^2 \\ &\leq \left\langle \begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \right\rangle_{\mathbf{A}} \left\langle \begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle_{\mathbf{A}} \\ &= \langle Tx, x\rangle_A \langle Ry, y\rangle_A. \end{aligned}$$

Conversely, let $\left|\langle Sx, y\rangle_A\right|^2 \leq \langle Tx, x\rangle_A \langle Ry, y\rangle_A$ for all $x, y \in \mathcal{H}$. Now,

$$\left\langle \begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{\mathbf{A}} = \langle Tx, x\rangle_A + \langle Ry, y\rangle_A + 2\mathrm{Re}\langle Sx, y\rangle_A$$

$$\geq 2\langle Tx, x\rangle_A^{1/2} \langle Ry, y\rangle_A^{1/2} + 2\mathrm{Re}\langle Sx, y\rangle_A \geq 2\left|\langle Sx, y\rangle_A\right| + 2\mathrm{Re}\langle Sx, y\rangle_A$$

$$\geq 2\left|\langle Sx, y\rangle_A\right| - 2\left|\langle Sx, y\rangle_A\right| = 0.$$

This completes the proof. □

Here we note that the inequality in above lemma is also observed in [3].

Lemma 2.6. [17] *Let T be an A-positive operator in $\mathcal{B}_A(\mathcal{H})$. Then, for any A-unit vector $x \in \mathcal{H}$ (i.e., $\|x\|_A = 1$),*

$$\langle Tx, x\rangle_A^n \leq \langle T^n x, x\rangle_A, \ \forall\ n \in \mathbb{N}^*.$$

Now, we prove our next theorem.

Theorem 2.7. *Let $T, S, R \in \mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive, where T and R are A-positive operators. Then*

$$\omega_A^n(S) \leq \frac{1}{2} \left\|T^n + R^n\right\|_A, \textit{ for all } n \in \mathbb{N}^*.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then from Lemma 2.5, we have

$$\begin{aligned} |\langle Sx, x\rangle_A| &\leq \langle Tx, x\rangle_A^{\frac{1}{2}} \langle Rx, x\rangle_A^{\frac{1}{2}} \leq \frac{1}{2}\left(\langle Tx, x\rangle_A + \langle Rx, x\rangle_A\right) \\ &\leq \left(\frac{\langle Tx, x\rangle_A^n + \langle Rx, x\rangle_A^n}{2}\right)^{\frac{1}{n}} \text{ (by Lemma 2.4)} \\ &\leq \left(\frac{\langle T^n x, x\rangle_A + \langle R^n x, x\rangle_A}{2}\right)^{\frac{1}{n}} \text{ (by Lemma 2.6).} \end{aligned}$$

Consequently, we infer that

$$|\langle Sx, x\rangle_A|^n \leq \frac{1}{2}\left\langle \left(T^n + R^n\right) x, x\right\rangle_A \leq \frac{1}{2}\left\|T^n + R^n\right\|_A.$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get the desired inequality. □

Considering $n = 1$ in Theorem 2.7, we have the following corollary.

Corollary 2.8. *Let $T, S, R \in \mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive where T and R are A-positive operators. Then*

$$\omega_A(S) \leq \frac{1}{2}\|T + R\|_A.$$

We next obtain the following inequality.

Theorem 2.9. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A^n(ST) \leq \frac{1}{2}\left\|\left(T^{\sharp_A} T\right)^n + \left(SS^{\sharp_A}\right)^n\right\|_A, \textit{ for all } n \in \mathbb{N}^*.$$

Proof. First, we will show that $\begin{pmatrix} T^{\sharp_A} T & (ST)^{\sharp_A} \\ ST & SS^{\sharp_A} \end{pmatrix}$ is $\mathbf{A}$-positve. We have that $T^{\sharp_A} T$ and $SS^{\sharp_A}$ are A-positive operators and we observe that

$$\begin{aligned} |\langle STx, x\rangle_A|^2 &= \left|\left\langle Tx, S^{\sharp_A} x\right\rangle_A\right|^2 \leq \|Tx\|_A^2 \left\|S^{\sharp_A} x\right\|_A^2 \\ &= \langle Tx, Tx\rangle_A \left\langle S^{\sharp_A} x, S^{\sharp_A} x\right\rangle_A = \left\langle T^{\sharp_A} Tx, x\right\rangle_A \left\langle SS^{\sharp_A} x, x\right\rangle_A, \end{aligned}$$

that is, $|\langle STx, x\rangle_A|^2 \leq \left\langle T^{\sharp_A} Tx, x\right\rangle_A \left\langle SS^{\sharp_A} x, x\right\rangle_A$ for all $x \in \mathcal{H}$.

Therefore, in view of Lemma 2.5 we conclude that $\begin{pmatrix} T^{\sharp_A} T & (ST)^{\sharp_A} \\ ST & SS^{\sharp_A} \end{pmatrix}$ is $\mathbf{A}$-positive. Now, by applying Theorem 2.7 we get

$$\omega_A^n(ST) \leq \frac{1}{2}\left\|\left(T^{\sharp_A} T\right)^n + \left(SS^{\sharp_A}\right)^n\right\|_A, \ \forall\ n \in \mathbb{N}^*.$$

This completes the proof. □

Remark 2.10. In [26], it is proved that $w(TS) \leq \frac{1}{2}\|T^*T + SS^*\|$, for $T, S \in \mathcal{B}(\mathcal{H})$. An alternative easy proof of the same inequality follows from Theorem 2.9 by considering $A = I$ and $n = 1$.

The following corollary is an immediate consequence of Theorem 2.9 by considering $S = T$.

Corollary 2.11. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A^n\left(T^2\right) \leq \frac{1}{2}\left\|\left(T^{\sharp_A}T\right)^n + \left(TT^{\sharp_A}\right)^n\right\|_A, \ \forall\ n \in \mathbb{N}^*.$$

Another application of Theorem 2.7 leads to the following inequality.

Theorem 2.12. *Let $T, S, R, X, Y \in \mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive where T and R are A-positive operators. Then*

$$\omega_A^n(YSX) \leq \frac{1}{2}\left\|\left(X^{\sharp_A}TX\right)^n + \left(YRY^{\sharp_A}\right)^n\right\|_A, \ \forall\ n \in \mathbb{N}^*.$$

Proof. Since $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix} \geq_{\mathbf{A}} 0$, so

$$\begin{pmatrix} X & 0 \\ 0 & Y^{\sharp_A} \end{pmatrix}^{\sharp_{\mathbf{A}}}\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y^{\sharp_A} \end{pmatrix} = \begin{pmatrix} X^{\sharp_A}TX & \left(Y^{\sharp_A^{\sharp_A}}SX\right)^{\sharp_A} \\ Y^{\sharp_A^{\sharp_A}}SX & Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A} \end{pmatrix} \geq_{\mathbf{A}} 0.$$

Applying Theorem 2.7, we obtain

$$\omega_A^n\left(Y^{\sharp_A^{\sharp_A}}SX\right) \leq \frac{1}{2}\left\|\left(X^{\sharp_A}TX\right)^n + \left(Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A}\right)^n\right\|_A.$$

Further, using the fact $\omega_A\left(W^{\sharp_A}\right) = \omega_A(W)$ for all $W \in \mathcal{B}_A(\mathcal{H})$, it can be observed that

$$\begin{aligned}\omega_A^n\left(Y^{\sharp_A^{\sharp_A}}SX\right) &= \omega_A^n\left(\left(Y^{\sharp_A^{\sharp_A}}SX\right)^{\sharp_A}\right) = \omega_A^n\left(X^{\sharp_A}S^{\sharp_A}Y^{\sharp_A}\right) \\ &= \omega_A^n\left((YSX)^{\sharp_A}\right) = \omega_A^n(YSX).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\left\|\left(X^{\sharp_A}TX\right)^n + \left(Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A}\right)^n\right\|_A = \left\|\left[\left(X^{\sharp_A}TX\right)^n + \left(Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A}\right)^n\right]^{\sharp_A}\right\|_A \\ &= \left\|\left(X^{\sharp_A}T^{\sharp_A}X^{\sharp_A^{\sharp_A}}\right)^n + \left(Y^{\sharp_A^{\sharp_A}}R^{\sharp_A}Y^{\sharp_A}\right)^n\right\|_A = \left\|\left[\left(X^{\sharp_A}TX\right)^n + \left(YRY^{\sharp_A}\right)^n\right]^{\sharp_A}\right\|_A \\ &= \left\|\left(X^{\sharp_A}TX\right)^n + \left(YRY^{\sharp_A}\right)^n\right\|_A.\end{aligned}$$

Consequently, we get $\omega_A^n(YSX) \leq \frac{1}{2}\left\|\left(X^{\sharp_A}TX\right)^n + \left(YRY^{\sharp_A}\right)^n\right\|_A$, as desired. □

Again, as an application of Theorem 2.7 we get the following inequality.

Theorem 2.13. *Let $T, S, R, Q \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A^n\left(TR^{\sharp_A} \pm SQ^{\sharp_A}\right) \leq \frac{1}{2}\left\|\left(TT^{\sharp_A} + SS^{\sharp_A}\right)^n + \left(RR^{\sharp_A} + QQ^{\sharp_A}\right)^n\right\|_A, \ \forall\, n \in \mathbb{N}^*.$$

Proof. Let $\begin{pmatrix} T & S \\ R & Q \end{pmatrix} \in \mathcal{B}_{\mathbf{A}}(\mathcal{H} \oplus \mathcal{H})$, then $\begin{pmatrix} T & S \\ R & Q \end{pmatrix}^{\sharp_{\mathbf{A}}^{\sharp_{\mathbf{A}}}} \begin{pmatrix} T & S \\ R & Q \end{pmatrix}^{\sharp_{\mathbf{A}}}$ is $\mathbf{A}$-positive. Now, it can be verified that

$$\begin{aligned}
\begin{pmatrix} T & S \\ R & Q \end{pmatrix}^{\sharp_{\mathbf{A}}^{\sharp_{\mathbf{A}}}} \begin{pmatrix} T & S \\ R & Q \end{pmatrix}^{\sharp_{\mathbf{A}}} &= \left[\begin{pmatrix} T & S \\ R & Q \end{pmatrix}\begin{pmatrix} T & S \\ R & Q \end{pmatrix}^{\sharp_{\mathbf{A}}}\right]^{\sharp_{\mathbf{A}}} \\
&= \begin{pmatrix} T^{\sharp_A^{\sharp_A}} T^{\sharp_A} + S^{\sharp_A^{\sharp_A}} S^{\sharp_A} & T^{\sharp_A^{\sharp_A}} R^{\sharp_A} + S^{\sharp_A^{\sharp_A}} Q^{\sharp_A} \\ R^{\sharp_A^{\sharp_A}} T^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} S^{\sharp_A} & R^{\sharp_A^{\sharp_A}} R^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} Q^{\sharp_A} \end{pmatrix} \\
&= \begin{pmatrix} T^{\sharp_A^{\sharp_A}} T^{\sharp_A} + S^{\sharp_A^{\sharp_A}} S^{\sharp_A} & \left(R^{\sharp_A^{\sharp_A}} T^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} S^{\sharp_A}\right)^{\sharp_A} \\ R^{\sharp_A^{\sharp_A}} T^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} S^{\sharp_A} & R^{\sharp_A^{\sharp_A}} R^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} Q^{\sharp_A} \end{pmatrix}.
\end{aligned}$$

Since $T^{\sharp_A^{\sharp_A}} T^{\sharp_A} + S^{\sharp_A^{\sharp_A}} S^{\sharp_A}$ and $R^{\sharp_A^{\sharp_A}} R^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} Q^{\sharp_A}$ are A-positive operators, applying Theorem 2.7, we get for all $n \in \mathbb{N}^*$,

$$\omega_A^n\left(R^{\sharp_A^{\sharp_A}} T^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} S^{\sharp_A}\right) \leq \frac{1}{2}\left\|\left(T^{\sharp_A^{\sharp_A}} T^{\sharp_A} + S^{\sharp_A^{\sharp_A}} S^{\sharp_A}\right)^n + \left(R^{\sharp_A^{\sharp_A}} R^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} Q^{\sharp_A}\right)^n\right\|_A.$$

Moreover, we have

$$\omega_A^n\left(R^{\sharp_A^{\sharp_A}} T^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} S^{\sharp_A}\right) = \omega_A^n\left(\left(TR^{\sharp_A} + SQ^{\sharp_A}\right)^{\sharp_A}\right) = \omega_A^n\left(TR^{\sharp_A} + SQ^{\sharp_A}\right),$$

and

$$\begin{aligned}
&\left\|\left(T^{\sharp_A^{\sharp_A}} T^{\sharp_A} + S^{\sharp_A^{\sharp_A}} S^{\sharp_A}\right)^n + \left(R^{\sharp_A^{\sharp_A}} R^{\sharp_A} + Q^{\sharp_A^{\sharp_A}} Q^{\sharp_A}\right)^n\right\|_A \\
&= \left\|\left[\left(TT^{\sharp_A} + SS^{\sharp_A}\right)^n + \left(RR^{\sharp_A} + QQ^{\sharp_A}\right)^n\right]^{\sharp_A}\right\|_A \\
&= \left\|\left(TT^{\sharp_A} + SS^{\sharp_A}\right)^n + \left(RR^{\sharp_A} + QQ^{\sharp_A}\right)^n\right\|_A.
\end{aligned}$$

Consequently, we get

$$\omega_A^n\left(TR^{\sharp_A} + SQ^{\sharp_A}\right) \leq \frac{1}{2}\left\|\left(TT^{\sharp_A} + SS^{\sharp_A}\right)^n + \left(RR^{\sharp_A} + QQ^{\sharp_A}\right)^n\right\|_A, \ \forall\, n \in \mathbb{N}^*.$$

Replacing S by $-S$ in the above inequality, we have

$$\omega_A^n\left(TR^{\sharp_A} - SQ^{\sharp_A}\right) \leq \frac{1}{2}\left\|\left(TT^{\sharp_A} + SS^{\sharp_A}\right)^n + \left(RR^{\sharp_A} + QQ^{\sharp_A}\right)^n\right\|_A, \ \forall\, n \in \mathbb{N}^*.$$

This completes the proof. $\square$

Remark 2.14. (i) By letting $R=S$ and $Q=\pm T$ in Theorem 2.13 we get the following inequality [34] :

$$\omega_A\left(TS^{\sharp_A}\pm ST^{\sharp_A}\right)\leq\left\|TT^{\sharp_A}+SS^{\sharp_A}\right\|_A.$$

(ii) Considering the case $A=I$, $n=1$, $R^{\sharp_A}=S$ and $Q^{\sharp_A}=T$ in Theorem 2.13, we obtain the following numerical radius inequality [26] for commutator of operators: For $T,S\in\mathcal{B}(\mathcal{H})$, $w(TS\pm ST)\leq\frac{1}{2}\|T^*T+TT^*+S^*S+SS^*\|$.

We next obtain the following inequalities.

Corollary 2.15. *Let $T,R\in\mathcal{B}_A(\mathcal{H})$. Then $\forall\ n\in\mathbb{N}^*$,*

(i) $\omega_A^n\left(TR^{\sharp_A}\right)\leq\frac{1}{2}\left\|\left(TT^{\sharp_A}\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A$,

(ii) $\omega_A^n(RT)\leq\frac{1}{2}\left\|\left(T^{\sharp_A}T\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A$.

Proof. (i) Putting $S=Q=0$ in the Theorem 2.13, we obtain

$$\omega_A^n\left(TR^{\sharp_A}\right)\leq\frac{1}{2}\left\|\left(TT^{\sharp_A}\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A,\forall\ n\in\mathbb{N}^*.$$

(ii) Using (i) and replacing T by $T^{\sharp_A}$, we get

$$\omega_A^n\left(T^{\sharp_A}R^{\sharp_A}\right)\leq\frac{1}{2}\left\|\left(T^{\sharp_A}T^{\sharp_A\sharp_A}\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A.$$

Now,
$$\omega_A^n\left(T^{\sharp_A}R^{\sharp_A}\right)=\omega_A^n\left((RT)^{\sharp_A}\right)=\omega_A^n(RT)$$

and
$$\left\|\left(T^{\sharp_A}T^{\sharp_A\sharp_A}\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A=\left\|\left(T^{\sharp_A}T^{\sharp_A\sharp_A}\right)^n+\left(R^{\sharp_A\sharp_A}R^{\sharp_A}\right)^n\right\|_A$$
$$=\left\|\left(\left(T^{\sharp_A}T\right)^n+\left(RR^{\sharp_A}\right)^n\right)^{\sharp_A}\right\|_A=\left\|\left(T^{\sharp_A}T\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A.$$

Hence, $\omega_A^n(RT)\leq\frac{1}{2}\left\|\left(T^{\sharp_A}T\right)^n+\left(RR^{\sharp_A}\right)^n\right\|_A$, $\forall\ n\in\mathbb{N}^*$, as desired. □

We next obtain the following inequalities.

Corollary 2.16. *Let $T,S,R,Q,X,Y\in\mathcal{B}_A(\mathcal{H})$. Then $\forall\ n\in\mathbb{N}^*$,*

(i) $\omega_A^n\left(Y\left(RT^{\sharp_A}+QS^{\sharp_A}\right)X\right)$
$\leq\frac{1}{2}\left\|\left(X^{\sharp_A}\left(TT^{\sharp_A}+SS^{\sharp_A}\right)X\right)^n+\left(Y\left(RR^{\sharp_A}+QQ^{\sharp_A}\right)Y^{\sharp_A}\right)^n\right\|_A$,

(ii) $\omega_A^n\left(Y\left(TT^{\sharp_A}\pm SS^{\sharp_A}\right)X\right)$
$\leq\frac{1}{2}\left\|\left(X^{\sharp_A}\left(TT^{\sharp_A}+SS^{\sharp_A}\right)X\right)^n+\left(Y\left(TT^{\sharp_A}+SS^{\sharp_A}\right)Y^{\sharp_A}\right)^n\right\|_A$,

(iii) $\omega_A^n(YX)\leq\frac{1}{2}\left\|\left(X^{\sharp_A}X\right)^n+\left(YY^{\sharp_A}\right)^n\right\|_A$.

Proof. (i) The proof follows from Theorem 2.12 by using the fact

$$\begin{pmatrix} T^{\sharp_A \sharp_A} T^{\sharp_A} + S^{\sharp_A \sharp_A} S^{\sharp_A} & \left(R^{\sharp_A \sharp_A} T^{\sharp_A} + Q^{\sharp_A \sharp_A} S^{\sharp_A} \right)^{\sharp_A} \\ R^{\sharp_A \sharp_A} T^{\sharp_A} + Q^{\sharp_A \sharp_A} S^{\sharp_A} & R^{\sharp_A \sharp_A} R^{\sharp_A} + Q^{\sharp_A \sharp_A} Q^{\sharp_A} \end{pmatrix}$$

is **A**-positive, (see Theorem 2.13).
(ii) Taking $R = T$ and $Q = \pm S$ in (i) we get the desired inequality.
(iii) Taking $T = S = R = Q = I$ in (i) we get the desired inequality. □

To prove our next results we need the following lemma.

Lemma 2.17. [7, Lemma 2.11] *Let $x, y, z \in \mathcal{H}$ with $\|z\|_A = 1$. Then*

$$|\langle x, z\rangle_A| \, |\langle z, y\rangle_A| \leq \frac{1}{2} \left(\|x\|_A \|y\|_A + |\langle x, y\rangle_A| \right).$$

Theorem 2.18. *Let $T, S, R \in \mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is **A**-positive where T and R are A-positive operators. Then*

$$\omega_A^{2n}(S) \leq \frac{1}{4} \left\| (T^{\sharp_A} T)^n + (R^{\sharp_A} R)^n \right\|_A + \frac{1}{2} \omega_A^n (R^{\sharp_A} T), \ \forall \ n \in \mathbb{N}^*.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then from Lemma 2.5 we have

$$\begin{aligned}
&|\langle Sx, x\rangle_A|^2 \leq \langle Tx, x\rangle_A \langle Rx, x\rangle_A \\
&\leq \frac{1}{2} \|Tx\|_A \|Rx\|_A + \frac{1}{2} |\langle Tx, Rx\rangle_A| \text{ (by Lemma 2.17)} \\
&\leq \left(\frac{\|Tx\|_A^n \|Rx\|_A^n + |\langle Tx, Rx\rangle_A|^n}{2} \right)^{\frac{1}{n}} \text{ (by Lemma 2.4)} \\
&\leq \left(\frac{1}{4} (\|Tx\|_A^{2n} + \|Rx\|_A^{2n}) + \frac{1}{2} |\langle Tx, Rx\rangle_A|^n \right)^{\frac{1}{n}} \\
&= \left(\frac{1}{4} (\langle T^{\sharp_A} Tx, x\rangle_A^n + \langle R^{\sharp_A} Rx, x\rangle_A^n) + \frac{1}{2} |\langle R^{\sharp_A} Tx, x\rangle_A|^n \right)^{\frac{1}{n}} \\
&\leq \left(\frac{1}{4} (\langle (T^{\sharp_A} T)^n x, x\rangle_A + \langle (R^{\sharp_A} R)^n x, x\rangle_A) + \frac{1}{2} |\langle R^{\sharp_A} Tx, x\rangle_A|^n \right)^{\frac{1}{n}} \text{ (by Lemma 2.6)} \\
&\leq \left(\frac{1}{4} \left\| (T^{\sharp_A} T)^n + (R^{\sharp_A} R)^n \right\|_A + \frac{1}{2} \omega_A^n (R^{\sharp_A} T) \right)^{\frac{1}{n}}.
\end{aligned}$$

Consequently, we infer that

$$|\langle Sx, x\rangle_A|^{2n} \leq \frac{1}{4} \left\| (T^{\sharp_A} T)^n + (R^{\sharp_A} R)^n \right\|_A + \frac{1}{2} \omega_A^n (R^{\sharp_A} T).$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get the desired inequality. □

Considering $n=1$ in Theorem 2.18 we obtain the following corollary.

Corollary 2.19. *Let $T,S,R\in\mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive where T and R are A-positive operators. Then*

$$\omega_A^2(S)\le \frac{1}{4}\left\|T^{\sharp_A}T+R^{\sharp_A}R\right\|_A+\frac{1}{2}\omega_A(R^{\sharp_A}T).$$

The next theorem is as follows.

Theorem 2.20. *Let $T,S,R\in\mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is $\mathbf{A}$-positive where T and R are A-positive operators. Then*

$$\omega_A^2(S)\le \frac{1}{2}\left\|T\right\|_A\left\|R\right\|_A+\frac{1}{2}\omega_A(R^{\sharp_A}T).$$

Proof. Let $x\in\mathcal{H}$ with $\|x\|_A=1$. Then it follows from Lemma 2.5 that

$$\begin{aligned}
\left|\langle Sx,x\rangle_A\right|^2 &\le \langle Tx,x\rangle_A\langle Rx,x\rangle_A\\
&\le \frac{1}{2}\|Tx\|_A\|Rx\|_A+\frac{1}{2}|\langle Tx,Rx\rangle_A| \text{ (by Lemma 2.17)}\\
&= \frac{1}{2}\|Tx\|_A\|Rx\|_A+\frac{1}{2}|\langle R^{\sharp_A}Tx,x\rangle_A|\\
&\le \frac{1}{2}\left\|T\right\|_A\left\|R\right\|_A+\frac{1}{2}\omega_A(R^{\sharp_A}T).
\end{aligned}$$

Taking supremum over all $x\in\mathcal{H}$ with $\|x\|_A=1$, we get the desired inequality. □

As an application of Theorem 2.20 we obtain the following corollary.

Corollary 2.21. *Let $T,S\in\mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A^2(ST)\le\frac{1}{2}\|S\|_A^2\|T\|_A^2+\frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right).$$

Proof. Since $\begin{pmatrix} T^{\sharp_A}T & (ST)^{\sharp_A} \\ ST & SS^{\sharp_A} \end{pmatrix}$ is $\mathbf{A}$-positive (see Theorem 2.9), by applying Theorem 2.20 we obtain that

$$\begin{aligned}
\omega_A^2(ST)&\le\frac{1}{2}\|T^{\sharp_A}T\|_A\|SS^{\sharp_A}\|_A+\frac{1}{2}\omega_A\left((SS^{\sharp_A})^{\sharp_A}T^{\sharp_A}T\right)\\
&=\frac{1}{2}\|T\|_A^2\|S\|_A^2+\frac{1}{2}\omega_A\left(S^{\sharp_A^{\sharp_A}}(TS)^{\sharp_A}T\right)\\
&=\frac{1}{2}\|T\|_A^2\|S\|_A^2+\frac{1}{2}\omega_A\left(T^{\sharp_A}(TS)^{\sharp_A^{\sharp_A}}S^{\sharp_A}\right)
\end{aligned}$$

(using the facts $\omega_A(X)=\omega_A\left(X^{\sharp_A}\right)$ and $\left(X^{\sharp_A^{\sharp_A}}\right)^{\sharp_A}=X^{\sharp_A}$ for every $X\in\mathcal{B}_A(\mathcal{H})$)

$$\begin{aligned}
&=\frac{1}{2}\|T\|_A^2\|S\|_A^2+\frac{1}{2}\omega_A\left(\left(S(TS)^{\sharp_A}T\right)^{\sharp_A}\right)\\
&=\frac{1}{2}\|T\|_A^2\|S\|_A^2+\frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right),
\end{aligned}$$

as desired. □

Also, by applying Corollary 2.19 we obtain the following.

Corollary 2.22. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\omega_A^2(ST) \leq \frac{1}{4}\left\|\left(T^{\sharp_A}T\right)^2 + \left(SS^{\sharp_A}\right)^2\right\|_A + \frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right).$$

Proof. Since $\begin{pmatrix} T^{\sharp_A}T & (ST)^{\sharp_A} \\ ST & SS^{\sharp_A} \end{pmatrix}$ is **A**-positive (see Theorem 2.9), by applying Corollary 2.19 we get,

$$\begin{aligned}
&\omega_A^2(ST)\\
\leq\ & \frac{1}{4}\left\|(T^{\sharp_A}T)^{\sharp_A}T^{\sharp_A}T + (SS^{\sharp_A})^{\sharp_A}SS^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left((SS^{\sharp_A})^{\sharp_A}T^{\sharp_A}T\right)\\
=\ & \frac{1}{4}\left\|T^{\sharp_A}T^{\sharp_A^{\sharp_A}}T^{\sharp_A}T + S^{\sharp_A{}^{\sharp_A}}S^{\sharp_A}SS^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right)\\
=\ & \frac{1}{4}\left\|T^{\sharp_A}T^{\sharp_A{}^{\sharp_A}}T^{\sharp_A}T^{\sharp_A{}^{\sharp_A}} + S^{\sharp_A{}^{\sharp_A}}S^{\sharp_A}S^{\sharp_A{}^{\sharp_A}}S^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right)\\
=\ & \frac{1}{4}\left\|\left(T^{\sharp_A}T^{\sharp_A{}^{\sharp_A}}\right)^2 + \left(S^{\sharp_A{}^{\sharp_A}}S^{\sharp_A}\right)^2\right\|_A + \frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right)\\
=\ & \frac{1}{4}\left\|\left(\left(T^{\sharp_A}T\right)^2 + \left(SS^{\sharp_A}\right)^2\right)^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right)\\
=\ & \frac{1}{4}\left\|\left(T^{\sharp_A}T\right)^2 + \left(SS^{\sharp_A}\right)^2\right\|_A + \frac{1}{2}\omega_A\left(S(TS)^{\sharp_A}T\right),
\end{aligned}$$

as required. □

Another application of Theorem 2.20 leads to the following corollary.

Corollary 2.23. *Let $T, S, R, X, Y \in \mathcal{B}_A(\mathcal{H})$ be such that $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is **A**-positive where T and R are A-positive operators. Then*

$$\omega_A^2(YSX) \leq \frac{1}{2}\left\|X^{\sharp_A}TX\right\|_A\left\|YRY^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left(YR^{\sharp_A}(XY)^{\sharp_A}TX\right).$$

Proof. Since $\begin{pmatrix} T & S^{\sharp_A} \\ S & R \end{pmatrix}$ is **A**-positive, so $\begin{pmatrix} X^{\sharp_A}TX & \left(Y^{\sharp_A^{\sharp_A}}SX\right)^{\sharp_A} \\ Y^{\sharp_A^{\sharp_A}}SX & Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A} \end{pmatrix}$ is **A**-positive (see Theorem 2.12), by applying Theorem 2.20, we have

$$\begin{aligned}
\omega_A^2\left(Y^{\sharp_A^{\sharp_A}}SX\right) &\leq \frac{1}{2}\left\|X^{\sharp_A}TX\right\|_A\left\|Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left(\left(Y^{\sharp_A^{\sharp_A}}RY^{\sharp_A}\right)^{\sharp_A}X^{\sharp_A}TX\right)\\
&= \frac{1}{2}\left\|X^{\sharp_A}TX\right\|_A\left\|YRY^{\sharp_A}\right\|_A + \frac{1}{2}\omega_A\left(YR^{\sharp_A}(XY)^{\sharp_A}TX\right).
\end{aligned}$$

This implies the desired inequality. □

Also, another application of Theorem 2.20 gives the following.

Corollary 2.24. *Let $T, S, R, Q, X, Y \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}&\omega_A^2\left(TR^{\sharp_A}+SQ^{\sharp_A}\right)\\&\le \frac{1}{2}\left\|TT^{\sharp_A}+SS^{\sharp_A}\right\|_A\left\|RR^{\sharp_A}+QQ^{\sharp_A}\right\|_A+\frac{1}{2}\omega_A\left((TT^{\sharp_A}+SS^{\sharp_A})(RR^{\sharp_A}+QQ^{\sharp_A})\right).\end{aligned}$$

Proof. The proof follows from Theorem 2.20 and by using the fact

$$\begin{pmatrix} T^{\sharp_A^{\sharp_A}}T^{\sharp_A}+S^{\sharp_A^{\sharp_A}}S^{\sharp_A} & \left(R^{\sharp_A^{\sharp_A}}T^{\sharp_A}+Q^{\sharp_A^{\sharp_A}}S^{\sharp_A}\right)^{\sharp_A} \\ R^{\sharp_A^{\sharp_A}}T^{\sharp_A}+Q^{\sharp_A^{\sharp_A}}S^{\sharp_A} & R^{\sharp_A^{\sharp_A}}R^{\sharp_A}+Q^{\sharp_A^{\sharp_A}}Q^{\sharp_A}\end{pmatrix}$$

is **A**-positive, (see Theorem 2.13). □

The following inequalities are immediate consequence of the above result.

Corollary 2.25. *Let $R, T \in \mathcal{B}_A(\mathcal{H})$, then*

(i) $$\omega_A^2(RT) \le \frac{1}{2}\|R\|_A^2\|T\|_A^2+\frac{1}{2}\omega_A\left(T^{\sharp_A}TRR^{\sharp_A}\right),$$

(ii) $$\omega_A^2(R+Q) \le 2\left\|RR^{\sharp_A}+QQ^{\sharp_A}\right\|_A.$$

We end the article with following new inequality on operator norm.

Theorem 2.26. *Let T be a bounded linear operator on $\mathcal{H}$, i.e., $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\frac{1}{\sqrt{2}}\|T^*T+TT^*\|^{1/2} \le \max\|Re(T)\pm Im(T)\| \le \|T^*T+TT^*\|^{1/2}.$$

Proof. Considering $A=I$ and $R=\mathrm{Re}(T)$, $Q=\pm\mathrm{Im}(T)$ in Corollary 2.25 (ii), we obtain $\|\mathrm{Re}(T)\pm\mathrm{Im}(T)\|^2 \le \|T^*T+TT^*\|$. Also it is easy to verify that

$$\begin{aligned}\|T^*T+TT^*\| &\le \|\mathrm{Re}(T)+\mathrm{Im}(T)\|^2+\|\mathrm{Re}(T)-\mathrm{Im}(T)\|^2\\&\le 2\max\|\mathrm{Re}(T)\pm\mathrm{Im}(T)\|^2.\end{aligned}$$

Hence

$$\frac{1}{\sqrt{2}}\|T^*T+TT^*\|^{1/2} \le \max\|\mathrm{Re}(T)\pm\mathrm{Im}(T)\| \le \|T^*T+TT^*\|^{1/2},$$

as desired. □

Acknowledgements. Dr. Pintu Bhunia would like to thank SERB, Govt. of India for the financial support in the form of National Post Doctoral Fellowship (N-PDF, File No. PDF/2022/000325) under the mentorship of Professor Apoorva Khare.

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