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Ky Fan's Lemma for Metric Spaces and an Approximation to the Goldbach's Problem

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Given a metric space (X, d) and a subset $K \subseteq X$ we say K is d-convex if for every $x, y \in K$, the segment between them defined as $[x, y] := \{z \in X : d(x, y) = d(x, z) + d(z, y)\}$ satisfy $[x, y] \subseteq K$. We generalize this notion to subsets where this condition is satisfied for a subset of segments that cover the subset. Then we show versions of a Ky Fan's Lemma on spaces with this property. As an application, we introduce an approximation to the Goldbach's problem.

Keywords: Metric space, d-convexity, Ky Fan's Lemma, Goldbach's problem.

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1. Introduction and preliminaries

Ky Fan's Lemma is a corner stone of mathematics and has its roots in the minimax principle on vector spaces, see [6]. This principle is based in the convexity and concavity of the subsets involved. The appearance of general concepts of convexity beyond vector spaces (see e.g. [1] and [11] for an overview of these ideas), suggest the possibility of generalizing or making versions of this result for metric spaces. For example, a version for hyperconvex metric spaces is given in [10] and another one for L-convex metric spaces in [19]. The notion of d-convexity we deal appeared in a J. de Groot's paper [3] and it was applied to functions in [18], although the term "d-convex set" was probably used for the first time by P. S. Soltan in [17]. On the other hand K. Menger introduced in [12] a similar, but more general convexity concept, where a convex subset must contains at least one (not necessarily all) point strictly between the end points of the segment. All the same, the priority of introducing the concept of d-convexity is not clear. As far as we know there is no versions of Ky Fan'n Lemma for this sort of convexity, so here we present some versions involving metric spaces. Furthermore, the minimax principle is usually applied to the existence of a fixed point (see for instance [11] for a survey on this idea) or for showing some inequalities in functional analysis (see for example [13, Lemma 6.13], [4, Theorem 9.7] or [14, Theorem 16.4.3]). In the present paper we propose an application to number theory by means of suitable inequalities.

The idea of applying Ky Fan's Lemma to the Goldbach's problem has some technical troubles mainly related to the discrete nature of the issue, which force us, in some sense, to the use of the non classical concept of δ-density for the range of the metric. Moreover, in the course of the investigation it has been appeared some sets which

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are not convex in the classical sense, but they preserve a kind of local convexity that we have called *segment convexity*. The idea is to cover the space by means of suitable segments and use the convexity property just in these segments. This allows us to split hairs with functions that are not convex for arbitrary segments, but they are so for adequate segments that cover the space.

We have chosen the following version given in [14, E.4], which we have adapted to our context.

Lemma 1.1. (Ky Fan) *Let E be a Hausdorff topological vector space, and let K be a compact convex subset of E. Let $\mathcal{K}$ be a concave family of lower semi-continuous convex real functions f in K. Assume that for each $f \in \mathcal{K}$ there is $x \in K$ such a that $f(x) \leq 0$. Then, there exists $x_0 \in K$ such that $f(x_0) \leq 0$ for every $f \in \mathcal{K}$.*

We have organized the paper as follows. After the introduction and preliminary section, we show three versions of this lemma in Section 2. The first one include the notion of d-convexity and density for the range of the metric. For the second version we introduce the notion of segment density, where density of the metric is restricted to a covering by segments. The third version is stated for δ-density instead of the classical density. Finally, we include two corollaries, one of them we need for our application to the Goldbach's problem. The last section (Section 3) is devoted to the approximation of such a problem, that we motivate and after some preliminary lemmas, we formulate our theorem.

Let us introduce some preliminaries. A *metric* on a non-empty set X is a mapping $d\colon X \times X \to [0, \infty)$ such that for every $x, y, z \in X$

(i) $d(x, y) = 0$ if and if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

The *metric space* defined by d is denoted as (X, d). For a metric space (X, d), let $x, y \in X$, the set

$$[x, y] := \{z \in X : d(x, y) = d(x, z) + d(z, y)\},$$

is called the *closed segment* or just *segment*. A set $A \subseteq X$ is *d-convex* if for any $x, y \in A$, $[x, y] \subseteq A$. A function $f\colon X \to \mathbb{R}$ is *d-convex* if for every $x, y \in X$

$$f(z) \leq \frac{d(y, z)}{d(x, y)} f(x) + \frac{d(z, x)}{d(x, y)} f(y)\,, \qquad z \in [x, y]\,.$$

See [18] for a survey on this concept. A set $\mathcal{K}$ of real-valued functions defined on a set K is *concave* if, for every $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{K}$ and $\alpha_1, \ldots, \alpha_n \geq 0$ such that $\sum_{j=1}^{n} \alpha_j = 1$, there exists $f \in \mathcal{K}$ such that

$$f(x) \geq \sum_{j=1}^{n} \alpha_j f_j(x) \text{ for every } x \in K\,.$$

Let (X, d) be a metric space with $d\colon X \times X \to I$ where I is an interval that contains 0. We say that *d has dense range* if $d(X \times X)$ is dense in I. Let (X, d) be a metric space where $d\colon X \times X \to I$. For a subset $A \subset X$ we use the notation $d|_A := d|_{A\times A}$. We also will deal with a non standard concept which was introduced in [9] (see also [5]).

Definition 1.2. Let $\delta > 0$, let (X, d) be a metric space and $A \subseteq X$. We say that A is *δ-dense* if for every $x \in X$ there exists $a \in A$ such that $d(x, a) < \delta$.

Let $\lambda > 0$ and $x \in X$, we denote the (open) *ball centered in x with radius λ* as $B_\lambda(x) := \{z \in X : d(x, z) < \lambda\}$ and the closed ball as its closure denoted by $\overline{B_\lambda(x)}$. It is well known that the set of all balls is a base of neighborhoods.

2. Ky Fan's lemma for metric spaces

In this section we prove three versions of Ky Fan's Lemma, each one of them with weak conditions from the previous one. The difference among other versions of Ky Fan's lemma underlies in the use of segments, this use arise from the existence of metric spaces and functions that are not convex, but there are some subsets (as segments) where this convexity works well. Next lemma is our first version of Ky Fan's Lemma for the case of metric spaces and d-convex functions, where we have made a version of the proof as was given in [4, Lemma 9.10]. Also notice we have obviated the Hausdorff hypothesis, since every metric space is so and in order to show the difference among the versions we have enumerated the hypothesis. We also introduce the following strong condition.

Definition 2.1. Let (X, d) be a metric space. We say that d has *segment dense range* if for every $[x, y] \subseteq X$, the mapping $d|_{[x,y]}$ has dense range.

Notice that if $d|_{[x,y]}$ has dense range, then the mappings $f, g\colon [x, y] \to [0, 1]$ defined as

$$f(z) := \frac{d(x, z)}{d(x, y)} \quad \text{and} \quad g(z) := \frac{d(y, z)}{d(x, y)}$$

have dense ranges in $[0, 1]$.

Lemma 2.2. *Let (X, d) be a metric space. Let K be a subset of X such that it is*

(1) *compact,*

(2) *d-convex and*

(3) *the restriction $d|_K$ has segment dense range.*

Let $\mathcal{K}$ be a family of real functions on K such that it is

(4) *concave*

and functions in $\mathcal{K}$ are

(5) *lower semi-continuous,*

(6) *d-convex and*

(7) *for each $f \in \mathcal{K}$ there exists $x \in K$ such that $f(x) \leq 0$.*

Then we can find $z_0 \in K$ such that $f(z_0) \leq 0$ for every $f \in \mathcal{K}$.

Proof. *Step 1.* Let $f \in \mathcal{K}$, $\varepsilon > 0$ and $S(f, \varepsilon) := \{x \in K : f(x) \leq \varepsilon\}$. Since f is lower semi-continuous, $S(f, \varepsilon)$ is closed in the compact set K for every $f \in \mathcal{K}$ and every $\varepsilon > 0$. Also, as there exists $x \in K$ such that $f(x) \leq 0$, $S(f,\varepsilon) \neq \emptyset$. Let us show that all finite intersections $\cap_{i=1}^n S(f_i,\varepsilon_i)$ are non-empty. Let $A := \prod_{i=1}^n (-\infty, \varepsilon_i]$ and let B be the convex hull in $\mathbb{R}^n$ of $W := \{(f_1(x),\ldots,f_n(x)) : x \in K\}$. Assume that $A \cap B \neq \emptyset$. Take $(\xi_1,\ldots,\xi_n) \in A \cap B$. Then

$$\varepsilon_i \geq \xi_i = \alpha f_i(x) + \beta f_i(y)\,,$$

where $i = 1,\ldots,n$, $x,y \in K$, $\alpha,\beta \geq 0$ and $\alpha+\beta = 1$.

Step 2. On the one hand $d|_K$ has segment dense range and K is d-convex, hence for each $j \in \mathbb{N}$ there exists $z_j \in [x,y] \subseteq K$ such that

$$\Big|\alpha - \frac{d(y,z_j)}{d(x,y)}\Big| < \frac{1}{j} \quad \text{and} \quad \Big|\beta - \frac{d(x,z_j)}{d(x,y)}\Big| < \frac{1}{j}.$$

On the other hand f_i is segment d-convex, hence it satisfies that

$$\frac{d(y,z_j)}{d(x,y)} f_i(x) + \frac{d(x,z_j)}{d(x,y)} f_i(y) \geq f_i(z_j) \quad \text{for} \quad i = 1,\ldots,n\,.$$

But K is compact, thus we can choose $(z_j)_j$ so that $z_j \to z_0 \in K$, then $\alpha = \frac{d(y,z_0)}{d(x,y)}$ and $\beta = \frac{d(x,z_0)}{d(x,y)}$. In consequence, there exists $z_0 \in [x,y] \subseteq K$ such that

$$\varepsilon_i \geq \alpha f_i(x) + \beta f_i(y) \geq f_i(z_0) \quad \text{for} \quad i = 1,\ldots,n\,.$$

This means that $z_0 \in S(f_i,\varepsilon_i)$ for $i = 1,\ldots,n$. Therefore, as K is compact and all finite intersections $\cap_{i=1}^n S(f_i,\varepsilon_i)$ are non-empty, there exists $z_0 \in K$ (not necessarily the same as above) such that $z_0 \in S(f,\varepsilon)$ for every $\varepsilon > 0$ and every $f \in \mathcal{F}$. Or equivalently $f(z_0) \leq 0$ for every $f \in \mathcal{F}$.

Step 3. Assume that $A \cap B = \emptyset$. Thanks to Hahn-Banach separation theorem in $\mathbb{R}^n$ there exists $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n |\mu_i| = 1$, and $\nu \in \mathbb{R}$ such that $\langle \mu, a\rangle < \nu \leq \langle \mu, b\rangle$ for each $a \in A$ and $b \in B$. Observe that $\lambda e_j \in A$ for every $\lambda \leq 0$, hence $\langle \mu, \lambda e_j\rangle = \lambda\mu_j \leq \nu$ for each $\lambda \leq 0$, thus $\mu_j \geq 0$ for $j = 1,\ldots,n$. By concavity of $\mathcal{K}$ there exists $f \in \mathcal{K}$ such that

$$f(x) \geq \langle \mu, \big(f_i(x)\big)_{i=1}^n\rangle \geq \nu > \langle \mu, (\varepsilon_i)_{i=1}^n\rangle \geq 0\,,$$

for every $x \in K$, which is a contradiction to the hypothesis (7). $\square$

In the following version we show it for a weak notion of convexity. In definitions below we restrict convexity to a subset of segments (among all possible segments) that covers all the space. This notion, clearly generalizes the classical convexity concept.

Definition 2.3. Let (X,d) be a metric space and a subset $U \subset X \times X$. We denote by $[U] := \{[x,y] : (x,y) \in U\}$ and we say that $[U]$ is a *segment covering of* X if $\cup_{[x,y]\in[U]}[x,y] = X$, i.e. for each $z \in X$ there is $[x,y] \in [U]$ such that $z \in [x,y]$.

Example 2.4. The family of all segments of a metric space is obviously a segment covering.

Example 2.5. Let $X := \mathbb{Q} \times \mathbb{Q}$, let $U := \{((0,0),(r,s)) \in X \times X : r,s \in \mathbb{Q}\}$ and let d be any metric. The family of segments $[U]$ covers X and represents all line segments starting in the origin.

Definition 2.6. Let (X,d) be a metric space and $[U]$ be a set of segments of X. We say that d has $[U]$-*segment dense range* if for every $[x,y] \in [U]$, the mapping $d|_{[x,y]}$ has dense range.

Let us show how we can adapt the proof above to this kind of convexity.

Lemma 2.7. *Let (X,d) be a metric space. Let K be a subset of X such that it is*

(1) *compact,*

(2) *admits a segment covering $[U]$ and*

(3) *the restriction $d|_K$ has $[U]$-segment dense range.*

Also let $\mathcal{K}$ be a family of real functions on K such that it is

(4) *concave*

and functions in $\mathcal{K}$ are

(5) *lower semi-continuous,*

(6) *d-convex and*

(7) *for each $f \in \mathcal{K}$ there exists $x \in K$ such that $f(x) \leq 0$.*

Then we can find $z_0 \in K$ such that $f(z_0) \leq 0$ for every $f \in \mathcal{K}$.

Proof. *Step 1.* Let $f \in \mathcal{K}$, $\varepsilon > 0$ and $S(f,\varepsilon) := \{x \in K : f(x) \leq \varepsilon\}$. Since f is lower semi-continuous, $S(f,\varepsilon)$ is closed in the compact set K for every $f \in \mathcal{K}$ and every $\varepsilon > 0$. Also, as there exists $x \in K$ such that $f(x) \leq 0$, $S(f,\varepsilon) \neq \emptyset$. Let us show that all finite intersections $\cap_{i=1}^n S(f_i,\varepsilon_i)$ are non-empty. Let $A := \prod_{i=1}^n (-\infty, \varepsilon_i]$, let $[U]$ be the segment covering of K and let

$$W := \{(\alpha f_k(x) + \beta f_k(y))_{k=1}^n : [x,y] \in [U],\ \alpha + \beta = 1,\ \alpha \geq 0, \beta \geq 0\} \subseteq \mathbb{R}^n .$$

Assume that $A \cap W \neq \emptyset$. Take $(\xi_1, \ldots, \xi_n) \in A \cap W$. Then

$$\varepsilon_i \geq \xi_i = \alpha f_i(x) + \beta f_i(y) ,$$

where $i = 1, \ldots, n$, $(x,y) \in U \subset K \times K$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Step 2. Repeat *Step 2* of Lemma 2.2, taking into account that $[x,y] \in [U]$.

Step 3. Assume that $A \cap W = \emptyset$ and denote B the convex hull of W. We define

$$W_i := \{\alpha f_i(x) + \beta f_i(y) : [x,y] \in [U],\ \alpha + \beta = 1, \alpha \geq 0, \beta \geq 0\} ,$$

hence $W_i \cap (-\infty, \varepsilon_i] = \emptyset$, then $\varepsilon_i < p = \alpha f_i(x) + \beta f_i(y)$ for every $(x,y) \in U$ and every $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta = 1$. We claim that the convex hull of W_i is disjoint with $(-\infty, \varepsilon_i]$. Let $q \in W_i$, since $W_i \cap (-\infty, \varepsilon_i] = \emptyset$, $q > \varepsilon_i$. Let $\gamma \geq 0$, $\delta \geq 0$ such that $\gamma + \delta = 1$, hence $\gamma p + \delta q \geq (\gamma + \delta) \min\{p,q\} > \varepsilon_i$. This means that the convex hull of W_i and $(\infty, \varepsilon_i]$ are disjoint sets. As $W = W_1 \times \cdots \times W_n$ and the product of convex hulls is the convex hull of the product, we get $A \cap B = \emptyset$. Observe that, since $[U]$ is a covering of K, B is actually the convex hull of the set $\{(f_i(x))_{i=1}^n : x \in K\}$. Finally repeat *Step 3* of Lemma 2.2, having in mind this last observation. $\square$

In the last step of this section we make a version of lemma above by changing density with δ-density (see Definition 1.2). Next definition is restricted to a set of segments.

Definition 2.8. Let $\delta > 0$, let (X, d) be a metric space and $[U]$ be a set of segments of X. We say that d has *$[U]$-segment δ-dense range* if for every $[x, y] \in [U]$, the mapping $d|_{[x,y]}$ has δ-dense range and we say that it is *segment δ-dense* if $U = X \times X$.

Obviously, density implies δ-density, but the converse is clearly not true.

Lemma 2.9. *Let $\delta \in (0, 1)$, (X, d) be a metric space and K be a subset of X such that it is*

(1) *compact,*
(2) *it admits a segment covering $[U]$ and*
(3) *the restriction $d|_K$ has $[U]$-segment δ-dense range.*

Also let $\mathcal{K}$ be a family of real functions on K such that it is

(4) *uniformly bounded,*
(5) *concave*

and functions in $\mathcal{K}$ are

(6) *lower semi-continuous,*
(7) *segment d-convex and*
(8) *for each $f \in \mathcal{K}$ there exists $x \in K$ such that $f(x) \leq \delta S$, where $S := \sup_{f \in \mathcal{K}} |f|$.*

Then we can find $z_0 \in K$ and $M \leq S$ such that $f(z_0) \leq \delta M$ for every $f \in \mathcal{K}$.

Proof. As K is compact and admits a segment covering, we can choose $[U]$ be a finite covering, i.e. $[U] = \{[x_1, y_1], \ldots, [x_m, y_m]\}$ for some $m \in \mathbb{N}$. Let $f \in \mathcal{K}$, $\varepsilon > 0$, $M_f := \max_{i=1,\ldots,m}\{|f(x_i)|, |f(y_i)|\}$ and define

$$S(f, \varepsilon) := \{x \in K : f(x) \leq \varepsilon + \delta M_f\}.$$

Since f is lower semi-continuous, $S(f, \varepsilon)$ is closed in the compact set K for every $f \in \mathcal{K}$ and every $\varepsilon > 0$. Also, as there exists $x \in K$ such that $f(x) \leq \delta S$, $S(f, \varepsilon) \neq \emptyset$. Let us show that all finite intersections $\cap_{i=1}^n S(f_i, \varepsilon_i)$ are non-empty. Let $A := \prod_{i=1}^n (-\infty, \varepsilon_i]$ and let

$$W := \{(\alpha f_k(x) + \beta f_k(y))_{k=1}^n : [x, y] \in [U],\, \alpha + \beta = 1,\, \alpha \geq 0, \beta \geq 0\} \subseteq \mathbb{R}^n .$$

Assume that $A \cap W \neq \emptyset$ and take $(\xi_1, \ldots, \xi_n) \in A \cap W$. Then

$$\varepsilon_i \geq \xi_i = \alpha f_i(x) + \beta f_i(y) ,$$

where $i = 1, \ldots, n$, $(x, y) \in U \subset K \times K$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. On the one hand $d|_K$ has $[U]$-segment δ-dense range, hence there exists $z_0 \in [x, y] \subseteq K$ such that $\left|\alpha - \frac{d(y,z_0)}{d(x,y)}\right| < \delta$ and $\left|\beta - \frac{d(x,z_0)}{d(x,y)}\right| < \delta$. Notice that $\frac{d(x,z_0)}{d(x,y)} + \frac{d(y,z_0)}{d(x,y)} = 1$ and $\alpha + \beta = 1$, hence either $\alpha \leq \frac{d(y,z_0)}{d(x,y)}$ or $\beta \leq \frac{d(y,z_0)}{d(x,y)}$. Assuming that $\alpha \leq \frac{d(y,z_0)}{d(x,y)}$, necessarily $\beta \geq \frac{d(y,z_0)}{d(x,y)}$ and also $\alpha \geq \frac{d(y,z_0)}{d(x,y)} - \delta$. We just prove this case, if in case the first inequality holds for β we just change α by β in such inequalities. On the other hand f_i is d-convex, so it satisfies that

$$\frac{d(y,z_0)}{d(x,y)} f_i(x) + \frac{d(x,z_0)}{d(x,y)} f_i(y) \geq f_i(z_0) \quad \text{for} \quad i = 1, \ldots, n\,.$$

In consequence we obtain

$$\varepsilon_i \geq \alpha f_i(x) + \beta f_i(y) \geq f_i(z_0) - \delta f_i(x)\,,$$

for $i = 1, \ldots, n$, which implies that $\varepsilon_i + \delta M_{f_i} \geq f_i(z_0)$. Moreover, $\mathcal{K}$ is uniformly bounded, thus there exists $M := \sup_{f \in \mathcal{K}} M_f$, so $\varepsilon_i + \delta M \geq f_i(z_0)$. This means that $z_0 \in S(f_i, \varepsilon_i)$ for $i = 1, \ldots, n$. Therefore, as K is compact and all finite intersections $\cap_{i=1}^n S(f_i, \varepsilon_i)$ are non-empty, there exists $z_0 \in K$ such that $z_0 \in S(f, \varepsilon)$ for every $\varepsilon > 0$ and every $f \in \mathcal{K}$. Or equivalently $f(z_0) \leq \delta M$ for every $f \in \mathcal{K}$.

Finally repeat *Step 3* of Lemma 2.7. □

Notice that in Lemmas 2.7 and 2.9 the segment covering of K can be fixed and convex properties of functions can be changed as follows:

Definition 2.10. Let (X, d) be a metric space, let $[U]$ be a set of segments of X and let $f\colon X \to \mathbb{R}$ be a real function. We say that f is $[U]$*-segment* d*-convex* if we have for every $[x, y] \in [U]$ and $z \in [x, y]$

$$f(z) \leq \frac{d(y,z)}{d(x,y)} f(x) + \frac{d(z,x)}{d(x,y)} f(y)\,.$$

Despite we do not use this notion for sets in the present paper, we believe that it is worthwhile to introduce it as follows.

Definition 2.11. Let (X, d) be a metric space, let $[U]$ be a set of segments of X and let a set $C \subseteq X$. We say that C is $[U]$*-segment* d*-convex* if for every $[x, y] \in [U]$, $[x, y] \subseteq C$.

Proposition 2.12. *Let* $[U_1]$,$[U_2]$ *be two families of segments in a metric space* (X, d) *so that* $[U_1] \subseteq [U_2]$*. Assume that* X *is* $[U_2]$*-segment* d*-convex, then it is* $[U_1]$*-segment* d*-convex.*

An easy example is the following one.

Example 2.13. Let d be the usual metric in $\mathbb{R}^2$. Assume $X_1 := \{0\} \times [-1, 1]$, $X_2 := [-1, 1] \times \{0\}$ and $X := X_1 \cup X_2$, which is clearly not convex. Let $U_1 := X_1^2$, $U_2 := X_2^2$ and $U_3 := U_1 \cup U_2$. Then (X_i, d) is $[U_i]$-segment d-convex but not $[U_j]$-segment d-convex where $i, j \in \{1, 2\}$ and $i \neq j$. Also, (X, d) is $[U_k]$-segment d-convex for $k = 1, 2, 3$.

With this, let us write our last versions of Ky Fan's Lemma as corollaries. From Lemma 2.7 we obtain

Corollary 2.14. *Let (X,d) be a metric space. Let K be a subset of X such that it is*

(1) *compact,*
(2) *it admits a segment covering $[U]$ and*
(3) *the restriction $d|_K$ has $[U]$-segment dense range.*

Also let $\mathcal{K}$ be a family of real functions on K such that it is

(4) *concave*

and functions in $f \in \mathcal{K}$ are

(6) *lower semi-continuous,*
(7) *$[U]$-segment d-convex and*
(8) *for each $f \in \mathcal{K}$ there exists $x \in K$ such that $f(x) \leq 0$.*

Then we can find $z_0 \in K$ such that $f(z_0) \leq 0$ for every $f \in \mathcal{K}$.

The following version is the one we apply in next section. From Lemma 2.9 we obtain the

Corollary 2.15. *Let $\delta \in (0,1)$, (X,d) be a metric space and K be a subset of X such that it is*

(1) *compact,*
(2) *it admits a segment covering $[U]$ and*
(3) *the restriction $d|_K$ has $[U]$-segment δ-dense range.*

Also let $\mathcal{K}$ be a family of real functions on K such that it is

(4) *uniformly bounded,*
(5) *concave*

and functions $f \in \mathcal{K}$ are

(6) *lower semi-continuous,*
(7) *$[U]$-segment d-convex and*
(8) *for each $f \in \mathcal{K}$ there exists $x \in K$ such that $f(x) \leq \delta S$, where $S := \sup_{f\in\mathcal{K}} |f|$.*

Then we can find $z_0 \in K$ and $M \leq S$ such that $f(z_0) \leq \delta M$ for every $f \in \mathcal{K}$.

3. Approximation to the Goldbach's problem

Next proposition describe our idea of approximation. It has been proved by means of the Generalized Bertrand Postulate [8], but it can be deduced from an observation of Sierpiński (see [16]) for dense subsets of rational numbers.

Proposition 3.1. *Any positive real number can be approximated from right as far as we want by the quotient of two prime numbers.*

Proof. Let $x \in \mathbb{R}^+$ and let $\varepsilon > 0$. We have to find p, q primes such that

$$0 \leq \frac{p}{q} - x < \varepsilon\,, \quad \text{and so} \quad x \leq \frac{p}{q} < x + \varepsilon\,,$$

or equivalently

$$qx \leq p < qx + q\varepsilon = qx\Big(1 + \frac{\varepsilon}{x}\Big)\,. \tag{1}$$

From the generalized Bertrand's Postulate (see [8]) we deduce that for each $\delta > 0$ there exists $m_\delta > 0$ such that for every $m \geq m_\delta$, there is a prime p such that

$$m \leq p < m(1+\delta)\,.$$

Let us take $\delta := \dfrac{\varepsilon}{x}$ and choose a prime q so that $qx \geq m_\delta$. Then, there exists two prime numbers p and q such that (1) holds. □

The following proposition justify the election of quotients of co-prime numbers given in Theorem 3.8.

Proposition 3.2. *Every positive integer greater than six can be written as the sum of two co-prime numbers greater than one.*

Proof. For the proof, the numbers a, b, u, v will be assumed positive integers. Let $r > 6$ be an integer. We proceed by contradiction. Assume r is odd and let $s := (r-1)/2$, hence $r = s + (s+1)$. If s and $s+1$ are not co-prime, then there is a prime p such that $s = pu$ and $s+1 = pv$, but $1 = (s+1) - s = pv - pu = p(v-u)$ which is a contradition on the election of p. If r is even, it can be written as $r = (s-1) + (s+1)$, where $s = r/2$. On the one hand, if $s-1$ and $s+1$ are not co-prime, then there is a prime p such that $s - 1 = pu$ and $s + 1 = pv$. Hence, $p(v-u) = 2$, thus $p = 2$ and $v - u = 1$, so $v = u+1$ and $r = p(u+v) = 2(2u+1)$. On the other hand, the same argument for $s-2$ and $s+2$ with $s-2 = qa$ and $s+2 = qb$ brings us to $q = 2$ and $b = a+2$, in consequence $r = 2(2a+2)$. This implies that $2u+1 = 2a+2$ which is a contradiction. □

In order to apply Corollary 2.15 we develop some theory.

Lemma 3.3. *Let (X,d) be a metric space, let $K \subseteq X$ be a compact subset. Given a real valued function $f\colon K \to \mathbb{R}$, the set of functions $\mathcal{K} := \{f - \varepsilon : \varepsilon > 0\}$ is concave.*

Proof. Let $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, hence $\alpha(f(x) - \varepsilon_1) + \beta(f(x) - \varepsilon_2) = (\alpha+\beta)f(x) - (\alpha\varepsilon_1 + \beta\varepsilon_2) < f(x) - \varepsilon_0$ for $\varepsilon_0 < \alpha\varepsilon_1 + \beta\varepsilon_2$. □

Given a metric space (X,d), we define the *diameter* of X as $\Delta(X) := \sup_{x,y\in X} d(x,y)$.

Lemma 3.4. *Let (X,d) be a non trivial bounded metric space, let $K \subset X$ be a non-empty proper subset. Then, there exists $\delta \leq \Delta(X)$ such that K is δ-dense in X.*

Proof. We define $\delta := \sup_{x\in X} d(x,K)$, thus having in mind that X is non-trivial

$$\delta = \sup_{x\in X}\{\inf_{y\in K} d(x,y)\} \leq \sup_{x\in X}\{\inf_{y\in X} d(x,y)\} \leq \sup_{x\in X}\{\sup_{y\in X} d(x,y)\} = \Delta(X)\,.$$

This δ computes for each $x \in X$ the distance to the subset K and takes the maximum among these distances, so for every $x \in X$ there exists $y \in K$ such that $d(x, y) \leq \delta$ which means that K is δ-dense in X. $\square$

Remark 3.5. Observe that if $d(z, a) \leq \min\{d(x, a), d(y, a)\}$ for every $x, y \in X$ and $z \in [x, y]$, then the function $f(z)$ is d-convex. For example for $x, y \in X$, if $z, a \in [x, y]$, then $d(z, a) \leq d(x, y)$

$$\begin{aligned} 2d(z,a) &= d(z,a) + d(z,a) \leq d(a,y) + d(y,z) + d(a,x) + d(x,z) \\ &= (d(a,x) + d(a,y)) + (d(y,z) + d(x,z)) = 2d(x,y) \,. \end{aligned}$$

Hence, if $z \in [x, a]$, $d(z, a) \leq d(x, a)$ and if $z \in [a, y]$, thus $d(z, a) \leq d(y, a)$, but we would need both conditions to obtain that $d(z, a) \leq \min\{d(x, a), d(y, a)\}$.

For a metric space (X, d) where all closed balls are d-convex, we say that it is *with d-convex closed balls*. There are several conditions for the convexity of open and closed balls in metric spaces, see e.g. [7, 2] and [15, Theorem 1.24].

Lemma 3.6. *Let (X, d) be a compact metric space with d-convex closed balls and balls has at least 3 elements. Then, for each $a \in X$ there exists a segment covering $[U_a]$ such that the mapping $f(x) := d(a, x)$ is $[U_a]$-segment d-convex.*

Proof. Let $y \in X \setminus \{a\}$ and consider the ball centered at a with radius $\lambda := d(a, y)$. Since balls has at least three elements and closed balls are compact, there exists

$$x \in \Lambda_y := \{z \in X : d(z,a) = \sup\{d(a,u) < \lambda : u \in \overline{B_\lambda(a)} \setminus \{y\}\}\} \,,$$

such that $x \neq y$. This means that any $z \in [x, y]$ satisfies $d(z, a) \leq \min\{d(x, a), d(y, a)\}$. In consequence the set

$$U_a = \bigcup_{y \in X} \Lambda_y \times \{y\}$$

is non empty and clearly $[U_a]$ covers X. Finally, because as balls are d-convex, given $[x, y] \in [U_a]$

$$d(a,z) = \tfrac{d(z,y)}{d(x,y)} d(a,z) + \tfrac{d(x,z)}{d(x,y)} d(a,z) \leq \tfrac{d(z,y)}{d(x,y)} d(a,x) + \tfrac{d(x,z)}{d(x,y)} d(a,y) \,,$$

for every $z \in [x, y]$. That is, the function $f(x) := d(a, x)$ is $[U_a]$-segment d-convex. $\square$

Remark 3.7. If in the previous Lemma 3.6 the element $a \in [x, y]$ can be chosen so that $d(x, a) = d(y, a)$, the the inequality $d(z, a) \leq \min\{d(x, a), d(y, a)\}$ holds. Therefore the set $U_a := \{(x, y) \in K \times K : d(x, a) = d(y, a)\}$ is non empty, and we can pick $[x, y] \in [U_a]$ and $\lambda := d(a, y) = d(a, x)$, thus $[x, y] \subseteq \overline{B_\lambda(a)}$ and so every $z \in [x, y]$ satisfies $d(z, a) \leq d(a, x) = d(a, y)$.

In the following theorem we show our approximation to the Goldbach's problem. As in Proposition 3.1 we show that we can be close to the quotient of rational numbers between 0 and 1 by means of the quotient of two primes, but in this case, prime numbers are bounded from above instead of from below as in such a proposition. Moreover, as we can observe in the proof, we gain certain control on this bound.

Theorem 3.8. *Let $1 < a < b$ be integers such that $\gcd(a, b) = 1$ and $\delta \in (0, 1)$. Then there exist a number $N \in \mathbb{N}$ and prime numbers $p, q \leq N$ such that*

$$\left|\frac{p}{q} - \frac{a}{b}\right| < \delta \,.$$

Proof. Let us define

$$X := \{(n, m) \in \mathbb{N} \times \mathbb{N} : \gcd(n, m) = 1 \text{ and } 1 < n < m\} \,.$$

We also define the metric on X

$$d((n, m), (r, s)) := \left|\frac{n}{m} - \frac{r}{s}\right| \,.$$

We now fix $a_0 := (a, b) \in X$. Notice that a and b have the same parity and so $a + b$ is even with $\gcd(a, b) = 1$. We denote by $\mathbb{P}$ the set of all prime numbers and. Let $k \in \mathbb{N}$ and define the set

$$K_k := \{(p, q) \in \mathbb{P} \times \mathbb{P} : 1 < p < q < k\} \subset X,$$

then $d(x, K_k) > d(x, K_{k+1})$ for every $x \in X$. So, the function $h(k) := \max_{x \in X} d(x, K_k)$ is a decreasing function in $k \in \mathbb{N}$. Let us choose $N := \max\{k \in \mathbb{N} : h(k) \leq \delta\}$ and define $K := K_N$ and $f_\varepsilon \colon K \to \mathbb{R}$ as $f_\varepsilon(x) := d(a_0, x) - \varepsilon$ for $\varepsilon > 0$. Then we denote the family of functions $\mathcal{K} := \{f_\varepsilon \colon K \to \mathbb{R} : \varepsilon \in (0, 1)\}$. From now on we are going to verify the hypothesis of Corollary 2.15.

(1) As K is finite, it is compact.

(4) Observe that $0 \leq d(x, y) < 1$ and so $\mathcal{K}$ is uniformly bounded where $\Delta(X) < 1$ and $f_\varepsilon(x) = d(a_0, x) - \varepsilon \leq 1$ for every $\varepsilon \in (0, 1)$.

(5) Thanks to Lemma 3.3, $\mathcal{K}$ is concave.

(6) The functions f_ε are continuous, since if $f_\varepsilon(x) > f_\varepsilon(y)$

$$\begin{aligned} f_\varepsilon(x) - f_\varepsilon(y) &= d(a_0, x) - \varepsilon - (d(a_0, y) - \varepsilon) = d(a_0, x) - d(a_0, y) \\ &\leq (d(x, y) + d(a_0, y)) - d(a_0, y) = d(x, y) \,, \end{aligned}$$

and if $f_\varepsilon(y) > f_\varepsilon(x)$, we change x by y and obtain the same result. In consequence, they are lower semi-continuous.

(2) and (7) Consider $(x, y) \in X$ and $B_\lambda(x, y)$ for some $\lambda > 0$. In order to apply Lemma 3.6 let us show that closed balls are d-convex and have more than two elements. Let $(x, y) \in B_\lambda(x, y)$, then the interval

$$\left(\frac{x}{y} - \lambda, \frac{x}{y} + \lambda\right)$$

contains infinite rational numbers c/d such that $\gcd(c, d) = 1$. Then the balls have at least three elements; so let $(n, m), (r, s) \in B_\lambda(x, y)$ such that $(n, m) \neq (r, s)$ and let $(u, v) \in [(n, m), (r, s)]$. It is clear that there is $t \in [0, 1]$ such that

$$\frac{u}{v} = t\frac{n}{m} + (1 - t)\frac{r}{s} = t(\frac{n}{m} - \frac{r}{s}) + \frac{r}{s} \,,$$

which is an afin function of t, as (n, m) and (r, s) are fixed and different, such a t is unique and the proof of d-convexity of the ball we are dealing, is now standard.

In consequence, Lemma 3.6 implies the existence of a covering $[U]$ and the $[U]$-segment d-convexity of f_ε.

(8) and (3): By construction of K, for each $\varepsilon \in (0,1)$, there exists $x \in K$ such that $d(a_0, x) - \varepsilon < \delta$. Lemma 3.4 shows that the converse is always possible. It is clear from the definition that d has δ-dense range in $[0,1)$.

In conclusion, by virtue of Corollary 2.15, there exist prime numbers $(p,q) \in K$ such that

$$d((a,b),(p,q)) = \left|\frac{a}{b} - \frac{p}{q}\right| < \delta. \qquad \square$$

Remark 3.9. Let us choose $r > 2$ be an even positive integer and consider the set of quotients

$$\left\{\frac{r-q}{q} : q \in \{r-2, \dots, r/2\} \cap \mathbb{P}\right\} = \left\{\frac{r-q_1}{q_1}, \dots, \frac{r-q_n}{q_n}\right\},$$

where $q_1, \dots, q_n$ are all the prime numbers in $\{r-2, \dots, r/2\}$. Then we choose $\delta := 1/(q_1 \cdots q_n)$. Thanks to theorem above we have

$$\left|\frac{r-q_i}{q_i} - \frac{p_i}{s_i}\right| < \frac{1}{q_1 \cdots q_n},$$

for $i = 1, \dots, n$, where $p_1, \dots, p_n$ and $s_1, \dots, s_n$ are prime numbers. In case that we were able to show that there exists some $i \in \{1, \dots, n\}$ such that

$$\left|\frac{r-q_i}{q_i} - \frac{p_i}{q_i}\right| \le \left|\frac{r-q_i}{q_i} - \frac{p_i}{s_i}\right|, \tag{2}$$

then Goldbach's problem should be solved. For example, if $s_i \overset{\text{(a)}}{>} q_i$ and $r \overset{\text{(b)}}{>} p_i + q_i$ for some $i \in \{1, \dots, n\}$, we obtain (2). By multiplying (b) by s_i and taking into account (a), we get $rs_i > p_i s_i + q_i s_i > p_i q_i + q_i s_i$. And by multiplying (b) by q_i we get $rq_i > p_i q_i + q_i^2$. Hence $rs_i - q_i s_i = (r - q_i)s_i > p_i q_i$ and $rq_i - q_i^2 = (r-q_i)q_i > p_i q_i$, thus we have respectively

$$\left|\frac{r-q_i}{q_i} - \frac{p_i}{s_i}\right| = \frac{r-q_i}{q_i} - \frac{p_i}{s_i}.$$

and

$$\left|\frac{r-q_i}{q_i} - \frac{p_i}{q_i}\right| = \frac{r-q_i}{q_i} - \frac{p_i}{q_i}$$

Moreover, $s_i > q_i$ implies that $(r-q_i)s_i q_i - p_i q_i^2 > (r-q_i)s_i q_i - p_i q_i s_i$. By dividing this inequality by $s_i q_i^2$ we get (2). Analogously, if $s_i < q_i$ and $r < p_i + q_i$ for some $i \in \{1, \dots, n\}$ we also obtain (2).

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