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Refining the Superadditivity Inequality for Weighted Jensen and Mercer Functionals

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The Jensen and Mercer functionals viewed as functions of their weighted vectors are superadditive as showed by S. S. Dragomir, J. E. Pečarić and L. E. Persson [*Properties of some functionals related to Jensen's inequality*, Acta Math. Hung. 70/1-2 (1996) 129–143] and by M. Krnić, N. Lovričević and J. Pečarić [*On some properties of Jensen-Mercer's functional*, J. Math. Inequalities 6/1 (2012) 125–139]. In the present paper we derive refinements of the corresponding superadditivity DPP/KLP inequalities. We establish Jensen and Mercer inequalities of second type. We introduce a preorder on the set of weighted vectors and show the monotonicity with respect to this preorder of a functional related to Jensen/Mercer inequality.

Keywords: Convex function, Jensen inequality, Mercer inequality, superadditive function, preorder, DPP/KLP inequalities, positive homogeneous function, monotone functional.

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1. Introduction

Convex functions play a significant role in many branches of mathematics, as mathematical analysis, optimization, nonlinear analysis, probability, statistics, mathematical programming, mathematical economics, theory of inequalities, and so on [3, 10, 14, 22, 24]. In the past decades inequalities for convex functions have received a great deal of attention from many researches (see e.g., [1, 2, 4, 6, 7, 5, 11, 13, 15, 16, 17, 21, 23, 25]). In the present paper we focus on some results about Jensen/Mercer inequalities and their refinements.

Throughout for given nonnegative numbers $p_1, p_2, \dots, p_k$ we denote $P_k = \sum\limits_{i=1}^{k} p_i$ and

$$\mathcal{P}_k^0 = \{\mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathbb{R}^k : p_i \geq 0, \quad P_k > 0\}.$$

Let $f : I \to \mathbb{R}$ be a function on an interval $I \subset \mathbb{R}$, $\mathbf{x} = (x_1, x_2, \dots, x_k) \in I^k$ and $\mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathcal{P}_k^0$. We define the *Jensen functional* by

$$J(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{k} p_i f(x_i) - P_k f\left(\frac{1}{P_k} \sum_{i=1}^{k} p_i x_i\right) \tag{1}$$

with the convention $0f(\frac{0}{0}) = 0$ (see [8]).

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Equivalently,

$$J(f,\mathbf{x},\mathbf{p}) = \langle \mathbf{p}, \mathbf{f}(\mathbf{x})\rangle - \langle \mathbf{p}, \mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \tag{2}$$

where $\mathbf{f}(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_k))$, $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^k$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^k$.

The celebrated Jensen's inequality says that

$$f \text{ is a convex function } \Rightarrow J(f,\mathbf{x},\mathbf{p}) \geq 0. \tag{3}$$

It is known by a result of Dragomir, Pečarić and Persson (see [8]) that if $f : I \to \mathbb{R}$ is a convex function then the function $\mathbf{p} \to J(f,\mathbf{x},\mathbf{p})$ is superadditive for any $\mathbf{x} \in I^k$, i.e.,

$$J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geq J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathcal{P}_k^0. \tag{4}$$

(For convenience, (4) is called *DPP inequality* in this paper.)

For a convex function $f : I \to \mathbb{R}$ and $\mathbf{x} \in I^k$ we define functional Φ by

$$\Phi(\mathbf{p},\mathbf{q}) = J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}) - J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathcal{P}_k^0. \tag{5}$$

It follows from (4) that

$$\Phi(\mathbf{p},\mathbf{q}) \leq 0 \quad \text{for } \mathbf{p},\mathbf{q} \in \mathcal{P}_k^0. \tag{6}$$

Given a function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $I = [a, b] \subset \mathbb{R}$, and k-tuples $\mathbf{x} = (x_1, x_2, \dots, x_k) \in [a, b]^k$ and $\mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathcal{P}_k^0$, we introduce the *Mercer functional* as follows:

$$M(f,\mathbf{x},\mathbf{p}) = P_k(f(a)+f(b)) - \sum_{i=1}^k p_i f(x_i) - P_k f\left(a+b-\frac{1}{P_k}\sum_{i=1}^k p_i x_i\right)$$

(see [9]). In other words,

$$M(f,\mathbf{x},\mathbf{p}) = (f(a)+f(b))\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p},\mathbf{f}(\mathbf{x})\rangle - \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{(a+b)\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right),$$

where $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^k$ and $\mathbf{f}(\mathbf{x}) = (f(x_1), f(x_2), \dots, f(x_k))$.

A. McD. Mercer [12] showed the following result (called Mercer's inequality):

$$f \text{ is a convex function } \Rightarrow M(f,\mathbf{x},\mathbf{p}) \geq 0. \tag{7}$$

Krnić, Lovričević and Pečarić [9] proved that if $f : I = [a, b] \to \mathbb{R}$ is a convex function then the function $\mathbf{p} \to M(f,\mathbf{x},\mathbf{p})$ is superadditive for any $\mathbf{x} \in I^k$, i.e.,

$$M(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geq M(f,\mathbf{x},\mathbf{p}) + M(f,\mathbf{x},\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathcal{P}_k^0. \tag{8}$$

(For convenience, (8) is called *KLP inequality* in this paper.)

For a convex function $f : I \to \mathbb{R}$ and $\mathbf{x} \in I^k = [a, b]^k$ we define the functional Ψ by

$$\Psi(\mathbf{p},\mathbf{q}) = M(f,\mathbf{x},\mathbf{p}) + M(f,\mathbf{x},\mathbf{q}) - M(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathcal{P}_k^0. \tag{9}$$

In light of (8) we see that

$$\Psi(\mathbf{p},\mathbf{q}) \le 0 \quad \text{for } \mathbf{p},\mathbf{q} \in \mathcal{P}_k^0. \tag{10}$$

In the present paper we built a framework for refining inequalities (6) and (10) by utilizing features of functionals Φ and Ψ defined by (5) and (9), respectively (see Theorem 2.2).

Secondly, we prove some refinements for inequalities (6) and (10) (see Theorem 2.3).

Finally, we introduce a preorder on the class of weighted k-tuples and prove the monotonicity of the functionals Φ and Ψ with respect to the preorder (see Corollary 3.2 and Remark 3.3).

2. Refining DPP/KLP inequalities

In the forthcoming lemma we demonstrate DPP/KLP inequalities of the second type (12)–(15) related to (6) and (16)–(19) related to (10). These inequalities are intended for a suitable configured quadruple of weighted vectors in $\mathcal{P}_k^0$. In fact, they are equivalent to corresponding superadditivity DPP/KLP inequalities of type (4) and (8), respectively. Nevertheless, they are convenient starting points for deriving refinements of (6) and (10).

Lemma 2.1. *Keep Φ and Ψ as defined in* (5) *and* (9)*. Let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{s}$ be k-tuples in $\mathcal{P}_k^0$. Denote*

$$\mathbf{r} = \mathbf{p} + \mathbf{q} \quad \textit{and} \quad \mathbf{t} = \mathbf{q} + \mathbf{s}. \tag{11}$$

If $f : I \to \mathbb{R}$ is a convex function on an interval $I \subset \mathbb{R}$ and $\mathbf{x} \in I^k$, then

$$\Phi(\mathbf{r},\mathbf{s}) \ge \Phi(\mathbf{q},\mathbf{s}) + \Phi(\mathbf{p},\mathbf{t}), \tag{12}$$

$$\Phi(\mathbf{p},\mathbf{q}) \ge \Phi(\mathbf{q},\mathbf{s}) + \Phi(\mathbf{p},\mathbf{t}), \tag{13}$$

$$\Phi(\mathbf{q},\mathbf{s}) \ge \Phi(\mathbf{p},\mathbf{q}) + \Phi(\mathbf{r},\mathbf{s}), \tag{14}$$

$$\Phi(\mathbf{p},\mathbf{t}) \ge \Phi(\mathbf{p},\mathbf{q}) + \Phi(\mathbf{r},\mathbf{s}). \tag{15}$$

If $f : I \to \mathbb{R}$ is a convex function on an interval $I = [a,b] \subset \mathbb{R}$ and $\mathbf{x} \in I^k$, then

$$\Psi(\mathbf{r},\mathbf{s}) \ge \Psi(\mathbf{q},\mathbf{s}) + \Psi(\mathbf{p},\mathbf{t}), \tag{16}$$

$$\Psi(\mathbf{p},\mathbf{q}) \ge \Psi(\mathbf{q},\mathbf{s}) + \Psi(\mathbf{p},\mathbf{t}), \tag{17}$$

$$\Psi(\mathbf{q},\mathbf{s}) \ge \Psi(\mathbf{p},\mathbf{q}) + \Psi(\mathbf{r},\mathbf{s}), \tag{18}$$

$$\Psi(\mathbf{p},\mathbf{t}) \ge \Psi(\mathbf{p},\mathbf{q}) + \Psi(\mathbf{r},\mathbf{s}). \tag{19}$$

Proof. We denote $\mathbf{u} = \mathbf{r} + \mathbf{s}$. Then, by (11),

$$\mathbf{u} = \mathbf{p} + \mathbf{t}. \tag{20}$$

In consequence, according to (5) and (11) we have

$$\begin{aligned}
&\Phi(\mathbf{p},\mathbf{q}) + \Phi(\mathbf{r},\mathbf{s}) \\
&= J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}) - J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) + J(f,\mathbf{x},\mathbf{r}) + J(f,\mathbf{x},\mathbf{s}) - J(f,\mathbf{x},\mathbf{r}+\mathbf{s}) \\
&= J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}) + J(f,\mathbf{x},\mathbf{s}) - J(f,\mathbf{x},\mathbf{u}).
\end{aligned}$$

Likewise, on account of (11) and (20), we also have

$$\begin{aligned}&\Phi(\mathbf{q},\mathbf{s})+\Phi(\mathbf{p},\mathbf{t})\\&=J(f,\mathbf{x},\mathbf{q})+J(f,\mathbf{x},\mathbf{s})-J(f,\mathbf{x},\mathbf{q}+\mathbf{s})+J(f,\mathbf{x},\mathbf{p})+J(f,\mathbf{x},\mathbf{t})-J(f,\mathbf{x},\mathbf{p}+\mathbf{t})\\&=J(f,\mathbf{x},\mathbf{q})+J(f,\mathbf{x},\mathbf{s})+J(f,\mathbf{x},\mathbf{p})-J(f,\mathbf{x},\mathbf{u}).\end{aligned}$$

Therefore we obtain the identity

$$\Phi(\mathbf{p},\mathbf{q})+\Phi(\mathbf{r},\mathbf{s})=\Phi(\mathbf{q},\mathbf{s})+\Phi(\mathbf{p},\mathbf{t}).\tag{21}$$

By using (6) and (21) it is not hard to verify that

$$\Phi(\mathbf{q},\mathbf{s})+\Phi(\mathbf{p},\mathbf{t})\quad\le\quad\begin{cases}\Phi(\mathbf{r},\mathbf{s})\\\Phi(\mathbf{p},\mathbf{q})\end{cases}\tag{22}$$

and

$$\Phi(\mathbf{p},\mathbf{q})+\Phi(\mathbf{r},\mathbf{s})\quad\le\quad\begin{cases}\Phi(\mathbf{q},\mathbf{s})\\\Phi(\mathbf{p},\mathbf{t}).\end{cases}\tag{23}$$

By virtue of (22)-(23) the inequalities (12), (13), (14) and (15) are fulfilled.

Furthermore, the inequalities (16)-(19) can be proved in a similar manner. □

We are now ready to state and prove some refinements of DPP/KLP inequalities of type (6) and (10).

Theorem 2.2. *Let* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{s}$ *be k-tuples in* $\mathcal{P}_k^0$. *Denote*

$$\mathbf{r}=\mathbf{p}+\mathbf{q}\quad\textit{and}\quad\mathbf{t}=\mathbf{p}+\mathbf{s}.\tag{24}$$

If $f:I\to\mathbb{R}$ *is a convex function on an interval* $I\subset\mathbb{R}$, $\mathbf{x}\in I^k$, *and* $\Phi(\mathbf{p},\mathbf{q})=0$, *then*

$$\Phi(\mathbf{r},\mathbf{s})\le\begin{cases}\Phi(\mathbf{q},\mathbf{t})\\\Phi(\mathbf{p},\mathbf{s})\end{cases},\tag{25}$$

where $\Phi(\cdot)$ *is defined by* (5).

If $f:I\to\mathbb{R}$ *is a convex function on an interval* $I=[a,b]\subset\mathbb{R}$, $\mathbf{x}\in I^k$, *and* $\Psi(\mathbf{p},\mathbf{q})=0$, *then*

$$\Psi(\mathbf{r},\mathbf{s})\le\begin{cases}\Psi(\mathbf{q},\mathbf{t})\\\Psi(\mathbf{p},\mathbf{s})\end{cases},\tag{26}$$

where $\Psi(\cdot)$ *is defined by* (9).

Proof. We define $\mathbf{u}=\mathbf{r}+\mathbf{s}$. Thanks to (24) we also have

$$\mathbf{u}=\mathbf{t}+\mathbf{q}.\tag{27}$$

Using (24) and (27) and proceeding as in the proof of Lemma 2.1, we find that

$$\Phi(\mathbf{p},\mathbf{q})+\Phi(\mathbf{r},\mathbf{s})=\Phi(\mathbf{p},\mathbf{s})+\Phi(\mathbf{q},\mathbf{t}).$$

Now, suppose that $\Phi(\mathbf{p}, \mathbf{q}) = 0$. Then we have

$$\Phi(\mathbf{r}, \mathbf{s}) = \Phi(\mathbf{p}, \mathbf{s}) + \Phi(\mathbf{q}, \mathbf{t}). \tag{28}$$

In addition, with the aid of (6) we get

$$\Phi(\mathbf{p}, \mathbf{s}) + \Phi(\mathbf{q}, \mathbf{t}) \leq \left\{ \begin{array}{c} \Phi(\mathbf{q}, \mathbf{t}) \\ \Phi(\mathbf{p}, \mathbf{s}) \end{array} \right. .$$

Hence, by (28), we deduce that

$$\Phi(\mathbf{r}, \mathbf{s}) \leq \left\{ \begin{array}{c} \Phi(\mathbf{q}, \mathbf{t}) \\ \Phi(\mathbf{p}, \mathbf{s}) \end{array} \right. ,$$

which is our result (25).

It remains to prove (26). In doing so, it is sufficient to apply the same method as for proving (25). □

By invoking to Theorem 2.2 we can derive some versions of refinements (25) and (26) of the inequalities $\Phi(\mathbf{r}, \mathbf{s}) \leq 0$ and $\Psi(\mathbf{r}, \mathbf{s}) \leq 0$, respectively (see Remark 2.4).

Theorem 2.3. *Let* $\mathbf{r}$ *and* $\mathbf{s}$ *be* k*-tuples in* $\mathcal{P}_k^0$. *If* $f : I \to \mathbb{R}$ *is a convex function on an interval* $I \subset \mathbb{R}$, $\mathbf{x} \in I^k$, *and* $J(f, \mathbf{x}, \mathbf{r}) \neq 0$ *and* $J(f, \mathbf{x}, \mathbf{s}) \neq 0$ *then*

$$\begin{gathered} J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) \\ - \left[2 - J\left(f, \mathbf{x}, \frac{\mathbf{r}}{J(f, \mathbf{x}, \mathbf{r})} + \frac{\mathbf{s}}{J(f, \mathbf{x}, \mathbf{s})} \right) \right] \min\{J(f, \mathbf{x}, \mathbf{r}), J(f, \mathbf{x}, \mathbf{s})\} \\ \leq J(f, \mathbf{x}, \mathbf{r} + \mathbf{s}). \end{gathered} \tag{29}$$

If $f : I \to \mathbb{R}$ *is a convex function on an interval* $I = [a, b] \subset \mathbb{R}$, $\mathbf{x} \in I^k$, *and* $M(f, \mathbf{x}, \mathbf{r}) \neq 0$ *and* $M(f, \mathbf{x}, \mathbf{s}) \neq 0$ *then*

$$\begin{gathered} M(f, \mathbf{x}, \mathbf{r}) + M(f, \mathbf{x}, \mathbf{s}) \\ - \left[2 - M\left(f, \mathbf{x}, \frac{\mathbf{r}}{M(f, \mathbf{x}, \mathbf{r})} + \frac{\mathbf{s}}{M(f, \mathbf{x}, \mathbf{s})} \right) \right] \min\{M(f, \mathbf{x}, \mathbf{r}), M(f, \mathbf{x}, \mathbf{s})\} \\ \leq M(f, \mathbf{x}, \mathbf{r} + \mathbf{s}). \end{gathered} \tag{30}$$

Proof. We start with the observation that our assertion (29) is symmetrical in $\mathbf{r}$ and $\mathbf{s}$. Therefore we can assume without lost of generality that $J(f, \mathbf{x}, \mathbf{s}) \leq J(f, \mathbf{x}, \mathbf{r})$. For this reason we have

$$0 < \frac{J(f, \mathbf{x}, \mathbf{s})}{J(f, \mathbf{x}, \mathbf{r})} \leq 1.$$

(The positivity of the last fraction follows from Jensen inequality (3).)

Let $\mathbf{v} \in \mathcal{P}_k^0$ and $0 < \lambda \in \mathbb{R}_+$ be arbitrarily fixed. By (2) we have

$$\begin{aligned} J(f, \mathbf{x}, \lambda\mathbf{v}) &= \langle \lambda\mathbf{v}, \mathbf{f}(\mathbf{x}) \rangle - \langle \lambda\mathbf{v}, \mathbf{e} \rangle f\left(\frac{\langle \lambda\mathbf{v}, \mathbf{x} \rangle}{\langle \lambda\mathbf{v}, \mathbf{e} \rangle} \right) \\ &= \lambda\langle \mathbf{v}, \mathbf{f}(\mathbf{x}) \rangle - \lambda\langle \mathbf{v}, \mathbf{e} \rangle f\left(\frac{\langle \mathbf{v}, \mathbf{x} \rangle}{\langle \mathbf{v}, \mathbf{e} \rangle} \right) = \lambda J(f, \mathbf{x}, \mathbf{v}). \end{aligned}$$

Hence,

$$J(f, \mathbf{x}, \lambda \mathbf{v}) = \lambda J(f, \mathbf{x}, \mathbf{v}) \quad \text{for } \mathbf{v} \in \mathcal{P}_k^0 \text{ and } \lambda > 0. \tag{31}$$

By putting

$$\mathbf{p} = \lambda \mathbf{r} \quad \text{and} \quad \mathbf{q} = (1-\lambda)\mathbf{r} \quad \text{for } \lambda \in (0, 1], \tag{32}$$

we get $\mathbf{r} = \mathbf{p} + \mathbf{q}$, as in Theorem 2.2 (see (24)).

Moreover, from (32) and (31) we conclude that

$$\begin{aligned} \Phi(\mathbf{p}, \mathbf{q}) &= J(f, \mathbf{x}, \mathbf{p}) + J(f, \mathbf{x}, \mathbf{q}) - J(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \\ &= J(f, \mathbf{x}, \lambda \mathbf{r}) + J(f, \mathbf{x}, (1-\lambda)\mathbf{r}) - J(f, \mathbf{x}, \mathbf{r}) \\ &= \lambda J(f, \mathbf{x}, \mathbf{r}) + (1-\lambda) J(f, \mathbf{x}, \mathbf{r}) - J(f, \mathbf{x}, \mathbf{r}) = 0. \end{aligned}$$

Thus the condition $\Phi(\mathbf{p}, \mathbf{q}) = 0$ is satisfied.

So, we are allowed to apply Theorem 2.2. From inequality (25) we obtain

$$\Phi(\mathbf{r}, \mathbf{s}) \le \Phi(\mathbf{p}, \mathbf{s}) \tag{33}$$

with $\mathbf{p} = \lambda \mathbf{r}$, $\lambda \in (0, 1]$.

By utilizing (5) and (33) we have

$$J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - J(f, \mathbf{x}, \mathbf{r} + \mathbf{s}) \le J(f, \mathbf{x}, \mathbf{p}) + J(f, \mathbf{x}, \mathbf{s}) - J(f, \mathbf{x}, \mathbf{p} + \mathbf{s}),$$

and next

$$J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - [J(f, \mathbf{x}, \lambda \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - J(f, \mathbf{x}, \lambda \mathbf{r} + \mathbf{s})] \le J(f, \mathbf{x}, \mathbf{r} + \mathbf{s}),$$

whence, by (31),

$$J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - [\lambda J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - J(f, \mathbf{x}, \lambda \mathbf{r} + \mathbf{s})] \le J(f, \mathbf{x}, \mathbf{r} + \mathbf{s}).$$

By inserting $\lambda = \frac{J(f, \mathbf{x}, \mathbf{s})}{J(f, \mathbf{x}, \mathbf{r})}$ into the last inequality we get

$$\begin{gathered} J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) \\ - \left[\frac{J(f, \mathbf{x}, \mathbf{s})}{J(f, \mathbf{x}, \mathbf{r})} J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - J\left(f, \mathbf{x}, \frac{J(f, \mathbf{x}, \mathbf{s})}{J(f, \mathbf{x}, \mathbf{r})} \mathbf{r} + \mathbf{s}\right) \right] \\ \le J(f, \mathbf{x}, \mathbf{r} + \mathbf{s}). \end{gathered}$$

Hence, equivalently, we have

$$\begin{gathered} J(f, \mathbf{x}, \mathbf{r}) + J(f, \mathbf{x}, \mathbf{s}) - \left[2 - J\left(f, \mathbf{x}, \frac{\mathbf{r}}{J(f, \mathbf{x}, \mathbf{r})} + \frac{\mathbf{s}}{J(f, \mathbf{x}, \mathbf{s})}\right) \right] J(f, \mathbf{x}, \mathbf{s}) \\ \le J(f, \mathbf{x}, \mathbf{r} + \mathbf{s}). \end{gathered}$$

Since $J(f, \mathbf{x}, \mathbf{s}) \le J(f, \mathbf{x}, \mathbf{r})$, the last inequality guarantees that (29) holds valid.

The proof of (30) is similar to the proof of (29), and therefore is omitted. $\square$

Remark 2.4. Inequality (29) in Theorem 2.3 is really a refinement of the DPP inequality $J(f,\mathbf{x},\mathbf{r}) + J(f,\mathbf{x},\mathbf{s}) \leq J(f,\mathbf{x},\mathbf{r}+\mathbf{s})$. Indeed, (4) and (31) imply that

$$2 - J\left(f, \mathbf{x}, \frac{\mathbf{r}}{J(f,\mathbf{x},\mathbf{r})} + \frac{\mathbf{s}}{J(f,\mathbf{x},\mathbf{s})}\right) \leq 0$$

and $\min\{J(f,\mathbf{x},\mathbf{r}), J(f,\mathbf{x},\mathbf{s})\} \geq 0$ is met by (3).

Similar remarks apply to inequality (30).

3. Preorder $\preccurlyeq$ and monotonicity of $\Phi(\cdot,\mathbf{s})$

In light of (31), for any $\mathbf{v} \in \mathcal{P}_k^0$ and $\lambda > 0$,

$$J(f,\mathbf{x},\lambda\mathbf{v}) = \lambda J(f,\mathbf{x},\mathbf{v}) \to 0 \quad \text{when } \lambda \to 0.$$

Therefore we are motivated to extend definition (1) by

$$J(f,\mathbf{x},\mathbf{0}) = 0, \tag{34}$$

which is in accordance with the convention $0f(\frac{0}{0}) = 0$.

It now follows from (4) and (34) that the function $J(f,\mathbf{x},(\cdot))$ is superadditive on the set $\mathcal{P}_k^0 \cup \{\mathbf{0}\}$.

For given k-tuples $\mathbf{p},\mathbf{r} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$, we write $\mathbf{p} \preccurlyeq \mathbf{r}$ if $\mathbf{r}-\mathbf{p} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$ and $\Phi(\mathbf{p},\mathbf{r}-\mathbf{p}) = 0$, i.e.,

$$J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{r}-\mathbf{p}) = J(f,\mathbf{x},\mathbf{r}). \tag{35}$$

Under (34) the relation $\preccurlyeq$ is reflexive on $\mathcal{P}_k^0 \cup \{\mathbf{0}\}$, i.e., $\mathbf{r} \preccurlyeq \mathbf{r}$ for any $\mathbf{r} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$.

Lemma 3.1. *The relation $\preccurlyeq$ is transitive on $\mathcal{P}_k^0 \cup \{\mathbf{0}\}$, i.e. for $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$, if $\mathbf{p} \preccurlyeq \mathbf{q}$ and $\mathbf{q} \preccurlyeq \mathbf{r}$ then $\mathbf{p} \preccurlyeq \mathbf{r}$.*

Proof. Let $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$ be such that $\mathbf{p} \preccurlyeq \mathbf{q}$ and $\mathbf{q} \preccurlyeq \mathbf{r}$. Then $\mathbf{q}-\mathbf{p}$ and $\mathbf{r}-\mathbf{q}$ are in $\mathcal{P}_k^0 \cup \{\mathbf{0}\}$. Hence we have $\mathbf{r}-\mathbf{p} = \mathbf{r}-\mathbf{q}+\mathbf{q}-\mathbf{p} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$.

Additionally, from (35) we get

$$J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}-\mathbf{p}) = J(f,\mathbf{x},\mathbf{q}) \text{ and } J(f,\mathbf{x},\mathbf{q}) + J(f,\mathbf{x},\mathbf{r}-\mathbf{q}) = J(f,\mathbf{x},\mathbf{r}).$$

By combining this and (4), we obtain

$$J(f,\mathbf{x},\mathbf{r}-\mathbf{p}) = J(f,\mathbf{x},\mathbf{r}-\mathbf{q}+\mathbf{q}-\mathbf{p}) \geq J(f,\mathbf{x},\mathbf{r}-\mathbf{q}) + J(f,\mathbf{x},\mathbf{q}-\mathbf{p})$$

$$= J(f,\mathbf{x},\mathbf{r}) - J(f,\mathbf{x},\mathbf{q}) + J(f,\mathbf{x},\mathbf{q}) - J(f,\mathbf{x},\mathbf{p}) = J(f,\mathbf{x},\mathbf{r}) - J(f,\mathbf{x},\mathbf{p}).$$

In consequence, we have

$$J(f,\mathbf{x},\mathbf{r}-\mathbf{p}) \geq J(f,\mathbf{x},\mathbf{r}) - J(f,\mathbf{x},\mathbf{p}). \tag{36}$$

By using again (4) we derive inequality

$$J(f,\mathbf{x},\mathbf{r}) = J(f,\mathbf{x},\mathbf{r}-\mathbf{p}+\mathbf{p}) \geq J(f,\mathbf{x},\mathbf{r}-\mathbf{p}) + J(f,\mathbf{x},\mathbf{p}).$$

Hence,

$$J(f,\mathbf{x},\mathbf{r}) - J(f,\mathbf{x},\mathbf{p}) \geq J(f,\mathbf{x},\mathbf{r}-\mathbf{p}). \tag{37}$$

By comparing (36) and (37) we get

$$J(f,\mathbf{x},\mathbf{r}-\mathbf{p}) = J(f,\mathbf{x},\mathbf{r}) - J(f,\mathbf{x},\mathbf{p}).$$

This and the proved fact $\mathbf{r}-\mathbf{p} \in \mathcal{P}_k^0 \cup \{\mathbf{0}\}$ give $\mathbf{p} \preccurlyeq \mathbf{r}$ by (35), which completes the proof. $\square$

On account of Lemma 3.1 the relation $\preccurlyeq$ is a preorder on $\mathcal{P}_k^0 \cup \{\mathbf{0}\}$. In Corollary 3.2 we show that the functional $\Phi(\cdot,\mathbf{s})$ is monotone with respect to $\preccurlyeq$.

Corollary 3.2. *Let* $\mathbf{p}$*,* $\mathbf{q}$ *and* $\mathbf{s}$ *be* k*-tuples in* $\mathcal{P}_k^0$*. Denote* $\mathbf{r} = \mathbf{p}+\mathbf{q}$.

If $f : I \to \mathbb{R}$ *is a convex function on an interval* $I \subset \mathbb{R}$ *and* $\mathbf{x} \in I^k$*, then*

$$\mathbf{p} \preccurlyeq \mathbf{r} \quad \textit{implies} \quad \Phi(\mathbf{p},\mathbf{s}) \geq \Phi(\mathbf{r},\mathbf{s}). \tag{38}$$

Proof. Assume $\mathbf{p} \preccurlyeq \mathbf{r}$. Hence $\Phi(\mathbf{p},\mathbf{r}-\mathbf{p}) = 0$ by (35), and next $\Phi(\mathbf{p},\mathbf{q}) = 0$, since $\mathbf{q} = \mathbf{r}-\mathbf{p}$. So, by (25) in Theorem 2.2, we get $\Phi(\mathbf{p},\mathbf{s}) \geq \Phi(\mathbf{r},\mathbf{s})$. This completes the proof of (38). $\square$

Remark 3.3. A preorder induced by $M(f,\mathbf{x},(\cdot))$ in place of $J(f,\mathbf{x},(\cdot))$ can be also defined (see (35)). Next, as a consequence of Theorem 2.2, the monotonicity of Ψ can be established.

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