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Viability of the Solution with a Perturbation and Feedback Control

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We consider a dynamic system with a perturbation in the right side. The perturbation is given by some geometrical restriction. We consider the question: find a feedback control that stabilizes a solution of the system required to be viable in a given compact convex subset? We construct minimal, in a certain sense, continuous feedback control which stabilizes the system. Some examples are considered.

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1. Introduction and main notations

Let (a, b) be an inner product of vectors $a, b \in \mathbb{R}^n$, $\|a\| = \sqrt{(a, a)}$. For a closed set $M \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ let $\varrho(x, M) = \inf_{y \in M} \|x - y\|$. For any compact subsets $M, N \subset \mathbb{R}^n$ the *Hausdorff distance* between M and N is defined as follows

$$h(M, N) = \max\{\max_{x \in M} \varrho(x, N), \max_{x \in N} \varrho(x, M)\}.$$

Define $B_r(a) = \{x \in \mathbb{R}^n \ : \ \|x - a\| \leq r\}$. For a set $M \subset \mathbb{R}^n$ we denote by $\operatorname{int} M$ and ∂M the *interior* and the *boundary* of the set, respectively.

Recall the Tallos viability theorem.

Proposition 1.1. [1, Theorem 11.7.1] *Let $M \subset \mathbb{R}^n$ be a closed set and assume that $F \colon [0, +\infty) \times M \to 2^{\mathbb{R}^n}$ is a nontrivial set-valued map satisfying*

(i) $\forall x \in M$ *the mapping* $t \to F(t, x)$ *is measurable,*

(ii) $\forall t \geq 0$ *the map* $x \to F(t, x)$ *is upper semicontinuous (in the Hausdorff metric) with compact convex images,*

(iii) $\exists\, c(\cdot) \in L_1([0, +\infty), [0, +\infty))$ *with* $h(\{0\},\ F(t, x)) \leq c(t)(1 + \|x\|)$.

If for almost all $t \geq 0$
$$F(t, x) \cap T_M(x) \neq \emptyset \tag{1}$$
then for any $t_0 \geq 0$ and $x_0 \in M$ there exists a viable solution of the Cauchy problem $x' \in F(t, x)$, $x(t_0) = x_0$: $x(t) \in M$ for all $t \geq t_0$.

Condition (1) is referred to as the *viability condition.*

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Here $T_M(x) = \{v \in \mathbb{R}^n \,:\, \liminf\limits_{t\to+0} \frac{\varrho_M(x+tv)}{t} = 0\}$ is the *contingent* (or the Bouligand) *cone* to the set M at the point $x \in M$. Another form for $T_M(x)$ is given through the Kuratowski upper limit

$$T_M(x) = \limsup_{h\to+0} \frac{M-x}{h} = \left\{ v \in \mathbb{R}^n \,:\, \forall N(v)\ \exists h_k \to +0 \quad N(v) \cap \frac{M-x}{h_k} \neq \emptyset \right\},$$

here $N(v)$ is any open set containing v.

Notice that (i) means that for the map $t \to F(t,x)$ the inverse image of any open subset $U \subset \mathbb{R}^n$, i.e. $\{t \in \mathbb{R} \,:\, F(t,x) \cap U \neq \emptyset\}$, is Lebesgue measurable. Item (iii) is the standard Caratheodory condition for prolongation of a solution to the right.

Sometimes the Caratheodory condition is unnatural for the right side. We can slightly modify Theorem 1.1 in the following way. Consider for a given $T > 0$

$$F_T(t,x) = \begin{cases} F(t,x), & t \in [0,T], \\ (T+1-t)F(T,x), & t \in (T, T+1], \\ 0, & t > T. \end{cases}$$

If $B = \sup\limits_{t\in[0,T],\ x\in M} h(\{0\}, F(t,x)) < +\infty$, then (iii) is valid for the upper semicontinuous map F_T with

$$c(t) = \begin{cases} (T+1)B, & t \leq T+1, \\ 0, & t > T. \end{cases}$$

Thus Theorem 1.1 guarantees viability of some solution for any measurable and uniformly bounded $F(t,x)$ on any finite time interval $[0,T]$. Hereafter by “a solution is extendible forwards/to the right” we shall mean that the solution is extendible on any time segment $[0,T]$.

We shall consider the dynamic system of the form

$$x' = f(t,x) + v(t,x), \qquad x(0) \in M \tag{2}$$

or even in a more simple form

$$x' = f(x) + v(t,x), \qquad x(0) \in M. \tag{3}$$

Here $M \subset \mathbb{R}^n$ is a closed compact set, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. The additive $v(t,x)$ is measurable in t, continuous in x and has the sense of a perturbation. We don't know the values of $v(t,x)$, we only know that $v(t,x) \in V(x)$ for all $t \geq 0$ and $x \in M$. We have full information about the convex compact set $V(x)$, in particular, $0 \in V(x)$ for all $x \in M$ and we can calculate its supporting function $s(p, V(x)) = \max_{y\in V(x)}(p,y)$ for all $p \in \mathbb{R}^n$. We shall also assume that $M \ni x \to V(x)$ is continuous in the Hausdorff metric, $\operatorname{int} M \neq \emptyset$ and the values $V(x)$ are uniformly bounded, i.e. $\sup_{x\in M} h(\{0\}, V(x)) \leq C$ for some $C > 0$. We shall expect that for some $B > 0$ we have the next estimate

$$\|f(x)\| \leq B(1 + \|x\|) \ \ \forall x \in \mathbb{R}^n. \tag{4}$$

Uniform boundedness of $V(x)$ together with (4) means that any solution of (3) is extendible to the right.

The questions about viability of solutions for differential equations and inclusions were considered in the monographs [1] and [2]. There are papers about viability with uncertainty [7] and in fractional differential inclusions [5]. The viability theory for stochastic differential inclusions was considered in [4]. Different applications, bibliography and state of art in the viability theory can be found in [6].

The question under consideration is to construct "minimal" feedback control $u(x)$, $x \in M$, such that the solution of the system

$$x' = f(x) + v(t,x) + u(x), \qquad x(0) \in M, \tag{5}$$

is viable in M: $x(t) \in M$ on any time interval. Minimality of $u(x)$ means that $\|u(x)\|$ pointwisely is as small as possible.

Recall that the *Minkowski difference* for sets M and N is the set

$$M \overset{*}{-} N = \{x \in \mathbb{R}^n \ : \ x + N \subset M\} = \bigcap_{x\in N} (M - x). \tag{6}$$

Define also by $P_M x$ the set-valued metric projection of a point $x \in \mathbb{R}^n$ onto a closed set $M \subset \mathbb{R}^n$. The *supporting function* is $s(p,M) = \sup_{x\in M}(p,x)$ for all $p \in \mathbb{R}^n$.

If we want validity of the inclusion $x(t) \in M$ in (5) for any perturbation $v(t,x) \in V(x)$ then by Theorem 1.1 we need the next inclusion to hold:

$$u(x) \in T_M(x) - f(x) - v \qquad \forall v \in V(x).$$

Put $G(x) = T_M(x) - f(x)$. We get $G(x) - v \supset G(x) \overset{*}{-} V(x)$ for all $v \in V(x)$ and hence one can take

$$u(x) \in P_{G(x) \overset{*}{-} V(x)} 0. \tag{7}$$

The situation here is quite similar with [1, Remark, p. 84].

There are at least two problems with control (7). First, $P_{G(x) \overset{*}{-} V(x)} 0$ is not continuous in the Hausdorff metric (as intersection, see [3, Theorem 8, Chapter 3, §1]).

Example 1.2. For the system

$$x' = Ax, \quad x = (x_1, x_2)^T, \qquad A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad a \in (-1, 0),$$

with $M = \mathrm{co}\{(1,1), (1,-1), (-1,-1), (-1,1)\}$ and $V(x) \equiv \{0\}$ for $u(x) = P_{G(x)}0$ we get $u(1,1) = (-1-a, 0)$ and $u(x_1, 1) = (0,0)$ for any $x_1 \in (0,1)$. □

Thus we can not take a continuous selector $u(x) \in P_{G(x) \overset{*}{-} V(x)} 0$ in general and problem (5) has a discontinuous right side.

Another challenge is that $\|u(x)\|$ can be very large.

Example 1.3. Consider an example in $\mathbb{R}^2$. Suppose that $V(x) = B_\gamma(0)$ and $G(x) = T_M(x) - f(x)$ is a convex cone with the angle 2α and vertex $-f(x)$. Assume that $P_{G(x)}0 = -f(x)$. Then for any $v \in B_\gamma(0)$ we have $P_{G(x)-v}0 \in B_\gamma(-f(x))$. This means that we can choose norm-small stability control for any fixed v. But $G(x) \overset{*}{-} B_\gamma(0)$ is a cone whose vertex w lies on the axis of symmetry inside $G(x)$ and $\|w + f(x)\| = \frac{\gamma}{\sin\alpha}$. If $\alpha > 0$ is small then w may be very far from $-f(x)$. □

Definition 1.4. We shall say that a set V is a *summand* of the set M if

$$M = V + V_1 = \{v + v_1 \ : \ v \in V,\ v_1 \in V_1\}. \qquad \square$$

The definition is mainly given for the case of convex and closed sets. Nevertheless, the case of nonconvex sets is also possible.

We shall consider a special situation when the perturbation set $V(x)$ is a summand of the set M in the Minkowskii sense for all $x \in M$, i.e. $M = V(x) + V_1(x)$ for all $x \in M$. The latter takes place for example when $V(x) \equiv V$ or $V(x)$ is a homothetic image of V with coefficient ≤ 1 and V is a summand of M.

Another important case is the set M with smooth boundary. In this case $T_M(x)$ for $x \in \partial M$ is a halfspace $H^-_{p(x)} = \{z \in \mathbb{R}^n \ : \ (p(x), z) \leq 0\}$ where $p(x)$ is the unit normal vector to the set M at the point x and the function $\partial M \ni x \to p(x)$ is continuous.

2. General properties of $G(X) \overset{*}{-} V(x)$ and continuous feedback control

Lemma 2.1. *Suppose that for closed sets $M, V, V_1 \subset \mathbb{R}^n$ we have $M = V + V_1$ and V is compact and convex. Then for any $x \in M$ the set V is a summand of $T_M(x)$.*

Proof. It is enough to consider $x \in \partial M$. Put $S_h = \left(\frac{1}{h} - 1\right) V + \frac{1}{h} V_1 - \frac{1}{h} x$ and $S = \limsup_{h \to +0} S_h$. From convexity of V we have the equality (for $0 < h < 1$)

$$T_h = \frac{M - x}{h} = \frac{V + V_1 - x}{h} = V + S_h$$

and for any $\varepsilon > 0$, $v_0 \in T_M(x)$ we have $(V + S_{h_k}) \cap B_\varepsilon(v_0) \neq \emptyset$ for some sequence $h_k \to +0$. There exists $v_k \in V$ such that $(v_k + S_{h_k}) \cap B_\varepsilon(v_0) \neq \emptyset$. From compactness of V we have without loss of generality that $v_k \to v \in V$. There exists k_ε such that $(v + S_{h_k}) \cap B_{2\varepsilon}(v_0) \neq \emptyset$ for all $k > k_\varepsilon$ and $S_{h_k} \cap B_{2\varepsilon}(v_0 - v) \neq \emptyset$. Hence $v_0 - v \in \limsup_{h \to +0} S_h = S$ and $v_0 \in v + S \subset V + S$. Thus $T_M(x) \subset V + S$.

Let $v \in V$. Then $v + S_h \subset T_h$, $v + S \subset T_M(x)$ and hence $V + S \subset T_M(x)$. $\square$

For a cone T the polar cone T^- is defined by $T^- = \{q \in \mathbb{R}^n \ : \ (p, q) \leq 0,\ \forall p \in T\}$.

Lemma 2.2. *Let $M \subset \mathbb{R}^n$ be a closed set, V is a compact convex set and V is a summand of $T_M(x)$ for a point $x \in \partial M$. Let $f \in \mathbb{R}^n$. Suppose that the cone $T_M(x)$ is convex. Put $N_M(x) = T^-_M(x)$ and $N^1_M(x) = N_M(x) \cap \partial B_1(0)$. Then for $G = T_M(x) - f$ we have*

$$\varrho(0, G \overset{*}{-} V) = \max\Big\{0, \max_{p \in N^1_M(x)} s(p, f + V)\Big\}. \tag{8}$$

Proof. By Lemma 2.1 $G = V + V_1$ and by the equality

$$G = \mathrm{cl}\ \mathrm{co}\, G = \mathrm{cl}\ \mathrm{co}\,(V + V_1) = V + \mathrm{cl}\ \mathrm{co}\, V_1$$

we may assume that the set V_1 is closed and convex. From compactness of V we have $V_1 = G \overset{*}{-} V$.

Assume that $\max_{p \in N^1_M(x)} s(p, f + V) \leq 0$. Then we obtain $f + V \subset T_M(x)$ and $0 \in T_M(x) - f \overset{*}{-} V = G \overset{*}{-} V = V_1$ by the separation theorem. Hence $\varrho(0, V_1) = 0$.

Now suppose that $\max_{p\in N^1_M(x)} s(p,f+V)>0$. From the equality $f+V+V_1=T_M(x)$ for any $p\in N^1_M(x)$ we have

$$s(p,f+V)+s(p,V_1)=s(p,T_M(x))=0 \quad \text{and} \quad s(p,V_1)=-s(p,f+V). \tag{9}$$

We have $\min_{p\in N^1_M(x)} s(p,V_1)<0$. Therefore $0\notin V_1$. Let $z=P_{V_1}0$, $p_0=-z/\|z\|$. Then we obtain $\varrho(0,V_1)=-(p_0,z)$ and for any $q\in N^1_M(x)$ we have $(q,z)\geq(p_0,z)$, $s(p_0,V_1)=(p_0,z)<+\infty$, so for the unit vector $p_0\in N^1_M(x)$ by (9)

$$-\varrho(0,V_1)=\min_{p\in N^1_M(x)} s(p,V_1)=-\max_{p\in N^1_M(x)} s(p,f+V)=-s(p_0,f+V). \qquad \square$$

Remark 2.3. Let $v\in V$ be a point such that $(p_0,v)=s(p_0,V)$. Then we have $\varrho(0,G-v)=\varrho(0,G\overset{*}{-}V)=\varrho(0,V_1)=\|z\|$. For $H^-_{p_0}=\{\xi\in\mathbb{R}^n\ :\ (p_0,\xi)\leq(p_0,z)\}$ we have

$$G-v=V+V_1-v\subset H^-_{p_0}$$

and hence $\varrho(0,G-v)\geq\varrho(0,H^-_{p_0})=\|z\|$. Thus the distance $\varrho(0,G\overset{*}{-}V)$ is reached for some $v\in V$. This is a straightforward consequence of Definition 1.4. $\square$

Denote by hypo $\varphi(x)$ the *hypograph* of a function $\varphi: M\to\mathbb{R}\cup\{-\infty\}$, i.e.

$$\operatorname{hypo}\varphi=\{(x,\mu)\ :\ \mu\leq\varphi(x),\ x\in M\}.$$

Recall for system (3) that $G(x)=T_M(x)-f(x)$ for all $x\in M$.

Notice that the map $M\ni x\to T_M(x)$ is lower semicontinuous with convex images in the case when the set M is convex [2, Theorem 1, Chapter 5, §1, p. 220]. If M is convex then $T_M(x)=\operatorname{cl}\bigcup\limits_{h>0}\frac{M-x}{h}$ [2, Proposition 1, Chapter 5, §1, p. 219].

Lemma 2.4. *Let $M\subset\mathbb{R}^n$ be a closed set, $M\ni x\to V(x)$ be a continuous set-valued map with compact convex images such that for any $x\in M$ the set $V(x)$ is a summand of $T_M(x)$. Suppose that the map $M\ni x\to T_M(x)$ is lower semicontinuous (lsc) with convex images. Then the map $M\ni x\to T_M(x)\overset{*}{-}V(x)$ is lsc.*

Proof. By Lemma 2.2 for any $x\in M$ (notice that $N_M(x)=\{0\}$ for $x\in\operatorname{int}M$ and we shall assume that $\max\limits_{p\in\emptyset}\cdots=-\infty$)

$$\varrho(0,G(x)\overset{*}{-}V(x))=\max\Big\{0,\max_{p\in N^1_M(x)} s(p,f(x)+V(x))\Big\}.$$

Changing $f(x)$ by $y+f(x)$ for an arbitrary $y\in\mathbb{R}^n$ we get

$$\begin{aligned}\varrho(0,(T_M(x)-y-f(x))\overset{*}{-}V(x))&=\varrho(y,G(x)\overset{*}{-}V(x))\\&=\max\Big\{0,\max_{p\in N^1_M(x)} s(p,y+f(x)+V(x))\Big\}.\end{aligned}$$

Consider the function $\varphi_y(x)=\max_{p\in N^1_M(x)} s(p,y+f(x)+V(x))$.
Let $(x_k,\mu_k)\in\operatorname{hypo}\varphi_y$ and $(x_k,\mu_k)\to(x_0,\mu_0)$, $\mu_0\in\mathbb{R}$. Let

$$p_k\in N^1_M(x_k):\quad s(p_k,y+f(x_k)+V(x_k))=\varphi_y(x_k).$$

Without loss of generality $p_k\to p_0$, $\partial M\ni x\to N_M(x)$ has a closed graph [2, Theorem 1, Chapter 5, §1, p. 220], hence $p_0\in N^1_M(x_0)$ and

$$\mu_0=\lim_{k\to\infty}\mu_k\leq\lim_{k\to\infty}\varphi_y(x_k)=s(p_0,y+f(x_0)+V(x_0))\leq\varphi_y(x_0).$$

Hence hypo φ_y is closed and φ_y is upper semicontinuous. From the equality

$$\text{hypo } \{\mu = \varrho(y, G(x) \overset{*}{-} V(x))\} = \text{hypo } \{\mu = 0\} \cup \text{hypo } \{\mu = \varphi_y(x)\}$$

we obtain the upper semicontinuity of the function $M \ni x \to \varrho(y, G(x) \overset{*}{-} V(x))$ for any $y \in \mathbb{R}^n$ and hence lsc of $M \ni x \to G(x) \overset{*}{-} V(x)$. $\square$

From Lemma 2.4 the function $M \ni x \to \varrho(0, G(x) \overset{*}{-} f(x))$ is upper semicontinuous.

Theorem 2.5. *Consider system* (3) *with convex closed set M, and a continuous (in the Hausdorff metric) map $M \ni x \to V(x)$ with convex compact and uniformly bounded values. Let $r : M \to [0, +\infty)$ be a lsc function satisfying the inequality*

$$r(x) \geq \max\left\{0, \max_{p \in N_M^1(x)} s(p, f(x) + V(x))\right\}$$

for all $x \in M$ (if $N_M^1(x) = \emptyset$ then $\max_{p \in \emptyset} \cdots = -\infty$). Suppose that we have $M = V(x) + \tilde{V}_1(x)$ for all $x \in M$ and some closed convex set $\tilde{V}_1(x)$. Then for any $\varepsilon > 0$ there exists a continuous feedback control u_ε with

$$\|u_\varepsilon(x)\| \leq r(x) + \varepsilon \qquad \forall x \in M.$$

Furthermore, the solution of system (5) *with $u(x) = u_\varepsilon(x)$ is viable in M.*

Proof. The function f is continuous, hence the map $G(x) = T_M(x) - f(x)$ is lsc. The map $M \ni x \to G(x) \overset{*}{-} V(x)$ is lsc due to Lemma 2.4. In view of Lemma 2.1, for any $x \in M$, we have $G(x) = V(x) + V_1(x)$ for some convex closed set $V_1(x)$. Noting that $V_1(x) = G(x) \overset{*}{-} V(x)$, from Lemma 2.2 (8) we have

$$\varrho(0, V_1(x)) = \max\left\{0, \max_{p \in N_M^1(x)} s(p, f(x) + V(x))\right\} \leq r(x),$$

and $r(x)$ is lsc. By [2, Proposition 5, Chapter 1, §1, p. 44] for any $\varepsilon > 0$ the map

$$F_\varepsilon(x) = (G(x) \overset{*}{-} V(x)) \cap \text{int}\, B_{r(x)+\varepsilon}(0) \neq \emptyset \qquad \forall x \in M$$

is lsc and the closure cl $F_\varepsilon(x)$ is lsc too. By the Michael convex selection theorem there exists a continuous selector $u_\varepsilon(x) \in$ cl $F_\varepsilon(x)$ with

$$u_\varepsilon(x) \in T_M(x) - f(x) - v \qquad \forall v \in V(x).$$

Thus the solution of (5) is viable in M.

We have $\|u_\varepsilon(x)\| \leq r(x) + \varepsilon$ and by (8), (4) and uniform boundedness of $V(x)$ we can take $r(x) \leq \|f(x)\| + h(\{0\}, V(x)) \leq B(1 + \|x\|) + C$. Thus any solution of (5) is extendible forwards. $\square$

Example 1.2 shows that one cannot use the function $r(x) = \varrho(0, G(x) \overset{*}{-} V(x))$ even in the case $V(x) \equiv \{0\}$.

The (discontinuous) metric projection $u(x) = P_{G(x) \overset{*}{-} V(x)}0$ for $x \in \partial M$ under conditions of Theorem 2.5 is

$$u(x)=\begin{cases} 0, & \max\limits_{p\in N_M^1(x)} s(p,f(x)+V(x))\le 0,\\ -p(x)s(p(x),f(x)+V(x)), & \max\limits_{p\in N_M^1(x)} s(p,f(x)+V(x))>0 \end{cases} \tag{10}$$

where $p(x)=\arg\max\limits_{p\in N_M^1(x)} s(p,f(x)+V(x))=\arg\min\limits_{p\in N_M^1(x)} s(p,G(x)\overset{*}{-}V(x))$.

The last arg-min exists and is unique in the case $\max_{p\in N_M^1(x)} s(p,f(x)+V(x))>0$ due to the uniqueness of the metric projection $P_{G(x)\overset{*}{-}V(x)}0$, $p(x)=p_0$ and p_0 is from the proof of Lemma 2.2.

Indeed, if $\max_{p\in N_M^1(x)} s(p,f(x)+V(x))\le 0$ then $0\in V_1(x)=G(x)\overset{*}{-}V(x)$ and $u(x)=0$. If $\max_{p\in N_M^1(x)} s(p,f(x)+V(x))>0$ then $p(x)\in N_M^1(x)$ is the unique unit vector such that

$$\varrho(0,V_1(x))=\max_{p\in N_M^1(x)} s(p,f(x)+V(x))=s(p(x),f(x)+V(x)).$$

Hence $\quad u(x)=P_{V_1(x)}0=-p(x)\varrho(0,V_1(x))=-p(x)s(p(x),f(x)+V(x))$.

3. Smooth boundary of the set M

Suppose that for any $x\in\partial M$ the cone $T_M(x)$ is a halfspace $\{y\in\mathbb{R}^n\,:\,(p(x),y)\le 0\}$ with a unit normal $p(x)$ is continuous function of x. It is obvious that in the considered case for any $x\in\partial M$ we have $T_M(x)\overset{*}{-}V(x)+V(x)=T_M(x)$ for an arbitrary convex compact set $V(x)$.

Theorem 3.1. *Consider system (3) with convex compact set M with smooth boundary and a continuous (in the Hausdorff metric) map $M\ni x\to V(x)$ with convex compact and uniformly bounded values. Suppose that $0\in\operatorname{int}M$ and $f(0)=0$. Then the continuous feedback control*

$$u(\alpha x)=-p(x)\max\{0,\ (p(x),f(\alpha x))+s(p(x),V(\alpha x))\},\ x\in\partial M,\ \alpha\in(0,1] \tag{11}$$

gives in system (5) *a viable solution in M.*

Note that by definition the function $u(\cdot)$ is continuous on the set $M\backslash\{0\}$. If we have $\lim_{x\to 0}V(x)=\{0\}$, then the function $u(\cdot)$ is continuous on M. Formula (11) also works for a compact star set M with $0\in\operatorname{int}M$.

Proof. Put first $x\in\partial M$ and $\alpha=1$. Then (11) follows from Formula (10).

If point $x=\alpha x_0$, $x_0\in\partial M$, $\alpha\in(0,1)$, then the unit normal to the point $x\in\partial(\alpha M)$ is equal to $p(x_0)$ and Formula (11) for this case is obtained similarly with the case $x\in\partial M$. $\square$

If the trajectory of (5) with control (11) will be at a point $x\in\alpha M$, $\alpha<1$, then further dynamic will be inside the set αM.

If $(p(x),f(x))+s(p(x),V(x))\le 0$ for all $x\in\partial M$ then trajectory of system (3) will be inside the set M.

There are some examples of perturbation sets (we assume that $0\in V$):

(1) $V(x)=V$ for all x; (2) $V(\alpha x)=g(\alpha)V$ for $x\in\partial M$, $\alpha\in[0,1]$, and the function g is continuous on the segment $[0,1]$, $0=g(0)<g(\alpha)$ for all $\alpha\in(0,1]$;

(3) $V(x)=\operatorname{co}(\{0\}\cup\{A_ix\}_{i=1}^m)$ where A_i are $n\times n$ matrices.

In cases (2) and (3) Formula (11) gives the desired continuous control. In case (1) we have

$$u(\alpha x) = -p(x) \max \{0,\ (p(x), f(\alpha x)) + s\,(p(x), V)\}\,, \quad x \in\ \partial M, \quad \alpha \in (0,1]$$

and the function $\partial M \times [0,1] \ni (x,\alpha) \to s\,(p(x),V)$ is not continuous at a point (x,α) for $\alpha = 0$. We can fix $\alpha_0 \in (0,1)$ and consider the control

$$u(\alpha x) = \begin{cases} -p(x) \max \{0,\ (p(x), f(\alpha x)) + s\,(p(x), V)\}\,, & \alpha \in (\alpha_0, 1], \\ 0, \quad \alpha \leq \alpha_0 \end{cases}$$

in the following sense: First, in the layer $x(t) \in M \backslash (\alpha_0 M)$, we take nonzero control and try to lead the phase vector $x(t)$ inside the set M with dynamic (5). In the case $x(t) \in \alpha_0 M$ we switch off control and realize dynamic (3) until vector $x(t) \in M$, $x(t) \notin \partial M$. When $x(t) \in \partial M$ we switch on nonzero control $u(\alpha x)$ and repeat the previous stage. In the described situation we shall have piecewise continuous solution which is consisted of solutions of (5) and (3), this solution is contained in M.

Example 3.2. Consider system (3) with $f(x) = Ax$, and $M = B_1(0)$. Suppose that $k_A(x) = (x, Ax) \leq 0$ for all $x \in \mathbb{R}^n$. Let $V(x) = [0, Bx]$ for some matrix $B \in \mathbb{R}^{n\times n}$. Then by (11) we have for $k_{A+B}(x) = (x, (A+B)x)$

$$u(x) = -x \max \left\{0, \left(\frac{x}{\|x\|}, (A+B)\frac{x}{\|x\|}\right)\right\} = \begin{cases} 0, & k_{A+B}\big(\frac{x}{\|x\|}\big) \leq 0, \\ -x k_{A+B}\big(\frac{x}{\|x\|}\big), & k_{A+B}\big(\frac{x}{\|x\|}\big) > 0. \end{cases}$$

Note that condition $k_A(x) \leq 0$ necessary and sufficient for a solution of (3) with $f(x) = Ax$ and $V(x) \equiv \{0\}$ to be viable in $M = B_1(0)$ for any $x(0) \in B_1(0)$. $\square$

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