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# Jensen Convex Functions on Semigroups

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We transfer some results concerning Jensen convex functions defined on linear topological spaces to Jensen convex functions defined on semigroups. We also show that the Bernstein-Doetsch theorem fails in some class of topological semigroups.

*Keywords:* Jensen convex function, Jensen convex set, semigroup.

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## 1. Introduction

We start with presentation of basic definitions which will be used in the paper. Let $(S,+)$ be a semigroup. A set $D \subset S$ is called *Jensen convex* if

$$2x = y + z \;\Rightarrow\; x \in D \text{ for any } x \in S \text{ and } y, z \in D. \tag{1}$$

Let $(S,+)$ be a semigroup and let $D \subset S$ be a Jensen convex set. A function $f\colon D \to \mathbb{R} \cup \{\infty\}$ is called *Jensen convex*, if

$$2x = y + z \;\Rightarrow f(x) \leq \frac{f(y)+f(z)}{2} \qquad \text{for any } x \in S \text{ and } y, z \in D. \tag{2}$$

Condition (2) (for functions taking finite values only) appeared in [2]. Clearly condition (2) generalizes the Jensen convexity. Some notion of convex set and convex function in semigroups appeared also in [1].

## 2. Results

In two lemmas below we present basic properties of Jensen convex sets and Jensen convex funcions.

**Lemma 2.1.** *Let $(S,+)$ be an abelian semigroup and let $D \subset S$ be Jensen convex. Then the following statements hold:*

(a) *if $m, n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{N}$ satisfy the condition $\sum_{i=1}^{n} k_i = 2^m$, $x_1, \ldots, x_n \in S$ and $2^m x_i \in D$ for every $i \in \{1, \ldots, n\}$ then $\sum_{i=1}^{n} k_i x_i \in D$;*

(b) *if $m \in \mathbb{N}$, $y, z \in S$, $2^m y, 2^m z \in D$ and $k \in \{1, \ldots, 2^m - 1\}$ then $ky + (2^m - k)z \in D$.*

*If, moreover, $S$ is uniquely divisible by 2, then $D$ is $\lambda$-convex for every dyadic number $\lambda \in (0,1)$.*

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**Proof.** We start by proving the statement ($a$) in case when $m \in \mathbb{N}$, $n = 2^m$ and $k_1 = \dots = k_{2^m} = 1$. We use induction with respect to $m$. So assume that $m = 1$, $x_1, x_2 \in S$ and $2x_i \in D$ for every $i \in \{1, 2\}$. As $2(x_1 + x_2) = 2x_1 + 2x_2$, using the Jensen convexity of $D$ we deduce that $x_1 + x_2 \in D$.

Now fix $m \in \mathbb{N}$ and assume that the condition mentioned above is valid. Let $x_1, \dots, x_{2^{m+1}} \in D$ be such that $2^{m+1}x_i \in D$ for every $i \in \{1, \dots, 2^{m+1}\}$. We have $2^m 2x_1, \dots, 2^m 2x_{2^m} \in D$ and $2^m 2x_{2^m+1}, \dots, 2^m 2x_{2^{m+1}} \in D$. Applying the inductive hypothesis we get $\sum_{i=1}^{2^m} 2x_i \in D$ and $\sum_{i=2^m+1}^{2^{m+1}} 2x_i \in D$. Clearly

$$2\sum_{i=1}^{2^{m+1}} x_i = \sum_{i=1}^{2^m} 2x_i + \sum_{i=2^m+1}^{2^{m+1}} 2x_i$$

whence on account of the Jensen convexity of $D$ we deduce that $\sum_{i=1}^{2^{m+1}} x_i \in D$.

Now we prove the statement (a) in the general case. Fix $m, n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}$, $x_1, ..., x_n \in D$ and assume that $\sum_{i=1}^n k_i = 2^m$ and $2^m x_i \in D$ for every $i \in \{1, ..., n\}$. We define points $y_1 = \dots = y_{k_1} := x_1$, $y_{k_1+1} = \dots = y_{k_1+k_2} := x_2, \dots$, up to $y_{k_1+\dots+k_{n-1}+1} = \dots = y_{k_1+\dots+k_n} := x_n$. Since

$$\{y_i : i \in \{1, \dots, 2^m\}\} \subset \{x_i : i \in \{1, \dots, n\}\},$$

we have $2^m y_i \in D$ for every $i \in \{1, \dots, 2^m\}$. Using the first part of the proof we obtain $\sum_{i=1}^{2^m} y_i \in D$ and thus $\sum_{i=1}^n k_i x_i \in D$.

To prove (b) fix $y, z \in S$, $m \in \mathbb{N}$, $k \in \{1, \dots, 2^m - 1\}$ and assume that $2^m y, 2^m z \in D$. Applying (a) with $n = 2, k_1 = k, k_2 = 2^m - k, x_1 = y$ and $x_2 = z$ we obtain the assertion.

If $S$ is uniquely divisible by 2, then inserting $\frac{y}{2^m}$ in place of $y$ and $\frac{z}{2^m}$ in place of $z$ (where $m \in \mathbb{N}$) in (b) we state that $D$ is $\lambda$–convex for every dyadic number $\lambda \in (0, 1)$. It completes the proof. □

**Lemma 2.2.** *Let $(S, +)$ be a semigroup, $D$ be a Jensen convex subset of $S$ and let $f \colon D \to \mathbb{R} \cup \{\infty\}$ be a Jensen convex function. Then the following statements hold:*

(a) *if $m, n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, $k_1, \dots, k_n \in \mathbb{N}$ satisfy the equality $k_1 + \dots + k_n = 2^m$ and $2^m x_i \in D$ for every $i \in \{1, \dots, n\}$, then*

$$f\Big(\sum_{i=1}^n k_i x_i\Big) \le \sum_{i=1}^n \frac{k_i}{2^m} f(2^m x_i); \tag{3}$$

(b) *if $x, y \in D$, $m \in \mathbb{N}$, $k \in \{1, \dots, 2^m - 1\}$, $2^m x, 2^m y \in D$, then*

$$f(kx + (2^m - k)y) \le \frac{k}{2^m} f(2^m x) + \Big(1 - \frac{k}{2^m}\Big) f(2^m y). \tag{4}$$

*If, moreover, $S$ is uniquely divisible by 2, then $f$ is $\lambda$–convex for every dyadic number $\lambda \in (0, 1)$.*

**Proof.** First we show that the statement ($a$) of the lemma holds in case when $m \in \mathbb{N}$, $n = 2^m$ and $k_1 = \dots = k_{2^m} = 1$. We use induction with respect to $m$.

Assume that $m = 1$. Fix $x_1, x_2 \in S$ such that $2x_1 \in D$ and $2x_2 \in D$. It follows from Lemma 2.1 that $x_1 + x_2 \in D$. Since $f$ is Jensen convex, we have $f(x_1 + x_2) \leq \frac{1}{2}(f(2x_1)+f(2x_2))$. Now fix $m \in \mathbb{N}$ and assume that the condition above is valid. Let $x_1, \dots, x_{2^{m+1}} \in S$ be such that $2^{m+1}x_i \in D$ for every $i \in \{1, \dots, 2^{m+1}\}$. Applying Lemma 2.1 we deduce that $\sum_{i=1}^{2^m} 2x_i \in D$, $\sum_{i=2^m+1}^{2^{m+1}} 2x_i \in D$ and $\sum_{i=1}^{2^{m+1}} x_i \in D$.

Clearly
$$2\sum_{i=1}^{2^{m+1}} x_i = \sum_{i=1}^{2^m} 2x_i + \sum_{i=2^m+1}^{2^{m+1}} 2x_i.$$

Using the Jensen convexity of $f$ and applying the inductive hypothesis we obtain

$$\begin{aligned} f\Big(\sum_{i=1}^{2^{m+1}} x_i\Big) &\leq \frac{1}{2}\Big(f\Big(\sum_{i=1}^{2^m} 2x_i\Big) + f\Big(\sum_{i=2^m+1}^{2^{m+1}} 2x_i\Big)\Big) \\ &\leq \frac{1}{2}\Big(\frac{1}{2^m}\sum_{i=1}^{2^m} f(2^{m+1}x_i) + \frac{1}{2^m}\sum_{i=2^m+1}^{2^{m+1}} f(2^{m+1}x_i)\Big) = \frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}} f(2^{m+1}x_i). \end{aligned}$$

Now fix any $m, n \in \mathbb{N}$, $x_1, \dots, x_n \in S$ and $k_1, \dots, k_n \in \mathbb{N}$ with $k_1 + \dots + k_n = 2^m$. Put $y_1 = \dots = y_{k_1} := x_1$, $y_{k_1+1} = \dots = y_{k_1+k_2} := x_2, \dots,$ $y_{k_1+\dots+k_{n-1}+1} = \dots = y_{k_1+\dots+k_n} := x_n$. Then we have
$$\sum_{i=1}^{2^n} y_i = \sum_{i=1}^{n} k_i x_i.$$

Hence
$$f\Big(\sum_{i=1}^{n} k_i x_i\Big) = f\Big(\sum_{i=1}^{2^m} y_i\Big) \leq \frac{1}{2^m}\sum_{i=1}^{2^m} f(2^m y_i) = \frac{1}{2^m}\sum_{i=1}^{n} k_i f(2^m x_i)$$

This completes the proof of (a).

Let $x, y \in S$, $m \in \mathbb{N}$, $k \in \{1, \dots, 2^m - 1\}$ be such that $2^m x \in D$ and $2^m y \in D$. Using $(a)$ with $n = 2^m$, $k_1 = \dots = k_{2^m} = 1$, $x_1 = \dots = x_k = x$, and $x_{k+1} = \dots = x_{2^m} = y$ we obtain
$$\begin{aligned} (kx + (2^m - k)y) &\leq \frac{k}{2^m} f(2^m x) + \frac{2^m - k}{2^m} f(2^m y) \\ &= \frac{k}{2^m} f(2^m x) + \Big(1 - \frac{k}{2^m}\Big) f(2^m y). \end{aligned} \tag{5}$$

If $S$ is uniquely divisible by 2, then inserting $\frac{x}{2^m}$ in place of $x$ and $\frac{y}{2^m}$ in place of $y$ (where $m \in \mathbb{N}$) in (4) we state that $f$ is $\lambda$-convex for every dyadic number $\lambda \in (0, 1)$. This completes the proof. □

Let $(S, +)$ be a semigroup, $D \subset S$ be Jensen convex and let $x \in D$. For every $n \in \mathbb{N}$ we define the set
$$A_{D,n}(x) = x + \left\{ \sum_{i=1}^{n} t_i : \begin{array}{l} t_1, \dots, t_n \in S \text{ and } x + \sum_{i=1}^{k} t_i + t_k \in D \\ \text{for every } k \in \{1, \dots, n\} \end{array} \right\}.$$

**Lemma 2.3.** *Let $(S, +)$ be an abelian semigroup, let $D \subset S$ be a Jensen convex set and assume $x \in D$. Then $A_{D,n}(x) \subset D$ for every $n \in \mathbb{N}$.*

**Proof.** We use induction with respect to $n$. Fix any $y \in A_{D,1}(x)$. There exists a $t \in S$ such that $y = x + t$ and $x + 2t \in D$. We have $2(x + t) = x + x + 2t$ whence on account of Jensen convexity of $D$ we deduce that $x + t \in D$. Now fix $n \in \mathbb{N}$ and assume that the assertion holds with every natural number less or equal to $n$. Fix $y \in A_{D,n+1}(x)$. There exist $t_1, \dots, t_{n+1} \in S$ such that $y = x + \sum_{i=1}^{n+1} t_i$ and $x + \sum_{i=1}^{k} t_i + t_k \in D$ for every $k \in \{1, \dots, n+1\}$. Hence on account of the inductive hypothesis we deduce that $x + \sum_{i=1}^{n} t_i \in D$. Moreover

$$2\Big(x + \sum_{i=1}^{n+1} t_i\Big) = x + \sum_{i=1}^{n+1} t_i + t_{n+1} + x + \sum_{i=1}^{n} t_i.$$

This in view of the Jensen convexity of $D$ means that $x + \sum_{i=1}^{n+1} t_i \in D$. □

**Lemma 2.4.** *Let $(S, +)$ be an abelian topological semigroup with a zero. Assume that $D \subset S$ is Jensen convex and $x_0 \in D$. Let $f : D \to \mathbb{R}$ be a Jensen convex function. Assume that there exists a neighbourhood of zero $U_0 \subset S$ such that $x_0 + U_0 \subset D$ and the function $f$ is bounded from above on $x_0 + U_0$. Then for every $n \in \mathbb{N}$ and for every $x \in A_{D,n}(x_0)$ there exists a neighbourhood of zero $W_0 \subset S$ such that $x + W_0 \subset D$ and the function $f$ is bounded from above on $x + W_0$.*

**Proof.** By the assumption, there exist a neighbourhood of zero $U_0 \subset S$ and an $M > 0$ such that $f(x_0 + x) \le M$ for every $x \in U_0$. We make the proof using the induction with respect to $n$. First we show that the assertion follows with $n = 1$. We have $A_{D,1}(x_0) = \{x_0 + t_1 : t_1 \in S \text{ and } x_0 + 2t_1 \in D\}$. Fix $z \in A_{D,1}(x_0)$. There exists a $t_1 \in S$ such that $z = x_0 + t_1$ and $x_0 + 2t_1 \in D$. We can find a neighbourhood of zero $W_0 \subset S$ such that $2W_0 \subset U_0$. Then $x_0 + 2x \in x_0 + 2W_0 \subset x_0 + U_0 \subset D$ for every $x \in W_0$. For every $x \in W_0$ the following condition holds:

$$2z + 2x = 2x_0 + 2t_1 + 2x = x_0 + 2t_1 + x_0 + 2x.$$

Hence on account of the Jensen convexity of $D$ we deduce that $z + x \in D$ for every $x \in W_0$. Consequently $z + W_0 \subset D$. Moreover, using Jensen convexity of $f$ we obtain

$$f(z + x) \le \frac{1}{2}\Big(f(x_0 + 2t_1) + f(x_0 + 2x)\Big) \le \frac{1}{2}\Big(f(x_0 + 2t_1) + M\Big).$$

for every $x \in W_0$. Therefore $f$ is bounded from above on the set $z + W_0$.

Now fix $n \in \mathbb{N}$ and assume that the statement of the lemma follows for every $m \le n$. Fix $z \in A_{D,n+1}(x_0)$. There exist $t_1, \dots, t_{n+1} \in S$ such that

$$z = x_0 + \sum_{i=1}^{n+1} t_i \text{ and } x_0 + \sum_{i=1}^{k} t_i + t_k \in D \quad \text{for every } k \in \{1, \dots, n+1\}. \tag{6}$$

Put $w = x_0 + \sum_{i=1}^{n} t_i$. It follows from (6) that $x_0 + \sum_{i=1}^{k} t_i + t_k \in D$ for every $k \in \{1, \dots, n\}$. Therefore $w \in A_{D,n}(x_0)$. Using the induction hypothesis we get that there exists a neighbourhood of zero $Z_0 \subset S$ and $c > 0$ such that $w + Z_0 \subset D$ and

$$f(w + x) \le c \quad \text{for every } x \in Z_0. \tag{7}$$

There exists a neighbourhood $T_0 \subset S$ of zero such that $2T_0 \subset Z_0$. Fix $x \in T_0$. We have $w + 2x \in w + 2T_0 \subset w + Z_0 \subset D$. Furthermore,

$$2z + 2x = x_0 + \sum_{i=1}^{n+1} t_i + t_{n+1} + w + 2x.$$

Hence in view of the Jensen convexity of $D$ we deduce that $z + x \in D$. Therefore $z + T_0 \subset D$. Making use of the Jensen convexity of $f$ we obtain

$$\begin{aligned} f(z+x) &\leq \frac{1}{2}\Big(f\Big(x_0 + \sum_{i=1}^{n+1} t_i + t_{n+1}\Big) + f(w+2x)\Big) \\ &\leq \frac{1}{2}\Big(f\Big(x_0 + \sum_{i=1}^{n+1} t_i + t_{n+1}\Big) + c\Big) \end{aligned}$$

for every $x \in T_0$. Therefore $f$ is bounded from above on the set $z + T_0$. The proof is completed. $\square$

**Theorem 2.5.** *Let $(S, +)$ be an abelian topological semigroup with a zero. Assume that $D \subset S$ is Jensen convex and $f\colon D \to \mathbb{R}$ is a Jensen convex function. Let $x_0 \in D$ and let $U_0 \subset S$ be a neighbourhood of zero such that $x_0 + U_0 \subset D$. Assume that $x_0 + W$ is open for every neighbourhood of zero $W \subset S$. If the function $f$ is bounded from above on $x_0 + U_0$, then $f$ is upper semicontinuous at $x_0$.*

**Proof.** First we consider the case when $x_0 = 0$ and $f(0) = 0$. Assume that $f$ is bounded from above on $U_0$. There exists an $M \in \mathbb{R}$ such that $f(x) \leq M$ for every $x \in U_0$. We inductively show that for every $n \in \mathbb{N}$ it holds the following condition:

$$\text{if } x \in S \text{ and } 2^n x \in D \text{ then } x \in D \text{ and } 2^n f(x) \leq f(2^n x). \tag{8}$$

Assume that $x \in S$ and $2x \in D$. Since $2x = 2x + 0$ and $0 \in D$, we have $x \in D$ and using the Jensen convexity of $f$ we get $f(x) \leq \frac{1}{2}(f(2x) + f(0)) = \frac{1}{2}f(2x)$. Now fix an $n \in \mathbb{N}$ and assume that condition (8) holds with every natural number less or equal to $n$. Fix $x \in S$ and assume that $2^{n+1}x \in D$. Applying the inductive hypothesis we get $2x \in D$. Hence on account of the first part of the proof we deduce that $x \in D$. Applying the inductive hypothesis again we obtain $f(2^{n+1}x) \geq 2^n f(2x) \geq 2^{n+1} f(x)$. This finishes the proof of (8).

Fix any $\varepsilon > 0$. There exists an $n \in \mathbb{N}$ such that $\frac{M}{2^n} < \varepsilon$. We can find a neighbourhood of zero $W_0 \subset S$ such that $2^n W_0 \subset U_0$. Fix $x \in W_0$. We have $2^n x \in U_0$ and

$$f(x) \leq \frac{f(2^n x)}{2^n} \leq \frac{M}{2^n} < \varepsilon.$$

Now we proceed to the general case. Assume that $f$ is bounded from above on the set $x_0 + U_0$. Put $D_0 = \{x \in S : x + x_0 \in D\}$. The set $D_0$ is Jensen convex. Define a function $g\colon D_0 \to \mathbb{R}$ by $g(x) = f(x + x_0) - f(x_0)$. The function $g$ is Jensen convex and $0 \in D_0$. Moreover $U_0 \subset D_0$ and $g$ is bounded from above on $U_0$. By the first part of the proof the function $g$ is upper semicontinuous at 0. Fix $\varepsilon > 0$. There exists a neighbourhood of zero $Z \subset U_0$ such that $g(x) \leq \varepsilon$ for every $x \in Z$. Hence $f(x + x_0) - f(x_0) \leq \varepsilon$ for every $x \in Z$.

Moreover, by the assumption, the set $x_0 + Z$ is open. Therefore the function $f$ is upper semicontinuous at $x_0$. This completes the proof. □

Applying Lemma 2.4 and Theorem 2.5 we obtain the following theorem.

**Theorem 2.6.** *Let $(S, +)$ be an abelian topological semigroup with a zero. Assume that $x + W$ is open for every $x \in S$ and for every neighbourhood of zero $W \subset S$. Let $D \subset S$ be Jensen convex, $x_0 \in D$ and let $f\colon D \to \mathbb{R}$ be a Jensen convex function. Assume that there exists a neighbourhood of zero $U_0 \subset S$ such that $x_0 + U_0 \subset D$ and $f$ is bounded from above on $x_0 + U_0$. Then for every $n \in \mathbb{N}$ and for every $x \in A_{D,n}(x_0)$ the function $f$ is upper semicontinuous at $x$.*

Using Theorem 2.6 with $D = S$ we obtain the following corollary.

**Corollary 2.7.** *Let $(S, +)$ be an abelian topological semigroup with a zero. Assume that $x + W$ is open for every $x \in S$ and for every neighbourhood of zero $W \subset S$. Let $f\colon S \to \mathbb{R}$ be Jensen convex. If $x_0 \in S$ and $f$ is bounded from above on some neighbourhood of $x_0$, then $f$ is upper semicontinuous on $x_0 + S$.*

The next two results concern functions defined on groups.

**Lemma 2.8.** *Let $(G, +)$ be an abelian topological group and let $D \subset G$ be Jensen convex and open. Assume that $f\colon D \to \mathbb{R}$ is a Jensen convex function and $x_0 \in D$. If $f$ is upper semicontinuous at $x_0$, then $f$ is lower semicontinuous at $x_0$.*

**Proof.** Assume that $f$ is upper semicontinuous at $x_0$. We can find a neighbourhood $U_0 \subset G$ of zero such such that $x_0 + U_0 \subset D$. Define the function $h\colon U_0 \to \mathbb{R}$ by $h(z) = f(x + x_0) - f(x_0)$. Then $h$ is Jensen convex. Moreover $h$ is upper semicontinuous at 0. Fix any $\varepsilon > 0$. There exists a neighbourhood $W_0 \subset U_0$ of zero such that

$$h(x) < \varepsilon \text{ for every } x \in W_0. \tag{9}$$

We can find a neighbourhood $Z_0 \subset G$ of zero such that $Z_0 \subset W_0$ and $-Z_0 = Z_0$. Fix $x \in Z_0$. Since $x + (-x) = 0$, using the Jensen convexity of $h$ we obtain $h(0) \leq \frac{1}{2}(h(x) + h(-x))$. Hence in view of (9) we obtain $-h(x) \leq h(-x) < \varepsilon$. Therefore $h(x) > -\varepsilon$. Consequently $f(x + x_0) - f(x_0) > -\varepsilon$ for every $x \in Z_0$. Therefore $f$ is lower semicontinuous at $x_0$. The proof is finished. □

**Theorem 2.9.** *Let $(G, +)$ be an abelian topological group, $D \subset S$ be Jensen convex and open and let $f\colon D \to \mathbb{R}$ be a Jensen convex function. Assume that there exist an $x_0 \in D$ and a neighbourhood of zero $U_0 \subset S$ such that $x_0 + U_0 \subset D$ and $f$ is bounded from above on $x_0 + U_0$. Then for every $n \in \mathbb{N}$ and for every $x \in A_{D,n}(x_0)$ the function $f$ is continuous at the point $x$.*

**Proof.** By the assumption, we can find an $x_0 \in D$ and a neighbourhood of zero $U_0 \subset S$ such that $x_0 + U_0 \subset D$ and $f$ is bounded from above on $x_0 + U_0$. It follows from Lemma 2.4 that $f$ is locally bounded from above at every point of $\bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. Using Theorem 2.5 we deduce that $f$ is upper semicontinuous on $\bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. Hence on account of Lemma 2.8 we obtain that $f$ is lower semicontinuous on $\bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. Finally $f$ is continuous on $\bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. Therefore the proof is complete. □

We recall the Bernstein-Doetsch theorem.

**Theorem 2.10.** (Bernstein-Doetsch) *Let $D \subset \mathbb{R}^m$ be an open convex set and let $f\colon D \to \mathbb{R}$ be a Jensen convex function. If $f$ is bounded from above on some non–empty open subset of $D$, then $f$ is continuous.*

Below we derive Theorem BD from Theorem 2.9.

**Proof of Theorem 2.10.** By the assumption there exists an $x_0 \in D$ and a neighbourhood of zero $U \subset \mathbb{R}^m$ such that $x_0 + U \subset D$ and $f$ is bounded from above on $x_0 + U$. We will show that $D \subset \bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. Fix $x \in D$. Fix can find a $\delta \in (0, 1)$ such that

$$x_0 + s(x - x_0) \in D \text{ for every } s \in (1 - \delta, 1 + \delta). \tag{10}$$

Since $\lim_{n\to\infty} \frac{2^n+1}{2^n} = 1$, there exists an $n \in \mathbb{N}$ such that $\frac{2^n+1}{2^n} \in (1 - \delta, 1 + \delta)$. Put $t_i = \frac{1}{2^n}(x - x_0)$ for every $i \in \{1, \dots, 2^n\}$. Then $x_0 + \sum_{i=1}^{2^n} t_i = x$. Moreover, for every $k \in \{1, \dots, 2^n\}$ it holds the condition

$$x_0 + \sum_{i=1}^{k} t_i + t_k = x_0 + k\frac{x - x_0}{2^n} + \frac{x - x_0}{2^n} = \frac{k + 1}{2^n}x + \Big(1 - \frac{k + 1}{2^n}\Big)x_0. \tag{11}$$

Fix $k \in \{1, \dots, 2^n\}$. If $k \in \{1, \dots, 2^n - 1\}$, then using the convexity of $D$ we deduce that $\frac{k+1}{2^n}x + (1 - \frac{k+1}{2^n})x_0 \in D$. Hence in view of (11) we obtain $x_0 + \sum_{i=1}^{k} t_i + t_k \in D$. Now assume that $k = 2^n$. It follows from the choice of $n$ that $x_0 + \frac{2^n+1}{2^n}(x - x_0) \in D$. On account of (11) we deduce that $x_0 + \sum_{i=1}^{2^n} t_i + t_{2^n} \in D$. Therefore $x \in A_{D,2^n}(x_0)$. Consequently $D \subset \bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. By Theorem 2.9 the function $f$ is continuous on $\bigcup_{n=1}^{\infty} A_{D,n}(x_0)$. Therefore $f$ is continuous. The proof is finished. □

Now we prove that the Bernstein–Doetsch theorem fails for Jensen convex functions defined on some classes of semigroups. We start by introducing a definition.

Let $(S, +)$ be an abelian semigroup with a zero. A point $x \in S$ is called *invertible* if there exists a $y \in S$ such that $x + y = 0$. By $I(S)$ we denote the set of all invertible elements of $S$.

Below we present some properties of invertible elements.

**Lemma 2.11.** *Let $(S, +)$ be an abelian semigroup with a zero. Then $0 \in I(S)$ and for every $x, y \in S$ it holds the condition*

$$x, y \in I(S) \Leftrightarrow x + y \in I(S). \tag{12}$$

*In particular, the set $I(S)$ is a subsemigroup of $S$. Moreover the set $I(S)$ is Jensen convex and if $I(S) \neq S$, then the set $S \backslash I(S)$ is a subsemigroup of $S$.*

**Proof.** Clearly $0 = 0+0$ and thus $0 \in I(S)$. Fix $x, y \in S$. Assume that $x, y \in I(S)$. There exist $z, t \in S$ such that $x+z = 0$ and $y+t = 0$. Hence $x+y+t+z = x+z = 0$. Therefore $x + y \in I(S)$. Now assume that $x + y \in I(S)$. There exists a $z \in S$ such that $x + y + z = 0$. Hence, by definition, $x \in I(S)$. The semigroup is abelian, and therefore $y + x + z = 0$. Consequently $y \in I(S)$.

Assume that $x, y \in I(S)$, $z \in S$ and $2z = x+y$. It follows from (12) that $x+y \in I(S)$. Hence $2z \in I(S)$. Using (12) again we deduce that $z \in I(S)$.

Condition (12) implies $S \backslash I(S) + S \backslash I(S) \subset S \backslash I(S)$. Therefore if $S \backslash I(S) \neq \emptyset$ then the set $S \backslash I(S)$ is a subsemigroup of $S$. This completes the proof. $\square$

Let $(S, +)$ be a semigroup. A set $A \subset S$ is called *absorbing*, if for every $x \in S$ there exists an $n \in \mathbb{N}$ such that $x \in nA$.

**Theorem 2.12.** *Let $(S, +)$ be an abelian topological semigroup with a zero. Assume that every neighbourhood of zero in $S$ is an absorbing set. Let $D \subset S$ be an open Jensen convex set and let $x_0 \in D \cap I(S)$. If the set $S$ is not a group, then there exists a bounded Jensen convex function $f\colon D \to \mathbb{R}$ which is discontinuous at $x_0$.*

**Proof.** Assume that the set $S$ is not a group. Then $I(S) \neq S$. Let $g$ denotes the characteristic function of the set $I(S)$. The function $g$ is Jensen convex. Define a function $f\colon D \to \mathbb{R}$ by $f(x) = g(x) - 1$. The function $f$ is Jensen convex. We show that $f$ is discontinuous at $x_0$. Suppose on the contrary, that $f$ is continuous at $x_0$. Making use of the supposition we can find a neighbourhood $W \subset S$ of zero such that $W + x_0 \subset D$ and $f(W + x_0) \subset (-1, 1)$. Hence $g(W + x_0) \subset (0, 2)$ and therefore $g(W + x_0) = \{1\}$. Consequently $W + x_0 \subset I(S)$. Using Lemma 2.11 we deduce that $W \subset I(S)$. The set $W$ is a neighbourhood of zero and hence, by the assumption, it is an absorbing set. Therefore

$$S = \bigcup_{n=1}^{\infty} nW \subset \bigcup_{n=1}^{\infty} nI(S) \subset I(S).$$

Consequently $I(S) = S$; a contradiction. Therefore $f$ is discontinuous at $x_0$. $\square$

We summarize our considerations in the following theorem.

**Theorem 2.13.** *Let $(S, +)$ be an abelian topological semigroup with a zero such that $(I(S), +)$ is a topological group. Assume that every neighbourhood of zero in $S$ is absorbing. Let $x + U$ be open for every $x \in S$ and for every neighbourhood of zero $U \subset S$. The following statements are equivalent:*

(a) *$S$ is a group;*

(b) *if $D \subset S$ is Jensen convex and open and $f\colon D \to \mathbb{R}$ is Jensen convex and upper semicontinuous, then $f$ is continuous;*

(c) *if $D \subset S$ is Jensen convex and open and $f\colon D \to \mathbb{R}$ is a Jensen convex function bounded from above, then $f$ is continuous.*

**Proof.** The implication (a) $\Rightarrow$ (b) is a consequence of Lemma 2.8. It follows from Theorem 2.5 that condition (b) implies (c). Applying Theorem 2.12 with $D = S$ and $x_0 = 0$ we see that the condition (c) $\Rightarrow$ (a) holds.

This completes the proof. $\square$

## References

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