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Fifty Years of Calculus of Variations

Giuseppe Buttazzo
Department of Mathematics, University of Pisa, Italy
giuseppe.buttazzo@unipi.it

This paper gives an overview of my mathematical life, in particular to my relationship with the Calculus of Variations.

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My first approach to the calculus of variations occurred exactly fifty years ago; in fact, in 1974 I was a third-year student at the Scuola Normale of Pisa and, according to the Scuola's regulations, third-year students had to carry out a small thesis on a topic, under the direction of a supervisor. I asked Ennio De Giorgi to be my supervisor and he proposed *"Hausdorff Measures"* as the topic. I remember the day of my thesis very well: in a classroom at the Scuola Normale there was only me with the commission made up of Ennio De Giorgi and Enrico Bombieri, the two most famous mathematicians in Italy at the time. The thesis went well and I was admitted to the fourth (and last for those times) year of the Scuola Normale.

In 1976 I carried out my diploma thesis at the University of Pisa, on Γ-convergence, under the direction of Ennio De Giorgi, who in that period had begun to be interested in this type of variational convergence. In the same year I returned to the Scuola Normale again as a doctoral student and began a long and fruitful collaboration with Gianni Dal Maso, on the semicontinuity of integral functionals and on the representation of them in integral form of the type

$$\int_\Omega f(x,u)\,dx \qquad \text{or} \qquad \int_\Omega f(x,u,\nabla u)\,dx.$$

The aim was to provide an explicit expression for the semicontinuous envelope (the *relaxed* functional in current terminology) of a given functional and for the Γ limit of a given sequence of functionals.

During my doctoral period at the Scuola Normale, thanks to an agreement with the École Normale Supérieure of Paris, I spent the academic year 1978–1979 in Paris. I thus had the opportunity to meet many French mathematicians from the research group that Jacques Louis Lions had built around variational problems and the theory that was called *Homogenization*, closely linked to the Γ-convergence of functionals of the type

$$\int_\Omega f(x/\varepsilon,u,\nabla u)\,dx.$$

The period spent in Paris was of great importance for my mathematical education, and the scientific relationships with many of the French colleagues I met during

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that period remained active for a long time. I want to remember in particular Hédy Attouch, who recently passed away, with whom I had an intense collaboration; our book

> H. Attouch, G. Buttazzo, G. Michaille: *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*, MPS-SIAM Book Series on Optimization **6**, SIAM, Philadelphia (2006)

continues to be highly regarded by the community working on variational problems.

Once I returned to Pisa I resumed my collaboration with Gianni Dal Maso; driven by the success of the homogenization theory, we dealt with other families of functionals dependent on a small parameter ε, such as

$$\int_\Omega f(x, u/\varepsilon, \nabla u)\, dx \qquad \text{and} \qquad \int_\Omega f(x, u, \varepsilon\nabla u)\, dx.$$

In 1980 I had my first permanent position as a researcher at the Scuola Normale; I began a fruitful collaboration with Emilio Acerbi and Danilo Percivale on various variational problems in which it was necessary to characterize the Γ-limit functional of sequences of functionals coming from continuum mechanics. We studied the limit problems in the presence of strong conductivity, weak conductivity, small thickness. Furthermore, with Emilio Acerbi we proved that the class of Riemannian metrics on a manifold is not closed with respect to the convergence of geodesic curves, and that its closure is given by the family of Finsler metrics.

Another important period of my education was the one spent in 1984 at the Institute for Mathematics and its Applications (IMA) of the University of Minnesota; at that time the IMA was the world center for variational problems and their applications to mechanics. I had the opportunity to meet a large number of colleagues, including Irene Fonseca, Luc Tartar, Robert Kohn; in particular, with the latter we studied the case of variational problems on domains with strongly oscillating boundaries.

In 1986 I won a full professor position at the University of Ferrara; even though the Mathematics Department was much smaller in size than those I had previously known, I have excellent memories of my colleagues and the friendly atmosphere that was created between us, thanks in particular to the presence of Mariolina Padula, Paolo Galdi and Giuseppe Toscani.

In 1990 there was the opportunity to return to Pisa as a full professor at the University. I began to supervise doctoral students, Loris Faina and Lorenzo Freddi were the first, on topics related to optimal control problems. On the same topic, with implications closer to the Hamilton-Jacobi-Bellman equations, I supervised Ariela Briani's doctoral thesis shortly afterwards.

In the same period I began a long collaboration with Guy Bouchitté, who had obtained a position as Maître de Conférences at Marseille and then as full professor at Toulon, on functionals defined on the space of measures. The interesting property that we highlighted is that, while on L^p spaces the semicontinuity for weak convergence necessarily implies convexity, this is no longer true in the space of measures, due to concave terms which appear naturally on the singular parts. In a series of papers we studied with Guy the semicontinuity, relaxation, and integral represen-

tation of this type of functionals. Soon after, Guy and I began to be interested in a class of variational problems in which Sobolev spaces are defined with respect to a generic measure μ instead of with respect to the Lebesgue measure; in this way it is possible to describe the cases in which there are concentration phenomena on lower dimensional structures. The usual gradient must be replaced in these cases with the *tangential gradient with respect to the measure* μ and Ilaria Fragal's doctoral thesis was aimed at the systematic study of these cases.

In 1993, following some results obtained by Gianni Dal Maso and Umberto Mosco, in which capacitary measures were introduced, we began with Gianni to study the functionals defined on this type of measures; since capacitary measures are a general framework that contains the class of domains where elliptic PDEs are defined, as a consequence we obtained an existence result for *shape optimization problems*, which is still widely cited due to its generality. This is where my interest in shape optimization problems began, which I continue to work on today.

Another collaboration that I remember fondly is the one with Victor Mizel, on variational problems that present the so-called *Lavrentiev phenomenon*. I didn't know about this topic and I thank Vic for explaining all the details to me. These are minimum problems for functionals of the type

$$\int_\Omega f(x, u, \nabla u)\, dx$$

where, due to the growth conditions (different from above and from below), the infimum on the class of regular functions is different from the infimum on Sobolev spaces. With Vic we gave an interpretation of the Lavrentiev phenomenon in terms of relaxation, and Marino Belloni's doctoral thesis brought together all the known cases.

Another interesting period of my career was the one spent at the University of Köln in 1996; with Bernhard Kawohl we studied the so-called *Newton problem of minimal resistance*, in which the profile offering the lowest resistance to motion has to be determined in a class of admissible profiles. Our research has been widely disseminated and has given rise to numerous scientific articles. The proximity of Köln to Bonn has also given me the opportunity to work with Stefan Hildebrandt on the book that, together with Mariano Giaquinta, we had started writing some time before:

> G. Buttazzo, M. Giaquinta, S. Hildebrandt: *One-dimensional Calculus of Variations: an Introduction.* Oxford University Press, Oxford (1998).

The book was very successful, with even a translation into Russian by Tamara Rozhkovskaya (Novosibirsk, 2002).

My interest in shape optimization problems continued to be very strong; thanks to Jean-Paul Zolésio I met Dorin Bucur who was dealing with very similar problems. Jean-Paul was the director of Dorin's doctoral thesis, with whom we began a long collaboration, which is still active today. Paola Trebeschi's doctoral thesis was on shape and obstacle optimization problems, and with Dorin we put our results together in a structured form in the book

D. Bucur, G. Buttazzo: *Variational Methods in Shape Optimization Problems*, Progress in Nonlinear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).

In the same years, thanks mainly to a publication by Craig Evans and Wilfrid Gangbo, I began to become interested in optimal transport problems. The doctoral theses of Luigi De Pascale, Luca Granieri, Alessio Brancolini, Filippo Santambrogio, Lorenzo Brasco address various issues related to transport theory. In particular, branched transport problems play a particular role, in which it is advantageous to cumulatively transport a quantity of mass rather than dividing it into smaller portions. On the contrary, in problems with congestion there is the opposite phenomenon and, as happens in urban traffic problems, a mass concentration produces a disadvantage in terms of transport. With Filippo Santambrogio we have developed a model of optimal planning of an urban area using transport theory and simultaneously putting together congestion terms for residential areas and concentration terms for commercial areas. Together with Guillaume Carlier we subsequently studied various models of optimization problems related to transport theory; in particular, in a work with Guillaume and Maxime Laborde we studied the problem of transport between singular measures, which we then refined into a model for *optimal parking areas* in a subsequent work with Guillaume and Katharina Eichinger.

With Guy Bouchitté we highlighted an unexpected connection between shape optimization problems and optimal transport: optimal structures have a material density that can be determined by solving an optimal transport problem. In this type of approach, the use of tangential gradients with respect to a measure is fundamental, a theory that we had developed with Guy several years earlier for completely different purposes. Once we understood the link between optimal transport and optimization problems for PDEs, in a series of works with Edoardo Mainini, Edouard Oudet, Eugene Stepanov, we dealt with elliptic equations for which the region where the Dirichlet datum is fixed is not prescribed but has to be chosen in an admissible class, in order to optimize a given cost functional, which we have called *average distance functional*. Xin Yang Lu's doctoral thesis was developed on these topics.

Another interesting class of variational problems is the one where the integral is replaced by the essential supremum; several problems from applications require this formulation and I supervised the PhD thesis of Francesca Prinari on this challenging topic.

The collaboration with Guy Bouchitté has continued over the years, also on different problems. More recently, Paola Gori Giorgi illustrated to us a series of variational problems derived from *Density Functional Theory* (DFT), developed as an alternative to the Schrödinger equation, considered too complex due to the high number of variables. With Guy, together with Thierry Champion and Luigi De Pascale, we dealt with functionals of the type

$$F(\rho) = T(\rho) + C(\rho) - U(\rho)$$

where ρ represents the electron density (a measure of probability), $T(\rho)$ the kinetic energy term, $C(\rho)$ the repulsive interaction between electrons, $U(\rho)$ the attractive potential of the nuclei. The existence of a minimizing probability ρ for F is a much

studied problem, which presents various difficulties, linked to the fact that the ambient space is the entire space $\mathbb{R}^d$. In addition, for the terms T, C, U there are various possible expressions, which make the problem very rich in different situations.

The search for potentials, for elliptic equations of the type

$$-\Delta u + V(x)u = f, \qquad u \in H_0^1(\Omega),$$

which optimize a given cost functional of the type

$$\int_\Omega j\big(x, u(x), V(x)\big)\, dx,$$

is another topic I have been interested in at various times. This is an optimal control problem, where the potential V is the control variable and the solution u is the state variable. Together with my former doctoral students Augusto Gerolin, Berardo Ruffini, Bozhidar Velichkov, we treated some particular cases in 2014; more recently, together with Juan Casado Diaz and Faustino Maestre, of the University of Seville, we worked to show the BV regularity of optimal potential. In some cases, when a constraint of the type $\alpha \le V(x) \le \beta$ is imposed a priori, the optimal potential is of a *bang-bang* type, that is, it assumes only the values α and β and therefore the problem can be reduced to a shape optimization problem, in which the optimal domains are sets of finite perimeter.

The collaboration with Bozhidar continued fruitfully on various issues arising from shape optimization problems: existence of an optimal domain in situations not previously explored, up to the study of the regularity properties of optimal domains. The most recent PhD theses I supervised, by Francesco Paolo Maiale and Luca Briani, are mainly devoted to these cases.

In addition to the research work on variational problems briefly described above, I have been involved in the organization of various scientific activities. I want to remember in particular three series of meetings that were very successful and continue to be so today.

- The series of conferences on *Optimal Transportation and Applications*, at the Ennio De Giorgi Center of the Scuola Normale of Pisa, started in 2001 and continued every two years; in 2024 we will be at edition n. 11. I share the organization with L. Ambrosio (Pisa), N. Gigli (SISSA), G. Savaré (Bocconi).
- The series of conferences on *Partial Differential Equations, Optimal Design and Numerics*, at the Pedro Pascual Research Center in Benasque (Spain), started in 2005 and continued every two years; in 2024 we will be at edition number 10. I share the organization with O. Glass (Paris), G. Leugering (Erlangen), and E. Zuazua (Erlangen).
- The series of conferences of the *International School of Mathematics "Guido Stampacchia"* at the *Ettore Majorana Center* in Erice. After the retirement of Franco Giannessi, who successfully directed the School for many years, together with Michel Théra we took over the direction of the School and for several years we have been organizing important conferences in different branches of mathematics.

I want to conclude by saying that I consider myself very satisfied with my career, and perhaps even a little lucky to have met some very valid collaborators and students.

In the previous few lines I could not mention all the doctoral students I have had, so I want to list them all with the dates and titles of their theses.

- Loris Faina: *Some nonconvex variational problems in BV spaces*, SISSA-Trieste, 1994.
- Lorenzo Freddi: *Rilassamento e convergenza di problemi di controllo ottimo*, Università di Pisa, 1994.
- Marino Belloni: *Rilassamento di problemi variazionali con fenomeno di Lavrentiev*, Università di Pisa, 1996.
- Paola Trebeschi: *Esistenza di soluzioni in problemi di ottimizzazione di forma e ostacoli*, Università di Pisa, 1998.
- Ariela Briani: *Hamilton-Jacobi-Bellman equations and* Γ*-convergence for optimal control problems*, Università di Pisa, 1999.
- Ilaria Fragalà: *Tangential calculus and variational integrals with respect to a measure*, Università di Pisa, 2000.
- Luigi De Pascale: *Morse-Sard theorem in Sobolev spaces, transport problems and applications*, Università di Pisa, 2001.
- Francesca Prinari: *Calculus of variations for supremal functionals*, Università di Pisa, 2003.
- Andrea Davini: *Finsler metrics in optimization problems and Hamilton-Jacobi equations*, Università di Pisa, 2004.
- Michele Gori: *Lower semicontinuity and relaxation for integral and supremal functionals*, Università di Pisa, 2004.
- Luca Granieri: *Mass transportation problems and minimal measures*, Università di Pisa, 2005.
- Alessio Brancolini: *Optimization problems for transportation networks*, Scuola Normale Superiore di Pisa, 2006.
- Filippo Santambrogio: *Variational problems in transport theory with mass concentration*, Scuola Normale Superiore di Pisa, 2006.
- Lorenzo Brasco: *Geodesics and PDE methods in transport models*, Università di Pisa, 2010.
- Al-hassem Nayam: *Shape optimization problems of higher codimension*, Università di Pisa, 2011.
- Berardo Ruffini: *Optimization problems for solution of elliptic equations and stability issues*, Scuola Normale Superiore di Pisa, 2013.
- Bozhidar Velichkov: *Existence and regularity results for some shape optimization problems*, Scuola Normale Superiore di Pisa, 2013.
- Xin Yang Lu: *Geometric and regularity properties of solutions of some problems related to the average distance functional*, Scuola Normale Superiore di Pisa, 2013.

– Serena Guarino Lo Bianco: *Some optimization problems in mass transport theory*, Università di Pisa, 2014.

– Augusto Gerolin: *Multimarginal optimal transport and potential optimization problems for Schrödinger operators*, Università di Pisa, 2016.

– Harish Shrivastava: *Shape optimisation problems for integral functionals and regularity properties of optimal domains*, Università di Pisa, 2018.

– Francesco Paolo Maiale: *Shape optimization problems and regularity of the free boundaries*, Scuola Normale Superiore di Pisa, 2022.

– Luca Briani: *Shape optimization problems with competing terms*, Università di Pisa, 2023.

Giuseppe Buttazzo
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