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On Poidge-Convexity

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Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ (always $d \geq 2$). A set $M \subset \mathbb{R}^d$ is called *$\mathcal{F}$-convex*, if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. We obtain the poidge-convexity, when $\mathcal{F}$ consists of all unions $\{x\} \cup \sigma$, called *poidges*, where x is a point, σ a line-segment, and $\operatorname{conv}(\{x\} \cup \sigma)$ a right triangle. In this paper we first present several new results on the poidge-convexity of various sets, such as unions of line-segments, fans, cones and cylinders, complements of some given sets and not simply connected sets. Then, we investigate the poidge-convex completion of compact convex sets, trying to determine the minimal number of points necessary to be added to make them poidge-convex.

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1. Introduction

The third author proposed a very general kind of convexity in 1974 at a meeting on convexity in Oberwolfach. Given a family $\mathcal{F}$ of sets in a certain space $\mathfrak{X}$, a set $M \subset \mathfrak{X}$ is called *$\mathcal{F}$-convex*, if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

Usual convexity, affine linearity, arc-wise connectedness are examples of $\mathcal{F}$-convexity, for suitably chosen families $\mathcal{F}$. Bruckner and Bruckner [3], and also Magazanik and Perles [4] investigated L_n sets, which are further examples of $\mathcal{F}$-convex sets, $\mathcal{F}$ consisting of all polygonal paths with at most n edges in the plane. Blind, Valette and the third author [1], and also Böröczky Jr. [2], studied the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles.

In [7], the third author investigated the right convexity, where $\mathcal{F}$ is the famliy of all non-degenerate right triangles, and the last two authors [6] presented a discretization of right convexity, the right triple convexity, for short *rt*-convexity, when $\mathcal{F}$ contains all triples $\{x, y, z\} \subset \mathbb{R}^d$ with $\angle xyz = \pi/2$.

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Wang and the last two authors [5], generalizing the right convexity once again, introduced and investigated the *poidge-convexity*, the case with $\mathcal{F}$ consisting of all poidges. The union $\{x\} \cup \sigma$ is called a *poidge*, if x is a point, σ a line-segment, and $\mathrm{conv}(\{x\} \cup \sigma)$ a non-degenerate right triangle [5].

In this paper, we further investigate the poidge-convexity.

If there is a poidge containing $u, v \in M$ and included in $M \subset \mathbb{R}^d$, we say that the pair u, v has the *poidge-property* in M [5]. Hence, a set $M \subset \mathbb{R}^d$ is called *poidge-convex* if each pair of points of M enjoys the poidge-property. Open sets are obviously poidge-convex.

2. Definitions and notation

For distinct points $x, y \in \mathbb{R}^d$, let xy be the line-segment from x to y, $\overline{xy}$ the line including xy and H_{xy} the hyperplane through x orthogonal to $\overline{xy}$. If L, $L^{'}$ are affine subspaces of $\mathbb{R}^d$, $L \parallel L^{'}$ means that they are parallel and $L \perp L^{'}$ that they are orthogonal.

For $M \subset \mathbb{R}^d$, we denote by $\complement M$ its complement, by $\overline{M}$ its affine hull, by $\mathrm{dim}M$ its Hausdorff dimension, and by $\mathrm{conv}M$ its convex hull; further, $\mathrm{int}M$, $\mathrm{cl}M$, $\mathrm{bd}M$, denote its topological interior, closure, boundary, respectively, considered in $\overline{M}$. In this paper, we always assume $d \geq 2$.

Let $\mathrm{diam}M = \sup_{x,y \in M} \|x-y\|$ and $d(x, M) = \inf_{y \in M} \|x-y\|$. If $d(x, M) = \|x-y\|$ for some unique point $y \in M$, put $\pi_M(x) = y$. A 2-point set $\{x, y\} \subset M$ with $\|x-y\| = \mathrm{diam}M$ is called a *diametral pair* of M, while the line-segment xy is a *diameter* of M.

For any compact set $C \subset \mathbb{R}^d$, let S_C be the smallest hypersphere containing C in its convex hull.

For $a_1, \cdots, a_n \in \mathbb{R}^d$, put $a_1 \cdots a_n = \mathrm{conv}\{a_1, \cdots, a_n\}$. Let

$$(a_1 a_2 \cdots a_n) = a_1a_2 \cup a_2a_3 \cup \cdots \cup a_{n-1}a_n \text{ and } \langle a_1a_2 \cdots a_n \rangle = (a_1a_2 \cdots a_n) \cup a_na_1.$$

Let Γ be a Jordan (simple) arc in $\mathbb{S}_1 \subset \mathbb{R}^2$, which is the unit circle centered at the origin $\mathbf{0}$ of $\mathbb{R}^2$. Then $\bigcup\{\mathbf{0}x : x \in \Gamma\}$ is called a *fan*, and its angle at $\mathbf{0}$ its *opening*.

Let h denote the Pompeiu-Hausdorff distance between compact sets.

3. Poidge-convex unions of line-segments

Theorem 3.1. *The union Σ of a family $\mathfrak{L}$ of line-segments in $\mathbb{R}^d$ is poidge-convex, if*

(1) *$\pi_{\overline{\sigma}}(\Sigma \setminus \sigma)$ contains an endpoint of σ for any $\sigma \in \mathfrak{L}$, and*

(2) *$\pi_{\overline{\sigma}}(\tau) \subset \sigma$ or $\pi_{\overline{\tau}}(\sigma) \subset \tau$ for any distinct $\sigma, \tau \in \mathfrak{L}$.*

Proof. Choose x, $y \in \Sigma$.

Case 1. $x \in \sigma$, $y \in \tau$ $(\sigma \neq \tau)$.
Without loss of generality assume $\pi_{\overline{\tau}}(\sigma) \subset \tau$, by (2). Let $w = \pi_{\overline{\tau}}(x)$. If $y = w$, take $w' \in \tau$ distinct from w; then $\{x\} \cup yw'$ is a suitable poidge in Σ. Otherwise, $\{x\} \cup yw$ is a suitable one.

Case 2. x, $y \in \sigma$.
Suppose $\sigma = ab$. Then a or b equals $\pi_{\overline{ab}}(z)$ for some $z \in \Sigma \setminus ab$. In any case, x, y belong to the poidge $\{z\} \cup ab$. □

Consider the quadrilateral closed curve $\langle abcd \rangle$, subject to $\|a-b\| = \|b-c\|$, $\|c-d\| = \|d-a\|$, $d \in abc$ and $\angle cda < \pi/2$. See Figure 1.

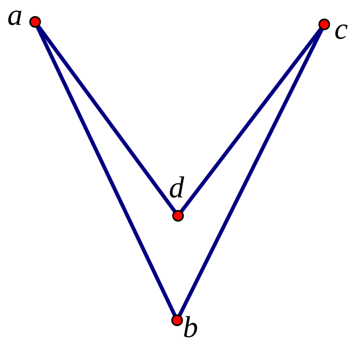


Figure 1: The quadrilateral closed curve $\langle abcd \rangle$.

This quadrilateral is poidge-convex. Notice that the hypothesis of Theorem 3.1 is not satisfied. Hence, Theorem 3.1 gives no characterization.

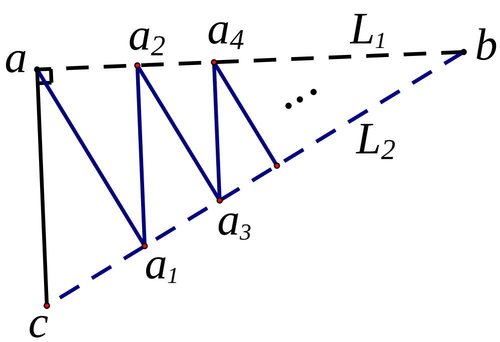
The following lemma is straightforward, but very useful.

Lemma 3.2. *A poidge-convex set in $\mathbb{R}^d$ contains at most one point not belonging to any line-segment included in the set.*

Problem. Find a poidge-convex Jordan closed curve possessing such a point.

Remark 3.3. The preceding problem receives a solution for Jordan arcs instead of Jordan closed curves, see Figure 2(a). There, L_1, L_2 are lines meeting at the point b, $\angle bac = \pi/2$, $\overline{aa_1} \perp L_2$, $\overline{a_1a_2} \perp L_1$, $\overline{a_2a_3} \perp L_2$, $\overline{a_3a_4} \perp L_1$, etc. Then $\{b\} \cup \bigcup_{n=1}^{\infty}(caa_1a_2 \cdots a_n)$ is a suitable example.

Remark 3.4. A poidge-convex Jordan closed curve doesn't need to be a polygon. See Figure 2(b). There, $\bigcup_{n=1}^{\infty}(bcaa_1a_2 \cdots a_n)$ is a suitable example.



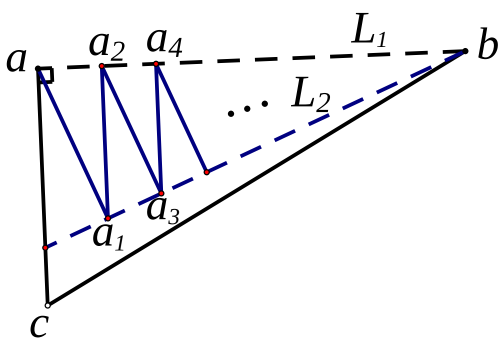


Figure 2(a): Illustration of Remark 3.3. Figure 2(b): Illustration of Remark 3.4.

Theorem 3.5. *The boundary of a compact convex set $M \subset \mathbb{R}^2$ is poidge-convex if and only if M is a non-obtuse triangle or a rectangle.*

Proof. The "if" part is obvious.

Suppose bd$M$ is poidge-convex.

Let L_1, L_2, L_3, L_4, L_5 be supporting lines of M, parallel to the sides of a regular pentagon. Choose $a_i \in L_i \cap M$ $(i = 1, \cdots, 5)$. Assume without loss of generality $a_1 \neq a_2$. Let A_1 be the arc in bd$M$ from a_1 to a_2 (not containing a_3). If $A_1 \neq a_1a_2$, $\overline{a_1a_2}$ is not a supporting line of M. Let $L \parallel \overline{a_1a_2}$ be a supporting line of M separated by $\overline{a_1a_2}$ from a_3, a_4, a_5. Clearly, $A_1 \setminus L$ has two components, B and C. Any pair of points chosen in the interiors of B and C do not enjoy the poidge-property in bd$M$. Hence M is a polygon.

Assume that an angle of the polygon M is obtuse. Then the midpoints of its sides don't enjoy the poidge-property in bd$M$. Hence, M has no obtuse angle, and the proof is complete. $\square$

Theorem 3.6. *A closed polygonal curve $\langle a_1a_2 \cdots a_n \rangle \subset \mathbb{R}^d$ is poidge-convex, if and only if $\pi_{\overline{a_ia_{i+1}}}(a_ja_{j+1}) \subset a_ia_{i+1}$ or $\pi_{\overline{a_ja_{j+1}}}(a_ia_{i+1}) \subset a_ja_{j+1}$ for any $i \neq j$ $(a_{n+1} = a_1)$.*

Proof. Put $A = \langle a_1a_2 \cdots a_n \rangle$ and suppose that we have $\pi_{\overline{a_ia_{i+1}}}(a_ja_{j+1}) \subset a_ia_{i+1}$ or $\pi_{\overline{a_ja_{j+1}}}(a_ia_{i+1}) \subset a_ja_{j+1}$ for any $i \neq j$. We claim that a_i, a_{i+1} enjoy the poidge-property in A for any i.

If $(A \setminus a_ia_{i+1}) \cap \overline{a_ia_{i+1}} \neq \emptyset$, let $b \in (A \setminus a_ia_{i+1}) \cap \overline{a_ia_{i+1}}$ with $\|b - a_i\| < \|b - a_{i+1}\|$. Then $H_{a_ia_{i+1}}$ separates a_{i+1} from b. Since A is a closed polygonal curve, there exists a point $c \in H_{a_ia_{i+1}} \cap A \setminus \{a_i\}$, whence $a_ia_{i+1} \cup \{c\}$ is a suitable poidge.

Otherwise, $A \setminus a_ia_{i+1}$ lies in one side of $\overline{a_ia_{i+1}}$. Both $\angle a_{i-1}a_ia_{i+1}$ and $\angle a_ia_{i+1}a_{i+2}$ are non-obtuse. If one of them, say $\angle a_{i-1}a_ia_{i+1} = \pi/2$, $a_ia_{i+1} \cup \{a_{i-1}\}$ is a poidge in A. Otherwise, both $\angle a_{i-1}a_ia_{i+1}$ and $\angle a_ia_{i+1}a_{i+2}$ are acute. Assume without loss of generality $\pi_{\overline{a_ia_{i-1}}}(a_{i+1}a_{i+2}) \subset a_ia_{i-1}$. Then $a_ia_{i+1} \cup \{d\}$ is suitable for $d = \pi_{\overline{a_ia_{i-1}}}(a_{i+1})$.

Hence, a_i, a_{i+1} enjoy the poidge-property in A.

Choose some points $x, y \in A$. If $x, y \in a_ia_{i+1}$ for some i, we only need to consider $x \in \text{int}a_ia_{i+1}$. Since A is closed and H_{xy} separates a_{i+1} from a_i, there exists a point $z \in H_{xy} \cap A \setminus a_ia_{i+1}$. Then $xy \cup \{z\}$ is suitable.

Consider $x \in a_ia_{i+1}$ and $y \in a_ja_{j+1}$. Assume without loss of generality that we have $\pi_{\overline{a_ia_{i+1}}}(a_ja_{j+1}) \subset a_ia_{i+1}$, $i < j$.

If $x = \pi_{\overline{a_ia_{i+1}}}(y)$, then $\{y\} \cup xw$ is suitable for any $w \in a_ia_{i+1} \setminus \{x\}$. Otherwise, $\{y\} \cup x\pi_{\overline{a_ia_{i+1}}}(y)$ is suitable.

On the other hand, suppose A is poidge-convex.

If there exist two line-segments a_ia_{i+1}, a_ja_{j+1} such that $\pi_{\overline{a_ia_{i+1}}}(a_ja_{j+1}) \not\subset a_ia_{i+1}$ and $\pi_{\overline{a_ja_{j+1}}}(a_ia_{i+1}) \not\subset a_ja_{j+1}$, then x_0, y_0 don't enjoy the poidge-property for the points $x_0 \in (\text{int}a_ia_{i+1}) \setminus \pi_{\overline{a_ia_{i+1}}}(a_ja_{j+1})$ and $y_0 \in (\text{int}a_ja_{j+1}) \setminus \pi_{\overline{a_ja_{j+1}}}(a_ia_{i+1})$. $\square$

Theorem 3.7. *A set $M \subset \mathbb{R}^d$ is poidge-convex if* bd$M$ *is poidge-convex.*

Proof. Choose $x, y \in M$. For $x, y \in \text{bd}M$, the poidge-property follows directly from the hypothesis. Otherwise, assume $x \in \text{int}M$. For $z \in H_{xy} \setminus \{x\}$, $\{y\} \cup xz$ is a suitable poidge in M, if $\|x - z\|$ is small enough. $\square$

4. Poidge-convex fans

In this and the following section, we consider infinite unions of line-segments. Here, we investigate the fans.

It turns out that some, but not all, fans are poidge-convex.

Theorem 4.1. *A fan is poidge-convex if and only if its opening belongs to the set* $]0, \pi/2] \cup [\pi, 2\pi[$.

Proof. Let α be the opening of the fan S, and let $a, b \in \mathbb{S}_1$ satisfy $\mathbf{0}a \cup \mathbf{0}b \subset \mathrm{bd}S$.

For the "only if" part, suppose $\alpha \in]\pi/2, \pi[$. We have $\angle a\mathbf{0}b > \pi/2$. Since $S \subset \mathrm{conv}S_{ab}$ and $S \cap S_{ab} = \{a, b\}$, $H_{ab} \cup H_{ba} \cap S = \{a, b\}$ and a, b don't enjoy the poidge-property in S.

For the "if" part, take $x, y \in S$.

Case 1. $\alpha \in]0, \pi/2]$.

If $x \in \overline{\mathbf{0}y}$, $S_{xy} \cap S \setminus \{x, y\} \neq \emptyset$. Choose $z \in S_{xy} \cap S \setminus \{x, y\}$. Then $\{x\} \cup yz$ is a poidge in S. See Figure 3(a). Otherwise, $\{x\} \cup y\pi_{\overline{\mathbf{0}y}}(x)$ or $\{y\} \cup x\pi_{\overline{\mathbf{0}x}}(y)$ is a poidge in S. See Figure 3(b).

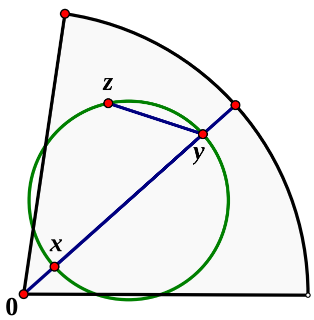


Figure 3(a): The poidge $\{x\} \cup yz$.

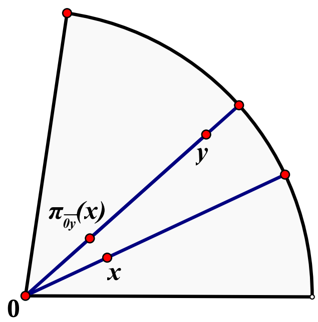


Figure 3(b): The poidge $\{x\} \cup y\pi_{\overline{\mathbf{0}y}}(x)$.

Case 2. $\alpha \in [\pi, 2\pi[$.

If x, y are in the same half-disc D with diameter $c(-c)$ included in S, where $c \in \mathbb{S}_1$, assume without loss of generality that x is not closer than y to $c(-c)$. In consequence, $S_{xy} \cap D \setminus \{x\}$ contains a point z close to x. Moreover, $xyz \subset S$.

Otherwise, there is a diameter $d(-d) \ni x$ in S. Then $\{y\} \cup x\pi_{\overline{d(-d)}}(y)$ is a poidge in S. $\square$

5. Poidge-convex cones and cylinders

Other well-known infinite unions of line-segments are the cones and the cylinders. Their right convexity and poidge-convexity have also been treated in [7], respectively [5]. Here we deal with more general objects.

A *cylinder* in $\mathbb{R}^d$ is here a set $Z = B + \mathbf{0}v = \{x + \lambda v : x \in B, \lambda \in [0, 1]\}$, where B is a set containing the origin $\mathbf{0}$, such that codim $\overline{B} = 1$, and $v \notin \overline{B}$.

The sets B and $B + v$ are called *bases* of Z.

Z is said to be a *right cylinder* if $\langle v, w\rangle = 0$ for all $w \in \overline{B}$.

Theorem 5.1. *The boundary of any right cylinder is poidge-convex.*

Proof. Let Z be a right cylinder with the bases B and $B+v$. Choose $x, y \in \mathrm{bd}Z$.

Case 1. $x \in B \cup (B+v)$.
Without loss of generality assume $x \in B$.

Subcase 1. $y \in \mathrm{int}B \cup \mathrm{int}(B+v)$.
A suitable poidge $\{x\} \cup yy'$ can be found in $B \cup (B+v)$.

Subcase 2. $y \notin \mathrm{int}B \cup \mathrm{int}(B+v)$.
If $x \neq \pi_{\overline{B}}(y)$, then $\{x\} \cup \pi_{\overline{B}}(y)(\pi_{\overline{B}}(y)+v)$ is a suitable poidge in $\mathrm{bd}Z$. Otherwise, $\{z\} \cup xy$ is suitable, for any $z \in B \setminus \{x\}$.

Case 2. $x \notin B \cup (B+v)$.
Let $H \ni x$ be the hyperplane parallel to $\overline{B}$. Without loss of generality assume $y \in \mathrm{conv}(B \cup H)$. If $x \neq \pi_H(y)$, then $\{x\} \cup \pi_H(y)\pi_{\overline{B}}(y)$ is a suitable poidge in $\mathrm{bd}Z$. Otherwise, $\{z\} \cup xy$ is suitable, for any $z \in H \cap \mathrm{bd}Z \setminus \{x\}$. □

The following result can be proved by using Theorems 3.7 and 5.1.

Theorem 5.2. *Any right cylinder is poidge-convex.*

Let B be a set such that codim $\overline{B} = 1$, and $p \notin \overline{B}$. The set $\bigcup_{x \in B} px$ is called a *cone* in $\mathbb{R}^d$, with B its *base* and p its *apex*.

P is said to be a *right cone* if $\pi_{\overline{B}}(p) \in \mathrm{conv}B$.

The *height* of a cone of apex p and base B is $d(p, \overline{B})$.

Theorem 5.3. *The boundary of a right cone with base B is poidge-convex, if its height is at least* $\mathrm{diam}B/2$.

Proof. Let P be a right cone with the base B, apex p and height h. Consider $x, y \in \mathrm{bd}P$. If $x, y \in B$,

$$\|p - \frac{x+y}{2}\| \geq h \geq \frac{\mathrm{diam}B}{2} \geq \frac{\|x-y\|}{2},$$

whence p lies on S_{xy} or outside it. Then $\angle xpy \leq \pi/2$.

For any $x, y \in P \setminus \{p\}$, consider $\{x'\} = \overline{px} \cap \overline{B}$ and $\{y'\} = \overline{py} \cap \overline{B}$. We have $\angle xpy = \angle x'py' \leq \pi/2$.

Case 1. $x, y \in P \setminus \{p\}$ and $p \notin \overline{xy}$.
Since $\angle xpy \leq \pi/2$, there exists a point $z \in S_{xy} \cap (px \cup py) \setminus \{x, y\}$. Assume without loss of generality $z \in px$. Then $xz \cup \{y\}$ is a suitable poidge.

Case 2. $x, y \in P \setminus \{p\}$ and $y \in px$.
Take $w \in P \setminus \overline{xy}$. Consider $u \in wp$, close to p. Then $x\pi_{\overline{xy}}(u) \cup \{u\}$ is a suitable poidge.

Case 3. $y = p$.
Choose $a \in B$ farthest from p. Obviously, $\pi_{\overline{B}}(p) \in S_{pa}$. Since $\pi_{\overline{B}}(p) \in \mathrm{conv}B$, we have $\pi_{\overline{B}}(p) \in \mathrm{bd}B$ or $(\mathrm{bd}B) \setminus S_{a\pi_{\overline{B}}(p)} \neq \emptyset$. Hence, there exists a point $b \in \mathrm{bd}P \cap S_{pa}$ distinct from p; only in one case we necessarily get $b = p$: if $d = 2$, card$B = 2$ and $h = \mathrm{diam}B/2$. But then, the conclusion of the theorem is obvious.

If $x \in pa$, $pa \cup \{b\}$ is suitable. Otherwise, since $\angle xpa \le \pi/2$ and $\|p-a\| \ge \|p-x\|$, $\pi_{\overline{pa}}(x) \in pa$. Then $\{x\} \cup p\pi_{\overline{pa}}(x)$ is suitable. □

Similarly, we obtain the next theorem.

Theorem 5.4. *A right cone with base B and height at least* diam$B/2$ *is poidge-convex.*

6. Unbounded poidge-convex sets

Theorem 6.1. *If $P \subset \mathbb{R}^d$ is a right cone, then $\complement(\mathrm{int}P)$ is poidge-convex.*

Proof. Choose $x, y \in \complement(\mathrm{int}P)$. Let p be the apex of P.

If $x \notin \mathrm{bd}P$, take $z \in H_{xy} \setminus P$ such that $xz \cap P = \emptyset$. Then $\{y\} \cup xz$ is a suitable poidge.

Now consider $x, y \in \mathrm{bd}P$.
If $x = p$, $S_{xy} \setminus \mathrm{int}P \ne \emptyset$. Then $\{y\} \cup xw$ is suitable for $w \in S_{xy} \setminus \mathrm{int}P$ close to x.

If $x \in \mathrm{bd}P \setminus (B \cup \{p\})$, there exists $u \in B$ such that $x \in pu$. If p, x, y are collinear, then $xy \cup \{w'\}$ is suitable for any $w' \in H_{xy} \setminus (P \cup \{x\})$. Otherwise, $0 < \angle xpy < \pi$. At most one of the lines $\overline{pu}$ and $\overline{py}$ is tangent to S_{xy}. Assume $\overline{pu} \cap S_{xy} \setminus \{x\} \ne \emptyset$. Then $\{y\} \cup xw''$ is suitable for a point $w'' \in S_{xy} \setminus \{x\}$ close to x.

Finally, assume $x, y \in B$. Let $H \parallel \overline{B}$ be a hyperplane such that $\overline{B}$ separates p from H. Then $xv \cup \{y\}$ is suitable for any $v \in H_{xy} \cap H$. □

Theorem 6.2. *The complement of a right cone with base B and height at least* diam$B/2$ *is poidge-convex.*

Proof. Let P be a right cone with base B and height $h \ge \mathrm{diam}B/2$. Choose $x, y \in \complement P$.

The poidge-property of x, y is evident, if x or y belongs to $\complement(\mathrm{cl}P)$. Thus, we only need consider $x, y \in \mathrm{bd}P$. Let p be the apex of P.

Take $u, v \in \mathrm{bd}B \setminus B$ such that $x \in pu$ and $y \in pv$. Obviously, $x \ne p$ and $y \ne p$. If $u = v$, take $z \in H_{xy} \setminus P$ distinct from x. Then $\{z\} \cup xy$ is a suitable poidge. If $u \ne v$, $\angle xpy \le \pi/2$. Assume without loss of generality that $\|x-p\| \ge \|y-p\|$. Then $\{y\} \cup x\pi_{\overline{xp}}(y)$ is suitable for $\angle xpy < \pi/2$ and $xw \cup \{y\}$ is suitable for $\angle xpy = \pi/2$, where $w \in S_{xy} \setminus P$ is close to x. □

Theorem 6.3. *The complement of any convex set $M \subset \mathbb{R}^d$ is poidge-convex.*

Proof. Choose $x, y \in \complement M$.

If $x \notin \mathrm{bd}M$, there exists a point $z \in H_{xy} \setminus (M \cup \{x\})$ such that $xz \subset \complement M$. Then $xz \cup \{y\}$ is a suitable poidge in $\complement M$.

If $x \in \mathrm{bd}M \setminus M$, the set $H_{xy} \setminus M$ includes an open half-flat H with $x \in \mathrm{bd}H$. Then $\{y\} \cup xw$ is a suitable poidge for any $w \in H \setminus \{x\}$. □

We can extend Theorem 6.3 to a certain kind of unions of open convex sets.

Theorem 6.4. *The complement of the union of a family of open convex sets with pairwise disjoint closures in $\mathbb{R}^d$ is poidge-convex.*

Proof. Let Γ be the union of a family $\mathfrak{L}$ of open convex sets with pairwise disjoint closures in $\mathbb{R}^d$. Choose $x, y \in \complement\Gamma$.

If $x \notin \mathrm{bd}\Gamma$, choose $z \in H_{xy} \setminus (\Gamma \cup \{x\})$ satisfying $xz \subset \complement\Gamma$. Then $xz \cup \{y\}$ is a poidge in $\complement\Gamma$.

Consider $x \in \mathrm{bd}M$ with $M \in \mathfrak{L}$. Let H_x be a supporting hyperplane of $\mathrm{cl}M$ at x. $\dim(H_x \cap H_{xy}) \geq 1$ when $d \geq 3$. Since $\mathrm{cl}(\Gamma \setminus M) \cap \mathrm{cl}M = \emptyset$, there is a point $z' \in H_x \cap H_{xy}$ with $xz' \subset \complement\Gamma$. Then $xz' \cup \{y\}$ is a suitable poidge in $\complement\Gamma$. $H_x \cap H_{xy} = \{x\}$ when $d = 2$. Let H_1, H_2 be the two half-lines of H_{xy} with endpoint x. At least one of them is disjoint from M, say H_1. Then take $z'' \in H_1$ such that $xz'' \subset \complement\Gamma$; $xz'' \cup \{y\}$ is a poidge in $\complement\Gamma$. □

Theorem 6.5. *The complement of the union of two disjoint bounded open convex sets is poidge-convex.*

Proof. Let M_1, M_2 be two bounded open convex sets, satisfying $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. Choose $x, y \in \complement M$.

If $\mathrm{cl}M_1 \cap \mathrm{cl}M_2 = \emptyset$, the conclusion follows from Theorem 6.4.

Now, assume $\mathrm{cl}M_1 \cap \mathrm{cl}M_2 = L \neq \emptyset$. If $\{x, y\} \setminus \mathrm{cl}M \neq \emptyset$, the points x, y obviously enjoy the poidge-property. So, consider $x, y \in \mathrm{bd}M$.

Case 1. $x, y \in L$.
Since M is bounded, there exists a point $z \in H_{xy} \setminus M$. Then $\{z\} \cup xy$ is suitable.

Case 2. $x \notin L$.
Since $H_{xy} \setminus M$ includes a half-hyperplane H' with x on its relative boundary, $\{y\} \cup xw$ is a suitable poidge, for any $w \in H'$. □

The preceding theorem is not true if the boundedness condition is dropped. Let L be a line in $\mathbb{R}^2$ and H^+, H^- the two open half-planes bounded by L. Then $\complement(H^+ \cup H^-) = L$. But L is not poidge-convex.

For $d \geq 3$, we can generalize Theorem 6.5 to unions of finitely many pairwise disjoint open convex sets.

Theorem 6.6. *For $d \geq 3$, the complement of the union of finitely many pairwise disjoint open convex sets in $\mathbb{R}^d$ is poidge-convex.*

Proof. Let M_i $(i = 1, 2, \cdots, n)$ be open convex sets and $M = M_1 \cup M_2 \cup \cdots \cup M_n$. $M_i \cap M_j = \emptyset$ for any $i \neq j$.

Choose $x, y \in \complement M$.

It is obvious that x, y enjoy the poidge-property in $\complement M$ if $x \notin \mathrm{bd}M$.

Now consider $x \in \mathrm{bd}M$. Then x lies in the boundary of one or more sets $\mathrm{cl}M_1$, $\mathrm{cl}M_2$, $\cdots$, $\mathrm{cl}M_m$, say. Hence x belongs to m supporting hyperplanes H_1, H_2, $\cdots$, H_m, respectively. Let B be a ball of centre x disjoint from $\mathrm{cl}M_{m+1}$, $\cdots$, $\mathrm{cl}M_n$. Then $H_{xy} \cap B \setminus M$ is a sector of a ball of codimension 2 in some $H_i (i \leq m)$, or 1. For any point z in this sector, $\{y\} \cup xz$ is a convenient poidge. □

Theorem 6.6 is not true for $d = 2$. Let $M = \complement(M_1 \cup M_2 \cup M_3)$, see Figure 4. There, x, y have not the poidge-property in M.

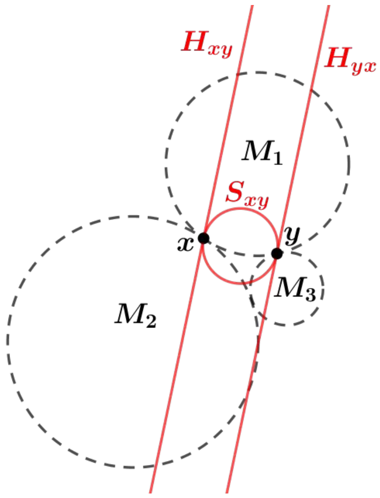


Figure 4: x, y without poidge-property in $\complement(M_1 \cup M_2 \cup M_3)$.

Let M be a set in $\mathbb{R}^d$. M is said to be *locally convex* at a point $x \in M$, if there exists an open ball N with center at x such that $M \cap N$ is convex.

Theorem 6.7. *Let $M \subset \mathbb{R}^d$ be connected. If $\operatorname{int}(\operatorname{cl}M) = M$ and $\operatorname{cl}M$ has at most one point at which it is not locally convex, then $\complement M$ is poidge-convex.*

Proof. Let $q \in \operatorname{cl}M$ be a point at which $\operatorname{cl}M$ is not locally convex. Choose some $x, y \in \complement M$. It is obvious that x, y enjoy the poidge-property if $x \notin \operatorname{bd}M$.

Consider $x, y \in \operatorname{bd}M$. Since $\operatorname{int}(\operatorname{cl}M) = M$, $x, y \in \operatorname{bd}\operatorname{cl}M$.

Assume $x \neq q$. Suppose N is an open ball with center at x such that $\operatorname{cl}M \cap N$ is convex. There exists a line-segment $L \subset (H_{xy} \cap N) \setminus M$ with $x \in L$. Then $xz \cup \{y\}$ is suitable, for any $z \in L \setminus \{x\}$. □

If there exists a point $y \in \operatorname{int}(\operatorname{cl}M) \setminus M$, then $\complement M$ may not be poidge-convex. See Figure 5.

If $\operatorname{cl}M$ has two points at which it is not locally convex, then $\complement M$ may not be poidge-convex. See Figure 6.

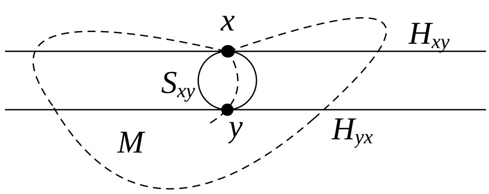


Figure 5: $y \in \operatorname{int}(\operatorname{cl}M) \setminus M$.

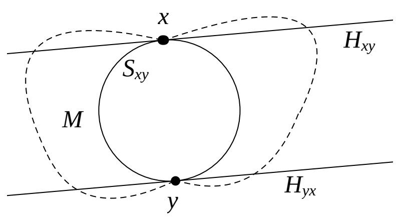


Figure 6: $\operatorname{cl}M$ is not locally convex at x, y.

Theorem 6.8. *The complement of a set included in a countable union of $(d-2)$-dimensional affine subspaces of $\mathbb{R}^d$ is poidge-convex.*

Proof. Let $M \subset \mathbb{R}^d$ be a set as described in the statement. Thus, $M \subset \bigcup_{n=1}^{\infty} A_n$, where each A_n is an affine subspace of $\mathbb{R}^d$ of codimension 2.

Choose $x, y \in \complement M$. Let $\sigma_n = \left\{ \frac{z-x}{\|z-x\|} : z \in A_n \right\} \subset \mathbb{S}_{d-1}$.

Since the $(d-1)$-dimensional measure of each σ_n is 0, we can find a direction $\tau \in \mathbb{S}_{d-1} \setminus \bigcup_{n=1}^{\infty} \sigma_n$, even with the additional condition that its angle with $y - x$ be acute. Thus, the half-line L_τ of direction τ starting at x meets no A_n. Now, $\{y\} \cup x\pi_{L_\tau}(y) \subset \complement M$. □

7. Not simply connected poidge-convex sets

Poidge-convex sets need not be simply connected.

Theorem 7.1. *If $K \subset \mathbb{R}^d$ is an open set and $L \subset K$ an open convex set different from K, then $K \setminus L$ is poidge-convex.*

Proof. Let $M = K \setminus L$. Choose $x, y \in M$.

Suppose $x, y \in \mathrm{bd}L$. Let H_x, H_y be a supporting hyperplane of $\mathrm{cl}L$ at x, y, respectively. If $H_x = H_{xy}$, then $xz \cup \{y\}$ is suitable for any $z \in H_{xy} \cap M \setminus \{x\}$.

Assume $H_x \neq H_{xy}$ and $H_y \neq H_{yx}$. For $d \geq 3$, $xw \cup \{y\}$ is suitable for any $w \in H_{xy} \cap H_x \cap M \setminus \{x\}$. When $d = 2$, H_{xy} includes an open half-line H disjoint from L with $x \in \mathrm{bd}H$. Then $\{y\} \cup xu$ is a suitable poidge for any $u \in H \cap M$.

For $\{x, y\} \not\subset \mathrm{bd}L$, the proof is immediate. □

Theorem 7.2. *If $L \subset \mathrm{int}\mathbb{B}_d$ is a closed set, then $\mathbb{B}_d \setminus L$ is poidge-convex.*

Proof. Let $M = \mathbb{B}_d \setminus L$. Choose $x, y \in M$.

It is obvious that x, y enjoy the poidge-property if one of them lies in $\mathrm{int}M$.

Suppose $x, y \in \mathrm{bd}\mathbb{B}_d$. Then $xz \cup \{y\}$ is suitable for any $z \in H_{xy}$ with $xz \subset M$, if x, y are not diametrally opposite. If they are, $xz \cup \{y\}$ is appropriate, if $z \in \mathrm{bd}\mathbb{B}_d$ is sufficiently close to x. □

Theorem 7.3. *If $L \subset \mathrm{int}\mathbb{B}_d$ is a convex body, then $\mathbb{B}_d \setminus \mathrm{int}L$ is poidge-convex.*

Proof. Let $M = \mathbb{B}_d \setminus \mathrm{int}L$. Choose $x, y \in M$. The only case we need to consider is $x, y \in \mathrm{bd}M$. If $x \in \mathrm{bd}\mathbb{B}_d$, $S_{xy} \cap M$ includes a whole circular arc C containing x. Then $xz \cup \{y\}$ is a suitable poidge in M, if $z \in C$ is sufficiently close to x.

Let $x \in \mathrm{bd}L$. Since L is convex, at least one half-hyperplane $H \subset H_{xy}$ with x on its relative boundary is disjoint from L. Then $xw \cup \{y\}$ is suitable for any $w \in H \cap M \setminus \{x\}$ close to x. □

8. Poidge-convex completion

Let M be a compact set in $\mathbb{R}^d$. If $M^* \supset M$ is poidge-convex, let

$$\kappa(M) = \inf_{M^*} \mathrm{card}(M^* \setminus M).$$

If $\kappa(M) < \aleph_0$, $\kappa(M)$ is called the *poidge-convex completion number* of M. The number

$$\delta(M) = \inf_{M^*} \dim(M^* \setminus M)$$

is called the *poidge-convex completion dimension* of M.

A set M^* realizing $\delta(M)$ is called a *poidge-convex completion* of M.

By Lemma 3.2, $\kappa(M)$ cannot be larger than 1 if $\kappa(M) < \aleph_0$, and if $\kappa(M) > 1$, $M^* \setminus M$ contains a whole line-segment, whence $\delta(M) \geq 1$.

Denote by $F_M(x)$ the set of all points of M farthest from $x \in \mathbb{R}^d$.

Lemma 8.1. *Let $M \subset \mathbb{R}^d$ be a compact convex set, $x \notin M$ and $y \in M$. If $y \notin \{\pi_M(x)\} \cup F_M(x)$, then x, y enjoy the poidge-property in $M \cup \{x\}$.*

Proof. By Lemma 4 in [6], there exists a right triple $\{x, y, z\}$ contained in $M \cup \{x\}$. Since M is convex and $y, z \in M$, $yz \subset M$. Thus, $\{x\} \cup yz$ is a suitable poidge in $M \cup \{x\}$. □

Theorem 8.2. *If $M \subset \mathbb{R}^d$ is a compact convex set which is neither strictly convex, nor poidge-convex, then $\kappa(M) = 1$.*

Proof. Let $\sigma \subset \mathrm{bd}M$ be a line-segment. If M has two diameters, then M is rightly convex, by Theorem 3 in [7].

Let ab be the unique diameter of M. Take $x' \in \mathrm{int}\sigma$. An outer normal at x' to $\mathrm{bd}M$ meets S_{ab} at x, say. Then $M \cup \{x\}$ is *rt*-convex; this follows from the proof of Theorem 3 in [6].

Since M is convex, the poidge-convexity of $M \cup \{x\}$ follows. □

Corollary 8.3. *If S is a fan the opening of which belongs to $]\pi/2, \pi[$, then $\kappa(S) = 1$.*

Theorem 8.4. *If $M \subset \mathbb{R}^d$ is compact and strictly convex, but not poidge-convex, then $\delta(M) = 1$.*

By Theorem 6 in [6], the *rt*-convex completion number of M is 2. No *rt*-convex completion can be poidge-convex. Since adding finitely many points doesn't improve the situation, $\kappa(M)$ is infinite. The claim inside the proof of Theorem 6 in [6] guarantees that there are two points $x \in S_M \setminus M$, $z \in (\mathrm{conv}S_M) \setminus M$, such that $x \in H_{\pi_M(z)z}$ and $z \in H_{\pi_M(x)x}$. One can use that proof to show that $M \cup x\pi_M(z) \cup \{z\}$ is poidge-convex. But we shall give a simpler, direct proof that $\delta(M) = 1$.

Proof of Theorem 8.4**.** Since the *rt*-convex completion number of M is 2, Lemma 3.2 implies that $\delta(M) \geq 1$. We now prove that $\delta(M) \leq 1$.

Let ab be the unique diameter of M. We have $M \subset S_{ab}$. Then choose the point $x \in S_{ab} \setminus \{a, b\}$ and the supporting hyperplane H of M, such that $x \in H$ and $H \cap ab = \emptyset$. Let $z \in H \cap M$. Assume without loss of generality that $\overline{x\pi_M(x)}$ separates z from b.

We claim that $M' = M \cup xz$ is poidge-convex. Let $x', y' \in M'$.

Case 1. $x', y' \in M$.
By Theorem 3 in [7], only diametral pairs may not enjoy the poidge-property in M. In M', a, b have the poidge-property, because $\angle axb = \pi/2$ and $ab \subset M$.

Case 2. $x' \in xz \setminus \{z\}, y' \in M$.
If $y' \in M \setminus (F_M(x') \cup \{\pi_M(x')\})$, the conclusion follows from Lemma 8.1.

If $y' \in F_M(x')$, then $\overline{ab}$ separates x' from y', and $ab \cap S_{x'y'} \neq \emptyset$. If y lies in this intersection, $\{x'\} \cup y'y$ is a good poidge.

For $y' = \pi_M(x')$, since $H_{\pi_M(x')x'}$ separates x' from z, $x'z' \cup \{y'\}$ is suitable, for $\{z'\} = H_{\pi_M(x')x'} \cap x'z$.

Case 3. $x', y' \in xz$.

A suitable poidge is $xz \cup \{w\}$, where $\{w\} = H_{xz} \cap ab$. This intersection is non-empty, because $\angle zxb > \angle axb = \pi/2$.

Therefore, M' is poidge-convex. Hence, $\delta(M) \leq 1$. □

Theorem 8.5. *If $M \subset \mathbb{R}^d$ is strictly convex or a polytope, then, for any $\varepsilon > 0$, there exists a poidge-convex completion M^* of M such that $h(M, M^*) < \varepsilon$.*

Proof. The proof uses Theorem 4 in [6] and its proof. □

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