Journal of Convex Analysis
Volume 31 (2024), No. 3, 779–786

A Property of Strictly Convex Functions which Differ from each other by a Constant on the Boundary of their Domain

Biagio Ricceri
Department of Mathematics and Informatics, University of Catania, Italy
ricceri@dmi.unict.it

Received: May 10, 2023
Accepted: July 29, 2023

We prove, in particular, the following result: Let E be a reflexive real Banach space and let $C \subset E$ be a closed convex set, with non-empty interior, whose boundary is sequentially weakly closed and non-convex. Then, for every function $\varphi : \partial C \to \mathbf{R}$ and for every convex set $S \subseteq E^*$ dense in E^*, there exists $\tilde{\gamma} \in S$ having the following property: for every strictly convex lower semicontinuous function $J : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C and $\lim_{\|x\| \to +\infty} (J(x)/\|x\|) = +\infty$ if C is unbounded, $\tilde{\gamma}$ is an algebraically interior point of $J'(\mathrm{int}(C))$ (with respect to E^*).

Keywords: Strictly convex function, derivative, minimax.

2020 Mathematics Subject Classification: 52A41, 26B25, 46G05, 47J05.

1. Introduction

When E is a topological space, a function $f : E \to \mathbf{R}$ is said to be *inf-compact* if, for every $r \in \mathbf{R}$, the set $f^{-1}(]-\infty, r])$ is compact. When E is a real vector space and $A \subseteq E$, a point $x_0 \in A$ is said to be an *algebraically interior point* of A (with respect to E) if, for every $y \in E$, there exists $\delta > 0$ such that $x_0 + \lambda y \in A$ for all $\lambda \in [0, \delta]$. The algebraic interior of A is the set of all its algebraically interior points.

After recalling these two notions, we can state the result which the title refers to:

Theorem 1.1. *Let E be a reflexive real Banach space and let $C \subset E$ be a closed convex set, with non-empty interior, whose boundary is sequentially weakly closed and non-convex. Then, for every function $\varphi : \partial C \to \mathbf{R}$ and for every convex set $S \subseteq E^*$ dense in E^*, there exists $\tilde{\gamma} \in S$ having the following property: for every strictly convex lower semicontinuous function $J : C \to \mathbf{R}$, Gâteaux differentiable in* $\mathrm{int}(C)$*, such that $J_{|\partial C} - \varphi$ is constant in ∂C and* $\lim_{\|x\| \to +\infty}(J(x)/\|x\|) = +\infty$ *if C is unbounded, $\tilde{\gamma}$ is an algebraically interior point of* $J'(\mathrm{int}(C))$ *(with respect to E^*).*

Now, some comments are in order. The main feature of Theorem 1.1 is the fact that $\tilde{\gamma}$ does not depend on J. But, for a moment, consider simply the following by-product of Theorem 1.1:

If C is as before, then, for every stricly convex lower semicontinuous function $J : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$ *and with* $\lim_{\|x\| \to +\infty}(J(x)/\|x\|) = +\infty$ *if C is unbounded, the algebraic interior of* $J'(\mathrm{int}(C))$ *(with respect to E^*) is non-empty.*

ISSN 0944-6532 / \$ 2.50

As far as we know, such a corollary itself is new when E is infinite-dimensional. To the contrary, if E is finite-dimensional, it can essentially be considered known, at least if J' is continuous. Indeed, in that case, J', being injective (due to the strict convexity of J), turns out to be open, thanks to the invariance of the domain theorem. The assumptions that ∂C is sequentially weakly closed and that $\lim_{\|x\|\to+\infty}(J(x)/\|x\|) = +\infty$ if C is unbounded cannot be removed. Actually, consider the following situation. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, increasing and bounded function. Define the functional $J : L^2([0,1]) \to \mathbf{R}$ by

$$J(u) = \int_0^1 \left(\int_0^{u(x)} f(t)dt \right) dx$$

for all $u \in L^2([0,1])$. Clearly, J is stricly convex and C^1, with

$$J'(u) = f \circ u$$

for all $u \in L^2([0,1])$ (after identifying $(L^2([0,1])^*$ to $L^2([0,1]))$. Notice that, since $J'(L^2([0,1])) \subseteq L^\infty([0,1])$, for each $A \subseteq L^2([0,1])$, the algebraic interior of $J'(A)$ (with respect to $L^2([0,1])$) is empty. Now, let C be any closed ball in $L^2([0,1])$. So, the restriction of J to C is weakly inf-compact. In this case, the conclusion of the corollary fails since ∂C is not sequentially weakly closed. On the other hand, when C is as in the corollary, the conclusion fails since the restriction of J to C is not weakly inf-compact.

Now, we come back to the full statement of Theorem 1.1. We observe that the assumption about the non-convexity of ∂C cannot be removed. In this connection, let U be the set of all C^1 strictly convex functions $J : [0,+\infty[\to \mathbf{R}$, with $J(0) = 0$ and $\lim_{x\to+\infty}(J(x)/x) = +\infty$. Fix $y \in \mathbf{R}$. Let $\mu > \max\{0, y\}$. Then, taking $J(x) = \mu(e^x - 1)$, we have $J \in U$ and $J'(x) > y$ for all $x > 0$. So, taking $C = [0, +\infty[$ and $\varphi(0) = 0$, the conclusion of the Theorem 1.1 does not hold: the boundary of C is convex.

Our proof of Theorem 1.1 is fully based on a minimax result obtained in [2]. More precisely, in the next section, we will establish a general abstract result (just based on [2]) from which we will derive a series of consequences among which there is Theorem 1.1.

2. Results

First, let us recall the following result from [2]:

Theorem 2.1. ([2], Theorem 1.1) *Let X be a topological space, F a topological vector space, $Y \subseteq F$ a non-empty convex set and $g : X \times Y \to \mathbf{R}$ a function satisfying the following conditions:*

(a) *for each $y \in Y$, the function $g(\cdot, y)$ is lower semicontinuous and inf-compact;*

(b) *for each $x \in X$, the function $g(x, \cdot)$ is continuous and quasi-concave.*

(c) $\sup_Y \inf_X g < \inf_X \sup_Y g$.

Then, there exists $y^* \in Y$ such that the function $g(\cdot, y^*)$ has at least two global minima.

In what follows, E is a topological space and Y is a convex set in a topological vector space. We introduce the two main classes of functions we deal with.

Let $X \subseteq C \subseteq E$ and let $\varphi : X \to \mathbf{R}$ be a given function. We denote by $\mathcal{B}(X, C, \varphi)$ the class of all functions $J : C \to \mathbf{R}$ such that $J_{|X} - \varphi$ is constant in X.

Let $C \subseteq E$ and $S \subseteq Y$. Let $f : C \times S \to \mathbf{R}$ be a given function. We denote by $\mathcal{C}(C, S, f)$ the class of all functions $J : C \to \mathbf{R}$ such that, for each $y \in S$, the function $f(\cdot, y) + J(\cdot)$ has at most one global minimum in C.

Our main abstract result is as follows:

Theorem 2.2. *Let $X \subseteq C \subseteq E$, let $S \subseteq Y$ be a convex set dense in Y, let $f : C \times Y \to \mathbf{R}$ and let $\varphi : X \to \mathbf{R}$. Assume that:*

(a) *for each $y \in S$, the function $f(\cdot, y) + \varphi(\cdot)$ is lower semicontinuous and inf-compact in X ;*

(b) *for each $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous in Y ;*

(c) $\sup_Y \inf_X (f + \varphi) < \inf_X \sup_Y (f + \varphi)$.

Then, there exists a point $y^* \in S$ such that

$$\inf_{x \in C} (f(x, y^*) + J(x)) < \inf_{x \in X} (f(x, y^*) + J(x))$$

for every $J \in \mathcal{B}(X, C, \varphi) \cap \mathcal{C}(C, S, f)$.

Proof. Consider the function $g : X \times S \to \mathbf{R}$ defined by

$$g(x, y) = f(x, y) + \varphi(x) \quad \text{for all } (x, y) \in X \times S.$$

For each $x \in X$, by the continuity of $f(x, \cdot)$ and density of S, we have

$$\sup_{y \in S} f(x, y) = \sup_{y \in Y} f(x, y) ,$$

and so

$$\sup_{y \in S} g(x, y) = \sup_{y \in Y} g(x, y) .$$

Therefore, in view of (c), we have

$$\sup_S \inf_X g \leq \sup_Y \inf_X g < \inf_X \sup_Y g = \inf_X \sup_S g .$$

The function g is lower semicontinuous and inf-compact in X, and quasi-concave and continuous in S. So, thanks to Theorem 2.1, there exists $y^* \in S$ such that the function $(f(\cdot, y^*))_{|X} + \varphi(\cdot)$ has at least two global minima in X.

Now, fix some $J \in \mathcal{B}(X, C, \varphi) \cap \mathcal{C}(C, S, f)$. Since $J_{|X} - \varphi$ is constant in X, the functions $(f(\cdot, y^*) + J(\cdot))_{|X}$ and $(f(\cdot, y^*))_{|X} + \varphi(\cdot)$ have the same global minima in X. Arguing by contradiction, assume that

$$\inf_{x \in C} (f(x, y^*) + J(x)) = \inf_{x \in X} (f(x, y^*) + J(x)) . \tag{1}$$

We know that $(f(\cdot, y^*) + J(\cdot))_{|X}$ has at least two global minima. But, in view of (1), they turn out to be global minima of $f(\cdot, y^*) + J(\cdot)$ in C, against the fact that $J \in \mathcal{C}(C, S, f)$. The proof is complete. □

Here is a remarkable corollary of Theorem 2.2.

Theorem 2.3. *Let $X \subseteq C \subseteq E$, let $S \subseteq Y$ be a convex set dense in Y, let $f : C \times Y \to \mathbf{R}$ and let $\varphi : X \to \mathbf{R}$. Assume that:*

(i) *for each $y \in S$, the function $f(\cdot, y) + \varphi(\cdot)$ is lower semicontinuous and inf-compact in X ;*

(ii) *for each $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous in Y ;*

(iii) $\inf_X \sup_Y f = +\infty$ *and there exists a finite set $A \subset X$ such that* $\sup_Y \inf_A f < +\infty$.

Then, there exists a point $y^* \in S$ such that

$$\inf_{x \in C}(f(x, y^*) + J(x)) < \inf_{x \in X}(f(x, y^*) + J(x))$$

for every $J \in \mathcal{B}(X, C, \varphi) \cap \mathcal{C}(C, S, f)$.

Proof. After observing that

$$\sup_Y \inf_X (f + \varphi) \leq \sup_Y \inf_A f + \sup_A \varphi < +\infty = \inf_X \sup_Y f = \inf_X \sup_Y (f + \varphi) ,$$

the conclusion follows directly from Theorem 2.2. □

In the next two results, E is also a real vector space (and the topology on E is still arbitrary).

Theorem 2.4. *Let $X \subseteq E$, let F be a real normed space, let $S \subseteq F^*$ be a convex set weakly-star dense in F^* let $I : \mathrm{conv}(X) \to \mathbf{R}$, let $\psi : \mathrm{conv}(X) \to F$ and let $\varphi : X \to \mathbf{R}$. Assume that that $\psi(X)$ is not convex and that the function $(I + \eta \circ \psi)_{|X} + \varphi$ is lower semicontinuous and inf-compact in X for all $\eta \in S$.*

Then, there exists $\tilde{\eta} \in S$ having the following property:

for every function $J : \mathrm{conv}(X) \to \mathbf{R}$ such that $J_{|X} - \varphi$ is constant in X and $I + J + \eta \circ \psi$ is strictly convex for all $\eta \in S$, one has

$$\inf_{x \in \mathrm{conv}(X)} (I(x) + J(x) + \tilde{\eta}(\psi(x))) < \inf_{x \in X}(I(x) + J(x) + \tilde{\eta}(\psi(x))) .$$

Proof. Fix $y_0 \in \mathrm{conv}\,(\psi(X)) \setminus \psi(X)$. We apply Theorem 2.3 with $C = \mathrm{conv}(X)$ and $Y = F^*$, considering F^* equipped with the weak-star topology and taking

$$f(x, \eta) = I(x) + \eta(\psi(x) - y_0)$$

for all $(x, \eta) \in C \times Y$.

Clearly, f satisfies conditions (i) and (ii).

Moreover, if $y_0 = \sum_{i=1}^n \lambda_i \psi(x_i)$, where $x_i \in X$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, by Proposition 2.1 of [3], we know that

$$\sup_{\eta \in Y} \inf_{1 \leq i \leq n} f(x_i, \eta) \leq \max_{1 \leq i \leq n} I(x_i) < +\infty .$$

On the other hand, for each $x \in X$, since $\psi(x) \neq y_0$, we have

$$\sup_{\eta \in Y} \eta(\psi(x) - y_0) = +\infty\,,$$

and so condition (iii) is satisfied too. Now, Theorem 2.3 ensures the existence of $\tilde{\eta} \in S$ such that, for every $J \in \mathcal{B}(X, C, \varphi) \cap \mathcal{C}(C, S, f)$, one has

$$\inf_{x \in \mathrm{conv}(X)} (f(x, \tilde{\eta}) + J(x)) < \inf_{x \in X} (f(x, \tilde{\eta}) + J(x))$$

which means

$$\inf_{x \in \mathrm{conv}(X)} (I(x) + J(x) + \tilde{\eta}(\psi(x))) < \inf_{x \in X} (I(x) + J(x) + \tilde{\eta}(\psi(x)))\,.$$

To finish the proof, it is enough to remark that if $J : \mathrm{conv}(X) \to \mathbf{R}$ is such that $I + J + \eta \circ \psi$ is strictly convex for all $\eta \in S$, then $J \in \mathcal{C}(C, S, f)$. $\square$

The following result is a particularly simple consequence of Theorem 2.4.

Theorem 2.5. *Let $X \subset E$ be a compact set, let F be a real normed space, and let $\psi : \mathrm{conv}(X) \to F$ be an affine operator, continuous with respect to the weak topology on F, such that $\psi(X)$ is not convex, and let $\varphi : X \to \mathbf{R}$ be a lower semicontinous function. Then, for every convex set $S \subseteq F^*$ weakly-star dense in F^*, there exists $\tilde{\eta} \in S$ having the following property: for every strictly convex function $J : \mathrm{conv}(X) \to \mathbf{R}$ such that $J_{|X} - \varphi$ is constant in X, one has*

$$\inf_{x \in \mathrm{conv}(X)} (J(x) + \tilde{\eta}(\psi(x))) < \inf_{x \in X} (J(x) + \tilde{\eta}(\psi(x)))\,.$$

Proof. For each $\eta \in F^*$, the function $\eta \circ \psi$ is continuous since η is weakly continuous. So, $\eta \circ \psi + \varphi$ is lower semicontinuous and inf-compact, since X is compact. Hence, the assumptions of Theorem 2.4 are satisfied, with $I = 0$. Now, our conclusion follows from that of Theorem 2.4 taken into account that if $\eta \in F^*$ and $J : \mathrm{conv}(X) \to \mathbf{R}$ is a strictly convex function, then so is $\eta \circ \psi + J$ since ψ is affine. $\square$

The next consequence of Theorem 2.4 can be considered as the central result: actually, Theorem 1.1 is a corollary of it.

Theorem 2.6. *Let E be a reflexive real Banach space, let $C \subset E$ be a closed convex set, with non-empty interior, such that ∂C is sequentially weakly closed and non-convex, let $S \subseteq E^*$ be a convex set dense in E^*, let $I : C \to \mathbf{R}$ be Gâteaux differentiable in $\mathrm{int}(C)$, and let $\varphi : \partial C \to \mathbf{R}$. Then, there exists $\tilde{\gamma} \in S$ having the following property: for every function $J : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C and $I + J$ is lower semicontinous and strictly convex, with $\lim_{\|x\| \to +\infty}((I(x) + J(x))/\|x\|) = +\infty$ if C is unbounded, and for every sequentially weakly lower semicontinuous function $G : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, there exists $\epsilon > 0$ such that, for each $\lambda \in [0, \epsilon[$, the equation*

$$I'(x) + J'(x) + \lambda G'(x) = \tilde{\gamma}$$

has at least one solution in $\mathrm{int}(C)$.

Proof. We apply Theorem 2.4 considering E equipped with the weak topology (but the interior of C is referred to the strong topology) and taking $X = \partial C$, $F = E$ and $\psi(x) = x$ for all $x \in C$ (notice that $\mathrm{conv}(\partial X) \subseteq C$). Of course, it is implicitly understood that there are functions $J_0 : C \to \mathbf{R}$ such that $J_{0_{|\partial C}} - \varphi$ is constant in ∂C, and $I + J_0$ is lower semicontinuous and strictly convex, with $\lim_{\|x\|\to+\infty}(I(x)+J_0(x))/\|x\|) = +\infty$ if C is unbounded. So, if J_0 is such a function, it follows that, for each $\eta \in E^*$, the function $I + J_0 + \eta$ is weakly inf-compact in C (in particular, notice that $\lim_{\|x\|\to+\infty}(I(x)+J_0(x)+\eta(x)) = +\infty$ if C is unbounded) and so $(I + \eta)_{|\partial C} + \varphi$ is weakly inf-compact in ∂C since ∂C is sequentially weakly closed (use also the Eberlein-Smulyan theorem).

Therefore, the assumptions of Theorem 2.4 are satisfied. Consequently, since $-S$ is convex and dense in E^*, there exists $\tilde{\eta} \in -S$, having the following property: for every function $J : C \to \mathbf{R}$ such that $J_{|\partial C} - \varphi$ is constant in ∂C and $I + J$ is lower semicontinuous and strictly convex, one has

$$\inf_{x\in C}(I(x) + J(x) + \tilde{\eta}(x)) < \inf_{x\in\partial C}(I(x) + J(x) + \tilde{\eta}(x)) .$$

In addition, let J be Gâteaux differentiable in $\mathrm{int}(C)$ and let $G : C \to \mathbf{R}$ be sequentially weakly lower semicontinuous in C and Gâteaux differentiable in $\mathrm{int}(C)$ with $\lim_{\|x\|\to+\infty}((I(x) + J(x))/\|x\|) = +\infty$ if C is unbounded. Fix σ so that

$$\inf_{x\in C}(I(x) + J(x) + \tilde{\eta}(x)) < \sigma < \inf_{x\in\partial C}(I(x) + J(x) + \tilde{\eta}(x)) . \tag{2}$$

Since the set $\{x \in C : I(x) + J(x) + \tilde{\eta}(x) \leq \sigma\}$ is sequentially weakly compact, in view of Theorem 2.1 of [1], there exists $\epsilon > 0$ such that, for every $\lambda \in [0, \epsilon[$, the restriction of the function $I + J + \tilde{\eta} + \lambda G$ to the set

$$\{x \in C : I(x) + J(x) + \tilde{\eta}(x) < \sigma\}$$

has a global minimum, say $\tilde{x}$. But, due to (2), we have

$$\{x \in C : I(x) + J(x) + \tilde{\eta}(x) < \sigma\} \subseteq \mathrm{int}(C)$$

and, since $I + J + \tilde{\varphi}$ turns out to be continuous in $\mathrm{int}(C)$, this implies that

$$I'(\tilde{x}) + J'(\tilde{x}) + \lambda G(\tilde{x}) + \tilde{\eta} = 0 ,$$

as claimed, with $\tilde{\gamma} = -\tilde{\eta}$. □

Now, we can give the

Proof of Theorem 1.1. Let $S \subseteq E^*$ be a convex set dense in the space E^* and let $\varphi : \partial X \to \mathbf{R}$. Apply Theorem 2.6 with $I = 0$. Let $\tilde{\gamma} \in S$ be as in the conclusion of Theorem 2.6. Fix any lower semicontinuous and strictly convex function $J : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C, with $\lim_{\|x\|_E\to+\infty}(J(x)/\|x\|) = +\infty$ if C is unbounded. Now, fix any $G \in E^*$. Then, there exists $\epsilon > 0$ such that the equation

$$J'(x) - \lambda G = \tilde{\gamma}$$

has a solution in $\mathrm{int}(C)$ for all $\lambda \in [0, \epsilon[$, and this means exactly that the point $\tilde{\gamma}$ is an algebraically interior point of $J'(\mathrm{int}(C))$. □

In a finite-dimensional setting, another consequence of Theorem 2.6 is as follows:

Theorem 2.7. *Let E be a finite-dimensional real Banach space and let $C \subset E$ be a compact convex set with non-empty interior. Then, for every function $\varphi : \partial C \to \mathbf{R}$, there exists $\tilde{\gamma} \in E^*$ having the following property: for every lower semicontinuous stricly convex function $J : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C, and for every lower semicontinuous, bounded below and Gâteaux differentiable function $H : \mathrm{int}(C) \to \mathbf{R}$, there exists $\epsilon > 0$ such that, for each $\lambda \in [0, \epsilon[$, the equation*

$$J'(x) + \lambda H'(x) = \tilde{\gamma}$$

has at least one solution in $\mathrm{int}(C)$.

Proof. Of course, since C is compact, ∂C is not convex. Fix $\varphi : \partial C \to \mathbf{R}$ and apply Theorem 2.6, with $I = 0$. Let $\tilde{\gamma} \in E^*$ be as in the conclusion of Theorem 2.6. Let $J : C \to \mathbf{R}$ be a lower semicontinuous strictly convex function, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C, and let $H : \mathrm{int}(C) \to \mathbf{R}$ be lower semicontinuous, bounded below and Gâteaux differentiable. Now, consider the function $G : C \to \mathbf{R}$ defined by

$$G(x) = \begin{cases} H(x) & \text{if } x \in \mathrm{int}(C) \\ a & \text{if } x \in \partial C \,, \end{cases}$$

where $a = \inf_C H$. Clearly, G is lower semicontinuous in C. Consequently, there exists $\epsilon > 0$ such that, for each $\lambda \in [0, \epsilon[$, the equation

$$J'(x) + \lambda G'(x) = \tilde{\gamma}$$

has at least one solution in $\mathrm{int}(C)$. The conclusion holds since $H = G$ in $\mathrm{int}(C)$. □

Finally, we highlight the following consequence of Theorem 2.7:

Theorem 2.8. *Let E be a finite-dimensional real Hilbert space and let $C \subset E$ be a closed ball, centered at 0.*

Then, for every function $\varphi : \partial C \to \mathbf{R}$, there exists $\tilde{\gamma} \in E^*$, having the following property: for every $P \in C^1(C)$ such that P' is Lipschitzian with Lipschitz constant $L \geq 0$, for every $\mu > L$, for every lower semicontinuous convex function $Q : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, such that $(P + Q)_{|\partial C} - \varphi$ is constant in ∂C, and for every lower semicontinuous, bounded below and Gâteaux differentiable function $H : \mathrm{int}(C) \to \mathbf{R}$, there exists $\epsilon > 0$ such that, for each $\lambda \in [0, \epsilon[$, the equation

$$\mu x + P'(x) + Q'(x) + \lambda H'(x) = \tilde{\gamma}$$

has at least one solution in $\mathrm{int}(C)$.

Proof. Fix $\varphi : \partial C \to \mathbf{R}$. Let $\tilde{\gamma}$ be as in the conclusion of Theorem 2.7. Observe that, since $\mu > L$, the function $\frac{\mu}{2}\| \cdot \|^2 + P(\cdot)$ turns out to be strictly convex since E is a Hilbert space. Consequently, the function

$$J(\cdot) := \frac{\mu}{2}\| \cdot \|^2 + P(\cdot) + Q(\cdot)$$

is lower semicontinuous, strictly convex, Gâteaux differentiable in $\mathrm{int}(C)$ and $J_{|\partial C}-\varphi$ is constant in ∂C. Hence, the conclusion of Theorem 2.7 applies with such a function J and we are done. $\square$

Acknowledgements. The author has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, PIACERI 2020-2022, Linea di intervento 2, Progetto "MAFANE". The author would like to thank the referee for his/her comments and remarks.

References

[1] B. Ricceri: *Nonlinear eigenvalue problems*, in: *Handbook of Nonconvex Analysis and Applications*, D. Y. Gao et al. (eds.), International Press, Somerville (2010) 543–595.

[2] B. Ricceri: *On a minimax theorem: an improvement, a new proof and an overview of its applications*, Minimax Theory Appl. 2 (2017) 99–152.

[3] B. Ricceri: *Multiplicity theorems involving functions with non-convex range*, Stud. Univ. Babeş-Bolyai Math. 68 (2023) 125–137.