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Some Recent Results on Premonotone Operators

Mohammad Hossein Alizadeh
Dept. of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran
m.alizadeh@iasbs.ac.ir

Alfredo N. Iusem
Escola de Matemática Aplicada, Fundação Getúlio Vargas, Rio de Janeiro, Brazil
alfredo.iusem@fgv.br

Wilfredo Sosa Sandoval
Graduate Program of Economics, Univ. Católica de Brasília, Brazil
sosa@ucb.br

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The notion of premonotone operator refers to a class of operators more general than monotone ones, which still enjoy a surjectivity property akin to Minty's Theorem for monotone operators. In this paper we prove that operators which are monotone outside a bounded set, are indeed premonotone. As a consequence, we show that premonotocity is preserved under rather arbitrary alterations of the operator values in a bounded subset of its domain. We also develop the basic elements of a polarity theory related to premonotonicity and we deepen the study of maximal premonotone operators. Finally, we prove the one-dimensional version of a previously stated conjecture, namely that any maximal premonotone operator contains a maximal monotone one.

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1. Introduction

We recall that a point-to-set operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be *monotone* whenever $\langle u-v, x-y\rangle \geq 0$ for all $x, y \in \mathbb{R}^n$, all $u \in T(x)$ and all $v \in T(y)$. Monotone operators have proved to be an essential concept in several pure and applied areas. A monotone operator is said to be *maximal* when its graph is not properly contained in the graph of another monotone operator. Maximal monotone operators enjoy several important properties. One of them, known as Minty's Theorem, states that the sum of a maximal monotone operator and the identity operator is onto, and its inverse is point-to-point (see [14]).

Several extensions of the class of monotone operators have been considered in the literature, like *submonotone* operators (see [16]), *hypo-monotone* operators (see [11], [15]), etc.

One of these extensions, the one of interest in this paper, is the notion of *premonotone* operators, which was introduced in [10], where it was shown that, under adequate assumptions, they enjoy a surjectivity property akin to Minty's Theorem for maximal monotone operators. They are defined as follows.

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Given a real-valued function $\sigma : \mathbb{R}^n \to \mathbb{R}_+$, a point-to-set operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be *σ-premonotone*, or just *premonotone*, when

$$\langle u - v, x - y\rangle \geq -\sigma(y)\, \|x - y\| \tag{1}$$

for all $(x, u), (y, v) \in T$.

It has been proved in [10] that, given a premonotone operator T, there exists a minimal function, namely σ_T, such that T is σ_T-monotone. This function is defined as

$$\sigma_T\,(y) = \max\left\{0,\ \sup_{v\in T(y)} \left\{\sup_{x\in\mathbb{R}^n\setminus\{y\}, u\in T(x)} \left\{\frac{\langle v - u, x - y\rangle}{\|x - y\|}\right\}\right\}\right\}. \tag{2}$$

An operator T is said to be *maximal premonotone* if whenever $T \subset T'$ for a premonotone operator T' with $\sigma_T = \sigma_{T'}$, it holds that $T = T'$.

In this paper we collect some new results on premonotone operators. In Section 2, we show that operators which are monotone outside a bounded set, are premonotone, with a precisely defined σ. As a consequence, we show that premonotonicity is preserved under rather arbitrary alterations of the operator values in a bounded subset of its domain. In Section 3 we develop the basic elements of a polarity theory related to premonotonicity. In Section 4 we deepen the study of maximal premonotone operators, started in [10]. In Section 5 we prove the one-dimensional version of a conjecture stated in [12], namely that any maximal premonotone operator contains a maximal monotone one.

2. Some examples of premonotone operators

In this section we prove that, under a weak assumption, operators that are monotone outside a bounded set (see a precise definition below) are σ-premonotone, with a precisely defined σ, and present some corollaries of this result.

Definition 2.1. An operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is *monotone outside a bounded set* $B \subset \mathbb{R}^n$, if $\langle u - v, x - y\rangle \geq 0$ for all $x, y \notin B$, all $u \in T(x)$ and all $v \in T(y)$.

We recall that an operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is *bounded on bounded sets* if $\cup_{x\in B} T(x)$ is bounded for all bounded set $B \subset \mathbb{R}^n$

For $M \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote $\operatorname{dist}(x, M) := \inf_{z\in M} \|z - x\|$. For a bounded set $B \subset \mathbb{R}^n$ and any positive $\varepsilon \in \mathbb{R}$, we define $B^\varepsilon \subset \mathbb{R}^n$ as

$$B^\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, B) \leq \varepsilon\}. \tag{3}$$

Since $\operatorname{dist}(\cdot, B)$ is continuous, it follows easily that if $y \in B, x \notin B^\varepsilon$, then there exists z in the segment between x and y such that z belongs to $B^\varepsilon \setminus B$. Note that boundedness of B implies boundedness of B^ε.

Theorem 2.2. *Consider an operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, with convex domain, bounded on bounded sets and monotone outside some bounded set $B \subset \mathbb{R}^n$. Then T is σ-premonotone, with*

$$\sigma(y) = \alpha + \sup_{v\in T(y)} \|v\|\,, \tag{4}$$

where, for an arbitrary $\varepsilon > 0$,

$$\alpha = \sup_{u\in T(x),\, x\in B^\varepsilon} \|u\|\,. \tag{5}$$

Proof. Note first that both suprema in the statement of the theorem are achieved because T is bounded on bounded sets.

Take $x, y \in \mathbb{R}^n, u \in T(x), v \in T(y)$. We will estimate $\langle u - v, x - y\rangle$ for all $(x, u), (y, v) \in T$, and we consider three cases.

(i) $x, y \notin B$. In this case

$$\langle u - v, x - y\rangle \geq 0 \tag{6}$$

because T is monotone outside B.

(ii) $x \in B$. In this case

$$\begin{aligned}\langle u - v, x - y\rangle &\geq -\|u - v\| \; \|x - y\| \geq -\left(\|u\| + \|v\|\right)\|x - y\| \\ &\geq -\left(\sup_{u \in T(x), x \in B^{\varepsilon}} \|u\| + \sup_{v \in T(y)} \|v\|\right)\|x - y\| \\ &= -\left(\alpha + \sup_{v \in T(y)} \|v\|\right)\|x - y\| = -\sigma(y)\|x - y\|,\end{aligned} \tag{7}$$

using (5) and (4) in the equalities and the fact that $x \in B \subset B^{\varepsilon}$ in the third inequality.

(iii) $y \in B$. We consider two options for x. If $x \in B^{\varepsilon}$, then (7) holds. Otherwise, $x \notin B^{\varepsilon}$. Since $y \in B$, as noted above there exists a point z in the segment between x and y such that $z \in B^{\varepsilon} \setminus B$. Let $z = (1-\eta)x + \eta y$, with $0 < \eta \leq 1$. By convexity of the domain of T, z belongs to it, so that $T(z)$ is nonempty. Take $s \in T(z)$. Note that both x and z do not belong to B, because $x \notin B^{\varepsilon}$ and $z \notin B$. Hence, since T is monotone outside B, we have

$$\langle u - s, x - z\rangle \geq 0 \tag{8}$$

for all $u \in T(x)$. Then

$$\begin{aligned}\langle u - v, x - y\rangle &= \langle u - s, x - y\rangle + \langle s - v, x - y\rangle \\ &= \frac{1}{\eta}\langle u - s, x - z\rangle + \langle s - v, x - y\rangle \geq \langle s - v, x - y\rangle\end{aligned} \tag{9}$$

using (8) in inequality. From (9)

$$\begin{aligned}\langle u - v, x - y\rangle &\geq \langle s - v, x - y\rangle \geq -\left(\|s\| + \|v\|\right)\|x - y\| \\ &\geq -\left(\sup_{w \in T(q), q \in B^{\varepsilon}} \|w\| + \sup_{v \in T(y)} \|v\|\right)\|x - y\| \\ &= -\left(\alpha + \sup_{v \in T(y)} \|v\|\right)\|x - y\| = -\sigma(y)\|x - y\|,\end{aligned} \tag{10}$$

where the third inequality holds because $s \in T(z)$ and $z \in B^{\varepsilon}$, and the last equality follows from (4) and (5).

We conclude from (6), (7) and (10) that

$$\langle u - v, x - y \rangle \geq -\sigma(y) \|x - y\| .$$

Hence, it follows from (1) that T is σ-premonotone with σ as in (4). $\square$

We continue with three corollaries of Theorem 2.2 dealing with operators in $\mathbb{R}^n$. The first corollary exhibits a remarkable robustness property of premonotonicity: it is preserved under rather arbitrary alterations of the operator values on a bounded subset of its domain. This property is not enjoyed by monotonicity, or any of its generalizations mentioned in Section 1.

Corollary 2.3. *Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be bounded on bounded sets and σ-premonotone. Consider a bounded set $B \subset \mathbb{R}^n$ and an operator $\bar{T} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ with convex domain which is bounded on bounded sets and such that $\bar{T}(x) = T(x)$ for all $x \notin B$. Then $\bar{T}$ is $\bar{\sigma}$-premonotone, with*

$$\bar{\sigma}(y) = \max\{\sigma(y), \widehat{\sigma}(y)\}, \tag{11}$$

where

$$\widehat{\sigma}(y) = \widehat{\alpha} + \sup_{v \in \bar{T}(y)} \|v\| , \tag{12}$$

and

$$\widehat{\alpha} = \sup_{u \in \bar{T}(x), x \in B^{\varepsilon}} \|u\| , \tag{13}$$

with B^{ε} as defined in (3) *for any $\varepsilon > 0$.*

Proof. As in the proof of Theorem 2.2 above, we estimate $\langle u - v, x - y \rangle$ for all $(x, u), (y, v) \in \bar{T}$, and we consider two cases. If either x or y belongs to B, we proceed as in items (ii) and (iii) in the proof of Theorem 2.2, obtaining

$$\langle u - v, x - y \rangle \geq -\widehat{\sigma}(y) \|x - y\| , \tag{14}$$

taking into account (12) and (13). For the remaining case, i.e., when both x and y are outside B, we use the fact that outside B the operator $\bar{T}$ coincides with T, which is σ-premonotone, so that

$$\langle u - v, x - y \rangle \geq -\sigma(y) \|x - y\| . \tag{15}$$

From (14), (15) we get, for all $x, y \in \mathbb{R}^n$, all $u \in T(x)$ and all $v \in T(y)$,

$$\langle u - v, x - y \rangle \geq -\max\{\widehat{\sigma}(y), \sigma(y)\} \|x - y\| = -\bar{\sigma}(y) \|x - y\| , \tag{16}$$

using (11) in the last equality. In view of (16), $\bar{T}$ is $\bar{\sigma}$-premonotone. $\square$

Corollary 2.4. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is point to point, continuous and monotone outside some bounded set $B \subset \mathbb{R}^n$, then T is σ-premonotone, with $\sigma(y) = \alpha + \|T(y)\|$, where $\alpha = \sup_{x \in B^{\varepsilon}} \|T(x)\|$ with B^{ε} as defined in* (3) *for any $\varepsilon > 0$.*

Proof. Follows from Theorem 2.2, since point-to-point and continuous operators are clearly bounded on bounded sets. $\square$

Definition 2.5. A continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, is said to be convex outside a compact set $B \subset \mathbb{R}^n$ if there exists a continuously differentiable and convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \notin B$.

Corollary 2.6. *If a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, is convex outside a compact set $B \subset \mathbb{R}^n$ then its gradient ∇f is σ-premonotone and satisfies $\sigma(y) = \alpha + \|\nabla f(y)\|$, where $\alpha = \sup_{x \in B^\varepsilon} \|\nabla f(x)\|$ and B^ε is as defined in* (3) *for any $\varepsilon > 0$.*

Proof. Follows from Corollary 2.4, because ∇f is point-to-point and continuous, and also monotone outside B, where it coincides with ∇g, which is monotone on the whole space by convexity of g. $\square$

We close this section with two one-dimensional corollaries of Theorem 2.2.

Corollary 2.7. *Consider a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, and an interval $[a, b] \subset \mathbb{R}$ such that φ is nondecreasing in $(-\infty, a]$, nondecreasing in $[b, \infty)$ and $\varphi(a) \le \varphi(b)$. Then φ is σ-premonotone with $\sigma(t) = \sup_{s \in [a-\varepsilon, b+\varepsilon]} |\varphi(s)| + |\varphi(t)|$, where ε is any positive real number.*

Proof. In view of Corollary 2.4 it suffices to check that φ is monotone outside $[a, b]$, which in the one-dimensional case means that if $s, t \notin [a, b]$ and $s \le t$ then

$$\varphi(s) \le \varphi(t). \tag{17}$$

Now, if both $s, t \le a$ then (17) holds because φ is nondecreasing in $(-\infty, a]$; if both $s, t \in [b, \infty)$ then (17) holds because φ is nondecreasing in $[b, \infty)$ and if $s \le a, t \ge b$, then we have $\varphi(s) \le \varphi(a) \le \varphi(b) \le \varphi(t)$, using the three assumptions on φ. We conclude that φ is monotone outside $[a, b]$ and the result follows from Corollary 2.4. $\square$

Corollary 2.8. *If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree, then p is σ-premonotone, with σ as in Corollary* 2.7.

Proof. We will look at p and its derivative p'. Note that p certainly has a real root. If p' has no real roots, then $p'(t) > 0$ for all t, so that p is increasing, i.e., monotone and *a fortiori* premonotone with any nonnegative σ. So, we assume that p' has real roots. Let $\underline{\lambda}, \bar{\lambda}$ be the smallest and the largest real root of p, respectively, and $\underline{\mu}, \bar{\mu}$ the smallest and the largest real root of p'. Let $a = \min\{\underline{\lambda}, \underline{\mu}\}, b = \max\{\bar{\lambda}, \bar{\mu}\}$. We claim that we are within the hypotheses of Corollary 2.7. Observe that p is nondecreasing in $(-\infty, a]$ because p', being of even degree, is nonnegative in this interval, since $a \le \underline{\mu}$. Noting that $\bar{\mu} \le b$, we conclude with the same argument that p is nondecreasing in $[b, \infty)$. Also, $p(a) \le p(\underline{\lambda}) = 0 = p(\bar{\lambda}) \le p(b)$. We have checked that all the assumptions of Corollary 2.7 hold, and hence the result follows from this corollary. $\square$

3. Polarity and premonotonicity

In this section we generalize some results presented in [8], [9] and [13] to premonotone polar sets. The polarity of premonotone operators and premonotonicity have some reciprocal relationships. In some recent papers, polarity has been studied for generalized monotone operators (see [2], [7], [6] and [5]).

Take $D \subset \mathbb{R}^n \times \mathbb{R}^n$. Define

$$D(x) := \{u : (x,u) \in D\}, \text{ and } \operatorname{dom} D := \{x \in \mathbb{R}^n : D(x) \neq \emptyset\}.$$

Following (2), we define σ_D as follows

$$\sigma_D(y) = \max\left\{0, \sup_{v \in D(y)} \left\{ \sup_{x \in \mathbb{R}^n \setminus \{y\}, u \in D(x)} \left\{ \frac{\langle v-u, x-y\rangle}{\|x-y\|} \right\}\right\}\right\}. \quad (18)$$

By convention, when $D(y)$ is empty, then the respective sup in (18) is $-\infty$. It implies that $\sigma_D(y) = 0\ \forall y \notin \operatorname{dom} D$. So, for each $y \in \mathbb{R}^n$ $\sigma_D(y) \in [0,+\infty]$. Moreover, $\langle u-v, y-x\rangle \leq \min\{\sigma_D(x), \sigma_D(y)\}\|x-y\|\ \forall (x,u) \in D\ \forall (y,v) \in D$. When σ_D is finite (i.e. $\sigma_D(x) < +\infty\ \forall x \in \mathbb{R}^n$), the set D is premonotone.

Definition 3.1. Let $D \subset \mathbb{R}^n \times \mathbb{R}^n$. Two pairs $(x,u),(y,v) \in \mathbb{R}^n \times \mathbb{R}^n$ are σ_D-*monotonically related* if

$$\langle u-v, y-x\rangle \leq \min\{\sigma_D(x), \sigma_D(y)\}\|x-y\|.$$

We consider the following reflexive and symmetric binary relation $\mu(\sigma, D)$ on D by

$$(x,u)^{\mu(\sigma,D)}(y,v) \text{ iff } (x,u) \text{ and } (y,v) \text{ are } \sigma_D\text{-monotonically related.}$$

Definition 3.2. Let D be a subset of $\mathbb{R}^n \times \mathbb{R}^n$. The σ_D-*monotone polar* of D is defined by

$$D^{\mu(\sigma,D)} := \left\{(x,u) \in \mathbb{R}^n \times \mathbb{R}^n : (x,u)^{\mu(\sigma,D)}(y,v), \forall (y,v) \in D\right\}.$$

or equivalently,

$$D^{\mu(\sigma,D)} = \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^n : \langle v-u, x-y\rangle \leq$$
$$\min\{\sigma_D(x), \sigma_D(y)\}\, \|x-y\|\, \forall (y,v) \in D\}.$$

In other words, $D^{\mu(\sigma,D)}$ is the set of all points in $\mathbb{R}^n \times \mathbb{R}^n$ which are σ_D-monotonically related with all $(y,v) \in D$.

Note that if $E, F \subset \mathbb{R}^n \times \mathbb{R}^n$ and $E \subset F$, then, by (18), we have $\sigma_E \leq \sigma_F$. Hence, $E^{\mu(\sigma,E)} \subset E^{\mu(\sigma,F)}$ and $F^{\mu(\sigma,F)} \subset E^{\mu(\sigma,F)}$.

We note that if $(x,u) \in D^{\mu(\sigma,D)}$ and $(y,v) \in D$, then $(x,tu)^{\mu(\sigma,D)}(y,tv)$ for all $t \in [0,t]$. Moreover if σ_D is positively homogeneous, then for each $t \geq 0$ we have

$$(x,u)^{\mu(\sigma,D)}(y,v) \iff t(x,u)^{\mu(\sigma,D)}t(y,v).$$

Example 3.3. Let $D = \{(0,0)\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Then $D^{\mu(\sigma,D)} = \mathbb{R}^n \times \mathbb{R}^n$. Indeed, D is monotone and so $\sigma_D(x) = 0$. Take $x \neq 0$ and

$$(x,u) \in D^{\mu(\sigma,D)} \iff \langle u-0, x-0\rangle \geq 0 \iff \langle u, \frac{x}{\|x\|}\rangle \geq 0$$
$$\iff \|u\| = \sup\langle u, \frac{x}{\|x\|}\rangle \geq 0.$$

In a similar way one can see that if $D = \{(0,v) : v \in \mathbb{R}^n\}$, then $D^{\mu(\sigma,D)} = \mathbb{R}^n \times \mathbb{R}^n$.

As for the monotone polar, [4], some polarity properties of σ-monotone sets hold. We present them next.

Proposition 3.4. *Let E, F and D_i for each $i \in I$ be subsets of $\mathbb{R}^n \times \mathbb{R}^n$. Then*

(i) $(\cup_{i\in I} D_i)^{\mu(\sigma,\cup_{i\in I} D_i)} = \cap_{i\in I} D_i^{\mu(\sigma,\cup_{i\in I} D_i)}$, $(\cup_{i\in I} D_i)^{\mu(\sigma,\cup_{i\in I} D_i)} \subset \cap_{i\in I} D_i^{\mu(\sigma,D_i)}$;

(ii) $D \subset \left[D^{\mu(\sigma,D)}\right]^{\mu(\sigma,D)}$;

(iii) $\left(\left[D^{\mu(\sigma,D)}\right]^{\mu(\sigma,D)}\right)^{\mu(\sigma,D)} = D^{\mu(\sigma,D)}$;

(iv) $\left(\left[D^{\mu(\sigma,D)}\right]^{\mu(\sigma,D)}\right)^{\mu(\sigma,D)} = D^{\mu(\sigma,D)}$;

(v) $\emptyset^{\mu(\sigma,E)} = \mathbb{R}^n \times \mathbb{R}^n$.

Proof. The proof is straightforward. □

In the remainder of this section, we assume that $\operatorname{int}(\operatorname{dom} D)$ is nonempty and convex.

As in the case of monotone sets, an easy consequence of Zorn's lemma establishes that if $D \subset \mathbb{R}^n \times \mathbb{R}^n$ is such that $\operatorname{int}(\operatorname{dom} D)$ is convex and it is σ_D-monotone, then it has a maximal σ_D-monotone extension (for the case of premonotone operators see [3, 1, 10]). Next, we present some properties of the σ_D-monotone polar which generalized some results from monotone operators (see [9], [12] and [13]).

Proposition 3.5. *Let D and E are two subsets of $\mathbb{R}^n \times \mathbb{R}^n$.*

(i) *D is σ_D-monotone iff $D \subset D^{\mu(\sigma,D)}$.*

(ii) *If $D \subset E$ and E is σ_D-monotone, then D is σ_D-monotone and $E^{\mu(\sigma,D)} \subset D^{\mu(\sigma,D)}$.*

(iii) *Suppose that D is σ_D-monotone and* $\operatorname{int}(\operatorname{dom} D)$ *is convex. Then $D^{\mu(\sigma,D)}$ is the union of all maximal σ_D-monotone extensions of D, that is*

$$D^{\mu(\sigma,D)} = \cup \left\{E \subset \mathbb{R}^n \times \mathbb{R}^n\text{: is maximal } \sigma_D\text{-monotone, } D \subset E\right\}. \tag{19}$$

(iv) *Let* $\operatorname{int}(\operatorname{dom} D)$ *be convex. Then D is maximal σ_D-monotone iff $D = D^{\mu(\sigma,D)}$.*

(v) *Suppose that D is σ_D-monotone and* $\operatorname{int}(\operatorname{dom} D)$ *is convex. Then*

$$\left[D^{\mu(\sigma,D)}\right]^{\mu(\sigma,D)} = \cap \left\{E \subset \mathbb{R}^n \times \mathbb{R}^n\text{: is maximal } \sigma_D\text{-monotone, } D \subset E\right\}.$$

(vi) *Let σ_D be upper semicontinuous. Then D is σ_D-monotone if and only if* $\operatorname{cl} D$ *is σ_D-monotone.*

(vii) *Suppose that σ_D is upper semicontinuous and* $\operatorname{int}(\operatorname{dom} D)$ *is convex. If D is maximal σ_D-monotone, then it is closed.*

(viii) *Suppose that* $\operatorname{int}(\operatorname{dom} D)$ *is convex and D is maximal σ_D-monotone. Then $D(x)$ is closed and convex for all $x \in \mathbb{R}^n$.*

Proof. The proofs of items (i) and (ii) are a direct consequence of the definition.

We prove item (iii). Denote by $\tilde{D}$, the right hand side of (19). Let $(x, u) \in D^{\mu(\sigma,D)}$. Then by definition, for each $(y, v) \in D$, we have

$$\langle u - v, y - x\rangle \le \min\{\sigma_D(x), \sigma_D(y)\} \|x - y\| .$$

Thus $D\cup\{(x,u)\}$ is σ_D-monotone. We can extend it to maximal σ_D-monotone, say D_1. Hence we get

$$(x,u)\in D_1\subset\tilde{D}.$$

Conversely, take $(x,u)\in\tilde{D}$. Then there exists E' such that $(x,u)\in E'$ which is maximal σ_D-monotone, and $D\subset E'$. Thus $(x,u)^{\mu(\sigma,D)}(y,v)$ for all $(y,v)\in D\subset E'$. This means that $(x,u)\in D^{\mu(\sigma,D)}$.

For the proof of item (iv), assume that D is maximal σ_D-monotone. According to item (iii), $D^{\mu(\sigma,D)}$ is a union of (maximal) σ_D-monotone extensions of D; since D is maximal σ_D-monotone, each of these extensions is equal to D; thus, $D^{\mu(\sigma,D)}=D$. Conversely, let $D=D^{\mu(\sigma,D)}$. If $(x,u)\in D$, then it is *σ_D-monotonically related to all* $(y,v)\in D$. Therefore, D is σ_D-monotone. From the fact that $D=D^{\mu(\sigma,D)}$ and item (iii), we get that the only σ_D-monotone extension of D is itself. This means that D is maximal σ_D-monotone.

For proving item (v), just take the σ_D-monotone polar from (iii).

For proving item (vi), since $D\subset\operatorname{cl}D$, the σ_D-monotonicity of $\operatorname{cl}D$ implies that D is σ_D-monotone. Conversely, assume that D is σ_D-monotone. Take $(x,u),(y,v)\in\operatorname{cl}D$. It follows that there exist sequences $\{(x_n,u_n)\},\{(y_n,v_n)\}\subset D$ such that we have $(x_n,u_n)\to(x,u)$ and $(y_n,v_n)\to(y,v)$. From the continuity of the inner product in $\mathbb{R}^n$ and upper semicontinuity of σ_D, we infer that

$$\begin{aligned}\langle u-v,y-x\rangle &= \lim_{n\to\infty}\langle u_n-v_n,y_n-x_n\rangle\\ &\le \limsup_{n\to\infty}\left(\min\{\sigma_D(x_n),\sigma_D(y_n)\}\|x_n-y_n\|\right)\\ &\le \min\{\sigma_D(x),\sigma_D(y)\}\|x-y\|.\end{aligned}$$

Therefore $\operatorname{cl}D$ is σ_D-monotone. Item (vii) is a consequence of items (iv) and (vi).

The proof of item (viii) is elementary. □

Corollary 3.6. *Let* $\operatorname{int}(\operatorname{dom}D)$ *be convex. Then D is σ_D-monotone if and only if $D^{\mu(\sigma,D)}$ contains a maximal σ_D-monotone set.*

Proposition 3.7. *Suppose that $D\subset\mathbb{R}^n\times\mathbb{R}^n$ is σ_D-monotone and σ_D is upper semicontinuous. Then $D^{\mu(\sigma,D)}$ is closed. Consequently,* $\operatorname{cl}D\subset D^{\mu(\sigma,D)}$.

Proof. Let $\{(x_n,u_n)\}$ be a sequence in $D^{\mu(\sigma,D)}$, converging to (x,u). For each $(y,v)\in D$, since (x_n,u_n) is σ_D-monotonically related to (y,v),

$$\langle u_n-v,y-x_n\rangle\le\min\{\sigma_D(x_n),\sigma_D(y)\}\|x_n-y\|.$$

Since $x_n\to x$, $u_n\to u$ and the inner product is continuous, we get the convergence $\langle u_n-v,y-x_n\rangle\to\langle u-v,y-x\rangle$. Also, by the upper semicontinuity of σ_D,

$$\begin{aligned}&\limsup_{n\to\infty}\left(\min\{\sigma_D(x_n),\sigma_D(y)\}\|x_n-y\|\right)\\ &\quad=\min\{\limsup_{n\to\infty}\sigma_D(x_n),\sigma_D(y)\}\lim_{n\to\infty}\|x_n-y\|\\ &\quad\le\min\{\sigma_D(x),\sigma_D(y)\}\|x-y\|.\end{aligned}$$

It follows that $\langle u-v, y-x\rangle \le \min\{\sigma_D(x), \sigma_D(y)\}\,\|x-y\|$.

So (x,u) is σ_D-monotonically related to every $(y,v)\in D$, i.e. $(x,u)\in D^{\mu(\sigma,D)}$. Thus, $D^{\mu(\sigma,D)}$ is closed. Proposition 3.5(i) and the closedness of $D^{\mu(\sigma,D)}$ imply the result. $\square$

4. Maximality of premonotone operators

For all $A\subset\mathbb{R}^n$, $\operatorname{co}A$ and $\overline{\operatorname{co}}A$ will denote the convex hull of A and the closed convex hull of A, respectively. Assume that $T:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is an operator. We define the operators $\operatorname{co}T$ and $\overline{\operatorname{co}}T$ respectively, as $(\operatorname{co}T)(x) := \operatorname{co}(T(x))$ and $(\overline{\operatorname{co}}T)(x) := \overline{\operatorname{co}}(T(x))$ for all $x\in X$.

Proposition 4.1. *If one of the operators $T, \operatorname{cl}T, \operatorname{co}T$ and $\overline{\operatorname{co}}T$ is premonotone, so are the other ones.*

Proof. The proof follows from the continuity and linearity of the inner product in $\mathbb{R}^n$. $\square$

Given a premonotone operator T, define the *premonotone hull* T^h of T as the operator $T^h:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ given by

$$T^h(x) := \{u\in\mathbb{R}^n : \langle u-v, x-y\rangle \le \min\{\sigma_T(x),\sigma_T(y)\}\,\|y-x\| \quad \forall (y,v)\in T\}. \tag{20}$$

The operator T^h has been introduced and studied in [12].

Assume that $A\subset\mathbb{R}^n$ and $y\in\mathbb{R}^n$. We define

$$\langle A, y\rangle := \{\langle a,y\rangle : a\in A\}.$$

Lemma 4.2. *Let $A,B\subset\mathbb{R}^n, A\subset B$ and assume that B is convex. Then*

$$\cup_{t\in[0,1]}(tA+(1-t)B) = B.$$

Proof. Obviously, $B\subset \cup_{t\in[0,1]}(tA+(1-t)B)$; just take $t=0$. For the reverse inclusion, by assumption B is convex so that $tB+(1-t)B=B$ for all $t\in[0,1]$. Thus, $\cup_{t\in[0,1]}(tA+(1-t)B)\subset\cup_{t\in[0,1]}(tB+(1-t)B)=B$. $\square$

Theorem 4.3. *Let $T:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ be a premonotone operator. Then $\sigma_{T^h}=\sigma_T$.*

Proof. Since $T\subset T^h$, according to the definition of σ_T, we get that $\sigma_T\le\sigma_{T^h}$. For proving the reverse inclusion, note that we have, by definition,

$$\langle T^h(x), y-x\rangle + \langle T(y), x-y\rangle \le \min\{\sigma_T(x),\sigma_T(y)\}\,\|x-y\|, \tag{21}$$

and

$$\langle T(x), y-x\rangle + \langle T^h(y), x-y\rangle \le \min\{\sigma_T(x),\sigma_T(y)\}\,\|x-y\|. \tag{22}$$

Multiply (21) and (22) by t and $(1-t)$, respectively, and then add them. We obtain

$$\begin{aligned}&\langle tT^h(x)+(1-t)T(x), y-x\rangle + \langle tT(y)+(1-t)T^h(y), x-y\rangle\\ &\le \min\{\sigma_T(x),\sigma_T(y)\}\,\|x-y\|.\end{aligned} \tag{23}$$

Using Lemma 4.2, we get

$$\begin{aligned}\cup_{t\in[0,1]}\left(tT^h(x)+(1-t)T(x)\right)&=T^h(x),\\ \cup_{t\in[0,1]}\left(tT(y)+(1-t)T^h(y)\right)&=T^h(y).\end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$\langle T^h(x),y-x\rangle+\langle T^h(y),x-y\rangle\leq\min\{\sigma_T(x),\sigma_T(y)\}\|x-y\|.$$

Therefore $\sigma_{T^h}\leq\sigma_T$. □

Corollary 4.4. *For all premonotone operator $T:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ it holds that*

$$\sigma_{(T^h)^h}=\sigma_{T^h}=\sigma_T. \tag{25}$$

Lemma 4.5. *Let $T,S:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ be two σ_T-premonotone operators with $D(T)=D(S)$. If $T\subset S$, then $S^h\subset T^h$.*

Proof. By definition, if $(x,u)\in S^h$, then (x,u) is σ_T-monotonically related to S. Since by assumption $T\subset S$, (x,u) is σ_T-monotonically related to T. Hence $(x,u)\in T^h$. □

Proposition 4.6. *Suppose that $T:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is a σ_T-premonotone operator and* $\operatorname{int}(\operatorname{dom}D(T))$ *is convex. Then $T^h=\left(T^h\right)^h$ and so T^h is maximal σ_T-premonotone.*

Proof. By definition T^h consists of all the pairs $(x,u)\in\mathbb{R}^n$ which are σ_T-monotonically related to T. Thus T^h is σ_T-premonotone and $T\subset T^h$. Applying Lemma 4.5 in the last inclusion, we obtain $T^{hh}=\left(T^h\right)^h\subset T^h$. On the other hand, by definition $T^h\subset T^{hh}$. These two inclusion imply that $T^h\subset T^{hh}\subset T^h$. Hence $T^h=T^{hh}$. Consequently, T^h is maximal σ_T-premonotone. □

5. One-dimensional version of a conjecture on maximal premonotone operators

The following conjecture has been proposed in [12]: the premonotone hull of any premonotone operator contains a maximal monotone operator, or equivalently, every maximal premonotone operator contains a maximal monotone one.

In this note we prove the conjecture in the one-dimensional case, i.e. when $n=1$. In this case, the formulas defining T and T^h, namely (2), (20), admit more manageable expressions, which we present next. For $x\in\mathbb{R}$, let

$$a(x):=\inf\{\cup_{y\geq x}T(y)\},\, b(x):=\sup\{\cup_{y\leq x}T(y)\},$$

$$p(x):=\inf T(x),\, q(x):=\sup T(x). \tag{26}$$

With the notation of (26), we have

$$\sigma_T(x)=\max\{q(x)-a(x),b(x)-p(x)\}. \tag{27}$$

The formula in (27) is obtained from (2) after noting that, in the one-dimensional case,

$$\frac{\langle u-v, x-y\rangle}{|x-y|} = \begin{cases} u-v & \text{if } x \geq y \\ v-u & \text{otherwise,} \end{cases}$$

using also the fact that $q(x) \geq a(x)$, $b(x) \geq p(x)$. With the same argument, we get the following one-dimensional formula for T^h:

$$T^h(x) = \left\{ \begin{array}{c} u : u \geq v - \min\{\sigma_T(x), \sigma_T(y)\} \, \forall y \leq x, \forall v \in T(y) \text{ and} \\ u \leq v + \min\{\sigma_T(x), \sigma_T(y)\} \, \forall y \geq x, \forall v \in T(y) \end{array} \right\}. \tag{28}$$

We proceed to establish the conjecture in this setting.

Proposition 5.1. *If $T : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is maximal premonotone, then it contains a maximal monotone operator.*

Proof. Define

$$c(x) := \frac{b(x)+a(x)}{2}, \quad d(x) := \frac{b(x)-a(x)}{2}, \tag{29}$$

with $a(\cdot)$, $b(\cdot)$ as in (26). We observe that $a(x) \leq b(x)$ for all $x \in \mathbb{R}$ because both sets over which the supremum and the infimum are taken in the definitions of a, b contain $T(x)$. It follows that $d(x) \geq 0$ for all $x \in \mathbb{R}$. We argue now that both $a(\cdot)$ and $b(\cdot)$ are non-decreasing functions. Indeed, if $x \leq x'$ then $a(x')$ is the infimum over a smaller set than the one in the definition of $a(x)$, and so $a(x) \leq a(x')$. By the same token, $b(x')$ is the supremum over a larger set than the one used for $b(x)$, and hence $b(x) \leq b(x')$. It follows that $c(\cdot)$ is nondecreasing, or equivalently, looking at c as a one-dimensional operator, that c is monotone.

We claim now that $c \subset T$, i.e., $c(x) \in T(x)$ for all $x \in \mathbb{R}$. Since T is maximal premonotone, we have that $T = T^h$, with T^h as in (20). So it suffices to prove that $c(x) \in T^h(x)$ for all $x \in \mathbb{R}$. In view of (28), we must prove that

$$c(x) - v \leq \min\{\sigma_T(x), \sigma_T(y)\} \ \ \forall y \geq x, \forall v \in T(y), \tag{30}$$

$$v - c(x) \leq \min\{\sigma_T(x), \sigma_T(y)\} \ \ \forall y \leq x, \forall v \in T(y). \tag{31}$$

In view of (27),

$$\sigma_T(x) = \max\{q(x)-a(x), b(x)-p(x)\} = d(x) + \max\{q(x)-c(x), c(x)-p(x)\}, \tag{32}$$

for all $x \in \mathbb{R}$, using (29). Combining (30), (31) and (32), we must verify that, for all $y \leq x$, and all $v \in T(y)$,

$$v - c(x) \leq d(x) + \max\{q(x)-c(x), c(x)-p(x)\}, \tag{33}$$

$$v - c(x) \leq d(y) + \max\{q(y)-c(y), c(y)-p(y)\}, \tag{34}$$

and for all $y \geq x$ and all $v \in T(y)$,

$$c(x) - v \leq d(x) + \max\{q(x)-c(x), c(x)-p(x)\}, \tag{35}$$

$$c(x) - v \leq d(y) + \max\{q(y)-c(y), c(y)-p(y)\}. \tag{36}$$

Before establishing (33)–(36), we note that both terms in the right hand sides of these four inequalities are nonnegative. This was already observed for $d(x), d(y)$. Regarding the"max" term, note that, since $q(x) \geq p(x)$ by definition, we have

$$\max\{q(x) - c(x), c(x) - p(x)\} \geq$$

$$\max\{p(x) - c(x), c(x) - p(x)\} = |p(x) - c(x)| \geq 0.$$

Thus, in order to verify any of the inequalities in (33)-(36), it suffices to check that the left hand side is no larger than any of the two terms in the right hand side.

We start with (33). It suffices to check that, for all $y \leq x$ and all $v \in T(y)$, we have $v - c(x) \leq d(x)$, or $v \leq c(x) + d(x) = b(x)$. Since $b(\cdot)$ is nondecreasing and $y \leq x$, we have $v \leq b(y) \leq b(x)$, using (26) and the fact that $v \in T(y)$.

We use the same argument for (35). It suffices to check that $v \geq c(x) - d(x) = a(x)$. Since $v \in T(y), v \geq a(y)$. Since $a(\cdot)$ is nondecreasing and in this case $y \geq x$, we have $a(y) \geq a(x)$ and hence $v \geq a(x)$.

Now we look at (34). It suffices to check that $v - c(x) \leq \max\{q(y) - c(y), c(y) - p(y)\}$. Since $c(\cdot)$ is nondecreasing and $y \leq x, v - c(x) \leq v - c(y)$. Since $v \in T(y), v \leq q(y)$, and so $v - c(y) \leq q(y) - c(y)$. It follows that

$$v - c(x) \leq q(y) - c(y) \leq \max\{q(y) - c(y), c(y) - p(y)\}.$$

Finally, we consider (36) with the same argument. It suffices to check that

$$c(x) - v \leq \max\{q(y) - c(y), c(y) - p(y)\}.$$

Since $c(\cdot)$ is nondecreasing and $y \geq x$ in this case, we have that $c(x) - v \leq c(y) - v$. Since $v \in T(y)$, $c(y) - v \leq c(y) - p(y)$. It follows that

$$c(x) - v \leq c(y) - p(y) \leq \max\{q(y) - c(y), c(y) - p(y)\},$$

and the claim holds also in this case.

We have proved that $c(x) \in T^h(x) = T(x)$ for all $x \in R$, i.e., that T contains a monotone operator. Consider now a maximal monotone extension of c, i.e., a maximal monotone operator $\bar{c}$ such that $c \subset \bar{c}$. It is immediate to check that all pairs $(x, u) \in \bar{c}$ are σ_T -premonotonically related to any pair $(y, c(y))$, and hence $\bar{c}$ is contained in c^h. Since $c \subset T$, we have that $c^h \subset T^h = T$. We conclude that $\bar{c} \subset T$, and hence T contains a maximal monotone operator. □

It is interesting to look now at some examples.

Consider the case of a bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$. Take $T = \varphi$. It has been proved in [10] that T is premonotone with $\sigma_T(x) \leq \sup_{t\in\mathbb{R}} \varphi(t) - \inf_{t\in\mathbb{R}} \varphi(t)$. In general, T is not maximal premonotone, and the maximal premontone operator of interest will be T^h. We will consider also the following assumption:

$$\limsup_{t\to\infty}\varphi(t) = \sup_{t\in\mathbb{R}} \varphi(t), \qquad \liminf_{t\to-\infty}\varphi(t) = \inf_{t\in\mathbb{R}} \varphi(t). \tag{37}$$

This assumption holds when φ is nonincreasing, but also in many other cases, e.g. for $\varphi(t) = \sin t$, $\varphi(t) = \cos t$. Under this assumption, both $a(\cdot)$ and $b(\cdot)$, as defined in (26), are constant, and the same holds for $c(\cdot)$, $d(\cdot)$, as defined in (29). These constants will be called a, b, c, d respectively. Since T is now point-to-point, we also have $p(\cdot) = q(\cdot) = \varphi(\cdot)$, with p, q as in (26). It follows then from (32) that $\sigma_{T^h}(x) = \sigma_T(x) = d + |\varphi(x) - c|$. Define $s(x) = \min\{d, |\varphi(x) - c|\}$. It is not difficult to check that $T^h(x) = \{c + t : |t| \leq s(x)\}$. Note that $c \in T^h(x)$ for all $x \in \mathbb{R}$, which corroborates the fact that the constant operator $c(\cdot) = c$, is contained in T.

This is the maximal monotone operator whose existence is ensured by Proposition 5.1, and the fact that it is constant is a consequence of the assumption (37). Let $Z = \{x \in \mathbb{R} : \varphi(x) = c\}$. Continuity of φ ensures that $Z \neq \emptyset$. For $x \in Z$, we have $s(x) = 0$ and so $T^h(x)$ is a singleton, namely $\{c\}$. It follows that if U is any monotone operator contained in T^h, we have $U(x) = \{c\}$ for all $x \in Z$. When $\inf Z = -\infty$, $\sup Z = +\infty$, we conclude that the unique monotone operator contained in T^h is the constant operator c.

We take now a few specific instances of φ. If φ is nondecreasing, and nonconstant, we have that T is maximal monotone (by continuity of φ), it coincides with T^h, and is also the maximal monotone operator contained in T^h. Of course, in this case assumption (37) does not hold.

Take now $\varphi(x) = \sin x$. Then $a = -1$, $b = 1$, $c = 0$, $d = 1$, and hence we have $\sigma_T(x) = 1 + |\sin x|$, $T^h(x) = [-|\sin x|, |\sin x|]$. Assumption (37) holds, and there is a unique maximal monotone operator contained in T^h, namely the constant 0.

Consider $\varphi(x) = -\arctan(x)$. We have $a = -\pi/2$, $b = \pi/2$, $c = 0$, $d = \pi/2$, $\sigma_T(x) = \pi/2 + |\arctan(x)|$, $T^h(x) = [-|\arctan(x)|, |\arctan(x)|]$, assumption (37) holds (note that φ is decreasing), and the maximal monotone operator contained in T^h, resulting from Proposition 5.1, is again the constant 0, but in this case the set Z is bounded (in fact, $Z = \{0\}$), and thus the argument for uniqueness of the maximal monotone operator contained in T^h fails; indeed, in this case there exist other maximal monotone operators contained in T^h, for instance $U(x) = \gamma \arctan(x)$ for any $\gamma \in (0, 1)$.

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