Journal of Convex Analysis
Volume 31 (2024), No. 3, 867–888

Lifting Approach to Integral Representations of Rich Multimeasures with Values in Banach Spaces II

Kazimierz Musiał
Institute of Mathematics, Wrocław University, Wrocław, Poland
kazimierz.musial@math.uni.wroc.pl

Received: July 17, 2023
Accepted: August 30, 2023

If X is a separable Banach space with RNP and Γ is an Effros measurable and Pettis integrable multifunction, then the conditional expectation of Γ with respect to a sub-σ-algebra is well described by the set of conditional expectations of selections of Γ. This is possible due to the Castaing representation of Γ that fails in case of non-separable X. Instead of RNP and separability of X we assume that the multimeasure defined by the Pettis integral of Γ is rich in selections possessing strongly measurable Radon-Nikodým derivatives. In general that cannot be reduced to a separable space (See Example 4.6). Then, using a lifting, we prove the existence of an Effros measurable conditional expectation of Γ in case of an arbitrary non-separable X and present its representation in terms of quasi-selections of Γ. We apply then the description to martingales of Pettis integrable multifunctions obtaining a scalarly equivalent martingale of measurable multifunctions with many martingale selections.

Keywords: Multimeasures, multifunctions, Pettis integral, Radon-Nikodým property, conditional expectation, martingales, rich multimeasures.

2020 Mathematics Subject Classification: Primary 28B20; secondary 28B05, 46G10, 46B22.

1. Introduction

In general investigation of multifunctions is based on the assumption of their measurability in the sense of Effros and separability of the Banach range space X. In particular, while investigating the problem of existence of conditional expectation of a Pettis integrable multifunction, as a rule it is assumed that the multifunction is Effros measurable, scalarly (quasi-) integrable and the spaces X and X^* are separable and have RNP. The conditional expectation of Γ with respect to a sub-σ-algebra is then well described by the set of conditional expectations of selections of Γ. This is possible due to the Castaing representation of Γ. The measurability of Γ is the most essential assumption because it guarantees existence of selections. The assumption of separability of X^* is postponed for multifunctions with weakly compact values. If X is non-separable, then Castaing representation in general fails and one has to use a different method.

This paper is a continuation of [25], where I prove that several multimeasures have Effros measurable (but without the separability assumption) Radon-Nikodým densities. In this paper, instead of separability of the range space X I assume riches of the Pettis integral of a Pettis integrable multifunction in strongly measurable

ISSN 0944-6532 / $ 2.50

Radon-Nikodým derivatives. This assumption is local and is essentially weaker than RNP of X and, in general cannot be reduced to a separable space (See Example 4.6). Among others, such an assumption makes it possible to prove that quite arbitrary Pettis integrable multifunction which takes as its values closed convex and bounded subsets of a Banach space with RNP, possesses the conditional expectation with respect to any sub-σ-algebra on which its integral is of σ-finite variation. Moreover, the conditional expectation can be chosen to be Effros measurable. That fact generalizes earlier known existence results. The conditional expectation is fully described by the conditional expectations of the quasi-selections of the multifunction under consideration. Then I apply the existence theorem to martingale theory. One of the consequences is that for each martingale of Pettis integrable multifunctions there exists a martingale of measurable multifunctions that are scalarly equivalent to the initial multifunctions.

To have a better representation of the results I introduce an extension of the ordinary lifting on the space $L_\infty(\mu)$ to a selection on $L_1(\mu)$ that coincides with the initial lifting on $L_\infty(\mu)$. I do not pretend to be the first person considering such an extension but I was unable to find any earlier reference.

A presentation of the general theory of Pettis integration of vector functions can be found in [20], [21] and [28]. For the general theory of Pettis integrable multifunctions I recommend [3], [14], [22] and [23].

2. Preliminaries.

Throughout (Ω, Σ, μ) is a complete probability space, $\mathcal{N}(\mu)$ is the collection of all sets of measure zero and $\mathcal{L}_0(\mu)$ is the space of all μ-measurable and a.e. finite real valued functions. $L_0(\mu)$ is the quotient space of μ-equivalent functions. $\mathcal{L}_\infty(\mu)$ is the space of all μ-measurable and a.e. bounded real valued functions. $L_\infty(\mu)$ is the ordinary quotient space of $\mathcal{L}_\infty(\mu)$. Similarly in case of $\mathcal{L}_1(\mu)$ and $L_1(\mu)$.

A family $W \subset L_1(\mu)$ (or just a family W of integrable functions, not equivalence classes) is *uniformly integrable* if W is bounded in $L_1(\mu)$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\sup_{f\in W} \int_A |f|\, d\mu < \varepsilon$.

ρ is an arbitrary lifting on $L_\infty(\mu)$ and on (Ω, Σ, μ) (formally these are two different liftings but each of them uniquely determines the other). The family of all liftings on $L_\infty(\mu)$ is denoted by $\Lambda(\mu)$.

X is a Banach space with its dual X^* and the closed unit ball of X is denoted by B_X. If $A \subset X^*$, then $\overline{A}^*$ is the weak*-closure of A and $\overline{A}$ is its norm closure. If $A \subset X$ is nonempty, then we write $|A| := \sup\{\|x\| : x \in A\}$. $\overline{\int}$ and $\underline{\int}$ denote respectively the upper and lower integrals.

$c(X)$ denotes the collection of all nonempty closed convex subsets of X, $cb(X)$ is the collection of all bounded elements of $c(X)$, $cwk(X)$ denotes the family of all weakly compact elements of $cb(X)$ and $cw^*k(X^*)$ denotes the family of all non-empty convex and weak*-compact subsets of X^*. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$. Notice that we have always $s(x^*, C) \neq -\infty$. d_H stands for the Hausdorff distance on the hyperspace $cb(X)$.

A function $f : \Omega \to \mathbb{R}$ is *quasi-integrable* (cf. [26]), if the integral $\int_\Omega f \, d\mu$ exists. A map $\Gamma : \Omega \to c(X)$ is called a *multifunction.* Γ is said to be *scalarly measurable* (*scalarly integrable, scalarly quasi-integrable*) if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable (integrable, quasi-integrable).

A multifunction $\Gamma : \Omega \to c(X^*)$ is said to be *weak***-scalarly measurable* (*weak***-scalarly integrable, weak***-scalarly quasi-integrable*) if for every $x \in X$, the map $s(x, \Gamma(\cdot))$ is measurable (integrable, quasi-integrable).

I associate with each Γ the trace of Γ on $\mathcal{L}_0(\mu)$:

$$\mathscr{Z}_\Gamma := \{s(x^*, \Gamma) : \|x^*\| \le 1\},$$

The set of equivalence classes of functions scalarly equivalent will be denoted by $\mathbb{Z}_\Gamma$.

Multifunctions $\Delta, \Gamma : \Omega \to c(X)[c(X^*)]$ are scalarly (weak* scalarly) equivalent, if for each $x^* \in X^*$ $[x \in X]$ we have $s(x^*, \Delta) = s(x^*, \Gamma)$ $[s(x, \Delta) = s(x, \Gamma)]$ a.e. We write then $\Delta \overset{s}{\approx} \Gamma$ $[\Delta \overset{w^*s}{\approx} \Gamma]$.

Similarly, we write $\Delta \subset \Gamma$, if $\Delta(\omega) \subset \Gamma(\omega)$ for every $\omega \in \Omega$.

A multifunction $\Gamma : \Omega \to c(X)$, $[c(X^*)]$ is [weak*-]scalarly bounded if there is a constant $M > 0$ such that for every $y \in X^*$ $[X]$

$$|s(y, \Gamma)| \le M\|y\| \qquad a.e.$$

Following the generally accepted terminology $\Gamma : \Omega \to c(X)$ is called measurable if

$$\{\omega \in \Omega \colon U \cap \Gamma(\omega) \neq \emptyset\} \in \Sigma. \tag{1}$$

for every norm open set $U \subset X$.

A function $\gamma : \Omega \to X$ is called a *selection of* $\Gamma : \Omega \to c(X)$, if for almost every $\omega \in \Omega$, one has $\gamma(\omega) \in \Gamma(\omega)$. $\gamma : \Omega \to X$ ($\gamma : \Omega \to X^{**}$) is called a *quasi-selection* (see [22]) of $\Gamma : \Omega \to c(X)$ (a *weak*** *quasi-selection of* Γ), if γ is scalarly measurable (weak* scalarly measurable) and for each $x^* \in X^*$ one has $\langle x^*, \gamma(\omega)\rangle \in \langle x^*, \Gamma(\omega)\rangle$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x^*). The collection of all quasi-selections of Γ (weak** quasi-selections of Γ) will be denoted by $\mathcal{QS}_\Gamma$ ($\mathcal{W}^{**}\mathcal{QS}_\Gamma$). A scalarly measurable selection $\gamma : \Omega \to X$ (a weak* scalarly measurable selection $\gamma : \Omega \to X^{**}$) is a (weak**) quasi-selection of $\Gamma : \Omega \to c(X)$ if and only if for each $x^* \in X^*$

$$\langle x^*, \gamma\rangle \le s(x^*, \Gamma) \quad a.e. \tag{2}$$

A function $\gamma : \Omega \to X^*$ is said to be a weak*-quasi-selection of a multifunction $\Gamma : \Omega \to c(X^*)$ if γ is weak*-scalarly measurable and we have $\langle x, \gamma(\omega)\rangle \in \langle x, \Gamma(\omega)\rangle$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x). The collection of all weak*-quasi-selections of Γ will be denoted by $\mathcal{W}^*\mathcal{QS}_\Gamma$. One can easily check that a weak*-scalarly measurable $\gamma : \Omega \to X^*$ is a weak*-quasi-selection of Γ if and only if for each $x \in X$

$$\langle x, \gamma\rangle \le s(x, \Gamma) \quad a.e. \tag{3}$$

If $\gamma \in \mathcal{W}^{**}\mathcal{QS}_\Gamma$ ($\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$, $\gamma \in \mathcal{QS}_\Gamma$) is Pettis integrable, then we will write it as $\gamma \in \mathcal{W}^{**}\mathcal{QS}^P_\Gamma$ ($\gamma \in \mathcal{W}^*\mathcal{QS}^P_\Gamma$, $\gamma \in \mathcal{QS}^P_\Gamma$).

If $\gamma : \Omega \to X^*$ is a weak*-scalarly measurable and weak*-scalarly bounded function, then $\rho(\gamma) : \Omega \to X^*$ is defined by $\langle x, \rho(\gamma)(\omega)\rangle := \rho(\langle x, \gamma\rangle)(\omega)$ (see [18, ch. IV.5]). If $\gamma : \Omega \to X$ is scalarly bounded and scalarly measurable, then $\rho(\gamma) : \Omega \to X^{**}$ is defined by $\langle x^*, \rho(\gamma)(\omega)\rangle := \rho(\langle x^*, \gamma\rangle)(\omega)$, $x^* \in X^*$.

A map $M : \Sigma \to c(X)$ is *additive*, if

$$M(A \cup B) = M(A) \oplus M(B) := \overline{M(A) + M(B)}$$

for each pair of disjoint elements of Σ.

An additive map $M : \Sigma \to c(X)$ is called a *multimeasure* if, for every $x^* \in X^*$, $s(x^*, M(\cdot)) : \Sigma \to (-\infty, +\infty]$ is a σ-finite measure. $M : \Sigma \to c(X^*)$ is called a *weak*-multimeasure* if $s(x, M(\cdot)) : \Sigma \to (-\infty, +\infty]$ is a σ-finite measure, for every $x \in X$. $\bigcup M(\Sigma) := \bigcup_{E\in\Sigma} M(E)$ denotes the range of M in X or X^*. One can easily see that if M is a weak*-multimeasure (multimeasure), and if for $y \in X(X^*)$ the equality $s(y, M) = s(y, M)^+ - s(y, M)^-$ is the Jordan decomposition of $s(y, M)$, then

$$s(y, M(\Omega))^- < \infty.$$

If $M : \Sigma \to cb(X)$ is countably additive in the metric d_H, then it is called an *h-multimeasure* (cf. [17], 8.Definition 4.3). A helpful tool to study h-multimeasures is the Rådström embedding $j : cb(X) \to \ell_\infty(B_{X^*})$, defined by $j(A) := s(\cdot, A)$, (see, for example [1, Theorem 3.2.9 and Theorem 3.2.4(1)] or [5, Theorem II-19]). It is known that B_{X^*} can be embedded into $\ell^*_\infty(B_{X^*})$ by the mapping $x^* \longrightarrow e_{x^*}$, where $\langle e_{x^*}, h\rangle = h(x^*)$, for each $h \in \ell_\infty(B_{X^*})$. Moreover, the range of B_{X^*} is a norming subset of $\ell^*_\infty(B_{X^*})$. The embedding j satisfies the following properties:

(a) $j(\alpha A \oplus \beta C) = \alpha j(A) + \beta j(C)$ for every $A, C \in cb(X)$, $\alpha, \beta \in \mathbb{R}^+$;

(b) $d_H(A, C) = \|j(A) - j(C)\|_\infty$, $A, C \in cb(X)$;

(c) $j(cb(X))$ is a closed cone in the space $\ell_\infty(B_{X^*})$ equipped with the norm of the uniform convergence.

If M is a point map, then we talk about measure and weak*-measure, respectively. If $m : \Sigma \to X^*$ is a weak*-measure such that $m(A) \in M(A)$, for every $A \in \Sigma$, then m is called a *weak*-selection* of M. $\mathcal{W}^*\mathcal{S}(M)$ will denote the set of all weak*-countably additive selections of M. Similarly, a vector measure $m : \Sigma \to X$ such that $m(A) \in M(A)$, for every $A \in \Sigma$, is called a *selection* of M. $\mathcal{S}(M)$ will denote the set of all countably additive selections of M.

A multimeasure $M : \Sigma \to c(X)$ is called *rich* if $M(A) = \overline{\{m(A), m \in S(M)\}}$. By a result of Costé, quoted in [16, Theorem 7.9], this is verified for multimeasures which take as their values weakly compact convex subsets of a Banach space X, and $cb(X)$-valued multimeasures, when X is a Banach space possessing RNP ([9, Théorème 1]).

If M is a weak*-multimeasure or a multimeasure, then M is called μ-continuous (we write then $M \ll \mu$), if $\mu(E) = 0$ yields $M(E) = \{0\}$. M is dominated by μ if there exists $\alpha > 0$ such that $|M(E)| \le \alpha\mu(E)$, for every $E \in \Sigma$.

A μ-continuous multimeasure $M : \Sigma \to c(X)$ is called *μ-rich in Pettis integrable strongly measurable densities* if M is rich and each $m \in S(M)$ has a strongly measurable density that is Pettis integrable with respect to μ. We will use the phrase "μ-rich in strong densities". If Γ is Pettis integrable and its integral M is μ-rich in strong densities, then we will say that Γ is μ-rich in strong densities.

If M is a weak*-multimeasure or a multimeasure, then the variation of M is defined by

$$|M|(E) := \sup_{\mathcal{P}} \left\{ \sum_{A\in\mathcal{P}} |M(A)| : \mathcal{P} \text{ is a finite measurable partition of } E \right\}.$$

One can easily see that $|M|$ is a measure (cf. [23, page 863]) and, $M \ll \mu$ if and only if $|M| \ll \mu$ (that is $\mu(E) = 0 \Rightarrow |M|(E) = 0$). If $M \ll \mu$ is of σ-finite variation, then there exists a partition $(\Omega_n)_n$ of Ω such that $|M(E)| \le n\mu(E)$ (equivalently $|M|(E) \le n\mu(E)$) for every $E \in \Omega_n \cap \Sigma$, $n \in \mathbb{N}$.

Definition 2.1. Denote by $\mathcal{C}$ a non-empty subfamily of $c(X^*)$. A weak*-scalarly quasi-integrable multifunction $\varGamma : \varOmega \to c(X^*)$ is *Gelfand integrable* in $\mathcal{C}$, if for each set $A \in \varSigma$ there exists a set $M_\varGamma^G(A) \in \mathcal{C}$ such that

$$s(x, M_\varGamma^G(A)) = \int_A s(x, \varGamma)\, d\mu\,, \text{ for every } x \in X. \tag{4}$$

$M_\varGamma^G$ is a weak*-multimeasure and $M_\varGamma^G(A)$ is called the Gelfand integral of $\varGamma$ on A: $(G)\int_E \varGamma\, d\mu := M_\varGamma^G(A)$. One should notice that the Gelfand integral is uniquely determined in the family $cw^*k(X^*)$ but not in $c(X^*)$ (see [24, Remark 3.12]).

Definition 2.2. Denote by $\mathcal{D}$ a non-empty subfamily of $c(X)$. A scalarly quasi-integrable multifunction $\varGamma\colon \varOmega \to c(X)$ is *Pettis integrable* in $\mathcal{D}$, if for each $A \in \varSigma$ there exists a set $M_\varGamma(A) \in \mathcal{D}$ such that

$$s(x^*, M_\varGamma(A)) = \int_A s(x^*, \varGamma)\, d\mu \quad \text{for every } x^* \in X^*. \tag{5}$$

$M_\varGamma : \varSigma \to \mathcal{D}$ is a multimeasure and $M_\varGamma(A)$ called the *Pettis integral of* $\varGamma$ over A. We write $(P)\int_A \varGamma\, d\mu := M_\varGamma(A)$. $\mathbb{P}(\mu, \mathcal{D})$ denotes the space of all multifunctions $\varGamma\colon \varOmega \to c(X)$ that are Pettis integrable in $\mathcal{D}$, with convention that scalarly equivalent multifunctions are identified. $\mathbb{P}(\mu, cb(X))$ is endowed with the Pettis metric

$$d_P(\varGamma, \varDelta) := \sup_{x^*\in B_{X^*}} \int_\varOmega |s(x^*, \varGamma) - s(x^*, \varDelta)|\, d\mu$$

generating the Pettis norm

$$\|\varGamma\|_P := \sup_{x^*\in B_{X^*}} \int_\varOmega |s(x^*, \varGamma)|\, d\mu.$$

We have (see [5, Lemme 7])

$$\sup_{E\in\varSigma} d_H(M_\varGamma(E), M_\varDelta(E)) \le d_P(\varGamma, \varDelta) \le 2 \sup_{E\in\varSigma} d_H(M_\varGamma(E), M_\varDelta(E)). \tag{6}$$

If $M_\varGamma$ is countably additive in the Hausdorff metric (i.e. it is an h-multimeasure), then we say that $\varGamma$ is *strongly Pettis integrable.*

The following well known property of lifting will be applied several times in this paper:

Proposition 2.3. *Let ρ be a lifting on the complete (Ω, Σ, μ) and $\{A_t : t \in T\} \subset \Sigma$ be a collection of sets such that $A_t \subset \rho(A_t)$, for each $t \in T$. Then $\bigcup_{t\in T} A_t \in \Sigma$ and $\bigcup_{t\in T} A_t \subset \rho(\bigcup_{t\in T} A_t)$.*

If $\{f_t : t \in T\}$ is a uniformly bounded family of real-valued functions such that $f_t \leq \rho(f_t)$ for each $t \in T$, then $\sup_{t\in T} f_t$ is a measurable function and we have $\sup_{t\in T} f_t \leq \rho(\sup_{t\in T} f_t)$ everywhere.

3. Pseudo-lifting on $L_0(\mu)$

In order to have a better shape of representations of multimeasures as Pettis integrals I introduce an extension of lifting from $L_\infty(\mu)$ to the whole $L_0(\mu)$. It is well known that no such an extension has the same properties as the ordinary lifting. This is in fact a selection $\rho^\# : L_0(\mu) \to \mathcal{L}_0(\mu)$ uniquely determined by the lifting ρ on $L_\infty(\mu)$. I introduce it not because of excelent properties but because it makes possible to write down several formulae in a consistent form. If $f \in \mathcal{L}_0(\mu)$, then the set

$$Dom(f) := \bigcup\{\rho(E) : f \text{ is a.e. bounded on } E\} = \bigcup_n \rho(E_n)$$

with $E_n := \{\omega : |f(\omega)| \leq n\}$, will be called the *domain* of f.

Definition 3.1. If ρ is a lifting on $L_\infty(\mu)$, then the mapping $\rho^\# : \mathcal{L}_0(\mu) \to \mathcal{L}_0(\mu)$ (or $\rho^\# : L_0(\mu) \to \mathcal{L}_0(\mu)$) is called a *pseudo-lifting* generated by ρ, if the following conditions are fulfilled:

(i) $\rho^\#(f) : Dom(f) \to \mathbb{R}$ for every $f \in \mathcal{L}_0(\mu)$;

(ii) $\rho^\#(f)\chi_{\rho(E)} = \rho(f\chi_E)$ and every $E \in \Sigma$ such that f is a.e. bounded on E.

Remark 3.2. If $f = g$ μ-a.e, then $\rho^\#(f) = \rho^\#(g)$ and so $\rho^\#$ is well defined. The pseudo-lifting is homogenous and one can easily see that $\rho^\#(f)(\omega) = \lim_n \rho(f\chi_{E_n})(\omega)$, where $E_n := \{|f| \leq n \text{ } a.e.\}$ and $\omega \in \bigcup_n \rho(E_n)$.

If f and g are a.e. bounded on $E \in \Sigma$ and $\alpha, \beta \in \mathbb{R}$, then

$$\rho^\#(\alpha f + \beta g) = \alpha\rho^\#(f) + \beta\rho^\#(g) \quad \text{on} \quad \rho(E).$$

but in general $\rho^\#$ is not additive. One can say only that $\rho^\#(f + g) = \rho^\#(f) + \rho^\#(g)$ a.e. More precisely, the pointwise equality holds true on $Dom(f) \cap Dom(g)$. If f is a.e. bounded on $E \in \Sigma$, then $\rho^\#(f) = \rho(f\chi_E) + \rho^\#(f\chi_{E^c})$.

If $E \in \Sigma$ is arbitrary, then $\rho^\#(f) = \rho^\#(f\chi_E) + \rho^\#(f\chi_{E^c})$. □

Definition 3.3. Let $\Gamma : \Omega \to cb(X^*)$ (in particular $\gamma : \Omega \to X^*$) be weak* scalarly measurable and

$$Dom(\Gamma) := \bigcup\{\rho(E) : \Gamma \text{ is weak}^* \text{ scalarly bounded on } E\} = \bigcup_n \rho(F_n),$$

where Γ is weak* scalarly bounded by n on F_n and F_n is maximal in the following sense: $\mu(F_n) = \sup\{\mu(E) : \forall\, x \in B_X\ |s(x, \Gamma)| \leq n\ \mu - a.e. \text{ on } F_n\}$. According to [22, Proposition 1.2] we have $\mu(Dom(\Gamma)) = 1$. We define a $cb(X^*)$-valued mapping $\rho^\#\Gamma = \rho^\#(\Gamma) : Dom(\Gamma) \to cb(X^*)$ by setting

$$\rho^\#(\Gamma)\chi_{\rho(E)} := \rho(\Gamma\chi_E) \quad \text{if } \Gamma \text{ is weak}^* \text{ scalarly bounded on } E\,,$$

where $\rho(\Gamma\chi_E)$ is defined by [24, Proposition 4.1].

Remark 3.4. If Γ is weak* scalarly bounded on E, then

$$\rho^{\#}\Gamma(\omega) = \rho(\Gamma\chi_E)(\omega) + \rho^{\#}(\Gamma\chi_{E^c})(\omega) \quad \text{if } \omega \in Dom(\Gamma). \tag{7}$$

If $\Gamma, \Delta : \Omega \to cb(X^*)$ are weak* scalarly bounded on $E \in \Sigma$ and $a, b \in [0, +\infty)$, then $\rho^{\#}(a\Gamma \oplus b\Delta) = a\rho^{\#}\Gamma \oplus b\rho^{\#}\Delta$ on $\rho(E)$.

In general one cannot say that $s(x, \rho^{\#}\Gamma) = \rho^{\#}[s(x, \Gamma)]$ everywhere. Simply, if $F_{nx} \in \Sigma$ is the μ-maximal set such that $s(x, \Gamma)$ is dominated by n on F_{nx}, then $Dom(s(x, \Gamma)) = \bigcup_n \rho(F_{nx}) \supset Dom(\Gamma)$ but it may happen that some of the sets $\rho(F_{nx})$ is not contained in any F_m. But we have the equality

$$s(x, \rho^{\#}\Gamma) = \rho^{\#}[s(x, \Gamma)] \quad \text{everywhere on } Dom(\Gamma)\,.$$

In particular, if $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$, then $\rho^{\#}(\gamma) \in \mathcal{W}^*\mathcal{S}_{\rho^{\#}\Gamma}$ and

$$\langle x, \rho^{\#}(\gamma)\rangle = \rho^{\#}\langle x, \gamma\rangle \quad \text{everywhere on } Dom(\Gamma)\,. \tag{8}$$

If $\gamma, \delta \in \mathcal{W}^*\mathcal{QS}_\Gamma$ and $a, b \in \mathbb{R}$, then we obviously have $\rho^{\#}(\gamma), \rho^{\#}(\delta) \in \mathcal{W}^*\mathcal{S}_{\rho^{\#}\Gamma}$ and $\rho^{\#}(a\gamma + b\delta) = a\rho^{\#}(\gamma) + b\rho^{\#}(\delta)$ on $Dom(\gamma) \cap Dom(\delta) \supset Dom(\Gamma)$. □

4. Radon-Nikodým Theorem

The subsequent result is a generalization of [24, Proposition 4.3].

Proposition 4.1. *If $\Gamma : \Omega \to cb(X^*)$ is a weak*-scalarly integrable multifunction and $M : \Sigma \to cb(X^*)$ is its Gelfand integral, then the multifunctions*

$$\Gamma_{\rho^{\#}},\ (\Gamma_M)_{\rho^{\#}} : \Omega \to cb(X^*) \cup \emptyset$$

are defined for every $\omega \in Dom(\Gamma)$ by the formulae

$$\Gamma_{\rho^{\#}}(\omega) := \overline{conv}\left\{\rho^{\#}(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\} = \overline{\{\rho^{\#}(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\}} \tag{9}$$

$$(\Gamma_M)_{\rho^{\#}}(\omega) := \overline{\left\{\rho^{\#}(\gamma)(\omega) : (G)\int_\bullet \rho^{\#}(\gamma)\, d\mu \in \mathcal{W}^*\mathcal{S}(M)\right\}} \tag{10}$$

and $\Gamma_{\rho^{\#}}(\omega) = (\Gamma_M)_{\rho^{\#}}(\omega) = \emptyset$, if $\omega \notin Dom(\Gamma)$.
Moreover, $\rho^{\#}\Gamma,\ \rho^{\#}(\Gamma_M) : \Omega \to cw^*k(X^*) \cup \emptyset$ are defined by

$$\rho^{\#}\Gamma(\omega) := \overline{\Gamma_{\rho^{\#}}(\omega)}^{\,*} \tag{11}$$

$$\rho^{\#}(\Gamma_M)(\omega) = \overline{(\Gamma_M)_{\rho^{\#}}(\omega)}^{\,*} \tag{12}$$

$\Gamma_{\rho^{\#}},\ (\Gamma_{M^*})_{\rho^{\#}},\ \rho^{\#}\Gamma$ and $\rho^{\#}(\Gamma_{M^*})$ are Gelfand integrable densities of M such that

$$s(x, \Gamma_{\rho^{\#}}(\omega)) = s(x, \rho^{\#}\Gamma(\omega)) = \rho^{\#}(s(x, \Gamma))(\omega) \tag{13}$$

for every $x \in X$ and every $\omega \in Dom(\Gamma)$.

Proof. If $\Omega = \bigcup_n E_n$ and Γ is weak*-scalarly bounded by n on E_n, then on each E_n the Proposition coincides with [24, Proposition 4.3]. The rest of the proof is standard. □

Remark 4.2. It follows from (13) that $\mathcal{W}^*\mathcal{QS}_\Gamma = \mathcal{W}^*\mathcal{QS}_{\rho^{\#}\Gamma} = \mathcal{W}^*\mathcal{QS}_{\Gamma_{\rho^{\#}}}$. □

The situation becomes more complicated when we investigate $cb(X)$-valued multifunctions and X is not necessarily a conjugate Banach space. We will use a result from [12] (see also [25, Lemma 3.2]): If $f : \Omega \to X$ is a scalarly bounded strongly measurable function, then $\rho(f)$ is a strongly measurable function with values in X^{**} but almost all values of $\rho(f)$ are in X.

If a multimeasure M is defined on Σ, then we denote by $\mathcal{D}_M^P$ the collection of all Pettis integrable densities of selections of M and by $\mathcal{D}_M^{mP}$ the collection of all strongly measurable members of $\mathcal{D}_M^P$.

Definition 4.3. Let $M : \Omega \to cb(X)$ be a multimeasure dominated by μ. If $\mathcal{D}_M^P \neq \emptyset$, then we define $\Gamma_{M\rho} : \Omega \to cb(X^{**})$ by

$$\Gamma_{M\rho}(\omega) := \overline{conv}\left\{\rho(\gamma)(\omega)\colon \gamma \in \mathcal{D}_M^P\right\} = \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{D}_M^P\}} \subset (\Gamma_M)_\rho(\omega)$$

and $\Gamma_{M\rho}^X : \Omega \to cb(X) \cup \emptyset$ by

$$\Gamma_{M\rho}^X(\omega) := \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{D}_M^P\} \cap X} \subset \Gamma_{M\rho}(\omega) \cap X\,.$$

It is clear that $\mu\{\omega \in \Omega : \Gamma_{M\rho}^X(\omega) = \emptyset\} = 0$. If X has RNP, then each element of $\mathcal{D}_M^P$ is scalarly equivalent to an element of $\mathcal{D}_M^{mP}$ and so everywhere in the above definition $\mathcal{D}_M^P$ may be replaced by $\mathcal{D}_M^{mP}$.

Proposition 4.4. [25, Proposition 3.7] *If $M : \Sigma \to cb(X)$ is a multimeasure dominated by μ, then $\Gamma_{M\rho}$ is weak* scalarly measurable and for each $x^* \in X^*$*

$$s(x^*, \Gamma_{M\rho}) \leq \rho[s(x^*, \Gamma_{M\rho})] \tag{14}$$

everywhere. Moreover, the function $|\Gamma_{M\rho}|$ is measurable.

If M is also μ-rich in strong densities (e.g. X has RNP) and $\rho(\Gamma_{M^{**}})\,, (\Gamma_{M^{**}})_\rho$ are defined by (10) *and* (12)*, then the multifunctions $\Gamma_{M\rho}^X\,, \Gamma_{M\rho}\,, \rho(\Gamma_{M^{**}})\,, (\Gamma_{M^{**}})_\rho$ are weak* scalarly equivalent and*

$$s(x^*, \Gamma_{M\rho}^X) \leq s(x^*, \Gamma_{M\rho}) \leq \rho[s(x^*, \Gamma_{M\rho}^X)] = \rho[s(x^*, \Gamma_{M\rho})]$$

everywhere on the set $\{\omega \in \Omega : \Gamma_{M\rho}^X(\omega) \neq \emptyset\}$. Moreover, the function $|\Gamma_{M\rho}^X|$ is measurable. If, moreover, M is the Pettis integral of $\Gamma : \Omega \to cb(X)$, then Γ is scalarly equivalent to $\Gamma_{M\rho}^X$ and $\Gamma_{M\rho} \cap X$.

The subsequent result is an extension of [24, Proposition 4.8] but under additional assumptions and is a generalization of [4, Theorem 4.1].

Theorem 4.5. [25, Theorem 3.8] *Let $M : \Sigma \to c(X)$ be a μ-continuous multimeasure of σ-finite variation. If M is μ-rich in strong densities (e.g. X has RNP and $M : \Sigma \to cb(X)$) and $\rho \in \Lambda(\mu)$, then there exists a measurable multifunction $\Gamma : \Omega \to cb(X)$ such that*

$$M(E) = \overline{\left\{(P)\!\int_E \gamma\, d\mu\colon \gamma \in \mathcal{D}_M^{mP}\right\}} = (P)\!\int_E \Gamma\, d\mu \quad \textit{for every } E \in \Sigma \tag{15}$$

and
$$\{\omega \in \Omega\colon \Gamma(\omega) \cap U \neq \emptyset\} \subset \rho\{\omega \in \Omega\colon \Gamma(\omega) \cap U \neq \emptyset\} \tag{16}$$

for every norm open $U \subset X$.

If M is dominated by μ, then one may set $\Gamma = X \cap \Gamma_{M\rho}$ or $\Gamma = \Gamma^X_{M\rho}$ and in each case the inequality $s(x^*, \Gamma) \leq \rho(s(x^*, \Gamma))$ holds true everywhere on the set $\{\omega : \Gamma^X_{M\rho}(\omega) \neq \emptyset\}$ and for every $x^* \in X^*$.

Below is an example of a multimeasure rich in strong densities whose range $\bigcup M(\Sigma)$ is non-separable.

Example 4.6. Let X be a non-separable Banach space with RNP. Let $f\colon \Omega \to X$ be a Pettis integrable function (not necessarily strongly measurable) and let $r\colon \Omega \to (0, \infty)$ be an integrable function. Define the function $\Gamma\colon \Omega \to cb(X)$ by $\Gamma(\omega)\colon = B(f(\omega), r(\omega))$, where $B(x, r)$ is the closed ball with its center in x and of radius r. One can easily check that Γ is Pettis integrable in $cb(X)$ and

$$M(E) := (P)\int_E \Gamma\, d\mu = B\left(\int_E f\, d\mu, \int_E r\, d\mu\right) \qquad \text{for every} \quad E \in \Sigma.$$

M is the required example.

5. Conditional expectation

The problem of existence of conditional expectation of Pettis integrable multifunctions and convergence of Pettis integrable martingales of multifunctions have been investigated in several papers. However the main results were obtained under the assumption of separability and RNP of X, separability of X^* and Castaing representation of measurable multifunctions. If X is non-separable, then there are several measurable multifunctions for which the classical Castaing representation fails. In this and the next chapter one may assume RNP of X but nothing is assumed about X^*. Under these assumptions we obtain an uncountable Castaing type representation of measurable multifunctions.

The following result is essential for our description of the conditional expectation of a multifunction:

Proposition 5.1. *Let X be an arbitrary Banach space and $M : \Sigma \to cb(X)$ be a multimeasure. Moreover, let $\Xi \subset \Sigma$ be a σ-algebra and $M_1 := M|_\Xi$. If M is rich, then M_1 is also rich and for each $m_1 \in \mathcal{S}(M_1)$ there exists an extension of m_1 to the whole Σ being a selection of M. If M is μ-rich in strong densities, then also M_1 is $\mu|_\Xi$-rich in strong densities.*

Proof. The riches of M_1 is a consequence of [17, 8. Proposition 4.35] and the existence of extensions of each selection of M_1 to a selection of M follows from the proof of [17, 8. Proposition 4.35]. If $m_1 \in \mathcal{S}(M_1)$ and $m \in \mathcal{S}(M)$ is its μ-continuous extension with $m(E) = (P)\int_E f\, d\mu$, for every $E \in \Sigma$, then $m_1(F) = (P)\int_F \mathbb{E}_\Xi(f)\, d\mu$ for every $F \in \Xi$ and $\mathbb{E}_\Xi(f)$ may be chosen to be strongly measurable. $\square$

Definition 5.2. Let $\Gamma : \Omega \to c(X)$ be a multifunction that is Pettis integrable in $c(X)$ and let $\Xi \subset \Sigma$ be a σ-algebra that is complete with respect to $\mu|_\Xi$. A multifunction $\mathbb{E}_\Xi(\Gamma) : \Omega \to c(X)$ that is scalarly measurable with respect to Ξ and Pettis integrable on Ξ in $c(X)$ is called the conditional expectation of Γ with respect to Ξ, if

$$(P)\int_E \Gamma\, d\mu = (P)\int_E \mathbb{E}_\Xi(\Gamma)\, d\mu \quad \text{for every } E \in \Xi\,. \tag{17}$$

Each multifunction $\varDelta$ that is scalarly measurable with respect to $\varXi$ and satisfies the equality (17) when placed instead of $\mathbb{E}_{\varXi}(\varGamma)$ is a version of the conditional expectation of $\varGamma$ with respect to $\varXi$. I will write sometimes that $\varDelta \in \mathbb{E}_{\varXi}(\varGamma)$. All versions are scalarly equivalent but not necessarily almost everywhere equal.

Theorem 5.3. *Assume that $\varGamma : \varOmega \to c(X)$ is Pettis integrable in $cb(X)$ and is μ-rich in strong densities (e.g. X has RNP). Assume also that $\varXi$ is a sub-σ-algebra of $\varSigma$ that is complete with respect to $\mu|_{\varXi}$ and the multimeasure $M_1 : \varXi \to cb(X)$ being the restriction of the multimeasure M to $\varXi$ is dominated by μ restricted to $\varXi$. If ρ_1 is a lifting on Ξ, then each of the following formulae defines a scalarly bounded and measurable version of the conditional expectation $\mathbb{E}_{\varXi}(\varGamma) : \varOmega \to cb(X)$ of $\varGamma$ with respect to $\varXi$:*

$$\mathbb{E}_{\varXi}(\varGamma)(\omega) = \overline{\{\rho_1[\mathbb{E}_{\varXi}(\gamma)](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\} \cap X} \quad \textit{for every } \omega \in \varOmega \tag{18}$$

$$\mathbb{E}_{\varXi}(\varGamma)(\omega) = X \cap \overline{\{\rho_1[\mathbb{E}_{\varXi}(\gamma)](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\}} \quad \textit{for every } \omega \in \varOmega \tag{19}$$

Moreover, if $x^* \in X^*$, then

$$s(x^*, \mathbb{E}_{\varXi}(\varGamma)(\omega)) \leq \rho_1[s(x^*, \mathbb{E}_{\varXi}(\varGamma))](\omega) \quad \textit{for every } \omega \in \varOmega$$

and if $U \subset X$ is open, then

$$\{\omega \in \varOmega : \mathbb{E}_{\varXi}(\varGamma)(\omega) \cap U \neq \emptyset\} \subset \rho_1\{\omega \in \varOmega : \mathbb{E}_{\varXi}(\varGamma)(\omega) \cap U \neq \emptyset\}\,.$$

Proof. By Theorem 4.5 there exists a measurable scalarly bounded multifunction $\varDelta : \varOmega \to cb(X)$ such that $M_1(E) = (P)\int_E \varDelta\, d\mu$ for every $E \in \varXi$, $\varDelta$ is measurable with respect to $\varXi$ and $\{\omega \in \varOmega\colon \varDelta(\omega) \cap U \neq \emptyset\} \subset \rho_1\{\omega \in \varOmega\colon \varDelta(\omega) \cap U \neq \emptyset\}$ for each open $U \subset X$. According to Theorem 4.5 we may assume that $\varDelta = X \cap \varDelta_{M_1\rho_1}$ or $\varDelta = \varDelta^X_{M_1\rho_1}$. That proves the first equalities in (18)–(19).

We have to prove validity of (18) and (19). We will concentrate on (19). According to Theorem 4.5 and Proposition 5.1 a measurable version of $\mathbb{E}_{\varXi}(\varGamma)$ can be described by the formula

$$\mathbb{E}_{\varXi}(\varGamma)(\omega) = X \cap \varDelta_{M_1\rho_1}(\omega) = X \cap \overline{\{\rho_1(\xi)(\omega)\colon \xi \in \mathcal{D}_{M_1}^{mP}\}}. \tag{20}$$

Assume now that $\xi \in \mathcal{D}_{M_1}^{mP}$ and let

$$\varTheta(\omega) := \overline{\{\rho_1[\mathbb{E}_{\varXi}(\gamma)](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\}}\,.$$

Let $\nu_\xi : \varXi \to X$ be the vector measure defined by $\nu_\xi(E) = (P)\int_E \xi\, d\mu = (P)\int_E \rho_1(\xi)\, d\mu$ for every $E \in \varXi$. Due to Proposition 5.1 there exists $\widetilde{\nu}_\xi \in \mathcal{S}(M)$ such that $\nu_\xi = \widetilde{\nu}_\xi|_{\Xi}$. As M is rich in strong densities there exists a strongly measurable γ such that $\widetilde{\nu}_\xi(A) = (P)\int_A \gamma\, d\mu$, for every $A \in \varSigma$. Due to strong measurability of ξ and γ we may assume that $\mathbb{E}_{\varXi}(\gamma) = \rho_1(\xi)$ μ-a.e. Hence, $\rho_1[\mathbb{E}_{\varXi}(\gamma)](\omega) \in \varTheta(\omega)$ for every $\omega \in \varOmega$. It follows that

$$X \cap \varDelta_{M_1\rho_1}(\omega) \subset X \cap \varTheta(\omega) \quad \text{for every } \omega \in \varOmega.$$

If $\gamma \in \mathcal{D}_M^{mP}$, then $\mathbb{E}_{\varXi}(\gamma)$ is strongly measurable and Pettis integrable on $\varXi$. Moreover, $\rho_1[\mathbb{E}_{\varXi}(\gamma)]$ is strongly measurable with respect to $\varXi$ and with almost all values in X.

Since $\gamma \in \mathcal{QS}_\Gamma$ we have $\langle x^*, \gamma\rangle \leq s(x^*, \Gamma)$ a.e. If $m(\bullet) = (P)\int_\bullet \mathbb{E}_\Xi(\gamma)\,d\mu$ and $E \in \Xi$, then

$$\begin{aligned}\langle x^*, m(E)\rangle &= \int_E \langle x^*, \mathbb{E}_\Xi(\gamma)\rangle\,d\mu = \int_E \langle x^*, \gamma\rangle\,d\mu \\ &\leq \int_E s(x^*, \Gamma)\,d\mu \stackrel{(17)}{=} \int_E s(x^*, \mathbb{E}_\Xi(\Gamma))\,d\mu = s(x^*, M_1(E))\,.\end{aligned}$$

Hence $m \in \mathcal{S}(M_1)$ and $\xi := \mathbb{E}_\Xi(\gamma) \in \mathcal{D}^{mP}_{M_1}$. Consequently,

$$X \cap \Theta(\omega) \subset X \cap \Delta_{M_1\rho_1}(\omega) \quad \text{for every } \omega \in \Omega\,. \qquad \square$$

Remark 5.4. Assume that X is separable, has RNP and $\Gamma: \Omega \to cb(X)$ is measurable and Pettis integrable in $cb(X)$. It is a consequence of [25, Remark 3.12] that if $\Delta := \mathbb{E}_\Xi(\Gamma)$, then $\Delta = \Delta^X_{M_1\rho_1} = \Delta_{M_1\rho_1} \cap X$ $\mu|_\Xi$-a.e. Moreover, there exists a family $\{\gamma_n = \rho_1(\gamma_n) : n \in \mathbb{N}\}$ of functions such that $\Delta(\omega) = \overline{\{\gamma_n(\omega) : n \in \mathbb{N}\}}$ $\mu|_\Xi$-a.e. This fact will be applied in Remark 6.6 to obtain a new proof of [13, Theorem 3.7]. $\square$

If the multimeasure M_1 above is only of σ-finite variation, then we obtain the following result:

Theorem 5.5. *Assume that $\Gamma : \Omega \to c(X)$ is Pettis integrable in $c(X)$ and its integral $M : \Sigma \to c(X)$ is μ-rich in strong densities (e.g. X has RNP and $M : \Sigma \to cb(X)$). Assume also that Ξ is a sub-σ-algebra of Σ that is $\mu|_\Xi$-complete, ρ_1 is a lifting on Ξ and the multimeasure M restricted to Ξ is of σ-finite variation. Then there exists a measurable conditional expectation $\mathbb{E}_\Xi(\Gamma) : \Omega \to cb(X)$ of Γ with respect to Ξ satisfying for each open $U \subset X$ the inclusion*

$$\{\omega \in \Omega : \mathbb{E}_\Xi(\Gamma)(\omega) \cap U \neq \emptyset\} \subset \rho_1\{\omega \in \Omega : \mathbb{E}_\Xi(\Gamma)(\omega) \cap U \neq \emptyset\}\,. \tag{21}$$

More precisely, the multifunctions defined by any of the following formulae:

$$\mathbb{E}_\Xi(\Gamma)(\omega) = \overline{\left\{\rho_1^\#[\mathbb{E}_\Xi(\gamma)](\omega) : \gamma \in \mathcal{D}^{mP}_M\right\} \cap X} \quad \text{if } \omega \in Dom(\Gamma) \tag{22}$$

$$\mathbb{E}_\Xi(\Gamma)(\omega) = X \cap \overline{\left\{\rho_1^\#[\mathbb{E}_\Xi(\gamma)](\omega) : \gamma \in \mathcal{D}^{mP}_M\right\}} \quad \text{if } \omega \in Dom(\Gamma) \tag{23}$$

are conditional expectations of Γ with respect to Ξ (please remember about (8)) . Moreover, if $x^* \in X^*$, then

$$s(x^*, \mathbb{E}_\Xi(\Gamma)(\omega)) \leq \rho_1^\#[s(x^*, \mathbb{E}_\Xi(\Gamma))](\omega) \quad \text{for every } \omega \in Dom(\Gamma). \tag{24}$$

Proof. If $m \in \mathcal{S}(M)$ and γ_m is a strongly measurable and Pettis integrable density of m, then $s(x^*, \Gamma - \gamma_m) \geq 0, \mu$-a.e. and $M - m$ is also μ-rich in strong densities. Thus, without loss of generality, we may assume at the beginning that $0 \in M(E)$ for every $E \in \Sigma$ and consequently $s(x^*, \Gamma) \geq 0, \mu$-a.e. for each $x^* \in X^*$ separately. If M_1 is the restriction of M to Ξ, then according to Theorem 4.5 M_1 has a measurable Radon-Nikodỳm density that is the conditional expectation of Γ with respect to Ξ and $s(x^*, \mathbb{E}_\Xi(\Gamma)) \geq 0$ $\mu|_\Xi$-a.e., for each x^* separately. We are going to describe its shape.

Let $h \geq 0$ be a Ξ-measurable function such that $|M_1|(E) = \int_E h\,d\mu$, for every $E \in \Xi$. Then, let $\Xi \ni F_n := \rho_1(\{\omega \in \Omega\colon h(\omega) < n\})$, $n \in \mathbb{N}$. Then $\mathbb{E}_\Xi(\Gamma)$ is scalarly bounded by n on each F_n. Moreover, $M|_{F_n} : F_n \cap \Sigma \to cb(X)$ and one may apply Theorem 5.3. If $\Gamma_n := \Gamma\chi_{F_n}$, then according to Theorem 5.3 a measurable $\mathbb{E}_\Xi(\Gamma_n) : \Omega \to cb(X)$ exists and can be given by any of the formulae

$$\mathbb{E}_\Xi(\Gamma_n)(\omega) = \overline{\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\} \cap X} \quad \text{for every } \omega \in \Omega\,;$$

$$\mathbb{E}_\Xi(\Gamma_n)(\omega) = X \cap \overline{\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\}} \quad \text{for every } \omega \in \Omega\,.$$

It is important to observe that if $\gamma \in \mathcal{D}_M^{mP}$, then $\gamma|_{F_n} \in \mathcal{D}_{M|_{F_n}}^{mP}$, and conversely, if $\xi_n \in \mathcal{D}_{M|_{F_n}}^{mP}$, then there exists $\gamma \in \mathcal{D}_M^{mP}$ such that $\gamma|_{F_n} = \xi_n$.

Since $\gamma = 0 \in \mathcal{D}_M^{mP}$ we have for every $\omega \in \Omega$ the following relations:

$$\begin{aligned}
&\big\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_{n+1}})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\big\} = \\
&= \big\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n} + \gamma\chi_{F_{n+1}\setminus F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\big\} \\
&= \big\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n}) + \mathbb{E}_\Xi(\gamma\chi_{F_{n+1}\setminus F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\big\} \\
&\supset \big\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\big\} \bigoplus \big\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_{n+1}\setminus F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\big\} \\
&\supset \big\{\rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n})](\omega)\colon \gamma \in \mathcal{D}_M^{mP}\big\}
\end{aligned}$$

It follows that $\mathbb{E}_\Xi(\Gamma_n)(\omega) \subset \mathbb{E}_\Xi(\Gamma_{n+1})(\omega)$ independently of its shape. We define now $\mathbb{E}_\Xi(\Gamma)$ by the formula

$$\mathbb{E}_\Xi(\Gamma)(\omega) := \mathbb{E}_\Xi(\Gamma_n)(\omega) \quad \text{whenever } \omega \in \rho(F_n),\ n \in \mathbb{N}\,. \tag{25}$$

Due to the above monotonicity the multifunction $\mathbb{E}_\Xi(\Gamma)$ is well defined on $Dom(\Gamma)$. If $x^* \in X^*$, then $0 \leq s(x^*, \Gamma_n) \leq s(x^*, \Gamma_{n+1})$, $n \in \mathbb{N}$ and so if $E \in \Xi$, then

$$\begin{aligned}
\int_E s(x^*, \mathbb{E}_\Xi(\Gamma))\,d\mu &= \int_E \lim_n s(x^*, \mathbb{E}_\Xi(\Gamma_n))\,d\mu \\
&= \lim_n \int_E s(x^*, \mathbb{E}_\Xi(\Gamma_n))\,d\mu = \lim_n \int_E s(x^*, \Gamma_n)\,d\mu \\
&= \lim_n \int_{E\cap F_n} s(x^*, \Gamma)\,d\mu = \int_E s(x^*, \Gamma)\,d\mu
\end{aligned}$$

If $\gamma \in \mathcal{D}_M^{mP}$, then $\mathbb{E}_\Xi(\gamma)$ is scalarly bounded by n on F_n and so $\rho_1(\mathbb{E}_\Xi(\gamma))$ is also scalarly bounded by n on $\rho_1(F_n)$. Hence, $\rho_1^{\#}[\mathbb{E}_\Xi(\gamma)]$ is well defined on the set $\bigcup_n \rho_1(F_n)$. We have then

$$\begin{aligned}
\rho_1^{\#}[\mathbb{E}_\Xi(\gamma)] &= \rho_1^{\#}[\mathbb{E}_\Xi(\gamma\chi_{F_n} + \gamma\chi_{F_n^c})] = \rho_1^{\#}[\mathbb{E}_\Xi(\gamma\chi_{F_n}) + \mathbb{E}_\Xi(\gamma\chi_{F_n^c})] \\
&\overset{(7)}{=} \rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n})] + \rho_1^{\#}[\mathbb{E}_\Xi(\gamma\chi_{F_n^c})] = \rho_1[\mathbb{E}_\Xi(\gamma)\chi_{\rho(F_n)}] + \rho_1^{\#}[\mathbb{E}_\Xi(\gamma)\chi_{\rho(F_n^c)})]
\end{aligned}$$

what yields the equality

$$\rho_1^{\#}[\mathbb{E}_\Xi(\gamma)](\omega) = \rho_1[\mathbb{E}_\Xi(\gamma\chi_{F_n})](\omega) \quad \text{if } \omega \in \rho(F_n)\,.$$

That proves validity of the formulae (22)–(23).

Now let $\varGamma$ be arbitrary. Let $m \in \mathcal{S}(M)$ and its strongly measurable density γ_m be fixed. If $\gamma \in \mathcal{D}_M^{mP}$, then γ_m is scalarly bounded on each F_n and

$$\rho_1[\mathbb{E}_{\varXi}(\gamma)\chi_{\rho(F_n)}] = \rho_1[\mathbb{E}_{\varXi}(\gamma - \gamma_m)\chi_{\rho(F_n)}] + \rho_1[\mathbb{E}_{\varXi}(\gamma_m)\chi_{\rho(F_n)}].$$

Thus,

$$\rho_1^{\#}[\mathbb{E}_{\varXi}(\gamma)] = \rho_1^{\#}[\mathbb{E}_{\varXi}(\gamma - \gamma_m)] + \rho_1^{\#}[\mathbb{E}_{\varXi}(\gamma_m)] \quad \text{on } Dom(\varGamma)\,.$$

One can easily deduce validity of the theorem in the general case. □

Remark 5.6. Since every multifunction $\varGamma$ that is Pettis integrable in $cb(X)$ generates a multimeasure of σ-finite variation (see [23, Proposition 6.5]), if $M|_{\varXi}$ is not of σ-finite variation, the conditional expectation does not exist. □

6. Martingales

We begin with a general convergence theorem for martingales valid without particular assumptions about the range Banach space. Several other results can be found in [23].

Definition 6.1. Let $(\varSigma_n)_n$ be a non-decreasing sequence of sub-σ-algebras of $\varSigma$. For each $n \in \mathbb{N}$ let $\varGamma_n : \varOmega \to c(X)$ be a scalarly $\varSigma_n$-measurable multifunction. The sequence $(\varGamma_n, \varSigma_n)_n$ is called a *martingale* if each $\varGamma_n$ is Pettis integrable in $c(X)$ and $\mathbb{E}_{\varSigma_n}(\varGamma_{n+1}) \overset{s}{\equiv} \varGamma_n$. Equivalently, if $(P)\int_E \varGamma_{n+1}\, d\mu = (P)\int_E \varGamma_n\, d\mu$, for every $E \in \varSigma_n$.

A martingale $(\gamma_n, \varSigma_n)_n$ of X-valued Pettis integrable functions is a *martingale (quasi-)selection* of $(\varGamma_n, \varSigma_n)_n$ if each γ_n is a (quasi-)selection of $\varGamma_n$.

Definition 6.2. [23, p. 852] A sequence $(\varGamma_n)_n$ of scalarly measurable multifunctions $\varGamma_n : \varOmega \to c(X)$ is called *scalarly equi-Cauchy in measure* if for every $\eta > 0$

$$\lim_{m,n\to\infty} \sup_{\|x^*\|\le 1} \mu\{\omega \in \varOmega \colon |s(x^*, \varGamma_m(\omega)) - s(x^*, \varGamma_n(\omega))| > \eta\} = 0.$$

Replacing $\varGamma_m$ by $\varGamma$ one obtains scalar equi-convergence of $(\varGamma_n)_n$ in measure to $\varGamma$.

The subsequent result can be considered as a generalization of [23, Theorem 4.6].

Theorem 6.3. *Let $\varGamma : \varOmega \to c(X)$ be a scalarly integrable multifunction and let $(\varSigma_n)_n$ be an increasing sequence of sub-σ-algebras of $\varSigma$ such that $\varSigma$ is the μ-completion of $\sigma\left(\bigcup_n \varSigma_n\right)$. Assume that for each $n \in \mathbb{N}$ there exists a multifunction $\varGamma_n : \varOmega \to c(X)$ that is scalarly measurable with respect to $\varSigma_n$ and Pettis integrable in $cb(X)$ $[cwk(X), ck(X)]$ on $\varSigma_n$. Assume also that for each $x^* \in X^*$ and $n \in \mathbb{N}$*

$$\mathbb{E}_{\varSigma_n}(s(x^*, \varGamma)) = s(x^*, \varGamma_n) \quad \mu - a.e. \tag{26}$$

or $(\varGamma_n, \varSigma_n)_n$ is a martingale and

$$\lim_n s(x^*, \varGamma_n) = s(x^*, \varGamma) \quad \text{in } \mu - \text{measure}. \tag{27}$$

If $\bigcup_n \mathcal{Z}_{\Gamma_n}$ is uniformly integrable, then

$$\Gamma \in \mathbb{P}(\mu, cb(X))\ [cwk(X), ck(X)], \quad \mathbb{E}_{\Sigma_n}(\Gamma) \stackrel{s}{\approx} \Gamma_n \quad \textit{for every } n \in \mathbb{N}$$

and
$$\lim_n d_H(M_\Gamma(E), M_{\Gamma_n}(E)) = 0 \quad \textit{for every } E \in \Sigma.$$

Moreover,
$$\lim_n d_P(\Gamma, \Gamma_n)) = 0$$

if and only if the sequence $(\Gamma_n)_n$ is scalarly equi-Cauchy in measure.

Proof. The sequence $(\Gamma_n, \Sigma_n)_n$ is a martingale and due to the uniform integrability of $\bigcup_n \mathcal{Z}_{\Gamma_n}$, for each $x^* \in X^*$ the martingale $s((x^*, \Gamma_n), \Sigma_n)_n$ is bounded in $L_1(\mu)$ and so (27) implies the convergence $\lim_n \int_\Omega |s(x^*, \Gamma_n) - s(x^*, \Gamma)|\, d\mu = 0$ and the equality $\mathbb{E}_{\Sigma_n}(s(x^*, \Gamma)) = s(x^*, \Gamma_n)$ μ-a.e. According to [23, Proposition 3.2] there exists an h-multimeasure $M : \Sigma \to cb(X)$ $[cwk(X), ck(X)]$ with the property

$$\lim_n d_H(M_{\Gamma_n}(E), M(E)) = 0 \quad \text{for each } E \in \Sigma.$$

Consequently, Γ is Pettis integrable, $M = M_\Gamma$ and $\mathbb{E}_{\Sigma_n}(\Gamma) \stackrel{s}{\approx} \Gamma_n$ for every $n \in \mathbb{N}$.

Assume that the sequence $(\Gamma_n)_n$ is scalarly equi-Cauchy in measure. For each $x^* \in X^*$ and $n \in \mathbb{N}$ we have $\mathbb{E}_{\Sigma_n}(s(x^*, \Gamma)) = s(x^*, \Gamma_n)$ μ-a.e. and in $L_1(\mu)$. Hence

$$\lim_n \int_E s(x^*, \Gamma_n)\, d\mu = \int_E s(x^*, \Gamma)\, d\mu \quad \text{for each } E \in \Sigma.$$

Let us fix $E \in \Sigma$, $\varepsilon > 0$ and $0 < \eta < \varepsilon$. For each $n, m \in \mathbb{N}$ and $x^* \in B_{X^*}$ denote by H_{mn,x^*} the set

$$H_{m,n,x^*} := \{\omega \in \Omega \colon |s(x^*, \Gamma_m(\omega)) - s(x^*, \Gamma_n(\omega))| > \eta\}.$$

By the assumption the family $\bigcup_n \mathcal{Z}_{\Gamma_n}$ is uniformly integrable and so one may choose $\delta < \varepsilon$ be such that if $\mu(F) < \delta$, then

$$\sup_n \sup_{\|x^*\| \le 1} \int_F |s(x^*, \Gamma_n)|\, d\mu < \varepsilon. \tag{28}$$

The scalar equi-Cauchy condition in measure yields existence of $k \in \mathbb{N}$ such that for all $m, n \ge k$

$$\sup_{\|x^*\| \le 1} \mu(H_{m,n,x^*}) < \delta.$$

Let $\mathbb{N} \ni n(x^*) > k$ be such that $\sup_{n \ge n(x^*)} \int_\Omega |s(x^*, \Gamma_n) - s(x^*, \Gamma)|\, d\mu < \varepsilon$. Denote also – for simplicity – the set $H_{n,n(x^*),x^*}$ by H_{nx^*}. Then, for all $n \ge k$ and a fixed $x^* \in B_{X^*}$

$$\begin{aligned}
&\left| \int_E s(x^*, \Gamma_n)\, d\mu - \int_E s(x^*, \Gamma)\, d\mu \right| \\
&\le \left| \int_E s(x^*, \Gamma_n)\, d\mu - \int_E s(x^*, \Gamma_{n(x^*)})\, d\mu \right| + \left| \int_E s(x^*, \Gamma_{n(x^*)})\, d\mu - \int_E s(x^*, \Gamma)\, d\mu \right| \\
&\le \left| \int_{E \cap H_{nx^*}} s(x^*, \Gamma_n)\, d\mu - \int_{E \cap H_{nx^*}} s(x^*, \Gamma_{n(x^*)})\, d\mu \right| + \\
&\quad + \left| \int_{E \cap H^c_{nx^*}} s(x^*, \Gamma_n)\, d\mu - \int_{E \cap H^c_{nx^*}} s(x^*, \Gamma_{n(x^*)})\, d\mu \right| + \varepsilon.
\end{aligned}$$

Observe that by (28), we have

$$\left|\int_{E\cap H_{n(x^*)}} s(x^*,\varGamma_n)\,d\mu - \int_{E\cap H_{n(x^*)}} s(x^*,\varGamma_m)\,d\mu\right| \le$$
$$\le \left|\int_{E\cap H_{n(x^*)}} s(x^*,\varGamma_n)\,d\mu\right| + \left|\int_{E\cap H_{n(x^*)}} s(x^*,\varGamma_m)\,d\mu\right| \le 2\varepsilon$$

and then

$$\left|\int_{E\cap H^c_{n(x^*)}} s(x^*,\varGamma_n)\,d\mu - \int_{E\cap H^c_{n(x^*)}} s(x^*,\varGamma_{n(x^*)})\,d\mu\right| \le \eta\,\mu(E\cap H^c_{n(x^*)}) < \varepsilon.$$

Consequently, if $n>k$, then

$$\left|\int_E s(x^*,\varGamma_n)\,d\mu - \int_E s(x^*,\varGamma)\,d\mu\right| < 5\varepsilon.$$

Since k is independent of x^* and of $E\in\varSigma$, we have finally for $n>k$ and arbitrary $A\in\varSigma$

$$\begin{aligned} d_H(M_{\varGamma_n}(A),M(A)) &= \sup_{\|x^*\|\le 1} |s(x^*,M_{\varGamma_n}(A)) - s(x^*,M(A))| \\ &\le \sup_{\|x^*\|\le 1}\left|\int_A s(x^*,\varGamma_n)\,d\mu - \int_A s(x^*,\varGamma)\,d\mu\right| \le 5\varepsilon. \end{aligned}$$

An appeal to (6) completes the proof.

If $\lim_n d_P(\varGamma,\varGamma_n)=0$, then obviously $(\varGamma_n)_n$ is scalarly equi-convergent in measure to $\varGamma$, so it is also scalarly equi-Cauchy in measure. □

Remark 6.4. Instead of the uniform integrability of $\bigcup_n \mathcal{Z}_{\varGamma_n}$ one may assume that $\mathcal{Z}_\varGamma$ is uniformly integrable. Such an assumption yields the uniform integrability of $\bigcup_n \mathcal{Z}_{\varGamma_n}$ (see the proof of [23, Proposition 3.3]).

It is perhaps worth to recall here that contrary to Pettis integrable functions, if $\varGamma$ is Pettis integrable in $cb(X)$, then the set $\{s(x^*,\varGamma)\colon x^*\in B_{X^*}\}$ may be not uniformly integrable (see [22, Remark 1.13]). It is however so in case of strongly Pettis integrable $\varGamma$ and in case of X not containing any isomorphic copy of c_0. Indeed, since $M_\varGamma$ is then an h-multimeasure (see [2, Proposition 4.1] or [10, Theorem 3.1]) its range $j\circ M_\varGamma$ in the Banach space $Y=\ell_\infty(B_{X^*})$ via the Rådström embedding $j: cb(X)\to\ell_\infty(B_{X^*})$ is a vector measure. Consequently, the family $\{\langle y^*, j\circ M_\varGamma\rangle\colon y^*\in\ell^*_\infty(B_{X^*})\ \&\ \|y^*\|\le 1\}$ is uniformly μ-continuous and $\sup_{\|y^*\|\le 1}|\langle y^*, j\circ M_\varGamma\rangle|(\varOmega)<\infty$. It means in particular that the family $\mathcal{Z}_\varGamma$ is uniformly integrable. But then it follows from [23, Proposition 3.3] that also $\bigcup_n \mathcal{Z}_{\varGamma_n}$ is uniformly integrable and $\lim_n d_H(M_\varGamma(E), M_{\varGamma_n}(E))=0$ for each $E\in\varSigma$. □

There is an amount of papers where martingale selections of a $cb(X)$-valued multimartingale are constructed in case of separable X (cf. [15], [7, Proposition 2.7], [13, Theorem 3.7]). Unfortunately, if X is non-separable then – as a rule – Castaing representation fails and in fact it is unknown whether even one martingale selection exists. When quasi-selections are taken into account, then an uncountable Castaing type representations of scalarly equivalent martingales become possible.

Proposition 6.5. *Assume that X has RNP, $(\Gamma_n, \Sigma_n)_{n=1}^{\infty}$ is a martingale of multifunctions $\Gamma_n \in \mathbb{P}(\mu, cb(X))$, $n \in \mathbb{N}$, and $\rho \in \Lambda(\mu)$. For each $n \in \mathbb{N}$ let M_n be the Pettis integral of Γ_n. Assume that every Σ_n contains $\mathcal{N}(\mu)$, Σ is the μ-completion of $\sigma(\bigcup_n \Sigma_n)$ and there exists a partition $\{F_k : k \in \mathbb{N}\} \subset \Sigma_1$ of Ω such that*

$$\sup_n |M_n|(F_k) < \infty \quad \textit{for every } k \in \mathbb{N}\,. \tag{29}$$

Then

(6.5.i) *there exists a measurable $cb(X)$-valued martingale $(\widetilde{\Gamma_n}, \Sigma_n)_n$ that is scalarly equivalent to $(\Gamma_n, \Sigma_n)_n$;*

(6.5.ii) *$(\widetilde{\Gamma_n}, \Sigma_n)_n$ admits a family $\{\gamma_{nt} : t \in T\,, n \in \mathbb{N}\}$ of X-valued Pettis integrable functions such that for each $t \in T$ $(\gamma_{nt}, \Sigma_n)_n$ is a martingale selection of $(\widetilde{\Gamma_n}, \Sigma_n)_n$;*

(6.5.iii) *$\widetilde{\Gamma_n} = \overline{conv}\{\gamma_{nt} : t \in T\}$ on $Dom(\Gamma_n)$, for each $n \in \mathbb{N}$;*

(6.5.iv) *Each martingale selection of $(\widetilde{\Gamma_n}, \Sigma_n)_n$ is a.e. convergent. If we have $\sup_n \|s(x^*, \Gamma_n)\|_{L_1} < \infty$ for every $x^* \in X^*$, then the limit function of every martingale selection is Pettis integrable.*

Proof. Since Σ_n contains $\mathcal{N}(\mu)$ the restriction of ρ to Σ_n is a lifting on Σ_n. It follows then from Proposition 4.4 that for each n there exists a measurable with respect to Σ_n multifunction $\widetilde{\Gamma_n}$ that is scalarly equivalent to Γ_n. That solves (6.5.i). Moreover, $\widetilde{\Gamma}_n$ can be defined by any of the following equalities:

$$\widetilde{\Gamma}_n(\omega) = \begin{cases} \overline{\{\rho^{\#}(\gamma)(\omega) : \gamma \in \mathcal{D}^{mP}_{M_n}\} \cap X}, & \text{if } \omega \in Dom(\Gamma_n); \\ \\ X \cap \overline{\{\rho^{\#}(\gamma)(\omega) : \gamma \in \mathcal{D}^{mP}_{M_n}\}}, & \text{if } \omega \in Dom(\Gamma_n);. \end{cases} \tag{30}$$

Let us fix $k \in \mathbb{N}$. For each $m \in \mathbb{N}$ denote by M^k_m the restriction of M_m to the σ-algebra $\Xi^k_m := F_k \cap \Sigma_m$ on F_k. If $m \geq k$ and $\gamma_{mk} \in \mathcal{D}^{mP}_{M^k_m}$, then there exists $\gamma_{m+1,k} \in \mathcal{D}^{P}_{M^k_{m+1}}$ such that $\mathbb{E}_{\Xi^k_m}(\gamma_{m+1,k}) = \gamma_{mk}$. That fact follows from RNP of X and Proposition 5.1, because then each M^k_m is $\mu|_{\Xi^k_m}$-rich in strong densities. $(\rho^{\#}(\gamma_{nk}), \Xi^k_n)_{n \geq k}$ is a martingale selection of $(\widetilde{\Gamma_n}|_{F_k}, \Xi^k_n)_{n \geq k}$ bounded in $L_1(\mu|_{F_k}, X)$. We extend the martingale for $m < k$ by setting $\gamma_{mk} := \rho^{\#}[\mathbb{E}_{\Xi^k_m}(\gamma_{kk})]$. If $\gamma_m := \sum_k \gamma_{mk}\chi_{\rho(F_k)}$, then γ_m is a scalarly integrable selection of $\widetilde{\Gamma}_m$. Since $c_0 \not\subseteq X$ isomorphically, the strongly measurable function γ_m is Pettis integrable (cf. [11, Theorem II.3.7]) and $(\gamma_m, \Xi_m)_m$ defines a martingale. This proves (6.5.ii). The condition (6.5.iii) is a consequence of the construction of each $\widetilde{\Gamma}_m$ presented in Theorem 5.5.

Since X has RNP, each martingale selection of $(\widetilde{\Gamma_n}, \Sigma_n)_n$ is convergent μ-a.e. on F_k to a strongly measurable function. Hence, the same is valid for the whole Ω. By the assumption each martingale selection of $(\widetilde{\Gamma}_n, \Sigma_n)_n$ is bounded in $L_1(\mu)$. Since X does not contain any isomorphic copy of c_0 its limit function is Pettis integrable on Ω. □

Remark 6.6. One can apply Remark 5.4 to give a new proof of [13, Theorem 3.7] without assuming RNP of X^*. □

Remark 6.7. The assumption (29) can be weaken but for the price of a more complicated thesis. Namely, one could assume only that $F_k \in \Sigma_{n_k}$ with $n_k < n_{k+1}$ and $\sup_{n \geq n_k} |M_n|(F_k) < \infty$. Then, we apply the thesis of Proposition 6.5 for each F_k separately and for each k the set T may be different. □

Contrary to Theorem 6.3, assuming RNP of X one gets a chance to prove existence of a limit multifunction invoking to properties of the martingale under consideration only. We need yet one more definition. We say that a sequence $(A_n)_n$ of elements of $cb(X)$ is τ_L-converging to $A \in cb(X)$ if $s(x^*, A_n) \longrightarrow s(x^*, A)$ for every $x^* \in X^*$ and $d(x, A_n) \longrightarrow d(x, A)$ for every $x \in X$. $d(x, B)$ is the distance of x from B.

The subsequent result is a generalization of [13, Theorem 4.6] in its existence part.

Theorem 6.8. *Assume that X has RNP, $(\Gamma_n, \Sigma_n)_{n=1}^{\infty}$ is a martingale of multifunctions $\Gamma_n : \Omega \to c(X)$, $n \in \mathbb{N}$, that are Pettis integrable in $cb(X)\,[cwk(X), ck(X)]$ and $\rho \in \Lambda(\mu)$. Assume that every Σ_n contains $\mathcal{N}(\mu)$, Σ is the μ-completion of $\sigma\left(\bigcup_n \Sigma_n\right)$ and there exists a partition $\{F_k : k \in \mathbb{N}\} \subset \Sigma_1$ of Ω such that*

(6.8.i) $\sup_n |M_{\Gamma_n}|(F_k) < \infty$ *for every* $k \in \mathbb{N}$;

(6.8.ii) $\sup_n \|s(x^*, \Gamma_n)\|_{L_1} < \infty$ *for every* $x^* \in X^*$;

Then there exists a measurable multifunction $\Gamma \in \mathbb{P}(\mu, cb(X))\,[cwk(X), ck(X)]$ and a martingale $(\widetilde{\Gamma}_n, \Sigma_n)_n$ of measurable $cb(X)$-valued multifunctions such that

(6.8.iii) *For each* $k \in \mathbb{N}$ $|M_\Gamma|(F_k) < \infty$;

(6.8.iv) *For each $x^* \in X^*$ we have $\lim_n s(x^*, \Gamma_n) = s(x^*, \Gamma)$ a.e. and* $\lim_n \|s(x^*, \Gamma_n) - s(x^*, \Gamma)\|_{L_1} = 0$;

(6.8.v) *For each $n \in \mathbb{N}$ $\widetilde{\Gamma}_n$ is a measurable version of $\mathbb{E}_{\Sigma_n}(\Gamma)$ and $\widetilde{\Gamma}_n \overset{s}{\approx} \Gamma_n$.*

If moreover the set $\bigcup_n \mathcal{Z}_{\Gamma_n}$ is uniformly integrable, then

(6.8.vi) *For each $E \in \Sigma$ $\tau_L - lim_n (P)\int_E \Gamma_n \, d\mu = (P)\int_E \Gamma \, d\mu$;*

(6.8.vii) $\lim_n d_H(M_{\Gamma_n}(E), M_\Gamma(E)) = 0$ *for every* $E \in \Sigma$.

Proof. In virtue of Proposition 6.5 there exists a martingale $(\gamma_n, \Sigma_n)_n$ such that each γ_n is a Pettis integrable quasi-selection of Γ_n. Then $(\Gamma_n - \gamma_n, \Sigma_n)_n$ is a martingale of Pettis integrable multifunctions with the property $s(x^*, \Gamma_n - \gamma_n) \geq)$ μ-a.e. for each $x^* \in X^*$ separately. To simplify the proof we assume that we have already $s(x^*, \Gamma_n) \geq 0$ μ-a.e.

(6.8.iii)–(6.8.v). Denote the algebra $\bigcup_n \Sigma_n$ by Σ_0. Without loss of generality we may also assume that $F_k = \rho(F_k)$, $k \in \mathbb{N}$. According to Proposition 6.5 the formula (30) defines a martingale $(\widetilde{\Gamma}_n, \Sigma_n)_n$ that is scalarly equivalent to $(\Gamma_n, \Sigma_n)_n$ and each multifunction $\widetilde{\Gamma}_n$ is measurable with respect to Σ_n. In virtue of Proposition

4.4 the function $|\widetilde{\Gamma}_n|$ is measurable and so it follows from [22, Theorem 4.7] that $|M_{\Gamma_n}|(E) = \int_E |\widetilde{\Gamma}_n|\,d\mu$, for every $E \in \Sigma_n$.

Let $k \in \mathbb{N}$ be fixed. It follows then from Fatou's lemma and (6.8.i) that

$$\int_{E\cap F_k} \liminf_n |\widetilde{\Gamma}_n|\,d\mu \le \liminf_n \int_{E\cap F_k} |\widetilde{\Gamma}_n|\,d\mu < \infty\,.$$

But that yields $\liminf_n |\widetilde{\Gamma}_n|\chi_{F_k} < \infty$ a.e. Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be such that $\mu(E \cap F_k) < \delta$ yields $\int_{E\cap F_k} \liminf_n |\widetilde{\Gamma}_n|\,d\mu < \varepsilon$. If $x^* \in B_{X^*}$, then $(s(x^*,\widetilde{\Gamma}_n))_n$ is a bounded martingale in $L_1(\mu|_{F_k})$ and so it is convergent in $L_1(\mu|_{F_k})$ and μ-a.e. to a function in $L_1(\mu|_{F_k})$. Let

$$M^k_{x^*}(E\cap F_k) := \int_{E\cap F_k} \lim_n s(x^*,\widetilde{\Gamma}_n)\,d\mu = \lim_n s(x^*, M_{\Gamma_n}(E\cap F_k)) \quad \text{for } E \in \Sigma_0\,.$$

But $|s(x^*,\widetilde{\Gamma}_n)| \le |\widetilde{\Gamma}_n|,\ n \in \mathbb{N}$, on F_k and so $\lim_n |s(x^*,\widetilde{\Gamma}_n\chi_{F_k})| \le \liminf_n |\widetilde{\Gamma}_n|\chi_{F_k}$ μ-a.e. Consequently, if $E \in \Sigma_0$ and $\mu(E\cap F_k) < \delta$, then

$$\begin{aligned} |M^k_{x^*}(E\cap F_k)| &= \lim_n |s(x^*, M_{\Gamma_n}(E\cap F_k))| = \lim_n \left|\int_{E\cap F_k} s(x^*,\widetilde{\Gamma}_n)\,, d\mu\right| \\ &\le \int_{E\cap F_k} \lim_n |s(x^*,\widetilde{\Gamma}_n)| \le \int_{E\cap F_k} \liminf_n |\widetilde{\Gamma}_n|\,d\mu < \varepsilon, \end{aligned} \tag{31}$$

for an arbitrary $x^* \in B_{X^*}$. We have $M_{\Gamma_n}(E) = M_{\Gamma_{n+1}}(E)$ for every $n \in \mathbb{N}$ and $E \in \Sigma_n$ and so we can define $M^\dagger_k : \Sigma_0 \cap F_k \to cb(X)$ by $M^\dagger_k(E) := M_{\Gamma_n}(E\cap F_k)$ if $E \in \Sigma_n$. $M^\dagger_k$ is an additive set function of finite variation on F_k. This is a direct consequence of the martingale property of $(M_{\Gamma_n})_n$ and of (6.8.i). Indeed, we have

$$\begin{aligned} |M^\dagger_k|(F_k) &= \sup\{\sum_{i=1}^m |M_{\Gamma_{n_i}}(E_{n_i}\cap F_k)| \colon \forall\, i\ E_{n_i} \in \Sigma_{n_i}\ n_1 \le n_2 \le \ldots \le n_m,\ m \in \mathbb{N}\} \\ &= \sup\{\sum_{i=1}^m |M_{\Gamma_{n_m}}(E_{n_i}\cap F_k)| \colon \forall\, i\ E_{n_i} \in \Sigma_{n_i}\ n_1 \le n_2 \le \ldots \le n_m,\ m \in \mathbb{N}\} \\ &\le \sup_m |M_{\Gamma_{n_m}}|(F_k) < \infty. \end{aligned}$$

Due to the initial assumption of positivity of the support functionals of every Γ_n, we have $0 \in M^\dagger_k(E)$, for every $E \in \Sigma_0$. One of the consequences of (31) is absolute continuity of $M^k_{x^*}$ on the algebra $\Sigma_0 \cap F_k$ in the $\varepsilon - \delta$ sense. That means that each $M^k_{x^*}$ is σ-additive on the algebra $\Sigma_0 \cap F_k$. But if $E \in \Sigma_0$, then

$$M^k_{x^*}(E\cap F_k) = \lim_n \int_{E\cap F_k} s(x^*,\Gamma_n)\,d\mu = \lim_n s(x^*, M_{\Gamma_n}(E\cap F_k)) = s(x^*, M^\dagger_k(E\cap F_k)).$$

This gives countable additivity of $s(x^*, M^\dagger_k)$ on $\Sigma_0 \cap F_k$. Moreover, formula (31) proves also that the $\varepsilon - \delta$ absolute continuity of the family $\{s(x^*, M^\dagger_k) \colon \|x^*\| \le 1\}$ is uniform on B_{X^*}. It follows from [19, Proposition 2.5] that $M_k^\dagger$ has an extension to a multimeasure $M^k : \Sigma \cap F_k \to cb(X)$. M^k is of finite variation on F_k because $M_k^\dagger$ is of finite variation on F_k and Σ_0 is μ-dense in Σ. M^k is in fact countably additive in the Hausdorff metric of $cb(X)$. Moreover, $0 \in M^k(E \cap F_k)$ for every $E \in \Sigma$.

In virtue of Theorem 4.5 M^k can be represented as $M^k(E \cap F_k) = (P)\int_{E\cap F_k} \Gamma^k \, d\mu$, where $\Gamma^k : F_k \to cb(X)$ is measurable with respect to $\Sigma \cap F_k$. But we have $M^k|_{\Sigma_n \cap F_k} = M_{\Gamma_n}|_{\Sigma_n\cap F_k}$, what yields (Theorem 5.5)

$$\mathbb{E}_{\Sigma_n}(\Gamma^k) \overset{s}{\approx} \Gamma_n|_{F_k} \overset{s}{\approx} \widetilde{\Gamma}_n|_{F_k}. \tag{32}$$

It follows that for each $x^* \in X^*$ we have also

$$s(x^*, \mathbb{E}_{\Sigma_n}(\Gamma^k)) = s(x^*, \Gamma_n|_{F_k}) = s(x^*, \widetilde{\Gamma}_n|_{F_k}) \quad \mu\text{-a.e.}$$

But due to (17) $s(x^*, \mathbb{E}_{\Sigma_n}(\Gamma^k)) = \mathbb{E}_{\Sigma_n} s(x^*, \Gamma^k)$ μ-a.e. and as $(\mathbb{E}_{\Sigma_n} s(x^*, \Gamma^k), \Sigma_n)_n$ is a martingale that is on F_k μ-a.e. and in $L_1(\mu|_{F_k})$ convergent to $s(x^*, \Gamma^k)$, we obtain the equality

$$\lim_n s(x^*, \Gamma_n|_{F_k}) = \lim_n s(x^*, \widetilde{\Gamma}_n|_{F_k}) = s(x^*, \Gamma^k) \quad \mu - \text{a.e. on } F_k\,. \tag{33}$$

As $0 \in M^k(E \cap F_k)$ for every $E \in \Sigma$, the sequence $\left(\bigoplus_{k=1}^m M^k(E \cap F_k)\right)_m$ is non-decreasing. It follows from [27, Proposition 1] that

$$\begin{aligned}\lim_m \sum_{k=1}^m s(x^*, M^k(E \cap F_k)) &= \lim_m s\left(x^*, \bigoplus_{k=1}^m M^k(E \cap F_k)\right) \\ &= s\left(x^*, \overline{\bigcup_m \bigoplus_{k=1}^m M^k(E \cap F_k)}\right).\end{aligned}$$

We define now $M : \Sigma \to c(X)$ setting $M(E) := \overline{\bigcup_m \bigoplus_{k=1}^m M^k(E \cap F_k)}$. Clearly $M|_{F_k} = M^k$ for every $k \in \mathbb{N}$. It is a consequence of the next calculation that $M(E) \in cb(X)$.

$$\begin{aligned}|s(x^*, M(E))| &= \left|\sup_m \sum_{k=1}^m s(x^*, M^k(E \cap F_k))\right| = \left|\sup_m \sum_{k=1}^m \int_{E\cap F_k} s(x^*, \Gamma^k)\, d\mu\right| \\ &= \left|\sup_m \sum_{k=1}^m \int_{E\cap F_k} \lim_n s(x^*, \Gamma_n)\, d\mu\right| = \left|\sup_m \lim_n \int_{E\cap \bigcup_{k=1}^m F_k} s(x^*, \Gamma_n)\, d\mu\right| \\ &\le \sup_m \limsup_n \int_{E\cap \bigcup_{k=1}^m F_k} |s(x^*, \Gamma_n)|\, d\mu \le \|x^*\| \sup_n \|s(x^*, \Gamma_n)\|_{L_1} < \infty.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^m |s(x^*, M^k)|(E \cap F_k)) &= \sum_{k=1}^m \int_{E\cap F_k} |s(x^*, \Gamma^k)|\, d\mu \\ &= \sum_{k=1}^m \int_{E\cap F_k} |\lim_n s(x^*, \Gamma_n)|\, d\mu = \sum_{k=1}^m \lim_n \int_{E\cap F_k} |s(x^*, \Gamma_n)|\, d\mu \\ &= \lim_n \int_{E\cap \bigcup_{k=1}^m F_k} |s(x^*, \Gamma_n)|\, d\mu \le \|x^*\| \sup_n \|s(x^*, \Gamma_n)\|_{L_1} < \infty.\end{aligned}$$

Thus,

$$s(x^*, M(E)) = \sup_m \sum_{k=1}^{m} s(x^*, M^k(E \cap F_k)) \tag{34}$$
$$= \lim_m \sum_{k=1}^{m} s(x^*, M(E \cap F_k)) = \sum_{k=1}^{\infty} s(x^*, M(E \cap F_k))$$

and the series is absolutely convergent. Assume now that $E = \bigcup_n E_n$ and the sets E_n are pairwise disjoint. Then, applying (34) and the countable additivity of $s(x^*, M)$ on each F_k, we have

$$s\left(x^*, M\left(\bigcup_n E_n\right)\right) = \sum_{k=1}^{\infty} s\left(x^*, M\left(\bigcup_n E_n \cap F_k\right)\right) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} s(x^*, M(E_n \cap F_k))$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} s(x^*, M(E_n \cap F_k)) = \sum_{n=1}^{\infty} s(x^*, M(E_n)).$$

Thus, M is a μ-continuous multimeasure of σ-finite variation. $\varGamma : \varOmega \to cb(X)$ defined by $\varGamma|_{F_k} = \varGamma^k$ is an obvious candidate for the Pettis integrable density of M with respect to μ. Indeed, applying the absolute convergence of the series $\sum_k \int_{E \cap F_k} |s(x^*, \varGamma^k)|\, d\mu$ following from (34), we have

$$s(x^*, M(E)) = \sum_k s(x^*, M(E \cap F_k)) = \sum_k \int_{E \cap F_k} s(x^*, \varGamma^k)\, d\mu = \int_E s(x^*, \varGamma)\, d\mu.$$

Thus, $\varGamma \in \mathbb{P}(\mu, cb(X))$ and $M = M_\varGamma$. The scalar convergence required in (6.8.iv) is a consequence of (33) and of the convergence of the scalar martingales in $L_1(\mu)$. (6.8.v) follows from (32).

(6.8.vi)–(6.8.vii). Assume now that the set $\bigcup_n \mathcal{Z}_n$ is uniformly integrable. M is rich in strong densities and so if $E \in \varSigma$, then $M(E) = \overline{\{(P)\int_E \gamma\, d\mu \colon \gamma \in \mathcal{D}_M^{mP}\}}$. If γ is a Pettis integrable density of $m \in \mathcal{S}(M)$, then $(\gamma_n := \mathbb{E}_{\varSigma_n}(\gamma), \varSigma_n)_n$ is a martingale such that γ_n is a quasi-selection of $\varGamma_n$. It follows from [6] that the martingale is μ-a.e. convergent to γ and [22, Corollary 5.3] yields the convergence $\lim_n \|\gamma_n - \gamma\|_P = 0$.

Due to this convergence, if $x \in X$, then

$$\limsup_n d\left(x, (P)\int_E \varGamma_n\, d\mu\right) \le \lim_n \left\| x - (P)\int_E \gamma_n\, d\mu \right\| = \left\| x - (P)\int_E \gamma\, d\mu \right\|,$$

what yields

$$\limsup_n d\left(x, (P)\int_E \varGamma_n\, d\mu\right) \le d\left(x, (P)\int_E \varGamma\, d\mu\right). \tag{35}$$

We now apply [1, Corollary 1.5.3]:

If $\emptyset \neq B \subset X$, then $d(x, B) = \sup\{\langle x^*, x\rangle - s(x^*, B) : x^* \in B_{X^*}\}$.

The following calculation leads to the desired result:

$$
\begin{aligned}
d\Big(x,(P)\int_E \Gamma\,d\mu\Big) &= \sup\Big\{\Big|\langle x^*,x\rangle - s\Big(x^*,(P)\int_E \Gamma\,d\mu\Big)\Big| : x^*\in B_{X^*}\Big\} \qquad (36)\\
&= \sup\Big\{\Big|\langle x^*,x\rangle - \lim_n s\Big(x^*,(P)\int_E \Gamma_n\,d\mu\Big)\Big| : x^*\in B_{X^*}\Big\}\\
&= \sup\Big\{\liminf_n\Big|\langle x^*,x\rangle - s\Big(x^*,(P)\int_E \Gamma_n\,d\mu\Big)\Big| : x^*\in B_{X^*}\Big\}\\
&\le \liminf_n \sup\Big\{\Big|\langle x^*,x\rangle - s\Big(x^*,(P)\int_E \Gamma_n\,d\mu\Big)\Big| : x^*\in B_{X^*}\Big\}\\
&= \liminf_n d\Big(x,(P)\int_E \Gamma_n\,d\mu\Big).
\end{aligned}
$$

The inequalities (35) and (36) give the required convergence in (6.8.vi).

The condition (6.8.vii) follows from Theorem 6.3.

If $(\Gamma_n,\Sigma_n)_n$ is arbitrary and $(\gamma_n,\Sigma_n)_n$ is a martingale as at the beginning of the proof, then $(\gamma_n,\Sigma_n)_n$ fulfills the thesis of Theorem 6.8 and this completes the whole proof. □

References

[1] G. Beer: *Topologies on Closed and Closed Convex Sets*, Kluwer Academic Press, Dordrecht (1993).

[2] D. Candeloro, L. Di Piazza, K. Musiał, A. R. Sambucini: *Integration of multifunctions with closed convex values in arbitrary Banach spaces*, J. Convex Analysis 27 (2020) 1233–1246.

[3] B. Cascales, V. Kadets, J. Rodriguez: *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Analysis 256/3 (2009) 673–699.

[4] B. Cascales, V. Kadets, J. Rodriguez: *Radon-Nikodym theorems for multimeasures in non-separable spaces*, J. Math. Physics, Analysis, Geometry 9 (2013) 7–24.

[5] C. Castaing, M. Valadier: *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer, Berlin (1977).

[6] S. D. Chatterji: *Martingale convergence and the Radon-Nikodym theorem in Banach spaces*, Math. Scand. 22 (1968) 21–41.

[7] A. Choukairi-Dini: *M-convergence, et régularité des martingales multivoques: Epi-martingales*, J. Multivar. Analysis 33 (1990) 39–71.

[8] A. Costé: *Set-valued measures*, in: Proc. Conf. Topology and Measure I, Zinnowitz 1974, Wiss. Beiträge Ernst-Moritz-Arndt-Universität, Greifswald (1975) 55–74.

[9] A. Costé: *La propriété de Radon-Nikodym en intégration multivoque*, C. R. Acad. Sci. Paris 280 (1975) 1515–1518.

[10] L. Di Piazza, K. Musiał, A. R. Sambucini: *Representations of multimeasures via the multivalued Bartle-Dunford-Schwartz integral*, J. Convex Analysis 29 (2022) 1119–1148.

[11] J. Diestel, J. J. Uhl: *Vector Measures*, Mathematical Surveys 15, American Mathematical Society, Providence (1977).

[12] S. Eder: *Some aspects of the lifting property of $\mathcal{L}_\infty(\mu, X)$*, Quaestiones Math. 21 (1998) 269–288.

[13] M. El Allali, M. El-Louh, F. Ezzaki: *On convergence and regularity of Aumann-Pettis integrable multivalued martingales*, J. Convex. Analysis 27 (2020) 1177–1194.

[14] K. El Amri, C. Hess: *On the Pettis integral of closed valued multifunctions*, Set-Valued Analysis 8 (2000) 329–360.

[15] C. Hess: *On multivalued martingales whose values may be unbounded:martingale selectors and Mosco convergence*, J. Multivar. Analysis 39 (1991) 175–201.

[16] C. Hess: *Set-valued integration and set-valued probability theory: an overview*, in: *Handbook of Measure Theory I*, North-Holland, Amsterdam (2002) 617–673.

[17] S. Hu, N. S. Papageorgiou: *Handbook of Multivalued Analysis I*, Kluwer Academic Publishers, Dordrecht (1997).

[18] A. Ionescu Tulcea, C. Ionescu Tulcea: *Topics in the Theory of Lifting*, Springer, Berlin (1969).

[19] D. A. Kandilakis: *On the extension of multimeasures and integration with respect to a multimeasure*, Proc. Amer. Math. Soc. 116 (1992) 85–92.

[20] K. Musiał: *Topics in the theory of Pettis integration*, Rend. Ist. Mat. Univ. Trieste XXIII (1991) 177–262.

[21] K. Musiał: *Pettis Integral*, in: *Handbook of Measure Theory I*, North-Holland, Amsterdam (2002) 532–586.

[22] K. Musiał: *Pettis integrability of multifunctions with values in arbitrary Banach spaces*, J. Convex Analysis 18/3 (2011) 769–810.

[23] K. Musiał: *Approximation of Pettis integrable multifunctions with values in arbitrary Banach spaces*, J. Convex Analysis 20/3 (2013) 833–870.

[24] K. Musiał: *Multimeasures with values in conjugate Banach spaces and the Weak Radon-Nikodým property*, J. Convex Analysis 28/3 (2021) 879–902.

[25] K. Musiał: *Lifting approach to integral representations of rich multimeasures with values in Banach spaces I*, J. Convex Analysis 31/1 (2024) 179–193.

[26] J. Neveu: *Bases Mathématiques du Calcul des Probabilités*, Masson, Paris (1964).

[27] Y. Sonntag, C. Zalinescu: *Scalar convergence of convex sets*, J. Math. Analysis Appl. 164 (1992) 219–241.

[28] M. Talagrand: *Pettis Integral and Measure Theory*, Memoirs of the AMS 307, American Mathematical Society, Providence (1984).