Journal of Convex Analysis
Volume 31 (2024), No. 3, 889–946

Generalised Young Measures and Characterisation of Gradient Young Measures

Tommaso Seneci
Mathematical Institute, University of Oxford, England
seneci@maths.ox.ac.uk

Received: February 25, 2023
Accepted: September 16, 2023

Given a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ with p-growth, we extend the framework developed by J.-J. Alibert and G. Bouchitté [*Non-uniform integrability and generalized Young Measure*, J. Convex Analysis 4 (1997) 129–148] obtaining a new way of representing accumulation points of

$$\int_\Omega f(v_i(z))\, d\mu(z),$$

where μ is a finite positive Borel measure on an open bounded set $\Omega \subset \mathbb{R}^n$, and $(v_i)_{i\in\mathbb{N}} \subset L^p(\Omega,\mu)$ is norm bounded. We call such representations *generalised Young Measures.*

With the help of the new representation, we then characterise these limits when they are generated by gradients, that is, when $v_i = Du_i$ for $u_i \in W^{1,1}(\Omega,\mathbb{R}^m)$, via a set of integral inequalities.

Keywords: Young measures, separable compactifications, functions of bounded variation, quasiconvexity.

2020 Mathematics Subject Classification: 49J45, 28B05, 35B05, 35B33, 46G10, 54D35.

1. Introduction

1.1. Background

Young Measures were first introduced by Young in [57] to study the minima of integral energies of the form

$$\inf\left\{\int_0^1 f(u(t),u'(t))dt : u \in C^1([0,1]), u(0)=a, u(1)=b, \|u'\|_\infty \le K\right\},$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and for related problems in optimal control theory. The author wanted to understand what conditions on f would guarantee the existence of a minimizing curve $u(t)$. Young had the intuition that minimising sequences $(u_j)_j$ in $C^1([0,1])$ always admit a subsequence (non-relabelled) and $u \in C([0,1])$ with $u' \in L^\infty([0,1])$ so that $u_j \to u$ uniformly and u_j' converges to a *generalised curve*

$$\nu = \nu_t\colon [0,1] \to \mathbb{M}_1^+(\mathbb{R}).$$

This translates to the following equality for every continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$:

$$\inf_{\substack{u(0)=a\\ u(1)=b}} \int_0^1 f(u(t),u'(t))dt = \lim_j \int_0^1 f(u_j(t),u_j'(t))dt = \int_0^1\int_{\mathbb{R}} f(u(t),z)d\nu_t(z)dt.$$

ISSN 0944-6532 / \$ 2.50

So the question of the existence of a minimiser can be reformulated in two different ways.

1. Are minimising sequences so well behaved that every generalised curve ν_t is, actually, just the gradient of a function u? In mathematical terms, we have to understand if $\nu_t = \delta_{u'(t)}$ for almost every $t \in (0,1)$.
2. Is f well behaved so that
$$\int_{\mathbb{R}} f(u(t),z)d\nu_t(z) \geq f(u(t),u'(t))$$
at $\mathcal{L}^1$ almost every $t \in (0,1)$?

Young's original work focused on the case $n = 1$ and was carried out via functional analytic methods. This approach was later extended by many authors, including [8, 9, 11, 16, 18, 44, 45] to higher dimensions. We call these generalised functions *oscillation Young Measures*. The method developed by Young is not powerful enough to tackle many problems arising in modern mathematics, as it essentially only describes oscillations in sequences $(v_j)_{j\in\mathbb{N}}$ that are bounded in L^∞. In [56], Tartar pioneered the use of Young Measures as a tool to study non-linear partial differential equations, which moved the development of classical Young Measures to the study of Young Measures in Lebesgue spaces, as they are the natural spaces for many questions related to partial differential equations. The first attempt to well represent generalised limits of integrable functions that also takes concentration effects into account is due to DiPerna and Majda, [21]. Functions $v_j \in L^p(\Omega,\mu,\mathbb{R}^d)$ - Borel measurable and so that $|v_j(x)|^p$ is integrable with respect to μ - are identified with the measures on $\Omega \times \mathbb{R}^d$ that result by integrating over their graph, namely
$$\int_\Omega \Phi(x, v_j(x))d\mu(x) = \int_{\Omega\times\mathbb{R}^d} \Phi(x,y)d\left(\big(\cdot, v_j(\cdot)\big)_{\#}\mu\right) =: \langle \xi_{v_j}, \Phi\rangle,$$
where "#" is called the *pushforward operator* and $\big(\cdot, v_j(\cdot)\big)_{\#}\mu \in \mathbb{M}(\Omega\times\mathbb{R}^d)$.
The product space $\Omega\times\mathbb{R}^d$ is subsequently embedded into a compact set $K \supset \Omega\times\mathbb{R}^d$, and assuming that the sequence $(v_j)_j$ is bounded in $L^p(\Omega,\mu,\mathbb{R}^d)$, an accumulation point, in a weak* sense, is then found
$$\xi_{v_{j_i}} \rightharpoonup^* \xi \in \mathbb{M}(K). \tag{1}$$
Such an accumulation point is a measure defined on K, where K is an abstract compact set containing $\Omega\times\mathbb{R}^d$, called *compactification*, and as such, it is not clear how to conveniently represent ξ. In [3], an explicit and natural formula for such accumulation points was obtained for a class of integrands that behave like homogeneous functions at infinity, whereas in other instances such as [40, 54], ξ as in (1) has been used in its bare form to prove results in the calculus of variations.

In what follows, we give a general representation formula for describing Young Measures for a large class of integrands in the spirit of [3]. The construction of Young Measures follows mainly the work by DiPerna and Majda [21], Alibert and Bouchitté [3], and a procedural approach developed in [35], see also [50, 53, 54]. This generalisation is based on the canonical way of constructing Hausdorff compactifications starting from continuous functions, see [19]. A small reduction lemma gives a clearer and more geometrical interpretation of such limits. This formula captures concentrations at infinity, which are now allowed to occur without restrictions. We also prove a few structure theorems that relate different compactifications and their Young Measure representations to each other. Formally, we show that given a bounded sequence $(v_j)_j$ in $L^p(\Omega,\mu,\mathbb{R}^d)$ and a family $\{f_i,\, i\in\mathbb{N}\}$ of integrands of p-growth – meaning $|f(z)|\le C(1+|z|^p)$ for some $C>0$ and all $z\in\mathbb{R}^d$ – there is a subsequence (non-relabelled) and a triple $(\nu_x,\lambda,\nu_x^\infty)$ with, letting $\mathbb{M}^+$ and $\mathbb{M}_1^+$ be, respectively, the set of positive Borel Measures and the set of probability measures,

$$\nu_x\colon \Omega\to\mathbb{M}_1^+(\mathbb{R}^d),\ \lambda\in\mathbb{M}^+(\overline{\Omega})\quad\text{and}\quad \nu_x^\infty\colon\overline{\Omega}\to\mathbb{M}_1^+(\partial e_{f_i,i\in\mathbb{N}}) \tag{2}$$

such that

$$\lim_j\int_\Omega f(v_j(x))d\mu(x)=\int_\Omega\int_{\mathbb{R}^d} f(z)d\nu_x(z)d\mu(x)+\int_{\overline{\Omega}}\int_{\partial e_{f_i,i\in\mathbb{N}}} f^\infty(w)d\nu_x^\infty(w)d\lambda(x).$$

Here $e_{f_i,i\in\mathbb{N}}$ is a compactification of $\mathbb{R}^d$ naturally derived from $\{f_i, i\in\mathbb{N}\}$ and f^∞ is the extension of f to such compactification. This compactification has the topology of a product space (in many cases of interest finite-dimensional) and so it is easier to work with compared to previous representations of generalised Young Measures.

It is important to mention that such generalised Young Measures, defined as the limit (1), were known to exist since DiPerna and Majda, and have been studied and used in applications ever since, see for example [30, 40, 54]. However, the lack of clear representation and geometric interpretation always hindered the use of generalised Young Measures to solve problems from the calculus of variations that involve gradients, one of which we solve in this paper.

This generalisation of Young Measures is then applied to study extensions and variations of energies that depend on gradients, the prototype of which is

$$u\mapsto\int_\Omega f(\nabla u(x))dx, \tag{3}$$

where u is initially taken in a small class of functions like $C^\infty(\Omega,\mathbb{R}^m)\cap W^{1,p}(\Omega,\mathbb{R}^m)$, $\Omega\subset\mathbb{R}^n$ is a bounded domain with Lipschitz boundary, and $f\colon\mathbb{R}^{m\times n}\to\mathbb{R}$ has linear growth. We notice that the space $C^\infty(\Omega,\mathbb{R}^m)\cap W^{1,1}(\Omega,\mathbb{R}^m)$ is dense in many spaces of interest in applications, such as $BV(\Omega,\mathbb{R}^m)$ with the weak* topology. This topology can be particularly hard to work with, but has very interesting outcomes in this class of problems. For example, it can be shown that for any non-linear $f\in C(\mathbb{R}^{m\times n})$ of linear growth, (3) is not sequentially weak* continuous in $BV(\Omega,\mathbb{R}^{m\times n})$, see [10, 12]. We can however find an extension which is lower semi-continuous for certain fs, under suitable growth conditions. In [47], Morrey established the equivalence of lower semi-continuity of (3) to a condition that he called *quasi-convexity*, which can be written as a Jensen-type inequality:

$$f(z)\le\frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f(z+\nabla\phi(x))dx\quad\forall\phi\in\mathcal{D}(\Omega,\mathbb{R}^m)\text{ and }z\in\mathbb{R}^{m\times n}. \tag{4}$$

The original result by Morrey applies to the setting of weak* convergence in the space $W^{1,\infty}(\Omega,\mathbb{R}^m)$ and weak convergence in $W^{1,p}(\Omega,\mathbb{R}^m)$ with $1\le p<+\infty$, when

f has p-growth and it is bounded from below. This was later extended to functions that depend on more than one variable in [46], and to Carathéodory integrands in [1,42,43]. The aforementioned results hold when f is positive and has appropriate growth. In the case of signed integrands, one must work within the Dirichlet class $W^{1,p}_g(\Omega,\mathbb{R}^m)$ for some $g \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)$, which is the set of functions in $W^{1,p}(\Omega,\mathbb{R}^m)$ so that

$$u\chi_\Omega + g\chi_{\Omega^c} \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^m),$$

χ_E being the indicator function of $E \subset \mathbb{R}^n$. In this case, [14] is one of the first examples where Young Measures are employed for proving lower semi-continuity for functionals of the type (3). To be more specific, (4) can be rephrased as a Jensen-type inequality for measures of the form

$$\left\{\nu : \nu = \frac{1}{\mathcal{L}^n(\Omega)}(\nabla\phi)_\# d\mathcal{L}^n \llcorner \Omega, \quad \phi \in \mathcal{D}(\Omega,\mathbb{R}^m)\right\}, \tag{5}$$

where the measure ν acts on f in the following way:

$$\frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f(z+\nabla\phi(x))dx = \frac{1}{\mathcal{L}^n(\Omega)}\int_{\mathbb{R}^{m\times n}} f(z+w)d\nu(w) \equiv \langle \nu, f(z+\cdot)\rangle.$$

It turns out that gradients of a weakly convergent sequence $(u_j)_j$ in $W^{1,p}(\Omega,\mathbb{R}^m)$ with $u_j \rightharpoonup u$ in $W^{1,p}$ if $1 \le p < +\infty$ or $u_j \rightharpoonup^* u$ in $W^{1,\infty}(\Omega,\mathbb{R}^m)$, subsequentially generate an oscillation Young Measure

$$\nu_x \colon \Omega \to \mathbb{M}^+_1(\mathbb{R}^{m\times n}),$$

where ν_x acts on functions $f \in C_0(\mathbb{R}^{m\times n})$ and $\phi \in C_0(\Omega)$ via the pairing

$$\lim_j \int_\Omega \phi(x) f(\nabla u_j(x))dx = \int_\Omega \phi(x)\int_{\mathbb{R}^m} f(z)d\nu_x(z)dx =: \int_\Omega \phi(x)\langle \nu_x, f\rangle dx,$$

and satisfies the moment condition

$$\int_\Omega \langle \nu_x, |\cdot|^p\rangle dx < +\infty$$

if $1 \le p < +\infty$, or $\operatorname{supp}(\nu_x) \subset \mathbb{B}_R(0)$ for some $R > 0$ and $\mathcal{L}^n$-a.e. $x \in \Omega$ if $p = +\infty$. The lower semi-continuity of (3) reduces to a Jensen-type inequality of the form

$$\langle \nu_x, f\rangle \ge f(\nabla u(x)) \quad \text{and} \quad \nabla u(x) = \int_{\mathbb{R}^{m\times n}} z d\nu_x \quad \mathcal{L}^n\text{-a.e.},$$

when f has p-growth and is quasi-convex. In this case, we call $(\nu_x)_{x\in\Omega}$ a *Gradient Young Measure*, which can be seen as an inhomogeneous version of the Young Measure (5). This class can be seen as the closure of the set (5) in a suitable weak* topology of measures over the graph of f. The opposite is also true and was proven in [31, 32] when $1 \le p \le +\infty$ (see also [30, 38, 39] for the case with concentration), i.e. every Young Measure $(\nu_x)_{x\in\Omega}$, $\nu_x \colon \Omega \to \mathbb{M}^+_1(\mathbb{R}^{m\times n})$ that satisfies the moment condition

$$\int_\Omega \langle \nu_x, |\cdot|^p\rangle dx < +\infty$$

if $1 \leq p < +\infty$ or $\operatorname{supp}(\nu_x) \subset \mathbb{B}_R(0)$ for some $R > 0$ and $\mathcal{L}^n$-a.e. $x \in \Omega$, and is $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ such that

$$\overline{\nu_x} = \nabla u(x) \quad \mathcal{L}^n\text{-a.e.}$$

for which a Jensen-type inequality holds against quasi-convex functions of suitable growth, is the limit of a sequence of gradients of $(u_j)_j$ in $W^{1,p}(\Omega, \mathbb{R}^m)$ so that $u_j \rightharpoonup u$ weakly in $W^{1,p}$ (weakly* in $W^{1,\infty}$), where we insist that the convergence is weak when $p = 1$.

Most of the above theory is not specific to gradients only. It is well known that for a matrix field $U \in L^1(\mathbb{B}^n, \mathbb{R}^{m\times n})$, then $\operatorname{curl}(U) = 0$ row-wise if and only if there is $u \in W^{1,1}(\mathbb{B}^n, \mathbb{R}^m)$ so that $\nabla u = U$ $\mathcal{L}^n$-almost everywhere. Inspired by this well-known result, the above theory has been largely extended to integral functionals

$$U \mapsto \int_\Omega f(U(x))dx$$

where $U \in L^1(\Omega, \mathbb{R}^d)$ solves some system of linear constant-coefficient homogeneous differential equations

$$P(\partial)(U) = 0, \tag{6}$$

see for example [6, 28, 37, 49].

The aforementioned results hold in the setting of weak convergence in the space $W^{1,p}$, $1 \leq p < +\infty$ and weak* convergence in $W^{1,\infty}$. This is a natural condition to assume when $p > 1$, but not when $p = 1$, as the Lebesgue space $L^1(\Omega, \mathcal{L}^n)$ is not the dual space of any Banach space. In particular, a bounded sequence in $L^1(\Omega, \mathcal{L}^n)$ can concentrate and converge to measures that are singular with respect to the Lebesgue measure. In terms of gradients, the closure of $W^{1,1}(\Omega, \mathbb{R}^m)$ so that its unit ball is weak* compact is the set of functions of bounded variations $BV(\Omega, \mathbb{R}^m)$.

This is the set of functions whose distributional derivatives are measures, meaning $u \in BV(\Omega, \mathbb{R}^m)$ if $u \in L^1(\Omega, \mathbb{R}^m)$ and $Du \in \mathbb{M}(\Omega, \mathbb{R}^{m\times n})$. If $(u_j)_j$ is bounded in $W^{1,1}(\Omega, \mathbb{R}^m)$, then there exists a subsequence $(u_{j_i})_i$ such that $u_{j_i} \rightharpoonup^* u$ in BV, and Du needs not be absolutely continuous with respect to the Lebesgue measure, as $\nabla u_{j_i}(x)dx$ could concentrate. Because this concentration phenomenon is exclusive of the case $p = 1$, it concerns integrands that have linear growth at infinity. It turns out that when f is non-negative, has linear growth and is quasi-convex, then if $(u_j)_j \subset W^{1,1}(\Omega, \mathbb{R}^m)$ is so that $u_j \rightharpoonup^* u$ in BV for some $u \in BV(\Omega, \mathbb{R}^m)$, then

$$\liminf_j \int_\Omega f(\nabla u_j(x))dx \geq \int_\Omega f(\nabla u(x))dx, \tag{7}$$

see [23, 26]. However, one can prove that the above is in general a strict inequality because for a sequence $u_j \rightharpoonup^* u$ in BV with $Du = \nabla u(x)dx + D^s u, D^s u \perp \mathcal{L}^n$ and $D^s u \neq 0$, formula (7) is unable to capture the singular part $D^s u$. Consider a function $f\colon \mathbb{R}^d \to \mathbb{R}$ having *regular recession* at infinity, meaning that the limit

$$f^\infty(z) = \lim_{t\to\infty, z'\to z} \frac{f(tz')}{t} \tag{8}$$

exists locally uniformly in $z \in \mathbb{R}^d$, the lower semi-continuous envelope of (3) in the space $BV(\Omega, \mathbb{R}^m)$ with respect to sequential weak* convergence is

$$u \mapsto \int_\Omega f(\nabla u(x))dx + f^\infty \left(\frac{D^s u}{|D^s u|}(x) \right) d|D^s u|(x). \tag{9}$$

References to this result with different conditions on f can be found in [4, 15, 27, 36, 52]. In this work, by writing $\frac{\eta}{\mu}$ where $\mu \in \mathbb{M}^+(\Omega)$ and $\eta \in \mathbb{M}(\Omega, \mathbb{R}^d)$ we mean the Radon-Nikodym derivative of η with respect to μ. In particular, for any vector-valued measure $\eta \in \mathbb{M}(\Omega, \mathbb{R}^d)$ we can consider its *polar decomposition* $\eta = \frac{\eta}{|\eta|} d|\eta|$.

In the above, Assumption (8) is, despite first appearances, not necessary. More precisely, two important theorems show that (9) makes sense without the assumption that f has regular recession, in other words without assuming that f^∞ exists everywhere:

- the rank-1 Theorem [2] states that every $u \in BV(\Omega, \mathbb{R}^m)$ is so that

$$\frac{D^s u}{|D^s u|}(x) \in \mathbb{R}^{m \times n} \quad \text{has rank 1 for} \quad |D^s u|\text{-a.e. } x \in \Omega;$$

- if f has linear growth and it is quasi-convex it is also rank-1 convex (convex along matrices of rank 1), and so $f^\infty(z)$ exists, in a classical sense, for all $\operatorname{rank}(z) = 1$.

These two facts show that (9) makes sense for all quasi-convex functions of linear growth and it turns out that (9) is still lower semi-continuous in this general setting. It is worth mentioning that some old results concerning the lower semi-continuity of (9) made use of the fact that f^∞ exists everywhere and had to be deeply changed to cover the general case of f that is just quasi-convex and of linear growth (in contrast, see [52] for a proof of lower semi-continuity that uses (8) but not the rank-1 property of BV functions). However, a Young Measure formulation of the Jensen-type inequality (4) has to take into account not only the singular part of Du – which has rank 1 – but also the concentration along any possible direction.

This can be made into a proper statement using the characterisation Theorem [35] together with the rank-1 auto convexity result [33]. In the spirit of the previous results, one is tempted to prove a duality-type characterisation of Young Measures with concentrations and quasi-convex functions. As just mentioned, one must be careful about the assumptions on f^∞ (8). As shown in [48], quasi-convex functions can oscillate at infinity, meaning that $f^\infty(z)$ defined as in (8) may not exist for some $z \in \mathbb{R}^m$. This result is improved in this paper as we show that quasi-convex functions of linear growth are not separable with respect to the canonical topology in use. This suggests that to obtain a Jensen-type inequality and characterisation result for Gradient Young Measures in the case $p = 1$, it is necessary to specify a compactification at infinity.

The characterisation for Gradient Young Measures when $p = 1$ has been already obtained on the so-called *sphere compactification* – functions for which $f^\infty(z)$ exists in the sense of (8) for all $z \in \mathbb{R}^{m \times n}$ – see [35] and [7] for a refinement concerning concentration on the boundary of Ω. We also refer to [29, 30] for the case with concentration when $p > 1$. In section 3, we prove the characterisation result for Gradient Young Measures on separable compactifications of subclasses of quasi-convex functions. Roughly, we show that given a compactification $e_{f_i, i\in\mathbb{N}}$ generated by a countable family of quasi-convex functions of linear growth, any triple $(\nu_x, \lambda, \nu_x^\infty)$ that satisfies (2),

$$\overline{\nu_x}dx + \overline{\nu_x^\infty}d\lambda = Du$$

for some $u \in BV(\Omega, \mathbb{R}^m)$, the moment condition

$$\int_\Omega \langle \nu_x, |\cdot|\rangle dx < +\infty,$$

and the Jensen-type inequality

$$\begin{cases} \langle \nu_x, f\rangle dx + \langle \nu_x^\infty, f^\infty\rangle \frac{\lambda}{\mathcal{L}^n}(x)dx \geq f(\nabla u(x))dx, \\ \langle \nu_x^\infty, f^\infty\rangle d\lambda \geq f^\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u| \end{cases}$$

in the sense of measures, for all quasi-convex functions f (suitably adapted to the compactification $e_{f_i, i\in\mathbb{N}}$), is generated by a sequence $(\nabla u_j)_j$ with $(u_j)_j$ bounded in $W^{1,1}(\Omega, \mathbb{R}^m)$. Here f^∞ is not defined as in (8), but it is the suitably defined extension of f to the remainder of the compactification $e_{f_i, i\in\mathbb{N}}$. Moreover, $e_{f_i, i\in\mathbb{N}}$ cannot contain all quasi-convex functions from the above family, so we are restricted to working with subfamilies of quasi-convex functions at once.

In the above characterisation result, we have confined our attention to first-order gradients. However, the argument can be extended to also cover the case of vector fields satisfying a system of linear constant-coefficient homogeneous differential equations as in (6).

1.2. Preliminaries and notation

We list here the notation and terminology that will be used in this paper.

- For a vector $v \in \mathbb{R}^m$, we write $|v| = \sqrt{\sum_{i=1}^m v_i^2}$ otherwise specified.
- Given a function $f\colon X \to Z$ and W an arbitrary set, we call

$$\mathrm{graph}_X(f) = \{(x, f(x)) \in X \times Z \text{ such that } x \in X\},$$
$$\mathrm{graph}_{X\times W}(f) = \{(x, w, f(x)) \in X \times W \times Z \text{ such that } x \in X, w \in W\}.$$

  When X is clear from the context we just write graph(f).
- We say that a function $f\colon \mathbb{R}^d \to \mathbb{R}$ has *p-growth* if there is a constant $C > 0$ such that

$$|f(z)| \leq C(1 + |z|^p) \quad \forall z \in \mathbb{R}^d.$$

- The identity matrix is indicated by $\mathbb{1}$, or $\mathbb{1}_d \in \mathbb{R}^{d\times d}$ if we need to specify the dimension.
- The Lebesgue measure is indicated by dx, or $\mathcal{L}^n$ depending on the context. The set of finite Borel d-vector measures is indicated by $\mathbb{M}(\Omega, \mathbb{R}^d)$. For $E \subset \mathbb{M}(\Omega)$, E^+ is the set of positive Borel measures that belong to E.
- The $n-1$-dimensional Hausdorff measure is indicated by $\mathcal{H}^{n-1}$.

- For a measure $\mu \in \mathbb{M}(\Omega)^+$, we write $L^p(\Omega, \mu, \mathbb{R}^d)$ to mean the space of Borel measurable functions $f \colon \Omega \to \mathbb{R}^d$ such that

$$\int_\Omega |f(x)|^p d\mu(x) < +\infty.$$

- For $\mu \in \mathbb{M}(\Omega, \mathbb{R}^d)$, we write its restriction to a μ-measurable set $E \subset \Omega$ as

$$\mu \llcorner E \colon A \text{ Borel set} \mapsto \mu(E \cap A).$$

- If $\mu \in \mathbb{M}^+(\Omega)$ and $f \in L^1(\Omega, \mu, \mathbb{R}^d)$, we call $f d\mu$ the measure in $\mathbb{M}(\Omega, \mathbb{R}^d)$ defined by

$$U \mapsto \int_U f d\mu,$$

  where U runs through all μ-measurable sets.
- If X is any space, the Dirac delta is indicated, for $x \in X$, by

$$\int_X f(y) d\delta_x(y) = f(x),$$

  where $f \colon X \to \mathbb{R}^d$ is an arbitrary function.
- Let X be a metric space, $\mu \in \mathbb{M}^+(\Omega)$ and $(f_j)_{j\in\mathbb{N}}$ in $L^p(\Omega, \mu, \mathbb{R}^d)$. We say that the sequence $(f_j)_j$ is p-equi-integrable (or simply equi-integrable if p is clear from the context) if it is norm-bounded and

$$\lim_{k\uparrow\infty} \sup_{j\in\mathbb{N}} \int_{|f_j|^p > k} |f_j|^p d\mu = 0.$$

- The set of $\mathbb{R}^d$-values test functions is

$$\mathcal{D}(\Omega, \mathbb{R}^d) = C_c^\infty(\Omega, \mathbb{R}^d),$$

  that is, infinitely differentiable functions with compact support in Ω. We do not insist on the topology this space is endowed with. We often abbreviate $\mathcal{D}(\Omega, \mathbb{R}^d) = \mathcal{D}(\Omega)$ when the target space is clear from the context.
- By the *derivative* of a function $u \in L^1(\Omega, \mathbb{R}^m)$ we mean the $m \times n$ matrix-valued distribution $Du = [\partial_j u_i]_{i,j} \in \mathcal{D}(\Omega, \mathbb{R}^{m\times n})^*$ such that

$$\int_\Omega u_i \partial_j \phi dx = -\langle \partial_j u_i, \phi \rangle \quad \forall \phi \in \mathcal{D}(\Omega).$$

- The Sobolev space of functions with integrable derivatives is

$$W^{1,1}(\Omega, \mathbb{R}^m) = \{u \in L^1(\Omega, \mathbb{R}^m) : Du \in L^1(\Omega, \mathbb{R}^{m\times n})\}.$$

  In this case, we write $Du = \nabla u$ in analogy with classical derivatives.
- The set of functions of bounded variation is

$$BV(\Omega, \mathbb{R}^m) = \{u \in L^1(\Omega, \mathbb{R}^m) : Du \in \mathbb{M}(\Omega, \mathbb{R}^{m\times n})\}.$$

- For a function $u \in BV(\Omega, \mathbb{R}^m)$ we can write

$$Du = \nabla u d\mathcal{L}^n \llcorner \Omega + D^s u, \quad D^s u = D^j u \llcorner J_u + D^c u$$

where $\nabla u d\mathcal{L}^n$ is the absolutely continuous part, $D^j u$ is the jump part concentrated on a $n-1$ rectifiable set J_u, and D^c is the Cantor part, which is absolutely continuous with respect to $\mathcal{H}^{n-1}$, see [5], Chapter 3.

- For a function $g \in BV(\Omega, \mathbb{R}^m)$, we write

$$BV_g(\Omega, \mathbb{R}^m) = \left\{ \begin{matrix} u \in BV(\Omega, \mathbb{R}^m) : \text{there is a sequence } (u_j)_j \text{ in } \mathcal{D}(\Omega, \mathbb{R}^m) \\ \text{such that } u_j \xrightarrow{\text{weak* in BV}} u - g \end{matrix} \right\}.$$

In this thesis, we will work with domains $\Omega \subset \mathbb{R}^n$ bounded and with a Lipschitz boundary, so the above definition is consistent with trace theorems and other literature on the subject, see for example [55] and [5], Chapter 3.

2. Generalised Young Measures on separable compactifications

In this section, we construct generalised Young Measures and provide a new geometric representation. Even though generalised Young Measures have been known to exist for a long time, our new representation captures how concentration "oscillates with different amplitudes at infinity" and gives a geometric insight into what these measures look like. To achieve such a result, we embed spaces of functions into bigger compact sets and use the theory of Hausdorff compactifications to study their topological properties.

2.1. Generalised limits and parametrised measures

Generalised Young Measures are objects that were known to exist since DiPerna and Majda [21], and have been used in a few instances, see for example [24, 30, 40, 54]. However, their existence per se does not give enough clarity on their properties, and so makes it hard to work with such objects.

We give a new interpretation and geometric representation that better captures oscillation and concentration effects that occur in limits of the form

$$\lim_j \int_\Omega f(v_j(x)) d\mu(x),$$

where $(v_j)_j$ is a norm-bounded sequence in $L^p(\Omega, \mu, \mathbb{R}^d)$ and $f \in C(\mathbb{R}^d)$ has p-growth. Under these assumptions, it is easy to see that, up to a subsequence, $f \circ v_j$ converges to a measure ν; formally, this means that

$$\int_\Omega f(v_j(x))\phi(x) d\mu(x) \to \int_\Omega \phi(x) d\nu(x)$$

for all $\phi \in C_0(\Omega)$. It is clear that $\nu = \nu(f)$ is a linear function of f. What is not clear is how such dependence can be represented in terms of μ and f. Without a clear representation, it is not possible to set up a system of calculus. This section is dedicated to working out a geometric interpretation of the relation between ν and f, which is called Young Measure.

Definition 2.1. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to have *p-growth* if there is a constant $C > 0$ such that
$$|f(z)| \leq C(1 + |z|^p) \quad \forall z \in \mathbb{R}^m.$$
When $p = 1$, we say that such functions have *linear growth.* □

We emphasise that when $v_j \to v$ strongly in $L^1(\mu)$, the limit Young Measure is trivial, meaning that

$$\int_\Omega f(v_j(x))\phi(x)d\mu(x) \to \int_\Omega f(v(x))\phi(x)d\mu(x)$$

for all $f \in C(\mathbb{R}^d)$ of linear growth and $\phi \in C_0(\Omega)$. This is a simple consequence of the Vitali convergence Theorem A.1 (or the generalised dominated convergence theorem).

When strong L^1 convergence fails, and the measure μ is finite, only two things can go wrong, namely *oscillation* and *concentration.* We illustrate oscillation and concentration in the toy case $n = m = 1$, $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\mu = \mathcal{L}^1$ and $p = 1$ with the two following examples.

- *Oscillation:* $u_j(x)$ assumes frequently multiple values around points $x \in \Omega$ and, in the limit, generates probability distributions
$$x \mapsto \nu_x \in \mathbb{M}_1^+(\mathbb{R}^d)$$
Example: Take $u(x) = -\chi_{[-\frac{1}{2},0]} + \chi_{[0,\frac{1}{2}]}$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, extended by 1-periodicity to the whole of $\mathbb{R}$ and define $u_j(x) = u\left(\frac{x}{j}\right)$. Then, if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, we immediately obtain that
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u_j(x))dx = \frac{f(-1)}{2} + \frac{f(1)}{2} = \int_{\mathbb{R}} f(z)d\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right).$$

- *Concentration:* $|u_j|^p$ concentrates to some measure $\lambda \in \mathbb{M}^+(\overline{\Omega})$, meaning
$$\lim_k \lim_j \int_{\{x \in U : |u_j| > k\}} \phi(x)|u_j(x)|^p d\mu(x) = \int_{\overline{\Omega}} \phi(x)d\lambda,$$
for all smooth $\phi\colon \mathbb{R}^n \to \mathbb{R}$. The values of $u_j(x)$, on the other side, escape to infinity on a set where λ is concentrated, and so their limit can be seen by embedding $u_j \in \mathbb{R}^d$ into a compact set K containing $\mathbb{R}^d$.

  Example: Think of u_j as a step function approximating a Dirac delta,
$$u_j(x) = j\chi_{[-\frac{1}{2j},\frac{1}{2j}]}.$$
  The integral becomes
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u_j(x))dx = \frac{f(j)}{j}.$$
  Because $p = 1$, we assume that f has linear growth and the above quantity is always bounded, so we can find a subsequence $(u_{j_i})_i$ – in fact, many different

subsequences – such that $\lim_i \frac{f(j_i)}{j_i} = f^\infty(\{j_i\}_i)$, where f^∞ depends on the particular subsequence we considered. But then we can write

$$\lim_i \int_{-\frac{1}{2}}^{\frac{1}{2}} f(u_{j_i}(x))dx = \lim_i \frac{f(j_i)}{j_i} = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]} f^\infty(\{j_i\}_i)d\delta_0.$$

In order to construct Young Measures, we regard functions as maps from a domain Ω into the set of probability measures over a target space $\mathbb{R}^d$. Ordinary functions $f\colon \Omega \to \mathbb{R}^d, x \mapsto f(x)$ are embedded into maps $\Omega \to \mathbb{M}_1^+(\mathbb{R}^d), x \mapsto \delta_{f(x)}$. Preliminary to the construction, we introduce two basic concepts that are at the core of this theory.

Definition 2.2. Let X and Z be locally compact, separable metric spaces and $\lambda \in \mathbb{M}^+(X)$. A map $\nu\colon X \to \mathbb{M}^+(Z)$ is said to be λ-*measurable* if for each $\phi \in C_0(Z)$ the function $x \mapsto \langle \nu(x), \phi\rangle$ is λ-measurable. □

We shall often write the measure-valued map ν as a *parametrized measure* $(\nu_x)_{x\in X}$, where $\nu_x := \nu(x)$. Given a measure ν on a product space $X \times Z$, it can always be decomposed as a product of its projection onto X and its cross section on Z.

Theorem 2.3. *(Disintegration of measures) Let X and Z be compact metric spaces and denote by $\pi\colon X \times Y \to Z$ the projection mapping onto the first coordinate $\pi(x, y) = x$. For $u \in \mathbb{M}^+(X \times Z)$ and $\lambda = \pi_\# \nu \in \mathbb{M}^+(X)$ (the pushforward of ν via π) there exists a unique λ-measurable parametrized measure $(\eta_x)_{x\in X}, \eta_x \in \mathbb{M}_1^+(Z)$ such that for all $\phi \in C(X), \psi \in C(Z)$ we have*

$$\langle \nu, \phi \otimes \psi\rangle = \int_X \langle \eta_x, \psi\rangle \phi(x)d\lambda(x) = \int_X \int_Z \psi(z)d\eta_x(z)\phi(x)d\lambda(x).$$

For a proof of this result see [5]. In this case, we write $\nu = \eta_x d\lambda$.

2.1.1. Generalised Young Measures

In what follows, we show how to obtain a good representation of Young Measures on separable compactifications. The procedure is adapted from [35], where it was developed to construct Young Measures on a very specific compactification, called *sphere compactification* (see 2.4 for the connection to our general setting). Some of the results can also be found in [53], Chapter 12, where they were also proven on the sphere compactification.

Throughout this section, $\Omega \subset \mathbb{R}^n$ is open and bounded, $\mu \in \mathbb{M}^+(\Omega)$ and $(v_j)_j$ is a bounded sequence in $L^p(\Omega, \mu, \mathbb{R}^d)$, with $1 \leq p < +\infty$. Assume

$$v_j \rightharpoonup v \ \text{ in } \ L^p \quad \text{when} \ \ 1 < p < +\infty$$

and

$$v_j \rightharpoonup^* v \ \text{ in } \ C_0(\Omega, \mathbb{R}^d)^* \quad \text{when} \ \ p = 1.$$

Given a continuous integrand $\Phi\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfying the p-growth condition

$$|\Phi(x, z)| \leq C(1 + |z|)^p \quad \forall (x, z) \in \Omega \times \mathbb{R}^d,$$

we seek to represent limits of $\left(\int_\Omega \Phi(x, v_j(x))dx\right)_j$ as $j \to \infty$, possibly passing through suitable subsequences.

Remark 2.4. For each j, the map Φ acts on the graph of v_j, i.e. $\Phi(x, v_j(x)) = \Phi \circ (x, v_j(x))$. Therefore, we look for the limiting distribution of $(x, v_j(x))$ as $j \to \infty$, and more precisely the Φ-moment of this limiting distribution. □

Morally speaking, since $(\Phi(\cdot, v_j))_j$ is bounded in $L^1(\Omega, \mu)$, $(\Phi(\cdot, v_j)d\mu)_j$ is bounded in $\mathbb{M}(\overline{\Omega}) \eqsim C(\overline{\Omega})^*$, so by the abstract compactness principle A.4, there exists a limit measure which depends on the integrand Φ. To better understand the way the limit measure depends on the integrand Φ, we need to change the setup of the problem.

Let $z \in \mathbb{R}^d \mapsto \hat{z} = \frac{z}{1+|z|} \in \mathbb{B}^d$ be a homeomorphism $\mathbb{R}^d \mapsto \mathbb{B}^d$. Define the class of functions of p-growth in the z variable to be

$$\mathbb{G}_p = \mathbb{G}_p(\Omega, \mathbb{R}^d) := \left\{ \Phi \in C(\Omega \times \mathbb{R}^d) : \sup_{(x,z)} \frac{|\Phi(x,z)|}{(1+|z|)^p} < +\infty \right\},$$

and for $\Phi \in \mathbb{G}_p$ put $(T\Phi)(x, \hat{z}) := (1 - |\hat{z}|)^p \, \Phi\Big(x, \dfrac{\hat{z}}{1-|\hat{z}|}\Big).$

Then $T \colon \mathbb{G}_p \to \mathrm{BC}(\Omega \times \mathbb{B}^d)$ is an isometric isomorphism provided $\mathbb{G}_p$ is normed by $\|T\Phi\|_\infty$ and $\mathrm{BC}(\Omega \times \mathbb{B}^d)$ by $\|\cdot\|_\infty$. The inverse operator is

$$(T^{-1}\Psi)(x,z) = (1+|z|)^p \Psi\Big(x, \frac{z}{1+|z|}\Big),$$

where $\Psi \in \mathrm{BC}(\Omega \times \mathbb{B}^d)$. The dual operator $T^* \colon \mathrm{BC}(\Omega \times \mathbb{B}^d)^* \to \mathbb{G}_p^*$ is again an isometric isomorphism. We are interested in the limits of $\int_\Omega \Phi(x, v_j(x))dx$ for $\Phi \in \mathbb{G}_p$ and may define $\xi_{v_j} \in \mathbb{G}_p^*$ by

$$\xi_{v_j}(\Phi) := \int_\Omega \Phi(x, v_j(x))dx, \Phi \in \mathbb{G}_p.$$

Note
$$\|\xi_{v_j}\| = \sup_{\|\Phi\|_{\mathbb{G}_p}} \xi_{v_j}(\Phi) = \int_\Omega (1+|v_j|)^p dx$$

so (ξ_{v_j}) is a bounded sequence in $\mathbb{G}_p^*$. But $\mathbb{G}_p^* \eqsim \mathrm{BC}(\Omega \times \mathbb{B}^d)^*$, and because BC (hence $\mathbb{G}_p$) is not separable we do not necessarily have sequential compactness. We must restrict the integrands Φ to a separable subspace of $\mathbb{G}_p$.

2.2. Hausdorff compactification

In this subsection, we present how to construct a compactification of the space $X = \Omega \times \mathbb{R}^d$ from a family of bounded and continuous functions $F \subset \mathrm{BC}(X)$. Roughly speaking, such compactification is a compact set $e_F X$ that contains X as a dense subset and on which all $f \in F$ admit a continuous extension. The idea behind such construction is to look at the graph of each $f \in F$. Because of the boundedness assumption, each function has its image contained in a closed bounded interval of $\mathbb{R}$. Therefore, the graph is embedded into a closed subset of an (infinite-dimensional) hypercube, which is compact in the product topology by the Tychonoff Theorem, A.3.

Most of the results will be stated without proof, which can be found in [19], Chapters 1 and 2, and [22], Chapter 4.

We first define three classes of functions that are rich enough to determine the topological structure of their domains:

Definition 2.5. Consider a family of functions $F \subset \mathrm{BC}(X)$, we say that F *separates points* from closed sets if for each $C \subset X$ closed and $x \in X \setminus C$ there exists $f \in C(X)$ such that $f(x) \notin \overline{f(C)}$.

Next, we define what a compactification of a topological space is.

Definition 2.6. A *compactification* of X is a compact Hausdorff space αX and an embedding $\alpha\colon X \to \alpha X$ (continuous and so that $\alpha^{-1}\colon \alpha X \to X$ exists and is continuous) such that $\alpha(X)$ is dense in αX.

It is useful to remark that because α is continuous, any function $f \in C(\alpha X)$ can be restricted to a continuous function on X. Indeed, $f \circ \alpha$ is the composition of a bounded continuous function with a continuous function, and thus it belongs to $\mathrm{BC}(X)$. On the other hand, because $\alpha(X)$ is dense in αX, each $f \in C(\alpha X)$ is uniquely recovered from $f \circ \alpha \in C(X)$.

Given a family $F \subset \mathrm{BC}(X)$ that separates points from closed sets, there is a canonical way of generating a compactification αX on which every $f \in F$ has a continuous extension.

Theorem 2.7. *To each family $F \subset \mathrm{BC}(X)$ that separates points from closed sets, we associate a canonical embedding*

$$e_F\colon X \to \Pi_{f\in F}\big[\inf f, \sup f\big],\ x \mapsto \{f(x)\}_{f\in F}.$$

$e_F X := \overline{e_F(X)}$ is a compactification of X

The above theorem is a direct implication of Tychonoff's Theorem. Namely, when $F \subset \mathrm{BC}(X)$ is a family that separates points from closed sets, then

$$e_F\colon X \mapsto \Pi_{f\in F}\big[\inf f, \sup f\big]$$

is open and continuous, and so it is an embedding.

However, the map e_F makes sense even if F does not separate points from closed sets, and $e_F X$ is always a compact subset of $\Pi_{f\in F}\big[\inf f, \sup f\big]$.

Lemma 2.8. *Let $F \subset \mathrm{BC}(X)$ be a family that separates points from closed sets and $e_F X$ its induced compactification. Each $f \in F$ embeds into $C(e_F(X))$ in an obvious way and admits a unique extension $\overline{f} \in C(e_F X)$.*

By definition of $e_F X$, for all $y \in e_F X$ there is a net $(y_\lambda)_\lambda \subset e_F(X) = \Pi_{f\in F} f(x_\lambda)$ such that $y_\lambda \to y$. To compute the extension of $f \in F$ on the compact set $e_F X$, we set

$$\overline{f}\colon e_F X \to \mathbb{R}, y\left(= \lim_\lambda \Pi_{f\in F} f(x_\lambda)\right) \mapsto \overline{f}(y) = \lim_\lambda f(x_\lambda).$$

Note that this definition is independent of the net chosen, because if y_γ also satisfies $y_\gamma = \Pi_{f\in F} f(x_\gamma)$ and $y_\gamma \to y$, then $\lim_\gamma f(x_\gamma) = \lim_\lambda f(x_\lambda)$ for each $f \in F$.

Suppose that we have given a family F and its associated compactification $e_F X$, and we consider the compactification of $F \cup \{f\}$, where $f \in \mathrm{BC}(X)$. We expect the latter compactification to be bigger than the former, i.e. to be a space where all the previous extensions can be further extended to continuous functions.

Definition 2.9. Given two compactifications αX and γX of X. Then we say that $\alpha X \geq \gamma X$ if there exists a continuous function $f\colon \alpha X \to \gamma X$ such that $f \circ \alpha = \gamma$. Moreover, we write $\alpha X \approx \gamma X$ if $\alpha X \geq \gamma X$ and $\gamma X \geq \alpha X$, or equivalently if $f\colon \alpha X \to \gamma X$ is a homeomorphism.

In the rest of the paper, we will sometimes refer to a generic compactification K without specifying the underlying family generating it. The reason why is contained in the following result.

Theorem 2.10. *Given a compactification αX of X, then there exists a family $F \subset \mathrm{BC}(X)$ that separates points from closed sets such that $e_F X \approx \alpha X$.*

2.2.1. Representation of compactifications

Consider a family $F \subset \mathrm{BC}(X)$ that separates points from closed sets. According to the Hausdorff compactification theory, its induced compactification can be written as a subset of the hypercube $\Pi_{f\in F}\big[\inf f, \sup f\big]$, where the sides of this cube are as many as the functions $f \in F$. Because F can be uncountable, its compactification could be hard to deal with from an analytical point of view. We seek a better representation of such space.

The idea behind the following result is that if we know the limits of functions $f, g \in \mathrm{BC}(\Omega, \mathbb{R})$, we also know the limits of $f^n + g^m$, $n, m \in \mathbb{N}$.

Definition 2.11. Let $\mathbb{F}$ be a family of functions $f\colon X \to \mathbb{R}$. We call $\mathbb{A}(\mathbb{F})$ the *algebra generated by* $\mathbb{F}$, i.e.

$$\mathbb{A}(\mathbb{F}) = \{f^n + g^m : f, g \in \mathbb{F},\ n, m \in \mathbb{N}\}.$$

Theorem 2.12. (Representation Theorem) *Let $F \subset \mathrm{BC}(X)$ be a closed sub-algebra that separates points from closed sets and let $F' \subset F$ be such that $\overline{\mathbb{A}(F')} = F$. Then $e_{F'}X$ is a compactification of X and $e_{F'}X \approx e_F X$.*

Proof. Let $e_{F'}X$ be the (formal) compactification of F'. Clearly $e_F X \geq e_{F'}X$. To prove the opposite inclusion, we must find a continuous function $T\colon e_{F'}X \to e_F X$ such that $T \circ e_{F'} = e_F$. Fix $y = \lim_\lambda \Pi_{f\in F'} f(x_\lambda) \in e_F X$. If $g \in \mathbb{A}(F')$, $\lim_\lambda g(x_\lambda)$ exists and coincides on all nets x_γ such that $y = \lim_\gamma \Pi_{f\in F'} f(x_\gamma)$.

Next, let $g \in \overline{\mathbb{A}(F')}$ and find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{A}(F')$ such that $f_n \to g$ uniformly. Because $\|f_n\|_\infty$ is bounded, so is $\{\lim_\lambda f_n(x_\lambda)\}_{n\in\mathbb{N}}$, and so we can extract a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_k \lim_\lambda f_{n_k}(x_\lambda) = L \in \mathbb{R}$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|f_{n_N} - g\|_\infty < \frac{\varepsilon}{3}$ and find $\tilde{\lambda} \in \Lambda$ such that $|f_{n_N}(x_\lambda) - \lim_\lambda f_{n_N}(x_\lambda)| < \frac{\varepsilon}{3}$ for all $\lambda \geq \tilde{\lambda}$. Finally

$$|g(x_\lambda) - L| \leq |g(x_\lambda) - f_{n_N}(x_\lambda)| + |f_{n_N}(x_\lambda) - \lim_\lambda f_{n_N}(x_\lambda)| + |\lim_\lambda f_{n_N}(x_\lambda) - L| < \varepsilon$$

for all $\lambda \geq \tilde{\lambda}$, i.e. the net $g(x_\lambda)$ converges to $L \in \mathbb{R}$.

In particular, by the uniqueness of $\lim_\lambda g(x_\lambda)$, we conclude that the original sequence $\{\lim_\lambda f_n(x_\lambda)\}_{n\in\mathbb{N}}$ converges to L, which also proves that the limit does not depend on the particular net x_γ as far as $\lim_\lambda f(x_\lambda) = \lim_\gamma f(x_\gamma)$ for all $f \in F'$. This shows that the map

$$T\colon e_{F'}X \to e_F X, y = \lim_\lambda \Pi_{f\in F'} f(x_\lambda) \mapsto Ty = \lim_\lambda \Pi_{f\in F} f(x_\lambda)$$

is a well-defined isomorphism. Because its inverse is the projection map $\pi_{F'}|_{T(e_{F'}X)}$, which is continuous and open, then

$$T \circ e_{F'}\colon x \mapsto \Pi_{f\in F'} f(x) \mapsto \Pi_{f\in F} f(x) = e_F(x)$$

is a homeomorphism. To prove that $e_{F'}X$ is a compactification of X, we notice that F separates points, and so does F'. If $U \subset X$ is open, so is

$$e_{F'}(U) = T^{-1}(e_F(U)),$$

and $e_{F'}$ is an injective continuous open map, thus an embedding onto its image. □

Let $F \subset \mathrm{BC}(\Omega\times\mathbb{B}^d)$ be a closed separable algebra that separates points from closed sets and let $e_F\Omega \times \mathbb{B}^d$ be its compactification. Because $e_F\Omega \times \mathbb{B}^d$ is a compact Hausdorff space we have the following isometric isomorphism of its dual

$$C(e_F\Omega \times \mathbb{B}^d)^* \approx \mathbb{M}(e_F\Omega \times \mathbb{B}^d).$$

Lemma 2.13. *If $F \subset \mathrm{BC}(X)$ is separable, so is $C(e_F X)$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be dense in F. By the Stone-Weierstrass Theorem, the algebra generated by $\{1\} \cup \{\overline{f}_n\}_{n\in\mathbb{N}}$ is dense in $C(e_F X)$, and so $C(e_F X)$ is separable. □

Because $C(e_F\Omega\times\mathbb{B}^d)$ is separable we also have the abstract sequential compactness principle A.4 on its dual $\mathbb{M}(e_F\Omega\times\mathbb{B}^d)$. Let $T^{-1}F \subset \mathbb{G}_p$ be the corresponding algebra, with respect to

$$(f \times^p g)(z) = \frac{f(z)g(z)}{1+|z|^p}$$

on the set of continuous functions with p-growth. There is an isometric isomorphism

$$T^{-1}F \overset{\tilde{T}}{\approx} C(e_F\Omega \times \mathbb{B}^d).$$

By the Riesz representation Theorem, we can write its adjoint as

$$\tilde{T}^*\colon \mathbb{M}(e_F\Omega \times \mathbb{B}^d) \to (T^{-1}F)^*, \nu \mapsto \Big(\Phi \mapsto (\tilde{T}^*\nu, \Phi) = (\nu, \tilde{T}\Phi) = \int_{e_F\Omega\times\mathbb{B}^d} \tilde{T}\Phi d\nu.\Big)$$

Lemma 2.14. *Let X, Z be completely regular Hausdorff spaces and let $F \subset \mathrm{BC}(X)$ and $G \subset \mathrm{BC}(Z)$ be closed sub-algebras that separate points from closed sets and contain the constant function. The following spaces are all isometrically isomorphic to each other*

$$C(e_{F\cup G}X \times Z) \approx C(e_F X \times e_G Z) \approx \overline{\mathbb{A}\Big(C(e_F X) \times C(e_G Z)\Big)} \approx \overline{\mathbb{A}(F \cup G)},$$

where each $f \in F$ and $g \in G$ is extended to a function on the product space by keeping constant the other variable. Moreover, $F \cup G \subset \mathrm{BC}(X \times Z)$ separates points from closed sets.

The proof of the above lemma is a straightforward application of the Stone-Weierstrass Theorem.

Remark 2.15. It is important to take families F, G defined exclusively on each respective space, and the theorem is false if we instead add $f = f(x, y)$ that is not of the form above. In this case, one has to work the compactification $e_f X \times Y$ which is still separable and metrisable, but does not have a product-like structure. □

Morally speaking, what the previous lemma says is that on product spaces it is enough to work with the compactifications in each coordinate separately. Moreover, their dual elements (generalised measures) can be tested against tensor products of functions that depend on each variable independently.

2.3. Representation of Young Measures

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $G' \subset \mathrm{BC}(\mathbb{B}^d)$ and $F' \subset \mathrm{BC}(\Omega)$ be closed separable sub-algebras that separate points from closed sets. Let $T^{-1}F = A \subset \mathbb{G}_p$ be the corresponding algebra, with respect to the $\times^p$ product, in the set of functions having p-growth. F is isometrically isomorphic to $C(e_F\Omega \times \mathbb{B}^d)$.

With abuse of notation, we are going to call T the isomorphism between A and $C(e_F\Omega \times \mathbb{B}^d)$. To each function $u \in L^p(\Omega, \mathbb{R}^d)$ we associate an *elementary Young Measure* $\xi_u \in A^*$ by setting

$$\xi_u \colon A \to \mathbb{R}, \Phi \mapsto \int_\Omega \Phi(x, u(x)) d\mu(x). \tag{10}$$

Next, consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega, \mathbb{R}^d)$ with $\sup_n \|u_n\|_p \leq C$. As $\Phi \in \mathbb{G}_p$, the sequence of elementary Young Measures is also bounded

$$\|\xi_{u_n}\| = \sup_{\|\Phi\|\leq 1} \left| \int_\Omega \Phi(x, u_n(x)) d\mu(x) \right| = \int_\Omega (1 + |u_n|^p) d\mu \leq \mu(\Omega) + C^p.$$

Because of the isomorphism $A^* \approx C(e_F\Omega \times \mathbb{B}^d)^* \approx \mathbb{M}(e_F\Omega \times \mathbb{B}^d)$, there exists a subsequence (non-relabelled) and $\nu \in A^*$ such that $\xi_{u_n} \rightharpoonup^* \nu$ in A^*. Set

$$L := (T^*)^{-1}\nu \in \mathbb{M}(e_F\Omega \times \mathbb{B}^d).$$

We now study the measure L to find a better representation for $\nu \in A^*$. Let $\psi \in C(e_F\Omega \times \mathbb{B}^d)$, we immediately notice that $L \in \mathbb{M}^+(e_F\Omega \times \mathbb{B}^d)$ as a consequence of the following equality

$$\ll \nu, T^{-1}\psi \gg = \lim_n \int_\Omega (T^{-1}\psi)(x, u_n(x)) d\mu(x).$$

Because the constant functions belong to G', we can plug $T^{-1}\psi = \phi(x)(1 + |z|)^p$, $\phi \in C(e_{F'}\Omega)$ into the previous equation and obtain the identity

$$\int_{e_F\Omega\times\mathbb{B}^d} \phi(x) dL(x, \overline{z}) = \lim_n \int_\Omega \phi(x)(1 + |u_n(x)|)^p d\mu(x) = \int_{e_{F'}} \phi(x) d\lambda(x).$$

By Lemma 2.14, the projection $\pi \colon e_F\Omega \times \mathbb{B}^d \to e_{F'}\Omega$ is well-defined, and we can write $\tilde{\lambda} = \pi_\# L$. Note that hereby $\tilde{\lambda} \in \mathbb{M}^+(e_{F'}(\Omega))$.

Consider now the unique $\tilde{\lambda}$-measurable parametrized family $\{\tilde{\nu}_x\}_{x\in e_{F'}\Omega}$ such that $\nu_x \in \mathbb{M}_1^+(e_{G'}\mathbb{B}^d)$ $\tilde{\lambda}$-almost every x and

$$\langle L, \Phi\rangle = \int_{e_{F'}\Omega} \langle \tilde{\nu}_x, \Phi(x,\cdot)\rangle d\tilde{\lambda}(x) \quad \forall\ \Phi \in C(e_F\Omega \times \mathbb{B}^d).$$

For any $\phi \in C_0(\Omega)$ take $\Phi = \phi(1-|\cdot|)^p$. We compute

$$\begin{aligned}\int_{e_{F'}\Omega} \phi(x)\langle \tilde{\nu}_x, (1-|\cdot|)^p\rangle d\tilde{\lambda}(x) &= \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-|z|)^p dL(x,z)\\ &= \lim_n \int_\Omega (\phi \mathbb{1}_{\mathbb{R}^d})(x, u_n(x))d\mu(x) = \int_\Omega \phi(x)d\mu(x).\end{aligned}$$

Because $\mu \in C_0(\Omega)^*$ we immediately conclude that

$$\mu = \langle \tilde{\nu}_x, (1-|\cdot|)^p\rangle \tilde{\lambda} \llcorner \Omega,$$

where μ is extended on $e_{F'}\Omega$ by $\mu(E) \equiv \mu(e_{F'}^{-1}(E)), E \subset e_{F'}\Omega$ Borel. Apply the Radon-Nikodym Theorem and write

$$\tilde{\lambda} = \frac{\tilde{\lambda}}{\mu}\, d\mu + \tilde{\lambda}^s.$$

From the previous identification, we get

$$\begin{cases} \langle \tilde{\nu}_x, (1-|\cdot|)^p\rangle \frac{\tilde{\lambda}}{\mu} = 1 & \mu - \text{a.e.}\\ \langle \tilde{\nu}_x, (1-|\cdot|)^p\rangle = 0 & \tilde{\lambda}^s - \text{a.e.,}\end{cases}$$

where the second condition implies that $\tilde{\nu}_x(e_{G'}(\mathbb{B}^d)) = 0$ $\tilde{\lambda}^s$-a.e., i.e. the measures are concentrated on the boundary $\partial e_{G'}(\mathbb{B}^d)$. On $e_{G'}(\mathbb{B}^d)$ we have $0 < (1-|z|)^p \le 1$, whereas $|z| = 1$ on $\partial e_{G'}(\mathbb{B}^d)$. In particular

$$\begin{cases} \frac{\tilde{\lambda}}{\mu} = \frac{1}{\langle \tilde{\nu}_x, (1-|\cdot|)^p\rangle} \ge 1 & \mu - \text{a.e.}\\ \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d)) = 1 & \tilde{\lambda}^s - \text{a.e.}\end{cases}$$

Now let $\phi \in \mathrm{BC}(\mathbb{R}^d)$ and define

$$\begin{aligned}\langle \nu_x, \phi\rangle &= \frac{\tilde{\lambda}}{\mu}(x) \int_{e_{G'}(\mathbb{B}^d)} (1-|z|)^p \phi\left(\frac{z}{1-|z|}\right) d\tilde{\nu}_x(z)\\ &= \frac{\tilde{\lambda}}{\mu}(x) \int_{\mathbb{B}^d} (1-|z|)^p \phi\left(\frac{z}{1-|z|}\right) d(e_{G'})_\# \tilde{\nu}_x(z)\end{aligned}$$

In particular $\nu_x \in \mathbb{M}_1^+(\mathbb{R}^d)$ and $\{\nu_x\}_{x\in\Omega}$ is μ-measurable. Let $\lambda = \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\tilde{\lambda}$. Then $\lambda \in \mathbb{M}^+(e_{F'}\Omega)$ and it decomposes into

$$\lambda = \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\,\frac{\tilde{\lambda}}{\mu}\,\mu + \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\tilde{\lambda}^s = \tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))\,\frac{\tilde{\lambda}}{\mu}\,\mu + \tilde{\lambda}^s.$$

For λ-almost every $x \in e_{F'}\Omega$ and for $\psi \in C(\partial e_{G'}\mathbb{B}^d)$ set

$$\langle \nu_x^\infty, \psi\rangle = \frac{1}{\tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d))} \int_{\partial e_{G'}(\mathbb{B}^d)} \psi(z)d\tilde{\nu}_x(z).$$

Hereby we have $\nu_x^\infty \in \mathbb{M}_1^+(\partial e_{G'}(\mathbb{B}^d))$.

For each $\Phi \in A$, its recession function is defined to be the restriction

$$\phi^\infty = \phi|_{e_{F'}\Omega \times \partial e_{G'}(\mathbb{B}^d)}.$$

Finally, we obtain the formula

$$\begin{aligned}
\langle L, T\Phi\rangle &= \int\limits_{e_{F'}(\Omega)} \langle \tilde{\nu}_x, T\Phi(x,\cdot)\rangle d\tilde{\lambda} \\
&= \int\limits_{e_{F'}\Omega} \int\limits_{e_{G'}(\mathbb{B}^d)} T\Phi d\tilde{\nu}_x \Big(\frac{\tilde{\lambda}}{\mu} d\mu + \overbrace{d\tilde{\lambda}^s}^{=0} \Big) + \int\limits_{e_{F'}\Omega} \fint\limits_{\partial e_{G'}(\mathbb{B}^d)} T\Phi d\tilde{\nu}_x \ (\tilde{\nu}_x(\partial e_{G'}(\mathbb{B}^d)) d\tilde{\lambda}) \\
&= \int\limits_{\Omega} \langle \nu_x, \Phi(x,\cdot)\rangle d\mu + \int\limits_{e_{F'}\Omega} \langle \nu_x^\infty, \Phi^\infty(x,\cdot)\rangle d\lambda
\end{aligned}$$

and the representation of the Young Measure as the triple

$$\nu = \big(\{\nu_x\}_{x\in\Omega}, \lambda, \{\nu_x^\infty\}_{x \in e_{F'}\Omega}\big), \quad \text{where}$$

oscillation:	$\nu_x \in \mathbb{M}_1^+(\mathbb{R}^d)$ μ-almost every $x \in \Omega$,
concentration:	$\lambda \in \mathbb{M}^+(e_{F'}\Omega)$,
angle-concentration:	$\nu_x^\infty \in \mathbb{M}_1^+(\partial e_{G'}(\mathbb{B}^d))$ λ-almost every $x \in \overline{\Omega}$.

We say that u_n converges in the sense of Young Measures to ν, and write

$$u_n \xrightarrow{Y^p(\mu, e_F)} \nu,$$

where μ is the measure that "regulates and weights" oscillation and concentration of u_n, and e_F is the compactification at infinity, generated by the family F.

From this point onwards the family F' in Ω will always be the set of functions $C(\overline{\Omega})$.

Remark 2.16. It is interesting to compare our representation with the classical formula by DiPerna and Majda [21] and later developments such as those in [30, 40, 54]. In these publications, the authors developed and used Young Measures by taking the object $L \in \mathbb{M}(\overline{\Omega} \times e_{f_i, i\in\mathbb{N}}\mathbb{B}^d)$ as in (10) and testing it against functions $f \in e_{f_i, i\in\mathbb{N}}$ and $\phi \in C_0(\Omega)$. Even though such measure L is, by all means, the same object as what we construct, our deeper analysis reveals a triple-like structure $(\nu_x, \lambda, \nu_x^\infty) \in Y(\mu, e_{f_i, i\in\mathbb{N}})$, where ν_x^∞ is the only part that reflects the use of the compactification $e_{f_i, i\in\mathbb{N}}$. Thanks to this representation, we are able to prove structural theorems about generalised Young Measures and systematically visualise and compare them when changing the measure $\mu \in \mathbb{M}^+(\Omega)$ and the family $\{f_i, i \in \mathbb{N}\}$ (and so the associated Hausdorff compactification $e_{f_i, i\in\mathbb{N}}$). □

2.4. Properties of generalised Young Measures

Here we show how the above construction generalises the more classical setting of Young Measures on the sphere, see [51] for the original idea behind their representation, and [3] for their modern implementation in the calculus of variations. We then study how the new representation for generalised Young Measures behaves geometrically, and its properties.

As a reminder, we state here the definition of integrands with a regular recession at infinity.

Definition 2.17. The set of integrands admitting a regular recession at infinity is

$$\mathbb{E}_p(\Omega, \mathbb{R}^d) = \Big\{ \Phi \in C(\overline{\Omega} \times \mathbb{R}^d) : \lim_{t\to\infty} \frac{\Phi(x,tz)}{t^p} \in \mathbb{R} \text{ loc. uniformly in } (x,z) \in \overline{\Omega} \times \mathbb{R}^d \Big\}.$$

Because we intend to generalise the theory of Young Measures on functions with a regular recession, we need to extend the above class and at the same time preserve good topological properties of such a larger class. To do so, consider countably many functions $(g_i)_i$ in $\mathrm{BC}(\mathbb{B}^d)$ and their representations as integrands of p-growth $g_i(\frac{z}{1+|z|})(1+|z|)^p$. We are interested in understanding how to represent, in a simple way, Young Measures relative to the compactification generated by

$$\mathbb{E}_p \cup \{g_i(\frac{z}{1+|z|})(1+|z|)^p\}.$$

In the language of Hausdorff compactifications, set $G' = C(\overline{\mathbb{B}^d}) \cup \{g_i, i \in \mathbb{N}\}$ and $F' = C(\overline{\Omega})$. Because $C(\overline{\mathbb{B}^d}) \subset G'$, the closure of the algebra generated by either family is separable, separates points from closed sets and contains the constants. Call $F = G' \cup F'$, and without loss of generality, we can assume that $\|g_i\| \leq 1$ for all i.

Lemma 2.18. *$C(e_F\Omega \times \mathbb{B}^d)$ is isometrically isomorphic to $C(\overline{\Omega} \times \overline{\mathrm{graph}(g_i)})$, where $(g_i)_i \colon \mathbb{B}^d \to [-1,1]^{\mathbb{N}}, z \mapsto (g_i(z))_{i\in\mathbb{N}}$ and the topology on the target space is the product topology. It is metrised by*

$$d(z,w) = |z-w| + \sum_i 2^{-i}|g_i(z) - g_i(w)|.$$

Proof. By Lemma 2.14 it is enough to prove that $C(e_{g_i, i\in\mathbb{N}}\mathbb{B}^d) \approx C(\overline{\mathrm{graph}(g_i)})$. Theorem 2.12 provides the isomorphism

$$\overline{\mathbb{A}\Big(C(\overline{\mathbb{B}^d}) \cup \{g_i, i \in \mathbb{N}\}\Big)} = \overline{\mathbb{A}\Big(1, z^1, \ldots, z^d, g_i, i \in \mathbb{N}\Big)},$$

and we conclude by noticing that $\overline{(1, z^1, \ldots, z^d, g_i, i \in \mathbb{N})(\mathbb{B}^d)}$ is homeomorphic to $\overline{\mathrm{graph}(g_i)}$. The topological equivalence between such metrics and the product topology is standard. $\square$

When $g_i \in C(\overline{\mathbb{B}^d})$ for all $i \in \mathbb{N}$, then $C(e_F\Omega \times \mathbb{B}^d) \approx C(\overline{\Omega} \times \overline{\mathbb{B}^d})$, and therefore we recover the usual sphere representation for the compactification induced by $\mathbb{E}_p$. This means the obvious, that we can add functions that have a regular recession and we still obtain the same space (up to homeomorphisms).

By definition, the compactification of $\mathbb{B}^d$ can be represented by the space Γ of sequences $\{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{B}^d$ such that $z_n \to z \in \overline{\mathbb{B}^d}$ and $g_i(z_n)$ converges for all $i \in \mathbb{N}$, and two such sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are identified provided

$$\lim_n |z_n - w_n| + \sum_i 2^{-i}|g_i(z_n) - g_i(w_n)| = 0.$$

Definition 2.19. We call $e_{g_i,i\in\mathbb{N}}$ the *compactification*, and

$$\partial e_{g_i,i\in\mathbb{N}} = e_{g_i,i\in\mathbb{N}} \setminus \operatorname{graph}(g_i, i \in \mathbb{N}),$$

we can write the triple Young Measure as

$$\nu = \big(\{\nu_x\}_{x\in\Omega}, \lambda, \{\nu_x^\infty\}_{x\in\overline{\Omega}}\big),$$

where $\nu_x \in \mathbb{M}_1^+(\mathbb{R}^d)$ for μ-almost every $x \in \Omega$, $\lambda \in \mathbb{M}^+(\overline{\Omega})$, and $\nu_x^\infty \in \mathbb{M}_1^+(\partial e_{g_i,i\in\mathbb{N}})$ for λ-almost every $x \in \overline{\Omega}$. □

Notice that $\partial e_{g_i,i\in\mathbb{N}}$ is an abuse of notation and refers to the boundary of the embedded space within the compactification.

So far we have constructed compactifications by glueing g_is on top of the functions $z^1, \ldots, z^d$; that is to say on top of the unit ball. It is sometimes useful to iterate this argument, to stack another countable family $\{f_i, i \in \mathbb{N}\}$ on top of the compactification $e_{g_i,i\in\mathbb{N}}$. This process gives the same compactification as if we were considering the two families at once, as the following lemma shows.

Lemma 2.20. *Let $e_{g_i,i\in\mathbb{N}}$ be a compactification of $\mathbb{B}^d$ and $f_i \in \mathrm{BC}(\mathbb{B}^d)$ for all $i \in \mathbb{N}$. Then*

$$e_{g_i,f_i,i\in\mathbb{N}} \approx \overline{\operatorname{graph}_{\operatorname{graph}(g_i)} f_i}.$$

Proof. This is a trivial consequence of the fact that, after extending f_i to a constant in the variable $w = (w^1, w^2, \ldots)$,

$$\begin{aligned}
&\{(z^1, \ldots, z^d, g_i(z), f_i(z)), z \in \mathbb{B}^d\} \\
&\quad = \{(z^1, \ldots, z^d, w, z) : w = g_i(z), y = f_i(z), z \in \mathbb{B}^d\} \\
&\quad = \{(z^1, \ldots, z^d, w, z) : y = f_i(z, w), w = g_i(z), z \in \mathbb{B}^d\} \\
&\quad = \operatorname{graph}_{\operatorname{graph}(g_i)}(f_i), z \in \mathbb{B}^d.
\end{aligned}$$

□

A standard application of the disintegration lemma yields the following.

Corollary 2.21. *Consider a compactification $e_{g_i,f_i,i\in\mathbb{N}}$ and $\nu^\infty \in \mathbb{M}(\partial e_{g_i,f_i,i\in\mathbb{N}})$, then*

$$\nu^\infty = P_{(z_n)_{n\in\mathbb{N}}} d\tilde{\nu}^\infty$$

where $\tilde{\nu}^\infty \in \mathbb{M}(\partial e_{g_i})$, $(z_n)_n \in \partial e_{g_i}$, and $P_{(z_n)_n}$ is a probability measure defined on the space of subsequences $(z_{n_i})_i$ of $(z_n)_n$ so that $f_i(z_{n_i})$ converges for all $i \in \mathbb{N}$ (with sequences being equivalent if all the limits are).

For the case of oscillating functions $\{f_i, i \in \mathbb{N}\}$, we also write the compactification as $e_{f_i,i\in\mathbb{N}}X \equiv e_{f_i,i\in\mathbb{N}}$ and the convergence as

$$v_j \xrightarrow{Y^p(\mu, e_{f_i,i\in\mathbb{N}})} \nu.$$

When working with the sphere compactification, we will simply write

$$v_j \xrightarrow{Y^p(\mu,\mathbb{B}^d)} \nu, \quad \text{or just } v_j \xrightarrow{Y^p(\mu)} \nu.$$

Also, when working with $p = 1$, we omit the superscript p in Y^p and write

$$v_j \xrightarrow{Y(\mu, e_{f_i,i\in\mathbb{N}})} \nu.$$

We now study the relation of Young Measures with respect to different compactifications and different underlying measures $\mu \in \mathbb{M}^+(\Omega)$. Using Chacon's Lemma A.5, we can prove the following structure theorems.

Lemma 2.22. *Let* $v_j \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} (\nu_x, \lambda, \nu_x^\infty)$. *Then for all* $\psi \in C_0(\mathbb{R}^d)$, *we have*

$$\psi(v_j) \rightharpoonup \langle \nu_x, \psi\rangle \quad \textit{weakly in } L^1(\mu).$$

Proof. Because $\|\psi(v_j)\|_\infty \leq \|\psi\|_\infty$, then the sequence is equi-integrable and there is a subsequence that converges weakly in L^1 to v. Because $\psi^\infty = 0$, testing against $\phi \in \mathcal{D}(\Omega)$ we get

$$\int_\Omega \phi\psi(u_j) d\mu \to \int_\Omega \phi v d\mu = \int_\Omega \phi \langle \nu_x, \psi\rangle d\mu. \qquad \square$$

We can improve the above weak convergence result to show the following.

Lemma 2.23. *Let* $u_j \xrightarrow{Y(\mu)} \big(\nu_x, 0, N/A\big)$. *For every* $a \in L^1(\Omega, \mu)$ *such that* $a > 0$ μ*-a.e. and for all* $\psi \in C_0(\mathbb{R}^d)$ *we have*

$$\psi\left(\frac{u_j}{a}\right) a \rightharpoonup \left\langle \nu_x, \psi\left(\frac{\cdot}{a(x)}\right)\right\rangle a(x) \quad \textit{in } L^1(\Omega, \mu). \qquad \square$$

As expected, this implies that oscillations do not depend on the particular compactification chosen.

Before proving the above results we show the following uniform approximation result:

Lemma 2.24. *Let* $\mu \in \mathbb{M}^+(\Omega)$ *and* $a \in L^1(\Omega, \mu), a > 0$ μ*-almost everywhere. There exists* $a_n \in L^1(\Omega), 0 < a_n < a$ *so that* $a_n(x) \in \mathbb{Q}$ *for all* $x \in \Omega$ *and*

$$\|a_n - a\|_\infty + \left\|\frac{a}{a_n} - 1\right\|_\infty \xrightarrow{n\to\infty} 0.$$

Proof. Let $a_n = \displaystyle\sum_{k\in\mathbb{N}, k\geq 1} \chi_{a^{-1}\left([\frac{k}{n}, \frac{k+1}{n})\right)} \frac{k}{n}.$

Because $a_n \leq a$ then $a_n \in L^1$ for all $n \in \mathbb{N}$ and it also assumes countably many values at a time. Also $|a_n(x) - a(x)| \leq \frac{1}{n}$ so it converges uniformly to a. Furthermore

$$1 = \frac{\frac{k}{n}}{\frac{k}{n}} \leq \frac{a(x)}{a_n(x)} \leq \frac{\frac{k+1}{n}}{\frac{k}{n}} = \frac{k+1}{k}$$

and so $\frac{a}{a_n}$ converges uniformly to 1. $\square$

Now we can prove Lemma 2.23

Proof. (of Lemma 2.23). Using the previous approximation, we write, for 1-Lipschitz $\psi \colon \mathbb{R}^d \to \mathbb{R}$,

$$\int_\Omega \psi\left(\frac{u_j}{a}\right) a = \int_\Omega \overbrace{\psi\left(\frac{u_j}{a}\right) a - \psi\left(\frac{u_j}{a_n}\right) a}^{=I} + \overbrace{\psi\left(\frac{u_j}{a_n}\right) a - \psi\left(\frac{u_j}{a_n}\right) a_n}^{=II} + \psi\left(\frac{u_j}{a_n}\right) a_n.$$

The first two terms are bounded by

$$|I| \leq \int_\Omega a \left|\frac{u_j}{a} - \frac{u_j}{a_n}\right| = \int_\Omega |u_j| \left|1 - \frac{a}{a_n}\right| \leq \sup_j \|u_j\| \left\|1 - \frac{a}{a_n}\right\|_\infty$$

$$|II| \leq \int_\Omega \left(1 + \frac{|u_j|}{a_n}\right) |a - a_n| \leq (1 + \sup_j |u_j|) \left\|1 - \frac{a}{a_n}\right\|_\infty,$$

which goes to 0 as $n \to \infty$ uniformly in j. With respect to the third term, calling $E_k = a^{-1}\big([\frac{k}{n}, \frac{k+1}{n})\big)$, we use the dominated convergence theorem to pass to the limit

$$\begin{aligned}\lim_j \int_\Omega \psi\left(\frac{u_j}{a_n}\right) a_n &= \lim_j \sum_k \int_{E_k} \psi\left(\frac{u_j}{k/n}\right) \frac{k}{n} \\ &= \sum_k \int_{E_k} \left\langle \nu_x, \psi\left(\frac{\cdot}{k/n}\right)\right\rangle \frac{k}{n} = \int_\Omega \left\langle \nu_x, \psi\left(\frac{\cdot}{a_n}\right)\right\rangle a_n.\end{aligned}$$

Another application of the dominated convergence theorem will let us conclude the result. $\square$

We can finally conclude with a structure theorem regarding concentration.

Proposition 2.25. *Consider two separable algebras (that separate points from closed sets) A and B of $\mathbb{G}_1$. Let $a \in L^1(\Omega, \mu)$, $a > 0$ μ-a.e. and $(v_j)_j$ in $L^1(\Omega, \mu)$ be so that*

$$v_j \xrightarrow{Y(\mu, e_A)} \big(\nu_x, \lambda_\nu, \nu_x^\infty\big) \quad \textit{and} \quad \frac{v_j}{a} \xrightarrow{Y(a\,d\mu, e_B)} \big(\eta_x, \lambda_\eta, \eta_x^\infty\big).$$

Then $\nu_x = \left(\frac{\cdot}{a(x)}\right)_\# \eta_x$, $\lambda_\nu = \lambda_\eta = \lambda$. *Moreover, decomposing*

$$\nu_x^\infty = P^\nu_{(z_n)_n} d\tilde{\nu}_x^\infty \quad \textit{and} \quad \eta_x^\infty = P^\eta_{(z_n)_n} d\tilde{\eta}_x^\infty,$$

where $\tilde{\nu}_x^\infty$ and $\tilde{\eta}_x^\infty$ are the projections on the sphere according to the decomposition 2.21, then $\tilde{\nu}_x^\infty = \tilde{\eta}_x^\infty$ λ-a.e. with

$$v_j \xrightarrow{Y(\mu, \mathbb{B}^d)} \big(\nu_x, \lambda, \tilde{\nu}_x^\infty\big).$$

Proof. For all $\psi \in C_0(\mathbb{R}^d)$ we have that $\psi(v_j)$ is equi-integrable so that

$$\psi(v_j) \rightharpoonup \langle \eta_x, \psi\rangle \quad \text{in } L^1(\mu)$$

and

$$\psi\left(\frac{v_j}{a}\right) \rightharpoonup \langle \nu_x, \psi\rangle \quad \text{in } L^1(ad\mu).$$

Next, identify v_j with its subsequence and find E_k so that $v_j \rightharpoonup v$ in $L^1(E_k, \mu)$ for all k. By inner approximation, we can assume that E_k is compact for all k. Consider now the sequence $v_j\chi_{E_k}$. Then

$$v_j\chi_{E_k} \xrightarrow{Y(\mu, e_A)} \big(\eta_x\chi_{E_k} + \delta_0\chi_{E_k^c}, 0, N/A\big).$$

By Lemma 2.23 we then have, for $\phi \in C_0(\Omega)$ and $\psi \in C_0(\mathbb{R}^d)$, because $\Omega \setminus E_k$ is open,

$$
\begin{aligned}
\int_\Omega \phi\langle \nu_x, \psi\rangle a(x) d\mu &= \lim_j \int_\Omega \phi\psi\left(\frac{v_j}{a}\right) a d\mu \\
&= \lim_j \int_{\Omega\setminus E_k} \phi\psi\left(\frac{v_j}{a}\right) a d\mu + \int_{E_k} \phi\psi\left(\frac{v_j}{a}\right) a d\mu \\
&= \int_{\Omega\setminus E_k} \phi\langle \nu_x, \psi\rangle a d\mu + \int_{E_k} \phi\left\langle \eta_x, \psi\left(\frac{\cdot}{a}\right)\right\rangle a d\mu.
\end{aligned}
$$

Next, let $\phi \in C(\overline{\Omega})$, then

$$
\begin{aligned}
\lim_j \int_\Omega \phi |v_j| d\mu &= \int_\Omega \langle \nu_x, |\cdot|\rangle \phi d\mu + \int_{\overline{\Omega}} \phi d\lambda_\nu \\
= \lim_j \int_\Omega \phi \left|\frac{v_j}{a}\right| a d\mu &= \int_\Omega \left\langle \eta_x, \frac{|\cdot|}{a(x)}\right\rangle \phi a(x) d\mu + \int_{\overline{\Omega}} \phi d\lambda_\eta.
\end{aligned}
$$

Using the previous part we conclude that $\lambda_\nu = \lambda_\eta = \lambda$.

Finally, let $f \in C(\partial\mathbb{B}^d)$ and extending by 1-homogeneity we obtain that

$$
\begin{aligned}
&\lim_j \int_\Omega \phi f(v_j) d\mu \\
&= \int_\Omega \phi\langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi\langle \nu_x^\infty, f^\infty\rangle d\lambda = \int_\Omega \phi\langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int\int f^\infty dP^\nu_{(z_n)} d\tilde{\nu}_x^\infty d\lambda_\nu \\
&= \int_\Omega \phi\langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int f^\infty d\tilde{\nu}_x^\infty d\lambda_\nu \\
&= \int_\Omega \phi\left\langle \nu_x, f\left(\frac{\cdot}{a(x)}\right)\right\rangle a(x) d\mu + \int_{\overline{\Omega}} \phi\langle \eta_x^\infty, f^\infty\rangle d\lambda \\
&= \int_\Omega \phi\langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int f^\infty dP^\eta_{(z_n)} d\tilde{\eta}_x^\infty d\lambda_\eta = \int_\Omega \phi\langle \nu_x, f\rangle d\mu + \int_{\overline{\Omega}} \phi \int f^\infty d\tilde{\eta}_x^\infty d\lambda_\eta,
\end{aligned}
$$

and the proof is complete. □

Notice that the previous identification with the concentration angle measure fails if we only consider $\Gamma = A \cap B$ which does not necessarily generate the sphere compactification. This is so because sequences $(u_j)_j$ can concentrate around values of a that are discontinuous in a measure-theoretic sense. However, equality holds if $a = 1$.

Lemma 2.26. *Under the assumptions of Proposition* 2.25 *above and* $a = 1$*, we let* $\Gamma = A \cap B$*. Writing* $\nu_x^\infty = P^\nu_{(z_n)_n} d(\gamma^\nu)_x^\infty$ *and* $\eta_x^\infty = P^\eta_{(z_n)_n} d(\gamma^\eta)_x^\infty$*, where* γ^η *and* γ^ν *are the projections onto the compactification generated by* Γ*, then*

$$
(\gamma^\nu)_x^\infty = (\gamma^\eta)_x^\infty \quad \lambda\text{-}a.e.
$$

Proof. Proven similarly at the end of Proposition 2.25 and testing against functions belonging in $\overline{\mathbb{A}(\Gamma)}$ and using the decomposition of angle Young Measures. □

Next, we show that the lack of concentration is equivalent to the equi-integrability of the generating sequence.

Theorem 2.27. *Let $(v_j)_j$ in $L^1(\Omega,\mu,\mathbb{R}^d)$ be so that*

$$v_j \xrightarrow{Y(\mu, e_{f_i,i\in\mathbb{N}})} \big(\nu_x, \lambda, \nu_x^\infty\big).$$

Then the sequence $(v_j)_j$ is equi-integrable if and only if $\lambda = 0$.
Moreover $v_j \to v$ strongly in $L^1(\Omega,\mu,\mathbb{R}^d)$ if and only if $\lambda = 0$ and $\nu_x = \delta_{v(x)}$ for μ-a.e. $x \in \Omega$.

Proof. Because λ does not depend on the compactification (see Proposition 2.25), we can apply the same theorem from the sphere compactification, [53], Lemma 12.14 and [53], Corollary 12.15, to conclude. □

Compare the above theorem to the Vitali convergence Theorem A.1 in the appendix.

Before stating the next two structure results, we prove that T^{-1} is a bounded operator from $\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})$ to $\mathrm{Lip}(\mathbb{R}^d)$, provided the compactification is generated by Lipschitz functions. In this case, by $\mathrm{Lip}(\mathbb{R}^d)$ we mean the weighted norm

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^d)} := \|Tf\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|} = \left\|\frac{f}{1+|\cdot|}\right\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}.$$

As for the compactification, the metric is always intended as in Lemma 2.18.

Lemma 2.28. *Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification metrised by the usual metric, where, for every $i\in\mathbb{N}$, $f_i \in \mathrm{Lip}(\mathbb{R}^d)$ are normalised so that $\|f\|_{\mathrm{Lip}(\mathbb{R}^d)} \le 1$.*

Then $$\sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|} \le 5\,\mathrm{Lip}(Tg, e_{f_i,i\in\mathbb{N}}) \quad \text{for all maps } g\colon \mathbb{R}^d \to \mathbb{R}$$

Proof. Without loss of generality assume that $\|Tg\|_{\mathrm{Lip}} \le 1$, i.e. for all $|x|,|y|<1$,

$$\left| g\left(\frac{x}{1-|x|}\right)(1-|x|) - g\left(\frac{y}{1-|y|}\right)(1-|y|)\right| \le |x-y| + \sum_i 2^{-i}|Tf_i(x)-Tf_i(y)|.$$

Then

$$\begin{aligned}
&|g(x)-g(y)| \\
&= \left|\frac{g(x)}{1+|x|}(1+|x|) - \frac{g(x)}{1+|x|}(1+|y|) + \frac{g(x)}{1+|x|}(1+|y|) - \frac{g(y)}{1+|y|}(1+|y|)\right| \\
&= \frac{|g(x)|}{1+|x|}\Big|1+|x|-1-|y|\Big| + (1+|y|)\left|\frac{g(x)}{1+|x|} - \frac{g(y)}{1+|y|}\right| \\
&\le |x-y| + (1+|y|)\Big(\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right| + \sum_i 2^{-i}\Big|Tf_i(\tfrac{x}{1+|x|}) - Tf_i(\tfrac{y}{1+|y|})\Big|\Big).
\end{aligned}$$

For all i we have that

$$\left|Tf_i\Big(\frac{x}{1+|x|}\Big) - Tf_i\Big(\frac{y}{1+|y|}\Big)\right| = \left|\frac{f_i(x)}{1+|x|} - \frac{f_i(y)}{1+|y|}\right|,$$

and therefore after multiplying by $1+|y|$ we obtain

$$\left|\frac{f_i(x)}{1+|x|}\Big(1+|x|+(|y|-|x|)\Big)-f_i(y)\right|=\left|f_i(x)-f_i(y)+\frac{f_i(x)}{1+|x|}(|y|-|x|)\right|$$
$$\leq|f_i(x)-f_i(y)|+\frac{|f_i(x)|}{1+|x|}|x-y|\leq 2|x-y|. \qquad \square$$

We now show that the above lemma allows us to test Young Measures on Lipschitz compactifications against Lipschitz functions of $\mathbb{R}^d$.

Definition 2.29. (Kantorovich semi-norm) Let X be a metric space and $\mu\in\mathbb{M}(X)$, then the (formal) *Kantorovich norm* of μ is

$$\|\mu\|_K=\sup_{\|\phi\|_{\mathrm{Lip}}\leq 1}\int_X \phi d\mu.$$

The above formula induces a pseudo-distance between measures by setting

$$d(\mu,\eta)_K=\|\mu-\eta\|_K.$$

It turns out that this is indeed a metric on the positive cone of non-negative measures, see Lemma A.6. In particular, by taking $\Psi\in\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})$ and the pull-back T^{-1} we deduce the following.

Lemma 2.30. *Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification. Then, every measure $\nu\in Y(\mu,e_{f_i,i\in\mathbb{N}})$ is defined by testing it against Lipschitz functions of the form*

$$\phi\otimes\psi,\quad \textit{where } \|\phi\|_{\mathrm{Lip}(\Omega)}\leq 1,\ \|\psi\|_{\mathrm{Lip}(\mathbb{R}^d)}\leq 1.$$

For this reason, we remind once again of the norm we will be using on the space $e_{f_i,i\in\mathbb{N}}$ throughout this paper.

Definition 2.31. We say that $e_{f_i,i\in\mathbb{N}}$ is a *Lipschitz compactification* if each f_i is Lipschitz continuous, and renormalised so that $\mathrm{Lip}(Tf)\leq 1$. The norm on $\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})$ will always be

$$\|g\|_{\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})}:=\sup_{x\in e_{f_i,i\in\mathbb{N}}}|g(x)|+\sup_{x\neq y}\frac{|g(x)-g(y)|}{d_{e_{f_i,i\in\mathbb{N}}}(x,y)}$$

where
$$d_{e_{f_i,i\in\mathbb{N}}}(x,y)=|x-y|+\sum_i 2^{-i}|Tf_i(x)-Tf_i(y)|.$$

To conclude this subsection, we state decomposition results for Young Measures regarding oscillation and concentration. Originally proven in the context of the sphere compactification, [34], we extend them here to separable compactifications.

Lemma 2.32. *Let $(v_j)_j$ in $L^1(\Omega,\mu)$ so that $v_j\xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})}\big(\nu_x,\lambda,\nu_x^\infty\big)$.*
We can write $v_j=o_j+c_j$, where for all $j\in\mathbb{N}$, $o_j\in L^1(\Omega,\mu)$ is equi-integrable,

$$o_j\xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})}\big(\nu_x,0,N/A\big)$$

and $c_j\in L^1(\Omega,\mu)$ so that $c_j\xrightarrow{Y(\mu,e_{f_i,i\in\mathbb{N}})}\big(\delta_0,\lambda,\nu_x^\infty\big)$.

Proof. A standard diagonal argument gives us $k_j \uparrow \infty$ so that $o_j = v_j\chi_{|v_j|} \leq k_j$ is equi-integrable and generates $o_j \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} \big(\nu_x, 0, N/A\big)$. Then, letting $c_j = v_j - o_j$, for $\eta \in C(\overline{\Omega})$, $T\psi \in C(e_{f_i,i\in\mathbb{N}})$ we have

$$\begin{aligned}\int_\Omega \eta(\psi(c_j) - \psi(v_j)) &= \int_{|v_j|\leq k_j} \eta(\psi(0) - \psi(o_j)) + \int_{|v_j|>k_j} \eta(\psi(v_j) - \psi(v_j)) \\ &= \int_\Omega \eta(\psi(0) - \psi(o_j)) \to \int_\Omega \eta(\psi(0) - \langle \nu_x, \psi\rangle).\end{aligned}$$

Writing $\psi(c_j) = \Big(\psi(c_j) - \psi(v_j)\Big) + \psi(v_j)$ and letting $j \to \infty$ we conclude. □

We remark here that when considering certain subsets of E of Y (for example Young Measures generated by gradients, see next section) o_j and c_j might, respectively, generate Young Measures that do not belong to E.

The next lemma shows that the opposite implication of 2.32 is also true.

Lemma 2.33. *Let $(v_j)_j$ and $(w_j)_j$ in $L^1(\mu)$ generate*

$$v_j \xrightarrow{Y(\mu, e_{f_i,i\in\mathbb{N}})} \big(\delta_{v(x)}, \lambda_\eta, \eta_x^\infty\big) \quad \textit{and} \quad w_j \xrightarrow{Y(\mu, e_{f_i,i\in\mathbb{N}})} \big(\nu_x, \lambda_\nu, \nu_x^\infty\big)$$

with $\lambda_\eta \perp \lambda_\nu$, for some $v \in L^1(\mu)$. Then the sum of the sequence generates

$$v_j + w_j \xrightarrow{Y(\mu, e_{f_i,i\in\mathbb{N}})} \big(\delta_{v(x)} * \nu_x, \lambda_\nu + \lambda_\eta, k_x^\infty\big),$$

where
$$k_x^\infty = \begin{cases} \nu_x^\infty & \lambda_\nu\text{-}a.e. \\ \eta_x^\infty & \lambda_\eta\text{-}a.e. \end{cases}$$

Proof. Write $w_j = o_j + c_j$ as in the previous lemma and put $b_j = v_j - v$. We claim

$$b_j + c_j \xrightarrow{Y(\mu, e_{f_i,i\in\mathbb{N}})} \big(\delta_0, \lambda_\nu + \lambda_\eta, k_x^\infty\big).$$

No oscillation is a consequence of the fact that $b_j + c_j \to 0$ in μ-measure. Next, let $\phi \in C(\overline{\Omega})$, $\|\phi\|_{\mathrm{Lip}} \leq 1$ and $\Psi \in e_{f_i,i\in\mathbb{N}}$, $\|T\Phi\|_{\mathrm{Lip}} \leq 1$ with $\Psi(0) = 0$. Let E_ν and E_η be sets where λ_ν and λ_η are concentrated, respectively. For $\varepsilon > 0$ find $C_\nu \subset E_\nu, C_\eta \subset E_\eta$ compact sets and $O_\nu \supset C_\nu, O_\eta \supset C_\eta$ open sets such that

$$\lambda_\eta(\overline{\Omega} \setminus C_\eta) + \lambda_\nu(O_\eta) + \lambda_\nu(\overline{\Omega} \setminus C_\nu) + \lambda_\eta(O_\nu) < \varepsilon.$$

Consider a function $\rho \in C(\mathbb{R}^n)$ with $\chi_{C_\eta} \leq \rho \leq \chi_{O_\eta}$. We write

$$\int_\Omega \phi\big(\Psi(b_j + c_j) - \Psi(b_j) - \Psi(c_j)\big)(\overbrace{\rho}^{=I} + \overbrace{1-\rho}^{II})d\mu.$$

We estimate the first term by

$$\limsup_j |I| \leq \limsup_j 2\int_\Omega \rho|b_j| = 2\int_{\overline{\Omega}} \rho d\lambda_\nu \leq 2\lambda_\nu(\overline{\Omega} \cap O_\eta) \leq 2\varepsilon.$$

Similarly for the second term

$$\limsup_j |II| \leq \limsup_j 2\int_\Omega (1-\rho)|c_j| \leq 2\lambda_\eta(\overline{\Omega}\setminus C_\eta) \leq 2\varepsilon.$$

But the first term converges to 0 and therefore we conclude for the representation of Young Measures. □

2.5. Characterisation of generalised Young Measures

We dedicate this part to clarifying which Young Measure can be generated by integrable functions. We start by giving a clear definition of what generalised Young Measures are.

Definition 2.34. Given a family $\{f_i, i \in \mathbb{N}\}$, with $f_i \in \mathbb{G}_p$ for all $i \in \mathbb{N}$, that separates points from closed sets, a *p-Young Measure* is a triple

$$\nu = \big((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\overline{\Omega}}\big)$$

where

1. $(\nu_x)_{x\in\Omega}$ is μ-measurable and $\nu_x \in \mathbb{M}_1^+(\mathbb{R}^d)$ μ-a.e. $x \in \Omega$. We call it *oscillation*;
2. $\lambda \in \mathbb{M}^+(\overline{\Omega})$. We call it *concentration*;
3. $(\nu_x^\infty)_{x\in\overline{\Omega}}$ is λ-measurable and $\nu_x^\infty \in \mathbb{M}_1^+(\partial e_{f_i, i\in\mathbb{N}})$ λ-a.e. $x \in \overline{\Omega}$. We call it *angle-concentration*;
4. the moment condition must hold:

$$\int_\Omega \int_{\mathbb{R}^d} |z|^p d\nu_x(z) < +\infty.$$

The collection of all such triples is denoted by $Y^p = Y^p(\Omega, \mu, e_{f_i, i\in\mathbb{N}})$. □

When obvious from the context, we will not specify the domain Ω, the measure μ, the family F or the target space $\mathbb{R}^d$. Also, when $\lambda = 0$, i.e. when there is no concentration, there is no point in specifying the compactification we are working with.

From every triple $\nu = \big(\nu_x, \lambda, \nu_x^\infty\big)$ one can construct the measure $L \in C(e_F\Omega \times \mathbb{R}^d)$ and vice-versa.

Lemma 2.35. *The set Y^p equals*

$$T^*\left\{L\in\mathbb{M}^+(e_F\Omega\times\mathbb{B}^d) : \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-|\hat{z}|)^p dL = \int_\Omega \phi(x)d\mu(x) \ \ \forall\phi\in C(e_G\Omega)\right\}.$$

In particular Y^p is a weak closed and convex subset of F^*.*

Proof. Let $L \in \mathbb{M}^+(e_F\Omega \times \mathbb{R}^d)^+$ as above. It was already shown that $T^*L \in Y^p$. On the other side, if $\nu = \big(\nu_x, \lambda, \nu_x^\infty\big)$, we let

$$L = \nu_x d\mu + \nu_x^\infty d\lambda.$$

Testing against a test function $\phi = \phi(x)$ that only depends on x,

$$\int_{e_F\Omega\times\mathbb{B}^d} T\phi dL = \int_{e_F\Omega\times\mathbb{B}^d} \phi(x)(1-|\hat{z}|)^p dL = \int_\Omega \phi d\mu.$$

Conclude by noticing that the above characterisation amounts to

$$Y^p = T^*\Big\{ \bigcup_{\phi\in C(e_F\Omega)} \Big\{\{L\in\mathbb{M}^+(e_F\Omega\times\mathbb{B}^d)^+ : \int_{e_F\Omega\times\mathbb{B}^d}\phi(x)(1-\hat{z})^p dL = \int_\Omega \phi d\mu\Big\}\Big\}. \quad \square$$

We now prove that every Young Measure $\nu = (\nu_x, \lambda, \nu_x^\infty)$ can be generated by a sequence of integrable functions. We will show this by constructing two sequences, one that generates oscillation and one that generates concentration only. We concentrate exclusively on the case of non-atomic measures $\mu \in \mathbb{M}^+(\Omega)$, which are measures so that for all $A \subset \Omega$ with $\mu(A) > 0$, there is $B \subset A$ such that $0 < \mu(B) < \mu(A)$. On the contrary, a purely atomic measure $\mu \in \mathbb{M}^+(\Omega)$ is a measure that can be written as $\mu = \sum_i a_i\delta_{x_i}$, where $(x_i)_i \in \Omega$ is a sequence of points and $(a_i)_i$ is summable (and so convergence is strong).

Lemma 2.36. *Let $\mu \in \mathbb{M}^+(\Omega)$ non-atomic and $\nu_x\colon \Omega \to \mathbb{M}_1^+(\mathbb{R}^d)$ weak* μ-measurable and so that*

$$\int_\Omega \langle \nu_x, |\cdot|\rangle d\mu < +\infty.$$

There exists a sequence $(v_j)_j$ in $L^1(\Omega,\mu,\mathbb{R}^d)$ such that $v_j \xrightarrow{Y} (\nu_x, 0, N/A)$ and $v_j(x) \in \cup_y \mathrm{supp}(\nu_y)$ for μ-a.e. $x \in \Omega$.

Proof. Notice that $\mathbb{M}_1^+(\mathbb{R}^d)$ is a metric space and denote by d_k its Kantorovich metric A.6, and define

$$d(\mu_1,\mu_2) = d_K(\mu_1,\mu_2) + \left|\int_{\mathbb{R}^d} |\cdot| d(\mu_1-\mu_2)\right|$$

Fix an arbitrary point $\tilde{z} \in \cup_y \mathrm{supp}(\nu_y)$. Find $K_j \subset \Omega$ compact sets such that $\mu(\Omega\setminus K_j) < j^{-1}$ and $x \mapsto \nu_x \in C(K_j, (\mathbb{M}_1^+, d))$.

For each $j\in\mathbb{N}$, by compactness, we can find finitely many sets $(E_s^j)_s, \mu(E_s^j) > 0$ so that $\cup_s E_s^j = K_j$ and $diam(E_s^j)$ is so small that, fixing an arbitrary point $x_s^j \in E_s^j$,

$$d(\nu_{x_j^s}, \nu_x) < j^{-1}\ \forall x \in E_s^j.$$

By the density of the span of Dirac measures, we can find finitely many values $(z_{s,t}^j, \lambda_{s,t}^j)_t \subset \mathbb{R}^d\times\mathbb{R}^+$ so that $\sum_t \lambda_{s,t}^j = 1$ and

$$d\Big(\sum_t \lambda_{s,t}^j \delta_{z_{s,t}^j}, \nu_{x_s^j}\Big) < j^{-1}.$$

In the above, we can assume that each $z_{s,t}^j \in \mathrm{supp}(\nu_{x_s^j})$. Finally, on each E_s^j, because $\mu(E_s^j) > 0$ and μ is non-atomic, we can divide the set into$t$ many subsets of measure $\mu(E_{s,t}^j) = \lambda_{s,t}^j\mu(E_s^j)$ and put v_j so that

$$v_j = \sum_s\sum_t \chi_{E_{s,t}^j} z_{s,t}^j + \tilde{z}\chi_{K_j^c}.$$

Because the above sets are disjoint, then $v_j(x) \in \cup_y \mathrm{supp}(\nu_y)$ for μ-a.e. $x \in \Omega$. To prove convergence, let $\Psi \in C_0(\mathbb{R}^d)$ with $\mathrm{Lip}(\Psi) \le 1$, or $\Psi(z) = |z|$, and compute

$$\left| \int_{E_j} \Psi(v_j(x)) d\mu(x) - \langle \nu_x, \Psi \rangle d\mu(x) \right| = \left| \sum_s \sum_t \int_{E^j_{s,t}} \Psi(z^j_{s,t}) d\mu(x) - \langle \nu_x, \Psi \rangle d\mu(x) \right|$$
$$= \left| \sum_s \int_{E^j_s} \int_{\mathbb{R}^d} \Psi(z) \left(\left(d \sum_t \lambda^j_{s,t} \delta_{z^j_{s,t}} \right) - d\nu_{x^k_s} + d\nu_{x^k_s} - d\nu_x \right) d\mu(x) \right| \le 2j^{-1}\mu(E),$$

where we remind that $d_K \le d$. Outside of E_j we have

$$\int_{\Omega \setminus E_j} \Psi(v_j(x)) d\mu(x) - \langle \nu_x, \Psi \rangle d\mu(x) = \Psi(\tilde{z})\mu(\Omega \setminus E_j) + \int_{\Omega \setminus E_j} \langle \nu_x, \Psi \rangle d\mu(x) \to 0.$$

By density, this holds for all $\Psi \in C_0(\mathbb{R}^d)$. To show that v_j is equi-integrable, we consider an approximation of the unity $\Psi \in C_c(\mathbb{B}_k), \mathbb{B}_k \subset \mathbb{R}^d$ so that $0 \le \Psi \le k-1$ and $\Psi(z) = |z|$ if $|z| \le k-1$.

$$\limsup_j \int_{|v_j|>k} |v_j| d\mu \le \limsup_j \int_\Omega |v_j| d\mu(x) - \int_\Omega \Psi_k(v_j) d\mu = \int_\Omega \langle \nu_x, |\cdot| - \Psi_k(\cdot) \rangle d\mu$$

which converges to 0 as $k \to \infty$ by the dominated convergence theorem. □

We now show that any angle concentration can also be generated by integrable functions. Notice that, in this case, we need to approximate λ and $(\nu^\infty_x)_{x \in \overline{\Omega}}$ at the same time.

Theorem 2.37. *Let $\mu \in \mathbb{M}^+(\Omega)$ be non-atomic. Let $\lambda \in \mathbb{M}^+(\overline{\Omega})$ be such that $\mathrm{supp}(\lambda) \subset \mathrm{supp}(\mu)$ and $\nu^\infty_x \colon \Omega \to \mathbb{M}^+_1(\partial e_{f_i, i \in \mathbb{N}})$ weak* λ-measurable, where, for all $i \in \mathbb{N}$, f_i are continuous and have linear growth. Then, there is a sequence $(v_j)_j$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ so that*

$$v_j \xrightarrow{Y(\mu, e_{f_i, i \in \mathbb{N}})} (\delta_0, \lambda, \nu^\infty_x).$$

Moreover, $v_j(x)$ satisfies

$$d\left(\frac{v_j(x)}{1 + v_j(x)}, \pi_{\partial \mathbb{B}^d}(\cup_y \mathrm{supp}(\nu^\infty_y)) \right) \xrightarrow{j \to \infty} 0 \quad or \quad v_j(x) = 0.$$

We first prove the result when λ is just a Dirac delta. In the following d is the Kantorovich metric (see Lemma A.6) on $\mathbb{M}^+_1(\partial e_{f_i, i \in \mathbb{N}})$.

Lemma 2.38. *Under the assumptions of the main theorem, let $\lambda = a\delta_z$, $a > 0$, $z \in \mathrm{supp}(\mu)$, then there is $\phi_j \in L^1(\mathbb{B}_{j^{-1}}, \mu, \mathbb{R}^d)$ so that*

$$\int_\Omega f(\phi_j) d\mu \to f(0) d\mu(\Omega) + a \langle \nu^\infty_0, f \rangle$$

uniformly in $Tf \in \mathrm{Lip}(e_{f_i, i \in \mathbb{N}})$ with $\mathrm{Lip}(Tf) \le 1$.

Proof. Assume $z = 0$. Find $(w_h^j)_h = (w_h)_h \in e_{f_i, i\in\mathbb{N}}$ and $\theta_h^j = \theta_h$ with $\sum_h \theta_h = 1$ so that

$$d\Big(\sum_h \theta_h \delta_{w_h}, \nu_0^\infty\Big) < j^{-1}.$$

Notice that $w_h = (T(\omega_n^h))_n$ for some $\omega_n \to \infty$ in $\mathbb{R}^d$, so can find $n = n_j$ so that $\tilde{d}(T(\omega_{n_j}^h), w_h) < j^{-1}$ for all h, where $\tilde{d}$ is the distance on the compact metric space $e_{f_i, i\in\mathbb{N}}$. By assumption, $\mu(\mathbb{B}_r) > 0$ for all $r > 0$. Therefore, because μ is non-atomic, we can find h many μ-measurable sets $E_h^j \subset \mathbb{B}_{j^{-1}}$ so that $\mu(E_h^j) > 0$ and

$$\mu(E_h^j) = \theta_h \, \frac{a}{|\omega_{n_j}^h|},$$

where, if necessary, we increased n_j. Finally, set $\phi_j = \displaystyle\sum_h \chi_{E_h^j} \omega_{n_j}^h$.

Taking f so that $\mathrm{Lip}_{e_{f_i, i\in\mathbb{N}}}(Tf) \le 1$ we compute

$$\int_\Omega f(\phi_j) d\mu = \int_{\Omega \setminus \cup_h E_h^j} f(0) d\mu + a \sum_h \theta_h \frac{f(\omega_{n_j}^h)}{|\omega_{n_j}^h|}.$$

The left term converges to $f(0)\mu(\Omega)$. Because Tf is one Lipschitz, we have that

$$\left| \frac{f(\omega_{n_j}^h)}{|\omega_{n_j}^h|} - Tf(w^h) \right| \le \tilde{d}(T(\omega_{n_j}^h), w^h) \le j^{-1}.$$

Therefore we conclude that

$$\left| a \sum_h \theta_h \frac{f(\omega_{n_j}^h)}{|\omega_{n_j}^h|} - a \sum_h \theta_h f^\infty(w^h) \right| \le j^{-1}$$

and the right-hand side converges to the desired quantity. Moreover,

$$\lim_h w_h \in \pi_{\partial \mathbb{B}^d} \cup_y \mathrm{supp}(\nu_y^\infty). \qquad \square$$

We now further approximate λ by Dirac deltas to prove the full statement.

Proof. (of Theorem 2.37). Upong metrising $\mathbb{M}_1^+(\partial e_{f_i, i\in\mathbb{N}})$ with the Kantorivich metric, which we just call d, we can find $K_j \subset \mathrm{supp}(\lambda) \subset \mathrm{supp}(\mu)$ compact so that $\lambda(\Omega \setminus K_j) \to 0$ and

$$x \mapsto \nu_x^\infty \in C(K_j, (\mathbb{M}_1^+(e_{f_i, i\in\mathbb{N}}), d)).$$

Then find finitely many disjoint closed balls $(\mathbb{B}_h^j)_h$ of radius bounded by $r > 0$ to be decided later, so that

$$\lambda(K^j \setminus \cup_h \mathbb{B}_h^j) < j^{-1}.$$

Call $K^j \cap \mathbb{B}_h^j = E_h^j$. By uniform continuity, letting $x_h^j \in E_h^j$, if $r > 0$ is small enough we have that

$$d(\nu_x^\infty, \nu_{x_h^j}) < j^{-1} \; \forall x \in E_h^j.$$

Then the following inequality holds uniformly for $\mathrm{Lip}(Tf) \le 1$:

$$\left| \int_{K^j} \langle \nu_x^\infty, f^\infty \rangle d\lambda - \sum_h \lambda(E_h^j) \langle \nu_{x_h^j}, f^\infty \rangle \right|$$
$$\le \left| \int_{\cup_h E_h^j} \langle \nu_x^\infty, f^\infty \rangle d\lambda - \sum_h \lambda(E_h^j) \langle \nu_{x_h^j}, f^\infty \rangle \right| + \lambda(K^j \setminus \cup_h E_h^j) \le 2j^{-1}.$$

Finally, because $\sum_h \lambda(E_h^j)\langle \nu_{x_h^j}, \cdot \rangle$, as a Young Measure, can be written as

$$\left(\delta_0, \sum_h \lambda(E_h^j) \delta_{x_h^j}, \nu_x^\infty \right),$$

we can use the lemma to obtain a uniform approximation on each x_h^j via $\sum_h v_h^j$ over $\mathrm{Lip}(Tf) \le 1$, where the terms in the summation have disjoint support. $\square$

Corollary 2.39. *Let μ be non-atomic and consider the Young Measure*

$$(\nu_x, \lambda, \nu_x^\infty) \in Y(\mu, e_{f_i, i \in \mathbb{N}}).$$

There exist two sequences $(v_j)_j$ and $(u_j)_j$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ so that

$$\xi_{v_j} \xrightarrow{Y(\mu)} (\nu_x, 0, N/A), \quad \xi_{u_j} \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} (\delta_0, \lambda, \nu_x^\infty), \quad \textit{and} \;\; \xi_{v_j + u_j} \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} \nu.$$

Moreover,
$$\begin{cases} v_j(x) \in \cup_y \mathrm{supp}(\nu_y) \quad \mu\text{-a.e. } x \in \Omega, \\ d\left(\frac{u_j(x)}{1 + u_j(x)}, \pi_{\partial \mathbb{B}^d}(\cup_y \mathrm{supp}(\nu_y^\infty)) \right) \xrightarrow{j \to \infty} 0 \quad \textit{or} \quad u_j(x) = 0. \end{cases}$$

Proof. Combine Lemma 2.36 and Theorem 2.37 via Lemma 2.33. $\square$

Remark 2.40. If $\mu = \mathcal{L}^n$, we can mollify the approximating sequence u_j so that $v_j := u_j \star \rho_{\varepsilon_j}$ belongs to $\mathcal{D}(\Omega, \mathbb{R}^d)$. In this case $v_j(x) \in \mathrm{cohull}(\cup_y \mathrm{supp}(\nu_y))$ for $\mathcal{L}^n$-a.e.

We already saw how to embed functions $v \in L^1(\mu)$ into generalised Young Measures on arbitrary compactifications. We write down such definitions and show how to embed measures $\mathbb{M}(\Omega, \mathbb{R}^d)$ as Young Measures on the sphere compactification.

Definition 2.41. Let $\mu \in \mathbb{M}^+(\Omega)$ and $v \in L^p(\Omega, \mu, \mathbb{R}^d), 1 \le p \le \infty$. The corresponding *elementary p-Young Measure* is

$$\xi_v := \big((\delta_{v(x)})_{x \in \Omega}, 0, \mathrm{N/A} \big) \in Y(\mu).$$

When $p = 1$, we extend the definition to $l \in \mathbb{M}(\Omega, \mathbb{R}^d)$ by setting, for $l = \frac{l}{\mu} d\mu + l^{s,\mu}$,

$$\xi_l := \left(\left(\delta_{\frac{l(x)}{\mu(x)}} \right)_{x \in \Omega}, |l^{s,\mu}|, \left(\delta_{\frac{l^{s,\mu}}{|l^{s,\mu}|}} \right)_{x \in \Omega} \right) \in Y(\mu, \mathbb{B}^d)$$

Note that for $\Phi \in \mathbb{E}_p$ we have

$$\ll \xi_v, \Phi \gg = \int_\Omega \Phi(x, v(x)) d\nu(x)$$

and
$$\ll \xi_l, \Phi \gg = \int_\Omega \Phi\left(x, \frac{l}{\mu}(x)\right) d\mu(x) + \int_\Omega \Phi^\infty\left(x, \frac{l^{s,\mu}}{|l^{s,\mu}|}(x)\right) d|l^{s,\mu}|(x).$$

In general, there is no clear way of defining elementary Young Measures on compactifications that are larger than the sphere. However, in the next section, we will see that this can be done in very specific cases when we have more information on both the family $\{f_i, i \in \mathbb{N}\}$ and the measure $l \in \mathbb{M}(\Omega, \mathbb{R}^d)$ we are trying to embed.

Definition 2.42. (Barycentre of a p-Young Measure) Let

$$\nu = \left((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\overline{\Omega}}\right) \in Y^p(\mu, e_{f_i, i\in\mathbb{N}}),$$

where $\{f_i, i \in \mathbb{N}\}$ is so that $e_{f_i, i\in\mathbb{N}} \geq \overline{\mathbb{B}^d}$ (in the sense of compactifications, see 2.9). We call its *barycentre*

$$\overline{\nu} = \begin{cases} \overline{\nu_x} & 1 < p < +\infty \\ \overline{\nu_x}\mu + \overline{\nu_x^\infty}\lambda & p = 1, \end{cases}$$

which is the following quantity

$$\overline{\nu_x} = \int_{\mathbb{R}^d} z d\nu_x(z), \qquad \overline{\nu_x^\infty} = \int_{\partial e_{f_i, i\in\mathbb{N}}} z d\nu_x^\infty.$$

In the above definition, "$\geq$" is the ordering over the set of Hausdorff compactifications of a topological space, see Definition 2.9. Moreover, the barycentre does not depend on the compactification, as far as $e_{f_i, i\in\mathbb{N}} \geq \overline{\mathbb{B}^d}$. Indeed, the map $z \mapsto z = (z^j)_{j=1,\ldots,d}$ extends to $e_{f_i, i\in\mathbb{N}}$ coordinate-wise, and so

$$\int_{\partial e_{f_i, i\in\mathbb{N}}} z d\nu_x^\infty = \int_{\partial\mathbb{B}^d} \int_{\{(w_n)_n\}} z dP_z((w_n)_n) d\pi_{\partial\mathbb{B}^d}\nu_x^\infty = \int_{\partial\mathbb{B}^d} z d\pi_{\partial\mathbb{B}^d}\nu_x^\infty,$$

as $z \mapsto z^j$ is constant on sequences $(w_n)_n$ that converge to the same value $w \in \partial\mathbb{B}^d$.

Notice that $x \mapsto \overline{\nu_x}$ is μ-measurable and $x \mapsto \overline{\nu_x^\infty}$ is λ-measurable. In particular,

$$\overline{\nu} \in L^p(\Omega, \mu, \mathbb{R}^d) \quad \text{for } 1 < p < +\infty$$

and
$$\overline{\nu} \in \mathbb{M}(\Omega, \mathbb{R}^d) \quad \text{for } p = 1.$$

It is easy to see that when $1 < p < +\infty$, $v_j \rightharpoonup v \in L^p(\Omega, \mu, \mathbb{R}^d)$ we have $v = \overline{\nu_v}$, and when $p = 1$, $\rho_j \rightharpoonup^* \rho \in C_0(\Omega, \mathbb{R}^d)^*$, then $\rho = \overline{\xi_\rho}$.

2.6. Stronger notions of convergence

To conclude the discussion about generalised Young Measures, we mention some stronger notions of convergence such as strict convergence and μ-strict convergence, where μ is any positive finite Borel measure. These modes of convergence explain why the above embedding of measures in the set of Young Measures is the canonical one. Moreover, we give a simple example of why such canonical embedding has no meaning when the compactification is larger than the sphere one.

Definition 2.43. Let $(\eta_j)_j$ and η in $\mathbb{M}(\Omega, \mathbb{R}^d)$, we say that $\eta_j \xrightarrow{s} \eta$ (η_j *converges strictly* to η) if $\eta_j \to \eta$ weakly* in $C_0(\Omega, \mathbb{R}^d)^*$ and $|\eta_j|(\Omega) \to |\eta|(\Omega)$. □

It is easy to see that if $\eta_j \to \eta$ strictly, then $|\eta_j| \rightharpoonup^* |\eta|$ in $C_0(\Omega, \mathbb{R}^d)^*$. Strict convergence prevents cancellations of sign and concentration at the boundary of Ω. However, it does not prevent oscillation. To prevent oscillation, we must choose a weight $\mu \in \mathbb{M}^+(\Omega)$ and ask for convergence of η_j on the graph (μ, η_j). We thus obtain a notion of μ-strict convergence.

Definition 2.44. We say that $\eta_j \xrightarrow{\mu-s} \eta$ (η_j *converges μ-strictly* to η) if $\eta_j \to \eta$ weakly* in $C_0(\Omega, \mathbb{R}^d)^*$ and $(\mu, \eta_j) \xrightarrow{s} (\mu, \eta)$ in $C_0(\Omega, \mathbb{R} \times \mathbb{R}^d)^*$. □

Similarly to what was observed in the case of strict convergence, such a notion implies that $|(\mu, \eta_j)| \rightharpoonup^* |(\mu, \eta_j)|$ in $C_0(\Omega)^*$. Moreover, μ-strict convergence of η_j to η simply amounts to weak* convergence and additional convergence of the following quantity: writing $\eta_j = \frac{\eta_j}{\mu} d\mu + \eta_j^{s,\mu}, \eta_j^{s,\mu} \perp \mu$,

$$\begin{aligned}|(\eta_j, \mu)|(\Omega) = \left| \left(\frac{\eta_j}{\mu} d\mu, \mu \right) + (\eta_j^{s,\mu}, 0) \right| (\Omega) &= \int_\Omega \sqrt{1 + \left| \frac{\eta_j}{\mu} \right|^2} d\mu + |\eta_j^{s,\mu}|(\Omega) \\ &\to \int_\Omega \sqrt{1 + \left| \frac{\eta}{\mu} \right|^2} d\mu + |\eta^{s,\mu}|(\Omega) = |(\eta, \mu)|(\Omega).\end{aligned}$$

When $\mu = \mathcal{L}^n$, we refer to such convergence as *area-strict convergence*, in analogy with the area formula for smooth functions.

Reshetnyak continuity Theorem shows that strict convergence is equivalent to convergence through 1-homogeneous functions.

Theorem 2.45. [51] *Let $f(x, z) \in C(\Omega \times \mathbb{R}^d)$ be 1-homogeneous in z. If $\eta_j \to \eta$ strictly in the sense of measures, then*

$$\int_\Omega f\left(x, \frac{\eta_j}{|\eta_j|}\right) d|\eta_j|(x) \to \int_\Omega f\left(x, \frac{\eta}{|\eta|}\right) d|\eta|(x)$$

In case f is not 1-homogeneous but has an extension on the sphere compactification, we can obtain the following continuity result by requiring μ-strict convergence instead.

Corollary 2.46. *Let $f \in \mathbb{E}(\Omega \times \mathbb{R}^d)$. If $\eta_j \xrightarrow{\mu-s} \eta$ in $C_0(\Omega, \mathbb{R}^d)^*$ then*

$$\int_\Omega f\left(x, \frac{\eta_j}{\mu}\right) d\mu + f^\infty\left(x, \frac{\eta_j^{s,\mu}}{|\eta_j^{s,\mu}|}\right) d|\eta_j^{s,\mu}| \to \int_\Omega f\left(x, \frac{\eta}{\mu}\right) d\mu + f^\infty\left(x, \frac{\eta^{s,\mu}}{|\eta^{s,\mu}|}\right) d|\eta^{s,\mu}|.$$

These are well-known results, but we write down a proof of the latter because it gives an insight into how to move from one type of convergence to the other.

Proof. Consider the so-called *perspective functional*

$$\tilde{f}(x,z,t) = \begin{cases} f(x,\frac{z}{t})|t| & t \neq 0 \\ f^\infty(x,z) & t = 0 \end{cases}$$

which is positively 1-homogeneous in the last variable. By the Reshetnyak continuity Theorem, we obtain that

$$\int_\Omega \tilde{f}(x,(\eta,\mu)) = \int_\Omega f\left(x,\frac{\eta}{d\mu}\right) d\mu + f^\infty\left(x,\frac{\eta^{s,\mu}}{|\eta^{s,\mu}|}\right) d|\eta^{s,\mu}|$$

is sequentially continuous in the μ-strict topology in $\eta \in \mathbb{M}(\Omega,\mathbb{R}^d)$. $\square$

Upon taking $f(x,z) = |z|$ we get that μ-strict convergence implies strict convergence, for every $\mu \in \mathbb{M}^+(\Omega)$.

In light of these results, for a fixed measure $\mu \in \mathbb{M}^+(\Omega)$, the canonical embedding of measures $\eta \in \mathbb{M}(\Omega,\mathbb{R}^d)$ into the set of Young Measures on the sphere

$$\eta \in \mathbb{M}(\Omega,\mathbb{R}^d) \mapsto \xi_\eta = \left(\delta_{\frac{\eta}{\mu}}, |\eta^{s,\mu}|, \delta_{\frac{\eta^{s,\mu}}{|\eta^{s,\mu}|}}\right)$$

is sequentially μ-strictly continuous. This is a solid justification for this choice of embedding. For the same reason, we can show why on larger compactifications we don't have, in general, a canonical embedding of measures.

Theorem 2.47. *Let $\mu \in \mathbb{M}^+(\Omega)$ with the property that there is $x \in \operatorname{supp}(\mu \llcorner \overline{\Omega})$ with $\delta_x \perp \mu$, and let $f \in C(\mathbb{R}^d)$ of linear growth, $f \notin \mathbb{E}(\mathbb{R}^d)$ (it doesn't have a regular recession). There is $(u_j)_j$ in $\mathcal{D}(\Omega,\mathbb{R}^d)$, $u_j \xrightarrow{\mu-strictly} z\delta_x$ in $\mathbb{M}(\overline{\Omega},\mathbb{R}^d)$ for some $z \in \partial\mathbb{B}^d$, but $\int_\Omega f(u_j(x))d\mu(x)$ does not converge.*

The above theorem implies that for all $e_{f_i,i\in\mathbb{N}} \geq e_f$ (in the sense of compactifications), ξ_{u_j} does not converge in $Y(\mu, e_{f_i,i\in\mathbb{N}})$.

Proof. Because of the assumptions on μ, we can find $x \in \operatorname{supp}(\mu \llcorner \overline{\Omega})$ so that $\delta_x \perp \mu$. Notice that the result is unchanged if we instead consider $f(x)+C_1|z|+C_2$, so taking $C_1, C_2 > 0$ big enough we can assume that $f \geq 0$ everywhere. Moreover, after multiplying by another function $g(z) \in C^\infty(\mathbb{R}^d)$ so that $g(\mathbb{B}_2^c) = 1$ and $g(\mathbb{B}_1) = 0$ we can also assume that $f(0) = 0$. After all such operations, it is still true that $f \notin \mathbb{E}(\mathbb{R}^d)$. Find $z \in \partial\mathbb{B}^d$ and $z_j, w_j \to z$ so that

$$\lim_n Tf(z_j) = M > m = \lim_j Tf(w_j)$$

Next, because $x \in \operatorname{supp}(\mu)$ then $\mu(\mathbb{B}_r(x)) > 0$ for all r. There are two possibilities. Either $x \in \Omega$, in which case we consider only those balls $\mathbb{B}_r(x)$ so that $\mathbb{B}_{2r}(x) \subset \Omega$, or $x \in \partial\Omega$, and we take a function $\delta(r)\colon (0,\infty) \to (0,\infty)$ so that $\delta(r) \downarrow 0$ as $r \downarrow 0$ and $\mu(\mathbb{B}_r(x) \cap \Omega_{-\delta(r)}) > 0$, where we remind that

$$\Omega_{-\delta(r)} = \{x \in \Omega : d(x,\partial\Omega) > \delta(r)\}.$$

Either way, we call $\mathbb{B}_r(x)$ or $\mathbb{B}_r(x) \cap \Omega_{-\delta(r)}$ simply $\mathbb{B}_r$. Furthermore, we can find an increasing and positive function $\varepsilon = \varepsilon(r)$ such that $\varepsilon(r) \to 0$ as $r \to 0$ and, setting

$$\mathbb{B}_r^{\varepsilon(r)} = (\mathbb{B}_r)_{\varepsilon(r)} = \{x \in \mathbb{R}^n : d(x, \mathbb{B}_r) < \varepsilon(r)\} \subset \Omega,$$

$\varepsilon(r)$ satisfies
$$\lim_{r\downarrow 0} \frac{\mu(\mathbb{B}_r^{\varepsilon(r)})}{\mu(\mathbb{B}_r)} = 1.$$

For each such r find $\phi_r \in \mathcal{D}(\mathbb{B}_r^{\varepsilon(r)}), 0 \le \phi_r \le 1$ so that $\phi_r(y) = 1$ for all $y \in \mathbb{B}_r$. Put $u_r = \frac{\phi_r}{\int \phi_r d\mu}$. Clearly $u_r \rightharpoonup^* \delta_x$ in $C(\overline{\Omega})^*$ as $r \to 0$. Next, refine the sequences $(z_j)_j$ and $(w_j)_j$ so that there is a sequence $r_j \downarrow 0$ so that

$$\int_{\mathbb{B}_{r_{2j}}} \phi_{2j} d\mu = 1 - |z_j| \quad \text{and} \quad \int_{\mathbb{B}_{r_{2j+1}}^{\varepsilon(r_{2j+1})}} \phi_{2j+1} d\mu = 1 - |w_j|,$$

and put
$$u_j = \begin{cases} u_{r_j} z_{\frac{j}{2}} & \text{if j is even,} \\ u_{r_j} w_{\frac{j-1}{2}} & \text{if j is odd.} \end{cases}$$

First we show that $u_j \xrightarrow{\mu-strictly} z\delta_x$ in $C(\overline{\Omega}, \mathbb{R}^d)^*$ as $j \to \infty$. Because $z_j, w_j \to z$, it is enough to show that $u_r \xrightarrow{\mu-strictly} \delta_x$ in $C(\overline{\Omega})^*$ as $r \downarrow 0$. Because $u_r \rightharpoonup^* \delta_x$ in $C(\overline{\Omega})^*$ then

$$\liminf_{r\to 0} |(u_r d\mu, \mu)|(\overline{\Omega}) \ge |(\delta_x, \mu)|(\overline{\Omega}).$$

To achieve the opposite inequality, we calculate

$$\begin{aligned}
&|(u_r d\mu, \mu)|(\Omega) = |(0, \mu)|(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + |(u_r, 1)d\mu|(\Omega \cap \mathbb{B}_r^{\varepsilon(r)}) \\
&= \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \int_{\mathbb{B}_r^{\varepsilon(r)}} \sqrt{1 + u_r^2} d\mu \\
&= \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \frac{\int_{\mathbb{B}_r^{\varepsilon(r)}} \sqrt{\left(\int \phi_r d\mu\right)^2 + \phi_r^2} d\mu}{\int \phi_r d\mu} \\
&\le \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \frac{\int_{\mathbb{B}_r^{\varepsilon(r)}} \sqrt{\mu(\mathbb{B}_r^{\varepsilon(r)})^2 + 1} d\mu}{\mu(\mathbb{B}_r)} \\
&= \mu(\Omega \setminus \mathbb{B}_r^{\varepsilon(r)}) + \frac{\mu(B_r^{\varepsilon}) \sqrt{\mu(\mathbb{B}_r^{\varepsilon(r)})^2 + 1}}{\mu(\mathbb{B}_r)} \xrightarrow{r\to 0} \mu(\Omega) + 1 = |(\delta_x, \mu)|(\overline{\Omega}).
\end{aligned}$$

Next, we study how the integral behaves on alternating integers of the sequence $(u_j)_j$. If $j = 2i$ then

$$\liminf_i \int_\Omega f(u_{2i}(x)) d\mu(x) \ge \liminf_i \int_{\mathbb{B}_{r_{2i}}} f\left(\frac{z_i}{1 - |z_i|}\right) d\mu = \lim_i Tf(z_i) = M.$$

On the other side, if $j = 2i + 1$ we get the upper bound

$$\limsup_i \int_\Omega f(u_{2i+1}(x)) d\mu(x) \le \limsup_i \int_{\mathbb{B}_{r_{2i+1}}^{\varepsilon(r_{2i+1})}} f\left(\frac{w_i}{1 - |w_i|}\right) d\mu = \lim_i Tf(w_i) = m. \quad \square$$

In the following remark, we show that the assumption of μ is sharp.

Remark 2.48. If for all $x \in \operatorname{supp}(\mu \llcorner \overline{\Omega})$ we have $\delta_x \not\ll \mu$, then all such xs belong to Ω. We can then consider the atomic decomposition of μ:

$$\mu = \mu^a + \mu^{n-a} = \sum_n \mu(x_n)\delta_{x_n} + \mu^{n-a},$$

see the atomic decomposition A.8. If $\mu(\Omega \setminus \{x_n, n \in \mathbb{N}\}) > 0$ then we could find $x \in \Omega \setminus \{x_n, n \in \mathbb{N}\}$ and $\delta_x \perp \mu$, so we immediately conclude that $\mu^{n-a} = 0$. Then $\mu = \sum_n \mu(x_n)\delta_{x_n}$, and the set $\{x_n, n \in \mathbb{N}\}$ contains its accumulation points. In particular $\{x_n, n \in \mathbb{N}\} = \operatorname{supp}(\mu)$ is a compact subset of Ω. It is easy to see that, in this setting, if $(\phi_j)_{j\in\mathbb{N}}$ in $L^1(\mu)$ is bounded in norm and

$$\phi_j \xrightarrow{Y(\mu, e_{f_i, i\in\mathbb{N}})} \left(\nu_x, \lambda, \nu_x^\infty\right),$$

then $\lambda \ll \mu$, i.e. $\lambda = \sum_n \lambda(x_n)\delta_{x_n}$ (because the space is countable and compact).

Also
$$\phi_j \rightharpoonup^* \phi = \left(\overline{\nu_x} + \overline{\nu_x^\infty}\frac{\lambda}{\mu}\right) d\mu$$

in $C(X)^*$, $X = \{x_n, n \in \mathbb{N}\}$. Assume also that $\phi_j \to \phi$ μ-strictly.

This amounts to the following

$$\int_X f(\phi_j) d\mu \to \int_X \langle \nu_x, f\rangle + \langle \nu_x^\infty, f^\infty\rangle \frac{\lambda}{\mu} d\mu = \int_X f(\phi) d\mu, \tag{11}$$

where $f(z) = \sqrt{1 + |z|^2}$. f is a strictly convex function, therefore the inequality

$$f(x+y) = f(x) + \left.\frac{f(x+ty) - f(x)}{t}\right|_{t=1} \leq f(x) + f^\infty(y)$$

is strict unless $y = 0$. We have

$$\langle \nu_x, f\rangle + \langle \nu_x^\infty, f^\infty\rangle \frac{\lambda}{\mu} \geq f(\overline{\nu_x}) + f^\infty\left(\overline{\nu_x^\infty}\frac{\lambda}{\mu}\right) > f\left(\overline{\nu_x} + \overline{\nu_x^\infty}\frac{\lambda}{\mu}\right) = f(\phi)$$

unless $\overline{\nu_x^\infty} = 0$ λ-a.e. So $\phi = \overline{\nu_x}$ μ-a.e., and using convexity once again in (11), and the fact that $f^\infty = 1$, we get

$$f(\phi) = \langle \nu_x, f\rangle + \langle \nu_x^\infty, f^\infty\rangle \frac{\lambda}{\mu} \geq f(\overline{\nu_x}) + \frac{\lambda}{\mu} = f(\phi) + \frac{\lambda}{\mu}.$$

So $\lambda = 0$, which implies that the sequence ϕ_j does not concentrate. Moreover, $\phi(x) = \overline{\nu_x}$, which means that $\phi_j \to \phi$ in measure. Then $\phi_j \to \phi$ strongly in $L^1(\mu)$, and Theorem 2.47 cannot be true. $\square$

3. Characterisation of Gradient Young Measure on separable compactifications

In this section, we show that a Young measure $\nu = (\nu_x, \lambda, \nu_x^\infty) \in Y(\partial e_{f_i, i\in\mathbb{N}})$, $f_i\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ quasi-convex and of linear growth for all i, that satisfies

$$\int_\Omega \langle \nu_x, |\cdot|\rangle dx < +\infty$$

is generated by a sequence of gradients (Du_i), $u_i \in C^\infty(\overline{\Omega}, \mathbb{R}^m)$, precisely when ν satisfies a set of integral inequalities that involve quasi-convex functions $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ and their extension f^∞ on $e_{f_i, i\in\mathbb{N}}$. A characterisation result was previously obtained in [35] the context of the sphere compactification and it is here extended to separable compactifications. See also [6, 34] for a similar characterisation in the setting of $\mathcal{A}$-free vector fields.

3.1. Non-separability of the space of quasi-convex functions

We start by showing that the set of quasi-convex functions having linear growth is not separable. The lack of separability prevents sequential compactness and other essential properties that were used to develop the theory of generalised Young Measures. Therefore, we are forced to consider only smaller countable collections of quasi-convex functions at the time, and cannot work with the entire class. To show that the class is not separable, we modify the example given by Muller in [48] and generate quasi-convex functions that oscillate at different amplitudes in the same direction.

Theorem 3.1. *The space of quasi-convex functions $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ having linear growth is not separable with respect to $\|T\cdot\|_{\infty,\mathbb{B}^{2\times 2}}$.*

We remind the reader that the map T is defined in the following way

$$T\colon \mathbb{G}_1(\mathbb{R}^{m\times n}) \to C(\mathbb{B}^{m\times n}), f \mapsto Tf = f\left(\frac{\cdot}{1-|\cdot|}\right)(1-|\cdot|).$$

The core of the proof is to construct an uncountable family $\{f_\Lambda\}_\Lambda$ so that we get $\|Tf_\Lambda - Tf_\Gamma\|_\infty \geq c$ for some universal constant $c > 0$ and all $\Lambda \neq \Gamma$. We split the proof of the above result into two different parts. First, we show that we have quasi-convex functions that oscillate along every possible sequence of natural numbers.

Proposition 3.2. *There is $c > 0$ such that for every $\Lambda \subset \mathbb{N}$ infinite with Λ^c also infinite there exists $f_\Lambda\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ quasi-convex and having linear growth such that*

1. $f_\Lambda(3^j\mathbb{1}) = 0$ *for all* $j \in \Lambda$ *and*
2. $\frac{f_\Lambda(3^j\mathbb{1})}{3^j} \geq c$ *for all* $j \in \Lambda^c$ *sufficiently large.*

To prove this theorem we first need the following preliminary lemma.

Lemma 3.3. ([48], Lemma 4) *For $k > 0$ we let*

$$g_k\colon \mathbb{R}^{2\times 2} \to \mathbb{R}, F \mapsto |F_{1,1} - F_{2,2}| + |F_{1,2} + F_{2,1}| + (2k - |F_{1,1} + F_{2,2}|)^+.$$

There exists $c_1 > 0$ such that $Qg_k(0) \geq c_1 k$ for all sufficiently large ks.

We can then prove Proposition 3.2.

Proof. Let $\Lambda \subset \mathbb{N}$ infinite and set $f_\Lambda = Qg_\Lambda$, where

$$g_\Lambda(F) = |F_{1,1} - F_{2,2}| + |F_{1,2} + F_{2,1}| + \inf\{|F_{1,1} + F_{2,2} - 2 \cdot 3^i|, i \in \Lambda\}.$$

We have that $g_\Lambda(3^j \mathbb{1}) = 0$ for all $j \in \Lambda$ and so $f_\Lambda(3^j \mathbb{1}) = 0$ as well. We first derive the following lower bound: compute

$$g_\Lambda(3^j \mathbb{1} + G) = |G_{1,1} - G_{2,2}| + |G_{1,2} + G_{2,1}| + \inf\{|G_{1,1} + G_{2,2} + 2(3^j - 3^i)| : i \in \Lambda\};$$

because 3^j is increasing we estimate, for arbitrary $\beta \in \mathbb{R}$,

$$\inf_{i \in \Lambda} |2(3^j - 3^i) + \beta| \geq \left(\inf_{i \neq j} |2(3^j - 3^i)| - |\beta| \right)^+ = \big(2(3^j - 3^{j-1}) - |\beta|\big)^+,$$

and so $g_\Lambda(3^j \mathbb{1} + G) \geq g_k(G)$ for all matrices G and $k = 3^j - 3^{j-1}$, and g_k given by

$$g_k(F) = |F_{1,1} - F_{2,2}| + |F_{1,2} + F_{2,1}| + (2k - |F_{1,1} + F_{2,2}|)^+$$

does not depend on Λ. Apply Lemma 3.3 to find $c > 0$ (independent of Λ) such that

$$f_\Lambda(3^j \mathbb{1}) = Qg_\Lambda(3^j \mathbb{1}) \geq Qg_k(0) \geq c_1(3^j - 3^{j-1})$$

for j big enough. Dividing everything by 3^j, we obtain the bound

$$\frac{f_\Lambda(3^j \mathbb{1})}{3^j} \geq \frac{2}{3} c_1 \equiv c$$

and conclude the proof. $\square$

It is not a priori clear if these functions are "far away from each other at infinity", given that different subsets of natural numbers could intersect infinitely many times. To show that there is a wide variety of sequences that differ at infinity, we start by defining the following relation.

Definition 3.4. Given two sequences Γ, Λ (not necessarily subsets of $\mathbb{N}$), we say that $\Gamma \leq \Lambda$ provided Γ is eventually a subset of Λ, i.e.

$$\Lambda = (a_i)_{,i \in \mathbb{N}}, \Gamma = (b_i)_{i \in \mathbb{N}}, \quad \Lambda \leq \Gamma$$

if and only if there exists $k > 0$ such that $(a_i)_{i \geq k}$ is a subsequence of $(b_i)_{i \geq 0}$.
Let $[\Lambda]$ be the equivalence class of Λ with respect to $\leq$, i.e.

$$\Gamma \in [\Lambda] \quad \text{if} \quad \Gamma \leq \Lambda \leq \Gamma.$$

If $\Lambda' \in [\Lambda]$ and $\Gamma' \in [\Gamma]$ then $\Lambda \leq \Gamma$ if and only if $\Lambda' \leq \Gamma'$. We use $\big(\mathbb{F}, \leq\big)$ to indicate the set of equivalence classes with the inherited order. $\square$

Our strategy is to use Zorn's Lemma to find a maximal set, and then show that such a set is uncountable. We remark that one could also reason that the Stone-Cech compactification of natural numbers is not metrisable and prove by contradiction using a suitably adapted version of the Young Measure representation 2.12.

Lemma 3.5. *There exists an uncountable family $G \subset \mathbb{F}$ such that for every different pair $\Lambda, \Gamma \in G$, Λ is not comparable to neither Γ nor Γ^c with respect to $\leq$.*

The above means that we can find a set G such that given any two sequences of natural numbers in $\Lambda, \Gamma \in G$, one is frequently in the other sequence and its complement, i.e. $\Lambda \cap \Gamma$ and $\Lambda \cap \Gamma^c$ are both infinite.

Lemma 3.6. *Given any countable or finite collection of infinite natural numbers $\{c_i^j, i \in \mathbb{N}\}_{j\in\mathbb{N}}$ in $\mathbb{N}$, there is $\{c_i, i \in \mathbb{N}\} \subset \mathbb{N}$ such that*

$$\{c_i^j, i \in \mathbb{N}\} \cap \{c_i, i \in \mathbb{N}\} \quad \textit{and} \quad \{c_i^j, i \in \mathbb{N}\} \setminus \{c_i, i \in \mathbb{N}\}$$

are both infinite subsets of $\mathbb{N}$.

The statement is equivalent to finding $\{c_i, \in \mathbb{N}\}$ such that, for all $j \in \mathbb{N}$,

$$\{c_i^j, i \in \mathbb{N}\} \setminus \{c_i, i \in \mathbb{N}\} < \{c_i^j, i \in \mathbb{N}\}$$

and
$$\{c_i^j, i \in \mathbb{N}\} \cap \{c_i, i \in \mathbb{N}\} < \{c_i^j, i \in \mathbb{N}\}.$$

Proof. Consider the isomorphism

$$N\colon \mathbb{F} \to \{0,1\}^{\mathbb{N}}, \{c_k, k \in \mathbb{N}\} \mapsto \left(Nc_i = \begin{cases} 1 & \text{if } i \in \{c_k, k \in \mathbb{N}\} \\ 0 & \text{otherwise} \end{cases} \right)_{i\in\mathbb{N}}$$

We are replacing subsets of natural numbers to sequences that take values 1 when the i-th number is in the subset, 0 otherwise, i.e. $Nc_i = \chi_{i\in\{c_k,k\in\mathbb{N}\}}$ is the indicator function.

Set initially $Nc_i = 0$ for all i. At i_1 so that $Nc_{i_1}^1 = 1$ for the first time put $Nc_{i_1} = 1$. We then iterate diagonally in the following way. At the n-th iteration find i_{n+1} so that $Nc_{k_j}^j = 1$ for all $1 \leq j \leq n$ and some $i_n < k_j$ with k_j increasing and $Nc_{t_j}^j = 1$ for all $1 \leq j \leq n+1$ and $k_n < t_j$ with t_j increasing and $t_{n+1} = i_{n+1}$. Set $Nc_{t_j} = 1$ for all $1 \leq j \leq n+1$. Iterate countably many times.

This way we guarantee that $Nc_{k_j} = 0$ for $i_n < k_j$ and $Nc_{k_j}^j = 1$, which means that Nc_i skips infinitely many 1s from each sequence $(Nc_i^j)_{i\in\mathbb{N}}$, for all $j \in \mathbb{N}$. On the other side $Nc_{t_j} = 1 = Nc_{t_j}^j$, $t_j \leq i_{n+1}$, so Nc_i is also frequently in every sequence $(Nc_i^j)_{i\in\mathbb{N}}$. $\square$

Now we can prove the main lemma.

Proof. (of Lemma 3.5). Consider the set of subsets

$$\mathcal{G} = \{G \subset \mathbb{F} : \text{no pair within } G \text{ is comparable according to } \leq\},$$

ordered by inclusion $\subset$. The above set is non-empty, which can be seen by taking $\Lambda = 2\mathbb{N}$ and $\Gamma = 4\mathbb{N} \cup (4\mathbb{N}+1)$. Every chain in $\mathcal{G}$ has an upper limit given by its union. By Zorn's Lemma, there exists a maximal element $\overline{G} \subset \mathcal{G}$, which we assume is countable or finite. Writing $\overline{G} = \big\{\{a_i^j, i \in \mathbb{N}\}, j \in \mathbb{N}\big\}$, we can apply the previous lemma to find $\{c_i, i \in \mathbb{N}\} \in \mathbb{F}$ generated by the countable family

$$\{c_i^j, i \in \mathbb{N}\}_{j\in\mathbb{N}} = \Big\{\{a_i^j, i \in \mathbb{N}\}\ ,\ \mathbb{N} \setminus \{a_i^j, i \in \mathbb{N}\}\Big\}_{j\in\mathbb{N}}$$

Because $\{c_i, i \in \mathbb{N}\}$ is frequently and properly in $\{a_i^j, i \in \mathbb{N}\}$ and its complement $\mathbb{N} \setminus \{a_i^j, i \in \mathbb{N}\}$ for all j, then $\{c_i, i \in \mathbb{N}\}$ is not comparable to any member of the family and this contradicts the maximality of our set. $\square$

We are now ready to prove Lemma 3.2.

Proof. (of Lemma 3.2). By the lemma, we can find an uncountable set of uncomparable subsequences of $\mathbb{N}$, call it $\overline{G}$. I claim that if $\Lambda, \Gamma \in \overline{G}, \Lambda \neq \Gamma$ then f_Λ and f_Γ have different recessions at infinity. Find $\{x_n, n \in \mathbb{N}\} \in e_{f_\Lambda}$ with $\{x_n, n \in \mathbb{N}\} \geq \{3^j\mathbb{1}, j \in \Lambda\}$, where the inequality could be strict given that there might be more zero points at infinity. Given our construction, we immediately have that $\{x_n, n \in \mathbb{N}\} \not\geq \{3^j\mathbb{1}, j \in \mathbb{N} \setminus \Lambda\}$ as $f_\Lambda(3^j\mathbb{1}) \geq c3^j$ for all $j \in \mathbb{N} \setminus \Lambda$. Also, Γ intersects $\mathbb{N} \setminus \Lambda$ and Λ infinitely many times, and vice versa, so that

$$\limsup_n \frac{f_\Gamma(x_n)}{|x_n|} \geq \limsup_{j\in\Lambda\cap\Gamma} \frac{f_\Gamma(3^j\mathbb{1})}{3^j} \geq c,$$

and

$$\liminf_n \frac{f_\Gamma(x_n)}{|x_n|} \leq \liminf_{j\in\Lambda\cap\Gamma} \frac{f_\Gamma(3^j\mathbb{1})}{3^j} = 0,$$

which shows that $\{x_n, n \in \mathbb{N}\} \notin e_{f_\Gamma}$. In terms of the non-separability, by the definition of x_n we have $\lim_n \frac{f_\Lambda(x_n)}{|x_n|} = 0$, thus

$$\|Tf_\Lambda - Tf_\Gamma\|_\infty \geq \limsup_n \left|\frac{f_\Gamma(x_n)}{|x_n|} - \frac{f_\Lambda(x_n)}{|x_n|}\right| = \limsup_n \left|\frac{f_\Gamma(x_n)}{|x_n|}\right| \geq c,$$

where c is independent of Λ or Γ. $\square$

Given that the space is metric, it can not be separable. We can end this section with the following corollary that covers the higher dimensional case.

Corollary 3.7. *The set of quasi-convex functions $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ having linear growth is separable in the topology induced by $\|T(\cdot)\|_\infty$ if and only if $min(m, n) = 1$.*

Proof. If m or n is 1, quasi-convex functions are convex and therefore the space is separable. This is because convex functions admit a limit at infinity in every direction; in this case, we actually recover the sphere compactification.

On the other side, for a function $g\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ we let

$$gP \equiv g \circ P\colon \mathbb{R}^{n\times m} \to \mathbb{R}, \quad P\colon \mathbb{R}^{m\times n} \to \mathbb{R}^{2\times 2}, M \mapsto \begin{bmatrix} M_{1,1}, M_{1,2} \\ M_{2,1}, M_{2,2} \end{bmatrix}.$$

If g is quasi-convex and locally bounded we let $\phi \in \mathcal{D}(Q, \mathbb{R}^m)$ and compute

$$\begin{aligned}\int_Q gP(\nabla\phi + z) &= \int_{Q\subset\mathbb{R}^{n-2}} d\mathcal{L}^{n-2} \int_{[0,1]^2} dx_1 x_2 g(P\nabla\phi + Pz)\\ &\geq \int_{Q\subset\mathbb{R}^{n-2}} d\mathcal{L}^{n-2} gP(z) = gP(z),\end{aligned}$$

which means that gP is quasi-convex. Then the set $f_\Lambda P$ is uncountable and

$$\|T(f_\Lambda P - f_\Gamma P)\|_{\infty,\mathbb{B}^{m\times n}} = \|T(f_\Lambda - f_\Gamma)\|_{\infty,\mathbb{B}^{2\times 2}} \geq c$$

if $\Gamma \neq \Lambda$, so the space cannot be separable. $\square$

3.2. Characterisation of Gradient Young Measures

In this section, we characterise Gradient Young Measures on separable compactification via certain Jensen-like inequalities. It is worth mentioning that the result for the sphere compactification, achieved in [35], can be easily improved in consideration of the fact that $\limsup_{t\to\infty} f(tz)t^{-1}$ is 1-homogeneous rank one convex and so convex at points $\operatorname{rank}(z) = 1$ (see Theorem A.9, taken from [33]). In what follows, we cannot use this type of auto-convexity. In our context, f^∞ lives on a separable compactification and convexity at points of rank one, in the form of a Jensen-type inequality, is not necessarily true anymore.

To prove our result, we will adopt the same strategy as in [34]. Preliminarily to stating the theorem, we define the upper recession of a function relative to a separable compactification.

Definition 3.8. Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification. For any f having linear growth, we define

$$f^{\sharp,e_{f_i,i\in\mathbb{N}}}((z_n)) = \sup_{(w_n)_n\in[(z_n)_n]} \limsup_n Tf(w_n),$$

where $(w_n)_n$ are sequences belonging to the equivalence class of $(z_n)_n$ within $e_{f_i,i\in\mathbb{N}}$.

The reason why we introduce this notion is that the strategy for proving the characterisation theorem makes use of the trivial fact that $f \geq f^{qc}$, where f^{qc} is the quasi-convex envelope of f. Dacorogna's formula [20] gives the representation

$$f^{qc}(z) = \inf_{\phi\in\mathcal{D}(\Omega)} \frac{1}{|\Omega|} \int_\Omega f(z + \nabla\phi(x))dx$$

under suitable assumptions on f. However, f^{qc} needs not to live in the same class of separable quasi-convex functions, so

$$\lim_n \frac{f(z_n)}{|z_n|} \text{ exists} \quad \not\Rightarrow \quad \lim_n \frac{f^{qc}(z_n)}{|z_n|} \quad \text{in general.}$$

Fix any compactification $e_{f_i,i\in\mathbb{N}}$ and a function $Tg \in e_{f_i,i\in\mathbb{N}}$. If $g \geq f$ then

$$g^\infty((z_n)_n) = \lim_n Tg(z_n) \geq \sup_{(z_n)_n\in[(z_n)_n]} \limsup_n Tf(z_n) = f^{\sharp,e_{f_i,i\in\mathbb{N}}}((z_n)_n).$$

$f^{\sharp,e_{f_i,i\in\mathbb{N}}}$ does not need to be continuous on $e_{f_i,i\in\mathbb{N}}$ with respect to its product topology. However, we can show that it is still upper semi-continuous.

Lemma 3.9. *Let $e_{f_i,i\in\mathbb{N}}$ be a separable compactification and $f\colon \mathbb{R}^d \to \mathbb{R}$ a continuous function having linear growth, then $f^{\sharp,e_{f_i,i\in\mathbb{N}}}$ is upper semi-continuous on $e_{f_i,i\in\mathbb{N}}$.*

Proof. Notice that $e_{f_i,i\in\mathbb{N}}$ is metrisable, so it is enough to check sequential upper semi-continuity. Moreover, we only have to prove continuity on $\partial e_{f_i,i\in\mathbb{N}}$.

Let $(z_n^j)_n \in e_{f_i,i\in\mathbb{N}}$ so that $(z_n^j)_n \xrightarrow{j} (z_n)_n \in \partial e_{f_i,i\in\mathbb{N}}$. By the very definition of $g^{\sharp,e_{f_i,i\in\mathbb{N}}}((z_n^j)_n)$, for fixed $\varepsilon > 0$ we can find $n_j \geq k_j$, where k_j is a natural number to be selected later, so that $g^{\sharp,e_{f_i,i\in\mathbb{N}}}((z_n^j)_n) \leq \varepsilon + Tg((z_{n_j}^j))$, where $(z_{n_j}^j)$ is a constant sequence and belongs to $e_{f_i,i\in\mathbb{N}}$. We want to show that we can select k_j so that $(z_{n_j}^j)_j$ belongs in the equivalence class of $(z_n)_n$. By applying the dominated convergence theorem we get

$$\lim_j \sum_i \lim_n 2^{-i}|f_i(z_n^j) - f_i(z_n)| = 0 = \lim_j \lim_n \sum_i 2^{-i}|f_i(z_n^j) - f_i(z_n)|,$$

and so can find k_j so that

$$\sum_i 2^{-i}|f_i(z_n^j) - f_i(z_n)| \leq \varepsilon_j \quad \forall n \geq k_j,$$

where $0 \leq \varepsilon_j \downarrow 0$. This shows that the above sequence $(z_{n_j}^j)_j \in [(z_n)_n]$ (the equivalence class), thus

$$\limsup_j g^{\sharp,e_{f_i,i\in\mathbb{N}}}((z_n^j)_n) \leq \varepsilon + \limsup_j Tg((z_{n_j}^j)) \leq \varepsilon + g^{\sharp,e_{f_i,i\in\mathbb{N}}}((z_n)_n).$$

By the arbitrariness of $\varepsilon > 0$ we conclude upper semi-continuity of $g^{\sharp,e_{f_i,i\in\mathbb{N}}}$ □

In particular, $g^{\sharp,e_{f_i,i\in\mathbb{N}}}$ is Borel measurable on $e_{f_i,i\in\mathbb{N}}$. The above statement can be generalised to extensions of functions over more general compact metric spaces, but this version suffices for our scopes.

We are interested in studying those Young Measures that are generated by gradients. So we define the following class.

Definition 3.10. Let $\{f_i, i \in \mathbb{N}\}$ be a family of functions so that $f_i\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ has linear growth and $f_i^\infty(z)$ exists for all $\operatorname{rank}(z) = 1$ and all $i \in \mathbb{N}$. We say that $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ is a *generalised Gradient Young Measure* if there exists a sequence $(u_j)_j$ in $BV(\Omega, \mathbb{R}^m)$ such that

$$Du_j \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \big(\nu_x, \lambda, \nu_x^\infty\big).$$

We write $GY(e_{f_i,i\in\mathbb{N}})$ to refer to this subset of generalised Young Measures. □

Because $f_i^\infty(z)$ exists for all $\operatorname{rank}(z) = 1$, the embedding of Du, with $u \in BV(\Omega, \mathbb{R}^m)$, into $Y(e_{f_i,i\in\mathbb{N}})$ is defined via the pairing

$$\left(\delta_{\nabla u(x)}, |D^s u|, \delta_{\frac{D^s u}{|D^s u|}}\right).$$

The convergence of measure derivatives has not been fully comprehended yet and it is still the subject of active research. This means that it is not so clear how rich the above class is, how gradients oscillate and concentrate – at least with respect to the weak* convergence of measures.

Remark 3.11. We can use the characterisation lemma for Young Measure on the sphere to show that the class of generalised Gradient Young Measures is numerous. Fix any $z = a \otimes b \in \mathbb{R}^{m\times n}$ and $\nu^\infty \in \mathbb{M}_1^+(\partial\mathbb{B}^{m\times n})$ with $\overline{\nu^\infty} = z$. The following Gradient Young Measure on the sphere

$$\big(\delta_0, \mathcal{H}^{n-1} \llcorner (\mathbb{B}^n \cap b^\perp), \nu^\infty\big)$$

satisfies the characterisation theorem from [35], with $u = a\chi_{x\cdot b\geq 0}$ and so it is generated by a sequence of gradients of $(u_j)_j$ in $BV(\mathbb{B}_1(0))$. Because Du_j is bounded in BV, we can find a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ such that

$$Du_{j_k} \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \big(\delta_0, \mathcal{H}^{n-1} \llcorner (\mathbb{B}^n \cap b^\perp), \eta^\infty\big),$$

where $\pi_{\partial\mathbb{B}^{m\times n}}\eta^\infty = \nu^\infty$.

Using Corollary 2.21 to decompose η^∞ as $\eta^\infty = P_z d\nu^\infty$, $z \in \partial\mathbb{B}^d$, it remains an open question to understand how many probabilities P_z over subsequences $z_n \to z$ can be generated by gradients. $\square$

We now state the main theorem of this section.

Theorem 3.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $e_{f_i,i\in\mathbb{N}}$ be a separable compactification of quasi-convex functions and consider a generalised Young Measure $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ that satisfies $\lambda(\partial\Omega) = 0$.*
Then $\nu \in Y(e_{f_i,i\in\mathbb{N}})$ is a Young Measure generated by a sequence

$$(\phi_j \star (Du \llcorner \Omega) + \nabla u_j) \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \big(\nu_x, \lambda, \nu_x^\infty\big),$$

where $u \in BV(\Omega, \mathbb{R}^m)$, $(u_j)_j$ is in $\mathcal{D}(\Omega, \mathbb{R}^m)$ and $\|u_j\|_1 \to 0$, and ϕ_j is any sequence of mollifiers with $\phi_j \rightharpoonup^* \delta_0$, if and only if there is $u \in BV(\Omega, \mathbb{R}^m)$ such that

1. $\displaystyle\int_\Omega \langle \nu_x, |\cdot|\rangle dx < +\infty$, *and for all f quasi-convex and having linear growth,*
2. $f(\nabla u(x))dx \leq \langle \nu_x, f\rangle dx + \langle \nu_x^\infty, f^{\sharp, e_{f_i,i\in\mathbb{N}}}\rangle \dfrac{\lambda}{\mathcal{L}^n} dx$ *and*
3. $f^\infty\Big(\dfrac{D^s u}{|D^s u|}\Big) d|D^s u| \leq \langle \nu_x^\infty, f^{\sharp, e_{f_i,i\in\mathbb{N}}}\rangle d\lambda^s$,

where the inequalities are in the sense of measures.

We divide the proof into different parts and put them together at the end of this section.

We can adjust the above theorem to fix the boundary of the converging sequence so that its trace equals u.

Lemma 3.13. *If $\nu \in Y(e_{f_i, i\in\mathbb{N}})$ is generated by a sequence*

$$\phi_j \star (Du \llcorner \Omega) + \nabla u_j$$

as above, there exists another sequence $(v_j)_j$ in $C^\infty(\Omega, \mathbb{R}^m) \cap W^{1,1}_u(\Omega, \mathbb{R}^m)$ such that

$$\nabla(v_j + u_j) \xrightarrow{Y(e_{f_i, i\in\mathbb{N}})} \nu.$$

In particular, $\nu \in GY(e_{f_i, i\in\mathbb{N}})$.

Proof. We can find $u_j \to u$ strictly in $BV(\Omega, \mathbb{R}^m)$ with

$$(u_j)_j \in C^\infty(\Omega, \mathbb{R}^m) \cap W^{1,1}_u(\Omega, \mathbb{R}^m),$$

see for example [35], Lemma 1 for a proof of this fact. In the construction of the u_js just mentioned, it is possible to select $\phi_j \star (Du \llcorner \Omega)$ on $\Omega_{-\varepsilon}$ for j big enough, ϕ_j as in Theorem 3.12. Also, without loss of generality, we can assume that $|Du|(\partial\Omega_{-\varepsilon}) = |Du_j|(\partial\Omega_{-\varepsilon}) = 0$. Because the f_is are all Lipschitz, it is enough to test against $f \in \mathrm{Lip}(\mathbb{R}^{m\times n})$ with $\mathrm{Lip}(f) \leq 1$. We then compute

$$\begin{aligned}\int_\Omega |f(\phi_j \star (Du \llcorner \Omega) + \nabla v_j) - f(\nabla u_j + \nabla v_j)| dx &\leq \int_\Omega |\phi_j \star (Du \llcorner \Omega) - \nabla u_j| \\ &\leq \int_{\Omega\setminus\Omega_{-\varepsilon}} |\phi_j \star (Du \llcorner \Omega)| + |\nabla u_j|.\end{aligned}$$

By strict convergence of both integrands, we have that

$$\limsup_j \int_\Omega |f(\phi_j \star (Du \llcorner \Omega) + \nabla v_j) - f(\nabla u_j + \nabla v_j)| dx \leq 2|Du|(\Omega \setminus \Omega_{-\varepsilon}),$$

and so by a diagonal argument conclude the existence and equality of the limit Young Measure. □

It is implicit in Theorem 3.12 that

$$Du = \overline{\nu} = \langle \nu_x, \cdot\rangle dx + \langle \nu_x^\infty, \cdot\rangle d\lambda.$$

Also, the above inequalities can be written as one inequality in the sense of distribution, in the form

$$\begin{aligned}&\int_\Omega \phi(x)\langle \nu_x, f\rangle dx + \int_\Omega \phi(x)\langle \nu_x^\infty, f^{\sharp, e_{f_i, i\in\mathbb{N}}}\rangle d\lambda \\ &\qquad \geq \int_\Omega \phi(x) f(\nabla u(x)) dx + \phi(x) f^\infty\left(\frac{D^s u}{|D^s u|}\right) d|D^s u|\end{aligned}$$

for all $\phi \in \mathcal{D}(\Omega), \phi \geq 0$.

To prove the characterisation result we will follow the same strategy as in [34]. We initially prove the result for homogeneous Gradient Young Measures and then extend the theorem to the inhomogeneous case, where the absolutely continuous part and the singular part of ν are approximated separately.

3.3. Homogeneous Gradient Young Measures

Notice that Young Measures that act on functions $f = f(z)$ that only depend on z can be represented by

$$\int_\Omega \langle \nu_x, f\rangle dx + \int_{\overline{\Omega}} \langle \nu_x^\infty, f^\infty\rangle d\lambda = \int_{\mathbb{R}^{m\times n}} f d\nu^0 + \int_{\partial e_{f_i,i\in\mathbb{N}}} f^\infty d\nu^\infty,$$

where $\nu^0 = \nu_x d\mathcal{L}^n \in \mathbb{M}^+(\mathbb{R}^{m\times n})$ and $\nu^\infty = \nu_x^\infty d\lambda \in \mathbb{M}^+(\partial e_{f_i,i\in\mathbb{N}})$.
The Kantorovich metric is then

$$\|(\nu^0,\nu^\infty)\|_K = \sup_{\Phi\in H, \|T\Phi\|_{\mathrm{Lip}(e_{f_i,i\in\mathbb{N}})}\leq 1} \left| \int_{\mathbb{R}^{m\times n}} \Phi d\nu^0 + \int_{\partial e_{f_i,i\in\mathbb{N}}} \Phi^\infty d\nu^\infty \right|.$$

For $z \in \mathbb{R}^{m\times n}$ we let $\mathcal{Y} = \mathcal{Y}(z)$ be the set of pairs

$$\big(\nu^0,\nu^\infty\big) \in \mathbb{M}_1^+(\mathbb{R}^{m\times n}) \times \mathbb{M}^+(\partial e_{f_i,i\in\mathbb{N}})$$

such that there is a sequence $(u_j)_j$ in $\mathcal{D}(Q,\mathbb{R}^m)$, where Q is the unit cube, so that $z + \nabla u_j \xrightarrow{Y(e_{f_i,i\in\mathbb{N}})} \big(\nu^0,\nu^\infty\big)$ and $\|u_j\|_1 \to 0$. The following proposition follows from simple variations of the proofs contained in [34], Lemmas 5.2, 5.3, and 5.4, which we combine in one lemma.

Lemma 3.14. *The family $\{\varepsilon_{z+\nabla u} : u \in \mathcal{D}(Q,\mathbb{R}^m)\}$ is weakly* dense in $\mathcal{Y}$, and $\mathcal{Y}$ is a weak* closed and convex subset of homogeneous Young Measures.*

We can now prove the main theorem in the case of homogeneous Gradient Young Measures.

Proposition 3.15. *Let $\nu = \big(\nu^0,\nu^\infty\big) \in M_1^+(\Omega) \times \mathbb{M}^+(\partial e_{f_i,i\in\mathbb{N}})$. Then $\nu \in \mathcal{Y}$ if and only if there is $z \in \mathbb{R}^{m\times n}$ such that*

$$\int_{\mathbb{R}^{m\times n}} f d\nu^0 + \int_{\partial e_{f_i,i\in\mathbb{N}}} f^{\sharp, e_{f_i,i\in\mathbb{N}}} d\nu^\infty \geq f(z)$$

for all $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ quasi-convex and having linear growth.

Proof. Suppose that $\nu \in \mathcal{Y}$ and let $(z+\nabla u_j)_j$ with $(u_j)_j$ in $\mathcal{D}(Q,\mathbb{R}^m)$ be the generating sequence, i.e. for all $\Phi\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ of linear growth and belonging to the compactification $e_{f_i,i\in\mathbb{N}}$ (in short, $\Phi \in T^{-1}e_{f_i,i\in\mathbb{N}}$),

$$\int_Q \Phi(z+\nabla u_j)dx \to \langle \nu, \Phi\rangle = \int_{\mathbb{R}^{m\times n}} \Phi d\nu^0 + \int_{\partial e_{f_i,i\in\mathbb{N}}} \Phi d\nu^\infty.$$

Fix an arbitrary f having linear growth and quasi-convex and let $e_{f,f_i,i\in\mathbb{N}}$ the bigger compactification containing both $\{f_i, i \in \mathbb{N}\}$ and f. Upon extracting a subsequence we have that

$$z + \nabla u_j \xrightarrow{Y(e_{f,f_i,i\in\mathbb{N}})} \big(\nu^0, \tilde{\nu}^\infty\big),$$

where we identify the Gradient Young Measure with its products as explained before.

By quasi-convexity, we have

$$\int_{\mathbb{R}^{m\times n}} f d\nu^0 + \int_{\partial e_{f,f_i,i\in\mathbb{N}}} f^\infty d\tilde{\nu}^\infty = \limsup_j \int_Q f(z+\nabla u_j)dx \geq f(z).$$

On the other side, using the decomposition of angle concentration Young Measure decomposition 2.21 we also obtain that

$$\int_{\partial e_{f,f_i,i\in\mathbb{N}}} f^\infty d\tilde{\nu}^\infty = \int_{\partial e_{f_i,i\in\mathbb{N}}} \int f^\infty dP_{(z_n)_n} d\nu^\infty \leq \int_{\partial e_{f_i,i\in\mathbb{N}}} f^{\sharp, e_{f_i,i\in\mathbb{N}}} d\nu^\infty.$$

For the other implication, because $\mathcal{Y}$ is weakly* closed and convex, we can write $\mathcal{Y} = \cap H$ where H are half-spaces containing $\mathcal{Y}$. By the Hahn-Banach Theorem, each H can be written as

$$H = \{l \in H^* : l(\Phi) \geq t\} \quad \text{ for some } \Phi \in T^{-1} e_{f_i,i\in\mathbb{N}}.$$

In particular, we can test the above inequality against $\varepsilon_{z+\nabla u}$ and get

$$t \leq \varepsilon_{z+\nabla u}(\Phi) \leq \int_Q \Phi(z+\nabla u)dx$$

for all $u \in \mathcal{D}(Q,\mathbb{R}^m)$. Passing to the infimum over all such u we deduce $t \leq \Phi^{qc}(z)$ and so

$$\begin{aligned}\langle \nu, \Phi\rangle &= \int_{\mathbb{R}^{m\times n}} \Phi d\nu^0 + \int_{\partial e_{f_i,i\in\mathbb{N}}} \Phi^\infty d\nu^\infty \\ &\geq \int_{\mathbb{R}^{m\times n}} \Phi^{qc} d\nu^0 + \int_{\partial e_{f_i,i\in\mathbb{N}}} (\Phi^{qc})^{\sharp, e_{f_i,i\in\mathbb{N}}} d\nu^\infty \geq \Phi^{qc}(z) \geq t\end{aligned}$$

which shows that $\nu \in H$. □

We remind the reader that, even if Φ belongs to the compactification $e_{f_i,i\in\mathbb{N}}$, it is not guaranteed that its convex envelope belongs to the same compactification, and so, in general,

$$\Phi \geq \Phi^{qc} \quad \Rightarrow \quad \Phi^\infty \geq (\Phi^{qc})^{\sharp, e_{f_i,i\in\mathbb{N}}},$$

where $(\Phi^{qc})^{\sharp, e_{f_i,i\in\mathbb{N}}}$ is defined as in 3.8.

3.3.1. Inhomogenization

In what follows, we will prove a semi-approximation result for the absolutely continuous and singular parts separately and then put them together via Lemma 2.33. In each case, we will use a covering argument to break it down so that we can use the result for the homogeneous case, which was solved in the above section. This part is adapted from [34], where it was worked out on the sphere compactification.

Consider a family standard mollifier $\phi_t(x) = t^{-n}\phi(\frac{x}{t}), t > 0$, where $\phi \in \mathcal{D}(Q)$ and let $M = \|\nabla\phi\|_\infty$. Also, unless otherwise specified, the norm on $\mathbb{R}^n$ is the maximum norm $\|x\| = \max_i |x_i|$.

Lemma 3.16. *Given $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that for all $t \in (0, t_\varepsilon]$ there is $\varphi_t \in \mathcal{D}(\Omega, \mathbb{R}^m)$ with $\|\varphi_t\|_1 \le \varepsilon$ so that*

$$\left| \int_\Omega \eta\Phi(0) + \eta\langle \Phi^\infty, \nu_x^\infty \rangle d\lambda^s - \int_\Omega \eta\Phi\big(\phi_t \star (\overline{\nu^\infty} d\lambda^s) + \nabla\varphi_t\big) dx \right| < \varepsilon$$

uniformly for all $\eta\colon \Omega \to \mathbb{R}$ and $\Phi\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying

$$\|\eta\|_{\mathrm{Lip}} \le 1 \quad \text{and} \quad \|T\Phi\|_{\mathrm{Lip,graph}(f)} \le 1.$$

The idea behind this approximation is the following. The singular part of the centre of mass $\overline{\nu} = Du \in \mathbb{M}(\Omega, \mathbb{R}^{m\times n})$ is approximated by mollification. Such a procedure generates area-strictly convergent smooth approximations. At the same time, we generate angle concentration and oscillation via compactly supported functions. Because the first type of convergence is very strong, and the latter doesn't concentrate, the two modes of convergence don't interfere with each other. Notice that this strategy would not be possible using only weak* convergence because of the lack of quantifiability, whereas the Kantorovich metric gives us an exact quantity to approximate.

Before proving the above statement we remind that according to Lemma 2.28, T pulls back bounded sets of Lipschitz functions on $e_{f_i, i\in\mathbb{N}}$ to bounded sets of Lipschitz functions on $\mathbb{R}^{m\times n}$ (provided $f_i\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ are all Lipschitz continuous). Therefore, all the functions in the following theorem can be taken to be, after renormalisation, of Lipschitz constant less than or equal to 1 in both spaces.

From now on, after fixing a compactification, we will always identify

$$\|T\Phi\|_{\mathrm{Lip}} = \|T\Phi\|_{\mathrm{Lip}(e_{f_i, i\in\mathbb{N}})} = \|T\Phi\|_\infty + \sup_{x\ne y} \frac{|T\Phi(x) - T\Phi(y)|}{|x-y| + \sum_i 2^{-i}|Tf_i(x) - Tf_i(y)|}.$$

Proof. Fix $\varepsilon > 0$ and apply Lusin's Theorem to the λ^s-measurable map

$$x \in \overline{\Omega} \to (\delta_0, \nu_x^\infty) \in \mathbb{M}_1^+(\mathbb{R}^{m\times n}) \times \mathbb{M}^+(\partial e_{f_i, i\in\mathbb{N}}) \hookrightarrow \big((T^{-1} e_{f_i, i\in\mathbb{N}})^*\big)^+$$

to find a compact set $C = C_\varepsilon \subset \overline{\Omega}$ with $\lambda^s(\overline{\Omega} \setminus C) < \lambda^s(\overline{\Omega})\varepsilon$ restricted to which the above map is uniformly continuous, and has modulus of continuity $\omega = \omega_\varepsilon$. Without loss of generality, assume that $\mathcal{L}^n(C^s) = 0$ and because $\lambda^s(\partial\Omega) = 0$ then

$$\Delta = \Delta_\varepsilon = d(C^s, \partial\Omega) > 0.$$

For the moment, fix two integers $a, b \in \mathbb{N}$ and put $t = 2^{-a}$, so that $\phi_t > 0$ if and only if $\|x\| < t$. Let a be so large that

$$2t \le \Delta \quad \text{and} \quad a \ge \log_2\left(\frac{2}{\Delta}\right).$$

Denote by $\mathbb{F}$ the collection of $a + b$-th generation dyadic cubes $\mathbb{Q}$ in $\mathbb{R}^n$ so that $d(\mathbb{Q}, \partial\Omega) > 2^{-a}$, and for each such $\mathbb{Q} \in \mathbb{F}$ we define

$$r_{\mathbb{Q}} = \fint_{\mathbb{Q}} \phi \star (\lambda^s \llcorner C^s) dx.$$

Notice that $r_{\mathbb{Q}} > 0$ means that $dist(\mathbb{Q}, C^s) < t$, and so for each such $\mathbb{Q}$ we can find $x_{\mathbb{Q}} \in C^s$ so that $d(x_{\mathbb{Q}}, \mathbb{Q}) < t$. Denote by $\mathbb{F}^s$ the set of those $\mathbb{Q} \in \mathbb{F}$ for which $r_{\mathbb{Q}} > 0$. In particular, if $\mathbb{Q} \in \mathbb{F}^s$ we can find $x_{\mathbb{Q}} \in C^s$ so that $\sup_{\mathbb{Q}} \|x - x_{\mathbb{Q}}\| < 2t$.

For every quasi-convex function having linear growth we have

$$f(z+w) \leq f(z) + f^\infty(w)$$

for all $z \in \mathbb{R}^{m\times n}$ and $\operatorname{rank}(w) = 1$, see Theorem A.10. So we have the upper bound

$$f(r_{\mathbb{Q}}\overline{\nu^\infty_{x_{\mathbb{Q}}}}) \leq f(0) + r_{\mathbb{Q}} f^\infty(\overline{\nu^\infty_{x_{\mathbb{Q}}}}) \leq f(0) + r_{\mathbb{Q}} \int_{\partial e_{f_i, i\in\mathbb{N}}} f^{\sharp, e_{f_i, i\in\mathbb{N}}} d\nu^\infty_{x_{\mathbb{Q}}}$$

for all f quasi-convex and having linear growth.

Going back to the homogeneous result 3.15, we can select $\varphi^{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}, \mathbb{R}^m)$ with $\|\varphi^{\mathbb{Q}}\|_1 < \varepsilon\lambda^s(\mathbb{Q})$ such that

$$\|(\delta_0, \nu^\infty_{x_{\mathbb{Q}}} r_{\mathbb{Q}}) - \varepsilon_{r_{\mathbb{Q}}\overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla\varphi^{\mathbb{Q}}}\|_K < \varepsilon.$$

Define $\varphi = \sum_{\mathbb{Q}\in\mathbb{F}^d} \varphi^{\mathbb{Q}} \in \mathcal{D}(\Omega, \mathbb{R}^m)$ and $\|\varphi\|_1 \leq \varepsilon\lambda^s(\Omega)$. We claim that the sought-after map is

$$\xi^s = \phi \star (\overline{\nu^\infty_x} d\lambda^s + \nabla\varphi) \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^{m\times n}).$$

To prove that this function is the desired one, we fix $\|\eta\|_{\mathrm{Lip}} \leq 1$, $\|\Psi\|_{\mathrm{Lip}(e_{f_i, i\in\mathbb{N}})} \leq 1$ as in the statement. We have

$$\begin{aligned} &\int_\Omega \eta\langle\Phi^\infty, \phi\star(\overline{\nu^\infty} d\lambda^s)\rangle dx \\ &= \int_\Omega \eta\langle\Phi^\infty, \phi\star(\overline{\nu^\infty} d\lambda^s \llcorner C^s)\rangle dx + \overbrace{\int_\Omega \eta\langle\Phi^\infty, \phi\star(\overline{\nu^\infty} d\lambda^s \llcorner \Omega\setminus C^s)\rangle dx}^{=\mathcal{E}_1}, \end{aligned}$$

and $|\mathcal{E}_1| \leq \varepsilon\lambda^s(\Omega)$. Notice that here

$$\int_\Omega \eta\langle\Phi^\infty, \phi\star(\nu^\infty d\lambda^s \llcorner U)\rangle dx = \int_\Omega \eta(x) \int_U \phi(x-y) \int_{\partial e_{f_i, i\in\mathbb{N}}} \Phi^\infty d\nu^\infty_y d\lambda^s(y) dx$$

where $U = \Omega$ or C^s. Since for each $\mathbb{Q} \in \mathbb{F}$ with $r_{\mathbb{Q}} = 0$

$$\int_{\mathbb{Q}} \eta\langle\Phi^\infty, \phi\star(\nu^\infty d\lambda^s \llcorner C^s)\rangle dx = 0$$

and $dist(\cup\mathbb{F}, \partial\Omega) > 2t$, we get

$$\begin{aligned} \int_\Omega \langle\Phi^\infty, \phi\star(\nu^\infty d\lambda^s \llcorner C^s)\rangle dx &= \sum_{\mathbb{Q}\in\mathbb{F}^s} \int_{\mathbb{Q}} \eta\langle\Phi^\infty, \phi\star(\nu^\infty d\lambda^s \llcorner C^s)\rangle dx + \mathcal{E}_2 \\ &= \sum_{\mathbb{Q}\in\mathbb{F}^s} \left(\int_{\mathbb{Q}} \eta dx \langle\Phi^\infty, \nu^\infty_{x_{\mathbb{Q}}}\rangle r_{\mathbb{Q}} + \mathcal{E}_3^{\mathbb{Q}} \right) + \mathcal{E}_2, \end{aligned}$$

where $|\mathcal{E}_2| \le \lambda^s(C^s \cap (\partial\Omega)_{2t})$. The third error is estimated in the following way:

$$
\begin{aligned}
|\mathcal{E}_3^Q| \le & \left| \int_{\mathbb{Q}} \left(\eta - \int_{\mathbb{Q}} \eta \right) \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx \right| \\
& + \left| \int_{\mathbb{Q}} \eta \left(\int_{\mathbb{Q}} \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner C^s) \rangle dx - \langle \Phi^\infty, \nu^\infty_{x_{\mathbb{Q}}} \rangle r_{\mathbb{Q}} \right) \right| \\
\le & \|\eta\|_{\mathrm{Lip}} \, \mathcal{L}^n(\mathbb{Q})^{\frac{1}{n}} \, \|\Phi^\infty\| \int_{\mathbb{Q}} \phi \star \lambda^s dx \\
& + \|\eta\|_{\mathrm{Lip}} \int_{\mathbb{Q}} \int_{C^s} \phi(x-y) \langle \Phi^\infty, \nu^\infty_y - \nu^\infty_{x_{\mathbb{Q}}} \rangle d\lambda^s(y) dx.
\end{aligned}
$$

In particular, we obtain

$$
\begin{aligned}
|\mathcal{E}_3^Q| &\le t \int_{\mathbb{Q}} \phi \star \lambda^s dx + \int_{\mathbb{Q}} \int_{C^s} \phi(x-y) \omega(\|y - x_{\mathbb{Q}}\|) d\lambda^s(y) dx \\
&\le (t + \omega(3t)) \int_{\mathbb{Q}} \phi \star \lambda^s dx.
\end{aligned}
$$

From each $\mathbb{Q} \in \mathbb{F}^s$ we get

$$
\Phi(0) + \langle \Phi^\infty, \nu^\infty_{x_{\mathbb{Q}}} \rangle r_{\mathbb{Q}} = \int_{\mathbb{Q}} \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx + \overbrace{\mathcal{E}_4^{\mathbb{Q}}}^{|\cdot| \le \varepsilon}.
$$

Further computations show that

$$
\begin{aligned}
\int_{\mathbb{Q}} |\Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) - \Phi(0)| dx &\le \|\varepsilon_{r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}}\|_K \\
&\le \|(\delta_0, \nu^\infty_{x_{\mathbb{Q}}} r_{\mathbb{Q}})\|_K + \varepsilon \le 1 + r_{\mathbb{Q}} + \varepsilon
\end{aligned}
$$

and consequently

$$
\int_{\mathbb{Q}} \eta \int_{\mathbb{Q}} \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx = \int_{\mathbb{Q}} \eta \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx + \mathcal{E}_5^{\mathbb{Q}},
$$

where the error term is upper bounded by

$$
|\mathcal{E}_5^{\mathbb{Q}}| \le \sup_{\mathbb{Q}} \left| \eta - \int_{\mathbb{Q}} \eta \right| \mathcal{L}^n(\mathbb{Q})(1 + r_{\mathbb{Q}} + \varepsilon) \le \mathcal{L}^n(\mathbb{Q})^{\frac{1}{n}} \int_{\mathbb{Q}} (2 + \phi \star \lambda^s) dx.
$$

Finally, we estimate the last term with

$$
\int_{\mathbb{Q}} \eta \Phi(r_{\mathbb{Q}} \overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla \phi^{\mathbb{Q}}) dx = \int_{\mathbb{Q}} \eta \Phi(\phi \star \overline{\nu^\infty} d\lambda^s) + \nabla \phi^{\mathbb{Q}}) dx + \mathcal{E}_6^{\mathbb{Q}}.
$$

To bound the 6th error term we introduce an extra quantity

$$
\begin{aligned}
&\left| \int_{\mathbb{Q}} \eta \Big(\Phi(\phi \star \overline{\nu^\infty} d\lambda^s) + \nabla \phi^{\mathbb{Q}}) dx - \Phi(\phi \star \overline{\nu^\infty} d\lambda^s \llcorner C^s) + \nabla \phi^{\mathbb{Q}}) \Big) dx \right| \\
&\qquad \le \int_{\mathbb{Q}} \phi \star (\lambda^s \llcorner \Omega \setminus C^s) dx,
\end{aligned}
$$

and
$$\begin{aligned}
&\left|\int_{\mathbb{Q}} \eta\Big(\Phi(r_{\mathbb{Q}}\overline{\nu^\infty_{x_{\mathbb{Q}}}} + \nabla\phi^{\mathbb{Q}})dx - \Phi(\phi\star\overline{\nu^\infty}d\lambda^s \llcorner C^s) + \nabla\phi^{\mathbb{Q}})\Big)dx\right| \\
&\le \int_{\mathbb{Q}} |r_{\mathbb{Q}}\overline{\nu^\infty_{x_{\mathbb{Q}}}} - \phi\star(\overline{\nu^\infty}d\lambda^s \llcorner C^s)|dx \\
&\le \left|\int_{\mathbb{Q}} \phi\star\big((\overline{\nu^\infty_{x_{\mathbb{Q}}}} - \overline{\nu^\infty}\big)d\lambda^s \llcorner C^s)dx\right| \\
&\qquad + \int_{\mathbb{Q}}\left|\fint_{\mathbb{Q}} \phi\star(\overline{\nu^\infty}d\lambda^s \llcorner C^s)dx' - \phi\star(\overline{\nu^\infty}d\lambda^s \llcorner C^s)\right|dx \\
&\le \int_{\mathbb{Q}}\int_{C^s} \phi(x-y)|\overline{\nu^\infty_{x_{\mathbb{Q}}}} - \overline{\nu^\infty_y}|d\lambda^s dx \\
&\qquad + \int_{\mathbb{Q}}\left|\fint_{\mathbb{Q}} \phi\star(\overline{\nu^\infty}d\lambda^s \llcorner C^s)dx' - \phi\star(\overline{\nu^\infty}d\lambda^s \llcorner C^s)\right|dx \\
&\le \omega(3t)\int_{\mathbb{Q}} \phi\star\lambda^s dx + \mathcal{E}_7^{\mathbb{Q}}.
\end{aligned}$$

The 7th error term is estimated by

$$\begin{aligned}
\mathcal{E}_7^{\mathbb{Q}} &\le \int_{\mathbb{Q}}\fint_{\mathbb{Q}}\int_{C^s} \big|\phi(x'-y) - \phi(x-y)\big||\overline{\nu^\infty_y}|d\lambda^s(y)dx'dx \\
&\le \sqrt{n}\int_{\mathbb{Q}}\fint_{\mathbb{Q}}\int_{\mathbb{Q}+tQ} \|x-x'\|\int_0^1 \big|\nabla\phi(x+(x'-x)\tau)(x'-x)\big|d\tau d\lambda^s(y)dx'dx.
\end{aligned}$$

Given the scaling of $\phi = \phi_t$ with respect to t, we have $|\nabla\phi| \le t^{-1}M(\chi_{\mathbb{Q}})_t$, so the above can be bounded by

$$\mathcal{E}_7^{\mathbb{Q}} \le \sqrt{n}M\frac{\mathcal{L}^n(\mathbb{Q})^{\frac{1}{n}}}{t}\int_{\mathbb{Q}} (\chi_{2Q})_t\star\lambda^s dx.$$

For our choice of a and b, we have that $\mathcal{L}^n(Q) = 2^{-n(a+b)}$. With ξ^s defined above we obtain that

$$\int_\Omega \eta\langle\Phi^\infty, \phi\star(\nu^\infty d\lambda^s)\rangle dx = \int_{\cup\mathbb{F}^s} \eta\Phi(\xi^s) + \mathcal{E},$$

where

$$\begin{aligned}
|\mathcal{E}| &\le \varepsilon\lambda^s(\Omega) + (t+\omega(3t))\int_{\cup\mathbb{F}^s}\Big(\phi\star\lambda^s + 2^{-a-b}(2+\phi\star\lambda^s) \\
&\qquad + \phi\star(\lambda^s \llcorner \Omega\setminus C^s) + \omega(3t)\ \phi\star\lambda^s dx + \sqrt{n}M2^{-b}(\chi_{2Q})_t\star\lambda^s\Big)dx \\
&\le \big(2\varepsilon + 2^{-a} + 2\omega(32^{-a}) + 2^{-a-b} + c_nM2^{-b}\big)\lambda^s(\Omega) + 2^{1-a-b}\mathcal{L}^n(\Omega).
\end{aligned}$$

To conclude, we add

$$\int_{\Omega\setminus\cup\mathbb{F}^s} \eta\Phi(\xi^s)dx = \int_{\Omega\setminus\cup\mathbb{F}^s} \eta dx\Phi(0)$$

to both sides and obtain

$$\int_\Omega \eta\big(\Phi(0) + \langle \Phi^\infty, \phi \star (\nu^\infty d\lambda^s \llcorner \Omega)\rangle\big)dx = \int_\Omega \Phi(\xi^s)dx + \mathcal{E} + \int_{\cup \mathbb{F}^s} \eta dx \Phi(0).$$

Because $\cup\mathbb{F}^s \subset (C^s)_{2t}$ and since $\mathcal{L}^n(C^s) = 0$ we can find $a_\varepsilon \geq a, b_\varepsilon \geq b$ such that

$$|\mathcal{E}| + \left|\int_{\cup\mathbb{F}^s} \eta dx \Phi(0)\right| \leq 3\varepsilon(\mathcal{L}^n + \lambda^s)(\Omega).$$

The left-hand side tends to

$$\int_\Omega \eta dx \Phi(0) + \int_\Omega \eta \langle \Phi^\infty, \nu_x^\infty \rangle d\lambda^s(x)$$

as $a \to \infty$, uniformly in η and Φ, and this concludes the proof. □

We now move on to the absolutely continuous part, which is proven similarly and is slightly easier to construct.

Lemma 3.17. *Let $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that for all $t \in (0, t_\varepsilon]$ there is $\psi_t \in \mathcal{D}(\Omega, \mathbb{R}^m)$ with $\|\psi_t\|_1 \leq \varepsilon$ so that*

$$\left|\int_\Omega \eta(\langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle)dx - \int_\Omega \eta\Phi\big(\phi_t \star (\overline{\nu} + \overline{\nu^\infty}\lambda^a(x))dx \llcorner \Omega + \nabla\psi_t\big)dx\right| < \varepsilon$$

uniformly for all $\eta\colon \Omega \to \mathbb{R}$ and $\Phi\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying

$$\|\eta\|_{\mathrm{Lip}} \leq 1 \quad \text{and} \quad \|T\Phi\|_{\mathrm{Lip}(e_{f_i, i\in\mathbb{N}})} \leq 1.$$

Proof. Fix $\varepsilon \in (0, 1)$ and apply Lusin's Theorem to the $\mathcal{L}^n$-measurable map

$$\Omega \ni x \mapsto (\nu_x, \lambda^a(x)\nu_x^\infty) \in \mathbb{M}_1^+(\mathbb{R}^{m\times n}) \times \mathbb{M}^+(\partial e_{f_i, i\in\mathbb{N}}) \hookrightarrow \big((T^{-1}e_{f_i, i\in\mathbb{N}})^*\big)^+$$

to find a compact set $C^a \subset \Omega$ such that

$$\int_{\Omega\setminus C^a} \overbrace{\langle \nu_x, |\cdot|\rangle + \lambda^a(x)}^{=M(x)} dx < \varepsilon$$

and such map is continuous on C^a, and let ω be the modulus of continuity

$$\|(\nu_x, \lambda^a(x)\nu_x^\infty) - (\nu_y, \lambda^a(y)\nu_y^\infty)\|_K \leq \omega(\|x - y\|) \quad \forall\, x, y \in C^a.$$

Fix $d \in \mathbb{N}$ and $s \in (0, 1)$ and let $\mathbb{F}^a$ be the family of dyadic cubes in $\mathbb{R}^n$ of side length $t = 2^{-d}$

$$\mathbb{F}^a = \{\mathbb{Q} \in \mathcal{D}_d : d(\mathbb{Q}, \partial\Omega) > t,\ \mathcal{L}^n(\mathbb{Q} \cap C^a) > s\mathcal{L}^n(\mathbb{Q})\},$$

where the distance is induced by $\|\cdot\|_\infty$ over vectors in $\mathbb{R}^n$, and d and s will be selected later in the proof. For every $\mathbb{Q} \in \mathbb{F}^a$ select $x_\mathbb{Q}$. By Proposition 3.15 we have $\psi^Q \in \mathcal{D}(Q, \mathbb{R}^{n\times m})$ with $\|\psi^\mathbb{Q}\|_1 < \varepsilon\big(\mathcal{L}^n(\mathbb{Q})/\mathcal{L}^n(\Omega)\big)$ and

$$\|(\nu_{x_\mathbb{Q}}, \lambda^a(x_\mathbb{Q})\nu_{x_\mathbb{Q}}^\infty) - \varepsilon_{\overline{\nu_{x_\mathbb{Q}}} + \lambda^a(x_\mathbb{Q})\overline{\nu_{x_\mathbb{Q}}^\infty} + \nabla\psi^\infty}\|_K < \varepsilon.$$

Let $\psi = \sum_{\mathbb{Q}} \psi^{\mathbb{Q}} \in \mathcal{D}(\Omega, \mathbb{R}^{m\times n})$ and $\|\psi\|_1 \leq \varepsilon$. Then

$$\int_\Omega \eta(\langle \Psi, \nu_x\rangle + \lambda^a(x)\langle \Psi^\infty, \nu_x^\infty\rangle)dx = \sum_{\mathbb{Q}\in\mathbb{F}^a} \int_{\mathbb{Q}} \eta(\langle \Psi, \nu_x\rangle + \lambda^a(x)\langle \Psi^\infty, \nu_x^\infty\rangle)dx + \mathcal{E}_1$$

with $|\mathcal{E}_1| \leq \int_{\Omega\setminus\cup\mathbb{F}^a} M(x)dx \leq \varepsilon$ for large enough d. Next

$$\begin{aligned}&\sum_{\mathbb{Q}\in\mathbb{F}^a} \int_{\mathbb{Q}} \eta(\langle \Psi, \nu_x\rangle + \lambda^a(x)\langle \Psi^\infty, \nu_x^\infty\rangle)dx \\ &\quad = \sum_{\mathbb{Q}\in\mathbb{F}^a} \fint_{\mathbb{Q}} \eta \int_{\mathbb{Q}} (\langle \Psi, \nu_x\rangle + \lambda^a(x)\langle \Psi^\infty, \nu_x^\infty\rangle)dx + \mathcal{E}_2,\end{aligned}$$

where $$|\mathcal{E}_2| \leq t \sum_{\mathbb{Q}\in\mathbb{F}^a} \int_{\mathbb{Q}} |\langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle| dx \leq t \int_\Omega M(x)dx.$$

We further estimate, on every set $\mathbb{Q} \in \mathbb{F}^a$,

$$\begin{aligned}&\left| \int_{\mathbb{Q}} \langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle - \mathcal{L}^n(\mathbb{Q})\Big(\langle \Phi, \nu_{x_{\mathbb{Q}}}\rangle + \lambda^a(x_{\mathbb{Q}})\langle \Phi^\infty, \nu_{x_{\mathbb{Q}}}^\infty\rangle\Big)\right| \\ &\leq \frac{\omega(t)}{s}\mathcal{L}^n(\mathbb{Q}) + \frac{1-s}{s}\int_{\mathbb{Q}} |\langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle| dx \\ &\qquad + \int_{\mathbb{Q}\setminus C^a} |\langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle| dx.\end{aligned}$$

By the linear growth of f, we get that

$$\begin{aligned}&\sum_{\mathbb{Q}\in\mathbb{F}^a} \fint_{\mathbb{Q}} \eta \int_{\mathbb{Q}} \langle \Phi, \nu_x\rangle + \lambda^a(x)\langle \Phi^\infty, \nu_x^\infty\rangle dx \\ &\quad = \sum_{\mathbb{Q}\in\mathbb{F}^a} \Big(\langle \Phi, \nu_{x_{\mathbb{Q}}}\rangle + \lambda^a(x_{\mathbb{Q}})\langle \Phi^\infty, \nu_{x_{\mathbb{Q}}}^\infty\rangle\Big) \int_{\mathbb{Q}} \eta + \mathcal{E}_3\end{aligned}$$

where $$|\mathcal{E}_3| \leq \frac{\omega(t)}{s}\mathcal{L}^n(\mathbb{Q}) + \frac{1-s}{s}\int_{\mathbb{Q}} M(x)dx + \varepsilon.$$

For every $\mathbb{Q} \in \mathbb{F}^a$ we have that

$$f(x_{\mathbb{Q}}) = \fint_{\mathbb{Q}} \Phi(\overline{\nu_{x_{\mathbb{Q}}}} + \lambda^a(x_{\mathbb{Q}})\overline{\nu_{x_{\mathbb{Q}}}^\infty} + \nabla\phi^{\mathbb{Q}}) + \mathcal{E}_4^{\mathbb{Q}}, \text{ with } |\mathcal{E}_4^{\mathbb{Q}}| \leq \varepsilon.$$

Set $v^a(x) = \overline{\nu_x} + \lambda^a(x)\overline{\nu_x^\infty}$. Letting $\Phi = z \cdot e^i$, where $(e^i)_{i=1}^{m\times n}$ is the canonical basis of $\mathbb{R}^{m\times n}$, we obtain, from continuity over C^a, $|v^a - v^a(x_{\mathbb{Q}})| \leq \omega(t)$ on $\mathbb{Q} \cap C^a$ for each $\mathbb{Q} \in \mathbb{F}^a$. Consequently, for all $\mathbb{Q} \in \mathbb{F}^a$,

$$\int_{\mathbb{Q}} |v^a - v^a(x_{\mathbb{Q}})| dx \leq \frac{\omega(t)}{s}\mathcal{L}^n(\mathbb{Q}) + \int_{\mathbb{Q}\setminus C^a} |v^a| dx + \frac{1-s}{s}\int_{\mathbb{Q}} |v^a| dx.$$

Because $|v^a| \leq M(x)$ and $\operatorname{Lip}(\Phi) \leq 1$, then

$$\sum_{\mathbb{Q}\in\mathbb{F}^a} \fint_{\mathbb{Q}} \eta dx \int_{\mathbb{Q}} \Phi(v^a(x_{\mathbb{Q}}) + \nabla\psi^{\mathbb{Q}})dx = \sum_{\mathbb{Q}\in\mathbb{F}^a} \fint_{\mathbb{Q}} \eta dx \int_{\mathbb{Q}} \Phi(v^a + \nabla\psi^{Q})dx + \mathcal{E}_5$$

with $$|\mathcal{E}_5| \leq \frac{\omega(t)}{s}\mathcal{L}^n(Q) + \varepsilon + \frac{1-s}{s}\int_\Omega M(x)dx.$$

Combining some of the previous estimates, we get

$$\sum_{\mathbb{Q}\in\mathbb{F}^a} \fint_{\mathbb{Q}} \eta dx \int_{\mathbb{Q}} \Phi(v^a + \nabla\psi^{\mathbb{Q}})dx = \sum_{\mathbb{Q}\in\mathbb{F}^a} \int_{\mathbb{Q}} \eta\Phi(v^a + \nabla\psi^{\mathbb{Q}})dx + \mathcal{E}_6,$$

and

$$\begin{aligned}
|\mathcal{E}_6| &\leq t\sum_{\mathbb{Q}\in\mathbb{F}^a}\int_{\mathbb{Q}} |\Phi(v^a + \nabla\psi^{\mathbb{Q}})|dx \leq t|\mathcal{E}_5| + t\sum_{\mathbb{Q}\in\mathbb{F}^a}\int_{\mathbb{Q}} |\Phi(v^a(x_{\mathbb{Q}}) + \nabla\psi^{\mathbb{Q}})|dx \\
&\leq t|\mathcal{E}_5| + t\varepsilon\mathcal{L}^n(\Omega) + t\sum_{\mathbb{Q}\in\mathbb{F}^a}\mathcal{L}^n(\mathbb{Q})(\langle|\Phi|,\nu_{x_{\mathbb{Q}}}\rangle + \lambda^a(x_{\mathbb{Q}})\langle|\Phi|^\infty,\nu^\infty_{x_{\mathbb{Q}}}\rangle) \\
&\leq t|\mathcal{E}_5| + t\varepsilon\mathcal{L}^n(\Omega) + t\frac{\omega(t)}{s}\mathcal{L}^n(\Omega) + \left(\int_{Q\setminus C^a} + \frac{1-s}{s}\int_\Omega\right) M(x)dx.
\end{aligned}$$

Finally, if ϕ_t is a standard mollifier, then $\phi_t \star v^a \llcorner \Omega \xrightarrow{L^1(\Omega)} v^a$ as $t \to 0$, and so

$$\int_\Omega \eta\Phi(v^a + \nabla\psi)dx = \int_\Omega \eta\Phi(\phi_t \star v^a \llcorner \Omega + \nabla\psi)dx + \mathcal{E}_7,$$

where using again that Φ is Lipschitz over $\mathbb{R}^{m\times n}$,

$$|\mathcal{E}_7| \leq \operatorname{Lip}(\Phi)\int_\Omega |\phi_t \star v^a \llcorner \Omega - v^a|dx \leq \int_\Omega |\phi_t \star v^a \llcorner \Omega - v^a|dx.$$

This concludes the proof. □

We are now ready to prove Theorem 3.12.

Proof. (Theorem 3.12) The proof of ⇒ is similar to that of 3.15, so we only show the reverse implication. Let $(\nu_x, \lambda, \nu_x^\infty)$ satisfy Assumptions 1,2 and 3 of the theorem and fix $j \in \mathbb{N}$. By Lemma 3.16 and Lemma 3.17, we can find $\phi_j \in \mathcal{D}(\Omega)$ and $u_j^{ac}, u_j^s \in \mathcal{D}(\Omega, \mathbb{R}^m)$ so that

$$\left|\int_\Omega \eta\Phi(0) + \eta\langle\Phi^\infty, \nu_x^\infty\rangle d\lambda^s - \int_\Omega \eta\Phi\big(\phi_j \star (\overline{\nu^\infty}d\lambda^s) + \nabla u_j^{ac}\big)dx\right| < \frac{1}{j}, \quad \text{and}$$

$$\left|\int_\Omega \eta(\langle\Phi,\nu_x\rangle + \lambda^a(x)\langle\Phi^\infty,\nu_x^\infty\rangle)dx - \int_\Omega \eta\,\Phi\big(\phi_j \star (\overline{\nu} + \overline{\nu^\infty}\lambda^a(x))dx \llcorner \Omega + \nabla u_j^s\big)dx\right| < \frac{1}{j},$$

where both inequalities hold uniformly for all η and Φ satisfying $\|\eta\|_{\mathrm{Lip}} \leq 1$ and $\|T\Phi\|_{\mathrm{Lip},\mathrm{graph}(f)} \leq 1$.

Let $o_j = \phi_j \star (\overline{\nu^\infty} d\lambda^s) + \nabla u_j^{ac}$ and $c_j = \phi_j \star (\overline{\nu} + \overline{\nu^\infty} \lambda^a(x)) dx \llcorner \Omega + \nabla u_j^s$.

Then $(o_j)_j$ and $(c_j)_j$ satisfy the assumption of Lemma 2.33, and because

$$o_j + c_j = \phi_j \star \overline{\nu} + \nabla(\overbrace{u_j^{ac} + u_j^s}^{=:u_j}) = \phi_j \star (Du \llcorner \Omega) + \nabla u_j,$$

we conclude that

$$\phi_j \star (Du \llcorner \Omega) + \nabla u_j \xrightarrow{Y(e_{f_i, i \in \mathbb{N}})} (\nu_x, \lambda^a(x)dx + \lambda^s, \nu_x^\infty) = (\nu_x, \lambda, \nu_x^\infty),$$

where we also used Lemma 2.30 to deduce full convergence to the Young Measure from uniform convergence against the Lipschitz functions η and Φ. □

A. Appendix

In this appendix, the references to the following theorems are not necessarily attributed to the original author of such results but are just references for the reader.

Vitali convergence Theorem.

Theorem A.1. ([17], Theorem 4.5.4) *Let $\mu \in \mathbb{M}^+(\Omega)$ and $(f_n)_n$ and f in $L^1(\Omega, \mu)$. Then $f_n \to f$ in $L^1(\Omega, \mu)$ if and only if $f_n \to f$ in measure and f_n is uniformly integrable.*

Stone-Weierstrass Theorem.

Theorem A.2. ([22], Theorem 4.45) *Let X be a compact Hausdorff space. If $\mathbb{A}$ is a closed subalgebra of $C(X)$ that separates points, then either $\mathbb{A} = C(X)$ or there is $x_0 \in X$ so that $\mathbb{A} = \{f \in C(X) : f(x_0) = 0\}$. In particular, $\mathbb{A} = C(X)$ if and only if $\mathbb{A}$ contains the constant functions.*

Tychonoff Theorem.

Theorem A.3. *If $\{X_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary family of compact spaces, $\Pi_{\lambda \in \Lambda} X_\lambda$ is compact in the product topology.*

Banach-Alaoglu sequential version.

Theorem A.4. ([41], Theorem 12) *Let X be a separable Banach space and $\mathbb{B} \subset X^*$ the closed unit ball of the dual. Then $\mathbb{B}$ is weakly* sequentially compact.*

Chacon biting Lemma.

Lemma A.5. ([13]) *Let $\mu \in \mathbb{M}^+(\Omega)$ and $(v_j)_j$ in $L^1(\Omega, \mu)$ be a sequence such that $\sup_j \|v_j\| < +\infty$. There are sets $E_k \subset E_{k+1}, \mu(\Omega \setminus E_k) \to 0$ and a subsequence $(v_{j_i})_i$ of $(v_j)_j$ and $v \in L^1(\mu)$ so that $v_{j_i} \rightharpoonup v$ in $L^1(E_k, \mu)$ for all k.*

Kantorovich metric.

Lemma A.6. ([34], Lemma 2.1) *Let (X, d) be a metric space. The Kantorich metric on $\mathbb{M}^+(X)$ generates the same topology as the weak* topology of measures.*

Lemma A.7. *Let $\eta \in \mathbb{M}(\Omega, \mathbb{R}^d)$, $\Omega \subset \mathbb{R}^n$ open bounded set, and $(\phi_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of standard mollifiers,* $\operatorname{supp}(\phi_1) \subset \mathbb{B}^n$. *Then*

$$\eta \star \phi_\varepsilon \xrightarrow{\textit{area-strictly}} \eta.$$

Proof. Because $(\mathcal{L}^n \llcorner \Omega, \eta_\varepsilon) \rightharpoonup^* (\mathcal{L}^n \llcorner \Omega, \eta)$, we immediately obtain the lower bound

$$\liminf_{\varepsilon\to 0} |(\mathcal{L}^n, \rho_\varepsilon)|(\Omega) \geq |(\mathcal{L}^n, \rho)|(\Omega).$$

To prove the upper semi-continuity of the above quantity, notice the equality:

$$(\mathcal{L}^n \llcorner \Omega, \rho_\varepsilon) = (\mathcal{L}^n \llcorner \Omega, \rho) \star \phi_\varepsilon - (\mathcal{L}^n \llcorner \Omega_{-\varepsilon} \star (\delta_0 - \phi_\varepsilon), 0),$$

where $\Omega_{-\varepsilon} = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$. Using the Jensen inequality we get

$$\begin{aligned}\limsup_{\varepsilon\to 0} |(\mathcal{L}^n, \rho_\varepsilon)|(\Omega) &\leq \limsup_{\varepsilon\to 0} |(\mathcal{L}^n \llcorner \Omega, \rho) \star \phi_\varepsilon|(\Omega) + |(\mathcal{L}^n \llcorner \Omega_{-\varepsilon} \star (\delta_0 - \phi_\varepsilon), 0)|(\Omega) \\ &\leq |(\mathcal{L}^n \llcorner \Omega, \rho)|(\Omega) + \limsup_{\varepsilon\to 0} 2\mathcal{L}^n(\Omega \setminus \Omega_{-\varepsilon}) = |(\mathcal{L}^n, \rho)|(\Omega). \quad \square\end{aligned}$$

Atomic decomposition.

Theorem A.8. ([25], Proposition 1.22) *Let $\mu \in \mathbb{M}(\Omega)$. Then there exists a purely atomic measure μ^a and a non-atomic measure μ^{n-a} such that $\mu = \mu^a + \mu^{n-a}$.*

Proof. See [25]. $\square$

Theorem A.9. ([33], Theorem 1.1) *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be quasi-convex and positively 1-homogeneous. Then f is convex at all $z \in \mathbb{R}^{m\times n}$ with* $\operatorname{rank}(z) \leq 1$.

When a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ does not have a regular recession at infinity we can still derive the following convex-like inequality.

Theorem A.10. ([33], Lemma 2.5) *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be quasi-convex. Then for all $z \in \mathbb{R}^{m\times n}$ and* $\operatorname{rank}(w) \leq 1$ *we have*

$$f(z + w) \leq f(z) + f^\infty(w).$$

Here $f^\infty(w)$ exists because f is convex at directions $\operatorname{rank}(w) \leq 1$

References

[1] E. Acerbi, N. Fusco: *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Analysis 86/2 (1984) 125–145.

[2] G. Alberti: *Rank one property for derivatives of functions with bounded variation*, Proc. Royal Soc. Edinburgh Sect. A Math. 123/2 (1993) 239–274.

[3] J.-J. Alibert, G. Bouchitté: *Non-uniform integrability and generalized Young Measure*, J. Convex Analysis 4 (1997) 129–148.

[4] L. Ambrosio, G. Dal Maso: *On the relaxation in $BV(\Omega;\mathbb{R}^m)$ of quasi-convex integrals*, J. Funct. Analysis 109/1 (1992) 76–97.

[5] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Science Publications, Clarendon Press, Oxford (2000).

[6] A. Arroyo-Rabasa: *Characterization of generalized Young Measures generated by $\mathcal{A}$-free measures*, Arch. Rational Mech. Analysis 242 (2021) 235–325.

[7] M. Baia, S. Kromer, M. Kruzik: *Generalized $W^{1,1}$-Young Measures and relaxation of problems with linear growth*, SIAM J. Math. Analysis 50/1 (2018) 1076–1119.

[8] E. J. Balder: *Lectures on Young Measures*, CEREMADE, Cahier 9517, Université de Paris-Dauphine, Paris (1995).

[9] E. J. Balder: *New fundamentals of Young Measure convergence*, in: *Calculus of Variations and Optimal Control*, A. Ioffe et al. (eds.), Res. Notes Math. 411, Chapman and Hall/CRC, London (2000) 24–48.

[10] J. M. Ball: *Convexity conditions and existence theorems in nonlinear elasticity* Arch. Rational Mech. Analysis 63 (1977) 337–403.

[11] J. M. Ball: *A version of the fundamental theorem for Young Measures*, in: *PDEs and Continuum Models of Phase Transitions*, M. Rascle et al. (eds.), Nice 1988, Lect. Notes Phys. 344, Springer, Berlin (1989) 207–215.

[12] J. M. Ball, J. C. Currie, P. J. Olver: *Null lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Analysis 41/2 (1981) 135–174.

[13] J. M. Ball, F. Murat: *Remarks on Chacon's biting lemma*, Proc. Amer. Math. Soc. 107/3 (1989) 655–663.

[14] J. M. Ball, K.-W. Zhang: *Lower semicontinuity of multiple integrals and the biting lemma*, Proc. Royal Soc. Edinburgh Sect. A Math. 114/3-4 (1990) 367–379

[15] B. Benešová, S. Krömer, M. Kružík: *Boundary effects and weak lower semicontinuity for signed integral functionals on BV*, ESAIM Control, Optim. Calc. Var. 21/2 (2015) 513–534.

[16] H. Berliocchi, J.-M. Lasry: *Intégrandes normales et mesures paramétrées en calcul des variations*, Bull. Soc. Math. France 101 (1973) 129–184.

[17] V. I. Bogachev, M. A. S. Ruas: *Measure Theory. Vol. 1*, Springer, Berlin (2007).

[18] C. Castaing, P. R. De Fitte, M. Valadier: *Young Measures on Topological Spaces. With Applications in Control Theory and Probability Theory*, Mathematics and its Applications Vol. 571, Kluwer, Dordrecht (2004).

[19] R. E. Chandler: *Hausdorff Compactifications*, Lecture Notes in Pure and Applied Mathematics Vol. 23, Marcel Dekker, New York (1976).

[20] B. Dacorogna: *Direct Methods in the Calculus of Variations*, Applied Mathematical Sciences Vol. 78, Springer, Berlin (2007).

[21] R. J. DiPerna, A. J. Majda: *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Physics 108/4 (1987) 667–689.

[22] G. B. Folland: *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Wiley, New York (1999).

[23] I. Fonseca: *Lower semicontinuity of surface energies*, Proc. Royal Soc. Edinburgh Sect. A Math. 120/1-2 (1992) 99–115.

[24] I. Fonseca, M. Kružík: *Oscillations and concentrations generated by $\mathcal{A}$-free mappings and weak lower semicontinuity of integral functionals*, ESAIM Control, Optim. Calc. Var. 16/2 (2010) 472–502.

[25] I. Fonseca, G. Leoni: *Modern Methods in the Calculus of Variations: L^p Spaces*, Springer, Berlin (2007).

[26] I. Fonseca, S. Müller: *Quasi-convex integrands and lower semicontinuity in* L^1, SIAM J. Math. Analysis 23/5 (1992) 1081–1098.

[27] I. Fonseca, S. Müller: *Relaxation of quasiconvex functionals in* $BV(O, \mathbb{R}^p)$ *for integrands* $F(x, u, \nabla u)$, Arch. Rational Mech. Analysis 123/1 (1993) 1–49.

[28] I. Fonseca, S. Müller: *$\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young Measures*, SIAM J. Math. Analysis 30/6 (1999) 1355–1390.

[29] I. Fonseca, S. Müller, P. Pedregal: *Analysis of concentration and oscillation effects generated by gradients*, SIAM J. Math. Analysis 29/3 (1998) 736–756.

[30] A. Kałamajska, M. Kružík: *Oscillations and concentrations in sequences of gradients*, ESAIM Control, Optim. Calc. Var. 14/1 (2008) 71–104.

[31] D. Kinderlehrer, P. Pedregal: *Characterizations of Young Measures generated by gradients*, Arch. Rational Mech. Analysis 115/4 (1991) 329–365.

[32] D. Kinderlehrer, P. Pedregal: *Gradient Young Measures generated by sequences in Sobolev spaces*, J. Geom. Analysis 4/1 (1994) 59–90.

[33] B. Kirchheim, J. Kristensen: *On rank-one convex functions that are homogeneous of degree one*, Arch. Rational Mech. Analysis 221/1 (2016) 527–558.

[34] J. Kristensen, B. Raiță: *Oscillation and concentration in sequences of PDE constrained measures*, Arch. Rational Mech. Analysis 246/2-3 (2022) 823–875.

[35] J. Kristensen, F. Rindler: *Characterization of generalized Gradient Young Measures generated by sequences in* $W^{1,1}$ *and* BV, Arch. Rational Mech. Analysis 197/2 (2010) 539–598.

[36] J. Kristensen, F. Rindler: *Relaxation of signed integral functionals in* BV, Calc. Var. Partial Diff. Equations 37/1-2 (2010) 29–62.

[37] S. Krömer: *On compactness of minimizing sequences subject to a linear differential constraint*, Z. Analysis Anwend. 30/3 (2011) 269–303.

[38] S. Krömer, M. Kružík: *Oscillations and concentrations up to the boundary*, J. Convex Analysis 20/3 (2013) 723–752.

[39] M. Kružík: *Quasiconvexity at the boundary and concentration effects generated by gradients**, ESAIM Control, Optim. Calc. Var. 19/3 (2013) 679–700.

[40] M. Kružík, T. Roubíček: *Explicit characterization of* L^p*-Young Measures*, J. Math. Analysis Appl. 198/3 (1996) 830–843.

[41] P. D. Lax: *Functional Analysis*, John Wiley & Sons, New York (2002).

[42] P. Marcellini: *Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals* Manuscripta Math .51/1-3 (1985) 1–28.

[43] P. Marcellini, C. Sbordone: *Semicontinuity problems in the calculus of variations*, Nonlinear Analysis: Theory, Meth. Appl. 4/2 (1980) 241–257.

[44] E. J. McShane: *Generalized curves*, Duke Math. J. 6/1 (1940) 513–536.

[45] E. J. McShane: *Relaxed controls and variational problems*, SIAM J. Control 5/3 (1967) 438–485.

[46] N. G. Meyers: *Quasi-convexity and lower semi-continuity of multiple variational integrals of any order*, Trans. Amer. Math. Soc. 119/1 (1965) 125–149.

[47] C. B. Morrey: *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. 2/1 (1952) 25–53.

[48] S. Müller: *On quasiconvex functions which are homogeneous of degree 1*, Indiana Univ. Math. J. 41/1 (1992) 295–301.

[49] F. Murat: *Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant*, Ann. Scuola Norm. Sup. Pisa Clas. Scienze 8/1 (1981) 69–102.

[50] P. Pedregal: *Parametrized Measures and Variational Principles*, Springer, Berlin (1997).

[51] Y. G. Reshetnyak: *Weak convergence of completely additive vector functions on a set*, Siberian Math. J. 9/6 (1968) 1039–1045.

[52] F. Rindler: *Lower semicontinuity and Young Measures in BV without Alberti's rank-one theorem*, Adv. Calc. Var. 5/2 (2012) 127–159.

[53] F. Rindler: *Calculus of Variations*, Universitext Vol. 5, Springer, Cham (2018).

[54] T. Roubiček: *Relaxation in Optimization Theory and Variational Calculus*, De Gruyter, Berlin (1997).

[55] T. Schmidt: *Strict interior approximation of sets of finite perimeter and functions of bounded variation*, Proc. Amer. Math. Soc. 143/5 (2015) 2069–2084.

[56] L. Tartar: *Compensated compactness and applications to partial differential equations*, in: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium Vol. 4*, Research Notes in Mathematics 39, Edinburgh (1979) 136–212.

[57] L. C. Young: *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, C. R. Soc. Sci. Lett. Varsovie 30 (1937) 212–234.