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Nonsmooth Multiobjective Fractional Programming and Second-order Optimality Conditions for Weak Efficiency

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Second-order Fritz John primal and dual necessary conditions for weak efficiency of multiobjective fractional programs involving equality, inequality and set constraints are established. With a constraint qualification of Mangasarian-Fromovitz type, Kuhn-Tucker necessary conditions are provided as well as sufficient conditions for weak efficiency.

Keywords: Multiobjective fractional programming, local weak efficient solution, Fritz John and Kuhn-Tucker necessary conditions.

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1. Introduction

In various applications of optimization a function is to be maximized or minimized which involves one or some ratios of functions. Such optimization problems are called fractional programming problems. For example, a related objective is Return per Risk to be maximized, models where Cost per Time to be maximized, maximization of Output per Input,.... Liang et al. [11] derived sufficient optimality conditions and duality for single fractional programming problems. In recent years, many authors have studied first and second-order optimality conditions for vector optimization problems and obtained Kuhn-Tucker optimality conditions via Lagrange multipliers rules (see, e.g., [4]–[6], [10]–[20]). The notion of nonconvex closed convexificator by Jeyakumar-Luc [9] is a generalization of some notions of known subdifferentials such as the subdifferentials of Clarke [4], Michel-Penot [18], Mordukhovich [19]. It has provided good calculus rules for establishing necessary efficiency conditions in nonsmooth optimization. Necessary conditions for efficiency via convexificators may be sharper than those expressed in terms of known subdifferentials as the Clarke, Michel-Penot and Mordukhovich subdifferentials. For a locally Lipschitz function which is regular in Clarke's sense, the closed convex hull of upper regular

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convexificator is contained in its Clarke subdifferential. This is shown in Example 2.1 by Jeyakumar-Luc [9]. Hence, optimality conditions in terms of convexifixicators will provide sharp results. Luu [12, 13] derived necessary conditions for local weak Pareto and Pareto minima of multiobjective programming problems involving inequality, equality and set constraints in terms of convexificators. Luu [14] established Fritz John and Kuhn-Tucker necessary conditions via convexificators.

Kuk et al. [10] established necessary and sufficient optimality conditions for weak efficiency of nonsmooth multiobjective fractional programming problems with inequality constraints via the Clarke subdifferential. Gadhi [5] derived optimality conditions for fractional multiobjective problems involving inequality constraints (without equality and set constraints) in terms of convexificators. Chuong [3], Hong et al. [8] studied optimality and duality for fractional robust multiobjective optimization problems. Luu and Linh [16] derived optimality conditions for weak efficiency and Luu and Linh [14] for weak quasi-efficiency of multiobjective fractional optimization problems involving equality, inequality and set constraints. Recently, Luu [15] established primal and dual second-order necessary conditions weak efficiency of nonsmooth multiobjective optimization problems in terms of the Palés-Zeidan second-order directional derivatives. In this paper, we derive second-order necessary and sufficient efficiency conditions via the Palés-Zeidan second-order directional derivatives for nonsmooth multiobjective fractional problems with equality, inequality and set constraints.

The paper is organized as follows. Section 2 gives some notions and preliminaries. Section 3 is devoted to derive Fritz John primal necessary optimality conditions for multiobjective fractional problems involving equality, inequality and set constraints via the Palés-Zeidan second-order directional derivatives. Section 4 deals with second-order Kuhn-Tucker dual necessary conditions for weak efficiency under a constraint qualification. Note that to derive dual sufficient optimality conditions in Section 6 it is not necessary to use assumptions on generalized convexity.

2. Preliminaries

Given a Banach space X, we denote by X^* its topological dual. For an extended-real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$, the expressions

$$f^-(x;v) = \liminf_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t},$$

and

$$f^+(x;v) = \limsup_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

stand for the *lower* and *upper Dini directional derivatives* of f at x in the direction of v, respectively. In case $f^+(x;v) = f^-(x;v)$ we shall denote their common value by $f'(x,v)$, which is called *directional derivative* of f at $\overline{x}$ in the direction v.

Consider the following multiobjective fractional optimization problem:

$$(MFP) \quad \begin{cases} \min\Big(\dfrac{f_1(x)}{g_1(x)},\dots,\dfrac{f_p(x)}{g_p(x)}\Big), \quad \text{subject to} \\ h_j(x) \leqslant 0,\ j\in I := \{1,\dots,m\},\ \ell_k(x)=0,\ k\in L := \{1,\dots r\},\ x\in D, \end{cases}$$

where $D \subset \mathbb{R}^p$ be a pointed closed convex cone with nonempty interior which induces a partial order in $\mathbb{R}^p$, f_i, g_i, h_j and ℓ_k are continuous functions defined on X such that $g_i(x) > 0$, for all i, j, k and all $x \in X$, D is a closed subset of X. Denote by M the feasible set of (MFP)

$$M := \{x \in D : h_j(x) \leq 0,\ j = 1, ..., m,\ \text{and}\ \ell_k(x) = 0,\ k = 1, ...n\},$$

and for $\overline{x} \in M$, $I(\overline{x}) := \{j \in I : h_j(\overline{x}) = 0\}$.

We set $\phi(x) := \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right)$. A point $\overline{x} \in M$ is said to be a *local weak efficient* (resp., *local efficient*) *solution* for (MFP) if there exists a neighborhood V of $\overline{x}$ such that there is no $x \in V \cap M$ satisfying

$$\phi_i(x) < \phi_i(\overline{x})\ (i = 1, \ldots, p),$$
$$\left[\text{resp., } \phi_i(x) \leqslant \phi_i(\overline{x})\ (i = 1, \ldots, p);\ \phi_s(x) < \phi_s(\overline{x}) \text{ for at least one } s \in \overline{\{1, p\}}\,\right].$$

In case $V = X$ we obtain the notions of weak efficient (resp. efficient) solutions.

For a set $D \subset X$, the *Clarke tangent cone* to D at $\overline{x}$ is defined as

$$T(D; \overline{x}) = \left\{ v \in X : \begin{array}{l} \forall x_n \in D,\ x_n \to \overline{x},\ \forall t_n \downarrow 0,\ \exists v_n \to v \\ \text{such that } x_n + t_n v_n \in D,\ \forall n \end{array} \right\}.$$

The *Clarke normal cone* to D at $\overline{x}$ is

$$N(D; \overline{x}) = \big\{\xi \in X^* : \langle \xi, v\rangle \leqslant 0,\ \forall v \in T(D; \overline{x})\big\}.$$

Let f be a real-valued function defined on $\mathbb{R}^n$, which is locally Lipschitz at $\overline{x} \in \mathbb{R}^n$. Recall [4] (see also the recent book [23]) that the *Clarke generalized derivative* of f at $\overline{x} \in X$ in a direction $v \in X$ is defined as

$$f^0(\overline{x}; v) := \limsup_{x \to \overline{x},\ t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Following [15], the *Páles-Zeidan second-order upper (lower) generalized directional derivative* of f at $\overline{x}$ in the direction v is, respectively, defined as

$$f^{00}_{+}(\overline{x}; v) := \limsup_{t \downarrow 0} 2\, \frac{f(\overline{x} + tv) - f(\overline{x}) - t f^0(\overline{x}; v)}{t^2},$$
$$f^{00}_{-}(\overline{x}; v) := \liminf_{t \downarrow 0} 2\, \frac{f(\overline{x} + tv) - f(\overline{x}) - t f^0(\overline{x}; v)}{t^2}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be Fréchet differentiable at $\overline{x} \in X$, and $\nabla f(\overline{x})$ be its gradient at $\overline{x}$. The following limit is called *second-order directionally derivative* at $\overline{x}$ in the direction $v \in \mathbb{R}^n$

$$f''(\overline{x}; v) := \lim_{t \downarrow 0} 2\, \frac{f(\overline{x} + tv) - f(\overline{x}) - t \nabla f(\overline{x}) v}{t^2}.$$

Remark 2.1. (a) If f is continuously Fréchet differentiable near $\overline{x}$ with the Fréchet derivative $\nabla f(\overline{x})$, then f is Lipchitz near $\overline{x}$, and $f^0(\overline{x}; v) = \nabla f(\overline{x}) v$ $(\forall v \in X)$. This is false, when $\nabla f(x)$ is not continuous at $\overline{x}$.

(b) If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously Fréchet differentiable near $\overline{x}$ and second-order directionally differentiable at $\overline{x}$ in a direction $v \in X$, then (see [5])

$$f_{+}^{00}(\overline{x}; v) = f''(\overline{x}; v). \qquad \square$$

Let f be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$. Recall that f is *Gâteaux differentiable* at $\overline{x}$ if there exists a continuous linear mapping Λ_1 from X into Y such that

$$f(\overline{x} + tv) = f(\overline{x}) + t\Lambda_1(v) + o(t) \ (\forall\ v \in \mathbb{R}^n),$$

where $\|o(t)\| / \mid t \mid \to 0$ as $t \to 0$. The mapping Λ_1 is said to be the *Gâteaux derivative* of f at $\overline{x}$ and is denoted by $f'_G(\overline{x})$. This concept is already known in the literature (see the book [22], p. 502 and also the book [2]). Note that a mapping which is Gâteaux differentiable at $\overline{x}$ may be not continuous at $\overline{x}$.

Let C be a nonempty subset of $\mathbb{R}^n$. Following [4], an element $u \in X$ is called a *tangent vector* to C at $\overline{x} \in \mathrm{cl}C$ iff

$$\lim_{t\downarrow 0} \frac{1}{t} d(\overline{x} + tu; C) = 0\,, \tag{1}$$

where $d(x; C)$ stands for the distance from x to C, $\mathrm{cl}\, C$ is the closure of C. Such a vector was already called "adjacent tangent vector" in [1] (consult p. 127 therein; see also [23], p. 776).

The set of all tangent vectors to C at $\overline{x}$ is denoted by $T_{\overline{x}}C$, and is called the *tangent cone* to C at $\overline{x}$. Note that $T_{\overline{x}}C$ is a nonempty closed cone containing $0 \in X$. Moreover, (1) is equivalent to the existence of a function $\gamma : (0, +\infty) \to X$ with $\gamma(t) \to 0$ as $t \downarrow 0$, and

$$\overline{x} + t(u + \gamma(t)) \in C \quad (\forall t > 0).$$

Adapting the definition in [4], an element $v \in X$ is said to be a *second-order tangent vector* to C at $\overline{x}$ iff there exist $u \in X$ such that

$$\lim_{t\downarrow 0} \frac{1}{t^2}\, d(\overline{x} + tu + \frac{t^2}{2}\, v; C) = 0\,. \tag{2}$$

The vector u satisfying (2) is said to be associated with v. Denote by $T^2_{\overline{x}}C$ the set of all second-order tangent vectors to C at $\overline{x}$. Observe that $v \in T^2_{\overline{x}}C$ with the associated vector u is equivalent to the existence of a function $\gamma_1 : (0, +\infty) \to X$ with $\gamma_1(t) \to 0$ as $t \downarrow 0$, and

$$\overline{x} + tu + \frac{t^2}{2}(v + \gamma_1(t)) \in C \ (\forall t > 0).$$

Moreover, $v \in T^2_{\overline{x}}C$ implies that $u \in T_{\overline{x}}C$, and $T^2_{\overline{x}}C$ is a nonempty cone containig $0 \in X$ (see [4]).

Remark 2.2. It is easy to see that for $\overline{x} \in C_1 \cap C_2$,

(i) $T_{\overline{x}}(C_1 \cap C_2) \subset T_{\overline{x}}C_1 \cap T_{\overline{x}}C_2$;

(ii) $T^2_{\overline{x}}(C_1 \cap C_2) \subset T^2_{\overline{x}}C_1 \cap T^2_{\overline{x}}C_2$.

The following lemma plays an important role in the proofs of second-order sufficient optimality conditions in Sention 6 in case that C is convex.

Lemma 2.3. *Assume that C is convex. Then, for $\overline{x} \in C$,*

(a) $C - \overline{x} \subset T_{\overline{x}}C$;

(b) $C - \overline{x} \subset T^2_{\overline{x}}C$; *for $x \in C$ we have $x - \overline{x} \in T^2_{\overline{x}}C$ with the associated vector $x - \overline{x} \in T_{\overline{x}}C$.*

Proof. (a) Since C in convex, for $x \in C, t \in [0,1], \overline{x} + t(x - \overline{x}) \in C$. Hence, for $\gamma(t) \equiv 0$, one gets $\overline{x} + t(x - \overline{x} + \gamma(t)) \in C$, which means that $x - \overline{x} \in T_{\overline{x}}C$.

(b) For t small enough, $t_1 := t + \frac{t^2}{2} \in [0,1]$, and so, for $\gamma_1(t) \equiv 0$, we have

$$\overline{x} + t(x - \overline{x}) + \frac{t^2}{2}(x - \overline{x} + \gamma_1(t)) \in C,$$

which means that $x - \overline{x} \in T^2_{\overline{x}}C$ with the associated vector $x - \overline{x} \in T_{\overline{x}}C$. $\square$

Let us consider the following vector optimization problem (FMP).

3. Primal necessary efficiency conditions

To derive second-order optimality conditions for (FMP) we introduce the following assumptions.

Assumption 3.1. (a) For $\overline{x} \in M$, f_k, g_k, h_i are locally Lipschitz and regular in the sense of Clarke, and Gâteaux differentiable at $\overline{x}$ with the directional derivatives $f'_k(\overline{x})$, $g'_k(\overline{x})$, $h'_i(\overline{x})$;

(b) h_j $(j \in L)$ are functions of the class $\mathcal{C}^2$ in a neighborhood of $\overline{x}$; C is convex.

Theorem 3.2. *Let $\overline{x} \in M$ be a weakly efficient solution of (MFP), and let Assumption* 3.1 *be fulfilled. Then, for every critical direction $u \in Ker\nabla\ell(\overline{x})$, there is no $v \in T^2_{\overline{x}}C$ with the associated vector u satisfying the following system:*

$$f'_k(\overline{x}; v) - \phi_k(\overline{x})g'_k(\overline{x}; v) + f^{00}_{k,-}(\overline{x}; u) - \phi_k(\overline{x})g^{00}_{k,+}(\overline{x}; u) < 0 \quad (\forall k \in J), \tag{1}$$

$$h'_i(\overline{x}; v) + h^{00}_{i,-}(\overline{x}; u) < 0 \quad (\forall i \in I_0(\overline{x}; u)), \tag{2}$$

$$\nabla\ell_j(\overline{x})v + \nabla^2\ell_j(\overline{x})(u,u) = 0 \quad (\forall j \in L). \tag{3}$$

Proof. Since $\overline{x} \in M$ is a local weak efficient solution of *(MFP)*, there exists a neighborhood U of $\overline{x}$ such that

$$\phi(x) - \phi(\overline{x}) \in \mathbb{R}^p \backslash -\mathbb{R}^p_{++}, \tag{4}$$

for all $x \in U \cap M$, where $\mathbb{R}^p_{++} = int\,\mathbb{R}^p_+$.

Let us prove that $\overline{x}$ is a local weak minimal solution of

$$(P_1) \qquad \begin{cases} \min\ (f_1(x) - \phi_1(\overline{x})g_1(x), ..., f_p(x) - \phi_p(\overline{x})g_p(x)), \\ \text{subject to } x \in M, \end{cases}$$

where $\phi_i(\overline{x}) = \frac{f_i(\overline{x})}{g_i(\overline{x})}$ $(i = 1, \ldots, p)$.

If this were not so, there would exist $x_1 \in U \cap M$ such that

$$(f_i(x_1) - \phi_i(\overline{x})g_i(x_1)) - (f_i(\overline{x}) - \phi_i(\overline{x})g_i(\overline{x})) \in -\mathbb{R}_{++} \;\; (i = 1, \ldots, p).$$

Since $f_i(\overline{x}) - \phi_i(\overline{x})g_i(\overline{x}) = 0$, one has

$$\frac{f_i(x_1)}{g_i(x_1)} - \frac{f_i(\overline{x})}{g_i(\overline{x})} \in -\mathbb{R}_{++} \;\; (i = 1, \ldots, p).$$

Hence, $\phi(x_1) - \phi(\overline{x}) \in \mathbb{R}^p_{++}$, which contradicts (4).

We set $\varphi_k(x) := f_k(x) - \phi_k(\overline{x})g_k(x)$. We invoke Theorem 3.1 by Luu [15] to obtain that for every critical direction $u \in Ker\nabla\ell(\overline{x})$ there is no $v \in T^2_{\overline{x}}C$ with the associated vector u satisfying the following system:

$$\varphi'_k(\overline{x}; v) + \varphi^{00}_{k,+}(\overline{x}; u) < 0 \qquad (\forall k \in J), \tag{5}$$

$$h'_i(\overline{x}; v) + h^{00}_{i,+}(\overline{x}; u) < 0 \qquad (\forall i \in I_0(\overline{x}; u)), \tag{6}$$

$$\nabla\ell_j(\overline{x})v + \nabla^2\ell_j(\overline{x})(u, u) = 0 \quad (\forall j \in L). \tag{7}$$

It is easy to see that $\forall k \in J$,

$$\varphi'_k(\overline{x}; v) = f'_k(\overline{x}; v) - \phi(\overline{x})g'_k(\overline{x}; v); \tag{8}$$

$$\varphi^{00}_{k,+}(\overline{x}; u) \geq f^{00}_{k,-}(\overline{x}; u) - \phi(\overline{x})g^{00}_{k,+}(\overline{x}; u). \tag{9}$$

It follows from (8),(9) and (5) that for every critical direction $u \in \mathrm{Ker}\nabla\ell(\overline{x})$, there is no $v \in T^2_{\overline{x}}C$ with the associated vector u satisfying the following system:

$$f'_k(\overline{x}; v) - \phi_k((\overline{x})g'_k(\overline{x}; v) + f^{00}_{k,-}(\overline{x}; u) - \phi_k(\overline{x})g^{00}_{k,+}(\overline{x}; u) < 0 \qquad (\forall k \in J),$$

$$h'_i(\overline{x}; v) + h^{00}_{i,-}(\overline{x}; u) < 0 \qquad (\forall i \in I_0(\overline{x}; u)),$$

$$\nabla\ell_j(\overline{x})v + \nabla^2\ell_j(\overline{x})(u, u) = 0 \quad (\forall j \in L).$$

4. Dual second-order Fritz John necessary condition for weak efficiency

Theorem 4.1. *Let $\overline{x} \in M$ be a weakly efficient solution of* (FMP). *Assume that Assumption* 3.1 *is fulfilled. Then, for every critical direction $u \in Ker\nabla h(\overline{x})$, there exist $\lambda_k \geqslant 0$ $(k \in J)$, $\mu_i \geqslant 0$ $(i \in I)$ and ν_j $(j \in L)$, not all zero, such that*

$$\sum_{k\in J}\lambda_k(f'_k(\overline{x}; v) - \phi_k(\overline{x})g'_k(\overline{x}; v)) + \sum_{i\in I(\overline{x})}\mu_i h'_i(\overline{x})v + \sum_{j\in L}\nu_j\nabla\ell_j(\overline{x})v \geq 0 \; (\forall v \in T^2_{\overline{x}}C), \tag{10}$$

$$\sum_{k\in J}\lambda_k(f^{00}_{k,-}(\overline{x}; u) - \phi_k(\overline{x})g^{00}_{k,+}(\overline{x}; u)) + \sum_{i\in I(\overline{x})}\mu_i h^{00}_i(\overline{x}; u) + \sum_{j\in L}\nu_j\nabla^2\ell_j(\overline{x})(u, u) > 0, \tag{11}$$

$$\mu_i h_i(\overline{x}) = 0 \; (\forall i \in I), \tag{12}$$

$$\mu_i h'_i(\overline{x})u = 0 \; (\forall i \in I(\overline{x})). \tag{13}$$

Moreover, if the following regularity condition holds:

$$\text{(RC)} \qquad \sum_{j\in L}\nu_j\nabla\ell_j(\overline{x})v \geqslant 0 \; (\forall v \in T^2_{\overline{x}}C) \implies \nu_1 = \cdots = \nu_l = 0,$$

then $(\lambda, \mu) \neq 0$, where $\lambda := (\lambda_1, \ldots, \lambda_r)$, $\mu := (\mu_1, \ldots, \mu_m)$.

Remark 4.2. In case that $C = X$, it follows that $T^2_{\overline{x}}C = X$. Therefore, condition (RC) becomes

$$\sum_{j\in L}\nu_j\nabla\ell_j(\overline{x})v = 0\ (\forall v\in X) \implies \nu_1=\cdots=\nu_\ell=0.$$

Hence, if the mapping $\nabla\ell(\overline{x}) := (\nabla\ell_1,\dots,\nabla\ell_l)$ is surjective, then the regularity condition (RC) holds. □

Proof of Theorem 4.1. Since $f_k\ (k\in J), h_i\ (i\in I)$ are regular in the Clarke sense, we have $f_k^0(\overline{x};v) = f_k'(\overline{x})v\ (k\in J,\ v\in X),\ h_i^0(\overline{x};v) = h_i'(\overline{x})v\ (i\in I_0(\overline{x};u),\ v\in X)$. We set

$$B := \prod_{k\in J}[f_k'(\overline{x})-\phi(\overline{x})g_k'(\overline{x})](T^2_{\overline{x}}C)\times\prod_{i\in I_0(\overline{x};u)}h_i'(\overline{x})(T^2_{\overline{x}}C)\times\prod_{j\in L}\nabla\ell_j(\overline{x})(T^2_{\overline{x}}C),$$

$$\begin{aligned} a := \Big(&-[f^{00}_{k,-}(\overline{x};u)-\phi_k(\overline{x})g^{00}_{k,+}(\overline{x};u)],\dots,-[f^{00}_{r,-}(\overline{x};u)-\phi_r(\overline{x})g^{00}_{r,+}(\overline{x};u)],\\ &-h_1^{00}(\overline{x};u),\dots,-h^{00}_{|I_0(\overline{x};u)|}(\overline{x};u),-\nabla^2\ell_1(\overline{x})(u,u),\dots,-\nabla^2\ell_l(\overline{x})(u,u)\Big),\end{aligned}$$

where $|I_0(\overline{x};u)|$ is the capacity of $I_0(\overline{x};u)$ (the number of elements of $I_0(\overline{x},u)$). Observe that $T^2_{\overline{x}}C$ is convex, as C is convex. Hence, B is a convex cone. Taking account of Theorem 3.2, we get $a\notin B$, where $a\in\mathbb{R}^r\times\mathbb{R}^{|I_0(\overline{x};u)|}\times\mathbb{R}^\ell$. It is obvious that ri$B$ is nonempty and $a\notin$ ri$B$. We invoke Corollary 11.4.2 [21] and Theorem 11.7 [21] to deduce that there exist real numbers $\lambda_k\ (k\in J)$, $\mu_i\ (i\in I_0(\overline{x};u))$, $\nu_j\ (j\in L)$, not all zero, such that, for all $v,w,z\in T^2_{\overline{x}}C)$

$$\sum_{k\in J}\lambda_k[f_k'(\overline{x})-\phi(\overline{x})g_k'(\overline{x})]v+\sum_{i\in I_0(\overline{x};u)}\mu_ih_i'(\overline{x})w+\sum_{j\in L}\nu_j\nabla\ell_j(\overline{x})z\geqslant 0 \tag{14}$$

$$> -\sum_{k\in J}\lambda_k[f^{00}_{k,-}(\overline{x};u)-\phi_k(\overline{x})g^{00}_{k,+}(\overline{x};u)]-\sum_{i\in I_0(\overline{x};u)}\mu_ih^{00}_{i,+}(\overline{x};u)-\sum_{j\in L}\nu_j\nabla^2\ell_j(\overline{x})(u,u).$$

Hence, (10) and (11) hold.

Let us see that $\lambda_k\geqslant 0\ (\forall k\in J)$. Indeed, suppose this is false, that is $\lambda_s<0$, for some $s\in J$. We take $v,w,z\in T^2_{\overline{x}}C$. For $t>0$, one has $tv\in T^2_{\overline{x}}C$, as $T^2_{\overline{x}}C$ is a cone. For t be sufficiently large, we get the following:

$$\sum_{k\in J}\lambda_kf_k'(\overline{x})(tv)+\sum_{i\in I_0(\overline{x};u)}\mu_ig_i'(\overline{x})w+\sum_{j\in L}\nu_j\nabla h_j(\overline{x})z<0,$$

which contradicts (14). Hence, $\lambda_s\geqslant 0\ (\forall s\in J)$. By an analogous argument, we obtain $\mu_i\geqslant 0\ (\forall i\in I_0(\overline{x};u))$. By taking $\mu_i=0\ (\forall i\notin I_0(\overline{x};u))$, we obtain (11) and (12).

We now suppose that condition (RC2) holds. Let us see that $(\lambda,\mu_{I(\overline{x})})\neq 0$. In fact, if it were not so, then by (14), we should have

$$\sum_{j\in L}\nu_j\nabla h_j(\overline{x})v\geqslant 0\ (\forall v\in T^2_{\overline{x}}C).$$

Hence, $\nu_1=\cdots=\nu_\ell=0$. Thus, we arrive a contradiction with the fact that $\lambda_k\ (k\in J),\mu_i\ (i\in I(\overline{x}))$ and $\nu_j\ (j\in L)$ are not all zero. The proof is complete. □

5. Second-order Kuhn-Tucker necessary conditions

To ensure that $\lambda \neq 0$, we introduce the following second-order constraint qualification (CQ): There exist $(u_0, v_0) \in \mathrm{Ker}\nabla h(\overline{x}) \times T^2_{\overline{x}}C$ such that

$$h_i'(\overline{x})v_0 + h^{00}_{i,+}(\overline{x}, u_0) < 0 \ (\forall i \in I_0(\overline{x}; u_0)),$$
$$\nabla \ell_j(\overline{x})v_0 + \nabla^2 \ell_j(\overline{x})(u_0, u_0) = 0 \ (\forall j \in L).$$

Remark 5.1. The constraint qualification (CQ) is a generalization of the second-order Ben-Tal constraint qualification [2] in the case that the inequality constraints are twice Fréchet differentiable. □

A second-order Karush-Kuhn-Tucker necessary condition for weak efficiency can be stated as follows.

Theorem 5.2. *Let $\overline{x} \in M$ be a weakly efficient solution of* (FMP). *Assume that Assumption* 3.1 *holds. Suppose also that the regularity condition* (RC) *and the constraint qualification* (CQ) *hold. Then for every critical direction $u \in Ker\nabla h(\overline{x})$, there exist $\lambda_k \geq 0$ $(\forall\, k \in J)$, $\lambda \neq 0$, $\mu_i \geq 0\,(i \in I)$ and ν_j $(j \in L)$ such that* (10)–(13) *hold.*

Proof. Since all hypotheses of Theorem 4.1 are fulfilled, for every critical direction $u \in \mathrm{Ker}\nabla \ell(\overline{x})$, there exist $\lambda_k \geq 0$ $(\forall\, k \in J)$, $\lambda \neq 0$, $\mu_i \geq 0\,(i \in I)$ and ν_j $(j \in L)$, not all zero, such that (10)–(13) hold. Hence,

$$\sum_{k \in J} \lambda_k[(f_k'(\overline{x}) - \phi_i((\overline{x})g_i'(\overline{x}))v_0 + (f^{00}_{k,-}(\overline{x}; u_0))] - \phi_k(\overline{x})g^{00}_{k+}(\overline{x}; u_0)]$$
$$+ \sum_{i \in I_0(\overline{x};u)} \mu_i[h_i'(\overline{x})v_0 + h^{00}_{i,+}(\overline{x}; u_0)] + \sum_{j \in L} \nu_j[\nabla \ell_j(\overline{x})v_0 + \nabla^2 \ell_j(\overline{x})(u_0, u_0)] \geqslant 0.$$

Since condition (RC) holds we have $(\lambda, \mu) \neq (0, 0)$ taking account of Theorem 4.1. Hence, if $\lambda = 0$, then

$$\sum_{i \in I_0(\overline{x};u)} \mu_i[h_i'(\overline{x})v_0 + h^{00}_{i,+}(\overline{x}; u_0)] + \sum_{j \in L} \nu_j[\nabla \ell_j(\overline{x})v_0 + \nabla^2 \ell_j(\overline{x})(u_0, u_0)] \geqslant 0. \quad (15)$$

On the other hand, by (CQ), we obtain

$$\sum_{i \in I_0(\overline{x};u)} \mu_i[h_i'(\overline{x})v_0 + h^{00}_{i,+}(\overline{x}; u_0)] + \sum_{j \in L} \nu_j[\nabla \ell_j(\overline{x})v_0 + \nabla^2 \ell_j j(\overline{x})(u_0, u_0)] < 0.$$

This stands in conflict with (15). Hence, the conclusion follows. □

In case that $C = X$, we obtain the following dual second-order Karush-Kuhn-Tucker necessary conditions for weak efficiency.

Corollary 5.3. *Let $C = X$, and let $\overline{x} \in M_1$ be a weakly efficient solution of* (VEP1). *Assume that all hypotheses of Theorem* 4.1 *are fulfilled. Suppose also that $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ are linearly independent and the constraint qualification* (CQ) *hold. Then for every critical direction $u \in Ker\nabla h(\overline{x})$, there exist $\lambda_k \geqslant 0$ $(\forall k \in J)$, $\lambda \neq 0$, $\mu_i \geqslant 0$ $(i \in I)$ and ν_j $(j \in L)$ such that* (10)–(13) *hold.*

Proof. For $C = X$, we have that $T^2_{\overline{x}} C = X$. Moreover, if $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ are linearly independent, then the mapping $\nabla h(\overline{x}) := (\nabla h_1, \ldots, \nabla h_\ell)$ is surjective, and so, the regularity condition (RC2) holds. We invoke Theorem 5.1 to deduce that (10)–(13) hold. □

Theorem 5.1 is illustrated by the following example

Example 5.4. Let $X = \mathbb{R}^2$, $D = [0,1] \times [0,1]$, $\overline{x} = (0,0)$. We consider the following functions:

$$\begin{aligned} &f_1(x) = |x_1| - x_2 + 2; && f_2(x) = \sin x_1; && f_3(x) = x_1^2 + 1; \\ &g_1(x) = -x_2 + 2; && g_2(x) = e^{-x_2}; && g_3(x) = -x_2 + 2; \\ &h_1(x) = x_1(x_1 - 2); && h_2(x) = x_1 - x_2; && \ell(x) = x_2 - 2x_1. \end{aligned}$$

Consider the following multiobjective fractional programming problem:

$$(P_2) \qquad \begin{aligned} &\min \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \frac{f_3(x)}{g_3(x)} \right), \\ &\text{s.t. } \ h_i(x) \leqslant 0 \ (i = 1,2), \ \ell(x) = 0, \ x \in D. \end{aligned}$$

The feasible set of (P_2) is

$$\begin{aligned} M &= \{x = (x_1, x_2) \in D : h_1(x) \leqslant 0, h_2(x) \leqslant 0, \ell(x) = 0\} \\ &= \{(x_1, x_2) : x_2 = 2x_1, 0 \leqslant x_1 \leqslant 1\}. \end{aligned}$$

Hence, $\phi_1(\overline{x}) = 1$, $\phi_2(\overline{x}) = 0$ and $\phi_3(\overline{x}) = 1/2$ and we have $f_1(x) - \phi_1(\overline{x}) g_1(x) = |x_1|$. Thus $\overline{x}$ is a solution of the problem

$$\min\{f_1(x) - \phi_1(\overline{x}) g_1(x) : \ x \in M\}.$$

Hence, $\overline{x}$ is a solution of the problem

$$\min\left\{ \frac{f_1(x)}{g_1(x)} \ : \ x \in M \right\}.$$

Therefore, $\overline{x}$ is a weak efficient solution of problem (P_2). It can be seen that for $v = (v_1, v_2)) \in \mathbb{R}^2$, we have

$$\begin{aligned} &f_1'(0; v) = |v_1| - v_2; \quad f_1^{00}(0; v) = 0; \quad g_1'(0; v) = -v_2; \quad g_1^{00}(0; v) = 0; \\ &f_2'(0; v) = v_1; \quad f_2^{00}(0; v) = 0; \quad g_3'(0; v) = -v_2; \quad g_3^{00}(0; v) = 0; \\ &h_1'(0; v) = -2v_1; \quad h_1^{00}(0; v) = 2v_1^2; \quad h_2'(0; v) = v_1 - v_2; \quad h_2^{00}(0; v) = 0; \\ &\ell'(0; v) = v_2 - 2v_1; \quad \ell^{00}(0; v) = 0. \end{aligned}$$

One has $T_{\overline{x}} D = T_{\overline{x}} D = \mathbb{R}^2_+$. The constraint qualification (CQ) is satisfied with $v_0 = (1/2, 1) \in \mathbb{R}^2_+ = T_{\overline{x}} D$. Thus all hypotheses of Theorem 5.1 are fulfilled, and Kuhn-Tucker necessary condition (10)-(13) holds with

$$\alpha_1 = \alpha_2 = 1, \quad \alpha_3 = 2, \quad \mu_1 = \mu_2 = 1, \quad \nu = 1.$$

6. Second-order sufficient optimality conditions

Theorem 6.1. *Let $\overline{x} \in M$. Assume that C is convex, Assumption 3.1 holds, and for every nonzero critical directions $u \in T^2_{\overline{x}}C$, there exist multipliers $\lambda_k \geq 0$, $k \in K$, $\mu_i \in I \geq 0$, $\nu_j \in \mathbb{R}$ not all zero such that $\forall v \in T^2_{\overline{x}}C$,*

$$\sum_{k\in J} \lambda_k[f'_k(\overline{x}; v) - \phi_k((\overline{x})g'_k(\overline{x}; v)]v + \sum_{i\in I_0(\overline{x};u)} \mu_i h'_i(\overline{x})v + \sum_{j\in L} \nu_j \nabla \ell_j(\overline{x})v_0 \geqslant 0, \tag{16}$$

$$\sum_{k\in J} \lambda_k(f^{00}_{k,-}(\overline{x}; u) - \phi_k(\overline{x})g^{00}_{k,+}(\overline{x}; u)) + \sum_{i\in I(\overline{x})} \mu_i h^{00}_{i,-}(\overline{x}; u) + \sum_{j\in L} \nu_j \nabla^2 \ell_j(\overline{x})(u, u) > 0. \tag{17}$$

Then $\overline{x}$ is a local weak efficient solution of problem (FMP).

Proof. Assume the contrary, that $\overline{x}$ is not a local weak efficient solution for (FMP). Then there exists sequences $\{x_n\}_{n\geq 1} \subset M \setminus \{\overline{x}\}$, $x_n \to \overline{x}$ such that

$$\frac{f_k(x_n)}{g_k(x_n)} < \frac{f_k(\overline{x})}{g_k(\overline{x})}.$$

Since $g(x_n) > 0$, it follows that

$$f(x_n) - \phi(\overline{x})g(x_n) < 0. \tag{18}$$

We easily see that $x_n - \overline{x}$ is a critical direction for (FMP). In fact, since f_k is regular in Clarke's sense, it follows from (18) that

$$f^0_k(\overline{x}; x_n - \overline{x}) = f'_k(\overline{x}; x_n - \overline{x}) \leq 0. \tag{19}$$

For $i \in I(\overline{x})$, one has $h_i(x_n) - h_i(\overline{x}) \leq 0$. Since h_i is regular in Clarke's sense, we get

$$h^0_i(\overline{x}; x_n - \overline{x}) = h'_i(\overline{x}; x_n - \overline{x}) \leq 0. \qquad (20).$$

By Lemma 2.3 and Proposition 3.1 [15] $C - \overline{x} \subset T_{\overline{x}}C \cap T^2_{\overline{x}}C$, and $x - \overline{x} \in T^2_{\overline{x}}C$ with the associated vector $x_n - \overline{x}$, and

$$\nabla \ell_j(\overline{x})(x_n - \overline{x}) = 0. \tag{21}$$

Hence, $x_n - \overline{x}$ is a critical direction for (FMP). By Proposition 3.1 [15], we get

$$\nabla^2 \ell(\overline{x})(x_n - \overline{x}, x_n - \overline{x}) = 0. \tag{22}$$

Setting $t_n = ||x_n - \overline{x}||$, one gets $x_n = \overline{x} + t_n v_n$, where $v_n = \dfrac{x_n - \overline{x}}{||x_- \overline{x}||}$.
In view of Lemma 2.3, one gets $C - \overline{x} \subset T^2_{\overline{x}}C$.

Hence, $v \in T^2_{\overline{x}}C$. Therefore, by (21), one gets that

$$\begin{aligned}
&\sum_{k\in J} \lambda_k[f^{00}{}_{k,-}(\overline{x}; x_n - \overline{x}) - \phi_k(\overline{x})g^{00}{}_{k,+}(\overline{x}; x_n - \overline{x})] + \sum_{i\in I(\overline{x})} \mu_i h^{00}{}_{i,-}(\overline{x}; x_n - \overline{x}) \\
&\qquad + \sum_{j\in L} \nu_j \nabla^2 \ell_j(\overline{x})(x_n - \overline{x}, x_n - \overline{x}) \\
&= \sum_{k\in J} \lambda_k[f^{00}{}_{k,-}(\overline{x}; x_n - \overline{x}) - \phi_k(\overline{x})g^{00}{}_{k,+}(\overline{x}; x_n - \overline{x})] + \sum_{i\in I(\overline{x})} \mu_i h^{00}{}_{i,-}(\overline{x}; x_n - \overline{x}).
\end{aligned}$$

Consequently, putting

$$E := \sum_{k\in J} \lambda_k [f^{00}{}_{k,-}(\overline{x}; x_n - \overline{x}) - \phi_k(\overline{x}) g^{00}{}_{k,+}(\overline{x}; x_n - \overline{x})] + \sum_{i\in I(\overline{x})} \mu_i h^{00}{}_{i,-}(\overline{x}; x_n - \overline{x}),$$

we see that

$$\begin{aligned} E \leq \sum_{k\in J} \lambda_k \liminf_{n\to\infty} \frac{2}{t_n^2} \Big\{ &[f_k(x_n) - \phi_k(\overline{x}) g_k(x_n)] - [f_k(\overline{x}) - \phi_k(\overline{x}) g_k(\overline{x})] - t_n [[f'_k(\overline{x})(v_n) \\ &- \phi_k(\overline{x}) g'_k(\overline{x})(v_n)] \Big\} \\ + \sum_{i\in I(\overline{x})} \mu_i \liminf_{n\to\infty} \frac{2}{t_n^2} &[h_i(x_n) - h_i(\overline{x}) - t_n h'_i(\overline{x})(v_n))], \end{aligned}$$

which in turn give

$$\begin{aligned} E &\leq \liminf_{n\to\infty} \frac{2}{t_n^2} \sum_{k\in J} \Big[\lambda_k \big[f_k(x_n) - \phi_k(\overline{x}) g_k(x_n)] - [f_k(\overline{x}) - \phi_k(\overline{x}) g_k(\overline{x})] \\ &\qquad - t_n [f'_k(\overline{x})(v_n) - \phi_k(\overline{x}) g'_k(\overline{x})(v_n)] \Big] \\ &\quad - \liminf_{n\to\infty} \frac{2}{t_n} \Big\{ \sum_{k\in J} \lambda_k \{ [f'_k(\overline{x})(v_n) - \phi_k(\overline{x}) g'_k(\overline{x})(v_n)] - \sum_{i\in I(\overline{x})} \mu_i h'_i(\overline{x})(v_n) \Big\} \\ &\leq 0 \end{aligned}$$

due to (16), (18) and (20), which contradicts (17). Therefore the proof is complete. Then $\overline{x}$ is a weakly efficient solution of (FMP). □

Conclusions

We derive second-order Fritz John necessary conditions via the Palés-Zeidan second-order directional derivatives for multiobjective fractional problems involving equality, inequality and set constraints. Under a constraint qualification, Kuhn-Tucker conditions for weak efficiency are also given. Second-order sufficient conditions without assumption on generalized convexity. The results obtained here are new.

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