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The Regularity Property of Compensated Convex Transforms for Semiconvex Functions of General Modulus

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We establish a general approximations theorem for semiconvex and semiconcave functions with general modulus by using the compensated convex transforms introduced by K. Zhang [*Compensated convexity and its applications*, Ann. Inst. H. Poincaré (C) Non Linear Analysis 25/4 (2008) 743–771]. For a semiconvex function f with general modulus, we show that the limit of the gradient of the upper compensated transform exists and is equal to the center of the minimal bounding sphere in the sense of H. Jung [*Über die kleinste Kugel, die eine räumliche Figur einschließt*, J. Reine Angew. Mathematik 123 (1901) 241–257] of the Fréchet subdifferential. We also prove a C^{1,ω_λ} regularity result for the upper transform of semiconvex functions with general modulus.

Keywords: Compensated convex transforms, convex function, semiconvex function, semiconcave function, linear modulus, general modulus, singularity extraction, minimal bounding sphere, Fréchet subdifferential.

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1. Introduction and main results

We apply compensated convex transforms to extract geometric singularities from general functions such as semiconvex/semiconcave functions. The terminology of semiconvex and semiconcave functions has been known for a long time and we refer to the context of Hamilton Jocobi equations [5] for further details and proofs. We study the approximations and geometric singular extractions for semiconvex and semiconcave functions by using compensated convex transforms. The question we consider in this paper is whether a general approximation theorem of semiconvex functions by the upper transform obtained in [21, Theorem 1.4 (ii)] for linear modulus still holds for semiconvex functions with general modulus.

The aim of this paper is to establish a general approximations theorem for semiconvex and semiconcave functions with general modulus by using the compensated convex transforms. The present work is motivated by [21] and our main results are more general than those obtained in [21] where only the case of linear modulus is considered. The main result is Theorem 1.3 for a semiconvex function f with general modulus, where we show that the limit of the gradient of the upper compensated transform $\lim_{\lambda\to+\infty} \nabla\mathcal{C}^u_\lambda(f)(x_0)$ exists and is equal to the centre of the minimal bounding sphere of the Fréchet subdifferential. A similar result holds for semiconcave functions with regards to the Fréchet superdifferential $\partial_+ f(x)$ of

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semiconcave functions. We also establish a C^{1,ω_λ} regularity result for the upper transform of semiconvex functions with general modulus (Theorem 1.4). Motivated by [21, Remark 3.1], which says that at that time the authors do not know whether that a version of Theorem 1.4(ii) in [21] holds for semiconvex functions with general modulus, we show that similar results hold for semiconvex functions with general modulus by using a more sophisticated approach.

Throughout the paper, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with the standard inner product $x \cdot y$ and norm $|x|$ for $x,\ y \in \mathbb{R}^n$. We denote by $B(x_0; r)$ and $\overline{B}(x_0; r)$ the open and closed balls centered at x_0 with radius $r > 0$, respectively. Also $\mathbf{co}[K]$ denotes the convex hull of K [10, 14, 15] which is the smallest convex set containing K. For a given function $f : \mathbb{R}^n \to \mathbb{R}$ bounded below, the convex envelope $\mathbf{Co}[f]$ of f is the largest convex function whose graph lies below f [10, 14, 15]. We denote the class of globally *Hölder* continuous functions by C^α, and the class of functions whose gradient is *Hölder continous* by $C^{1,\beta}$ [9] and the upper/lower semicontinuous envelope of f by $\overline{f}(x), \underline{f}(x)$ see [10, 14, 15].

Let us first recall the notions of quadratic compensated convex transforms for bounded functions in $\mathbb{R}^n$ defined in [1, 2, 17, 19, 20]. For the definition of compensated convex transforms for functions under more general growth conditions, see [17, Definition 1.1].

Let $f : \mathbb{R}^n \to \mathbb{R}$ satisfy the growth condition

$$|f(x)| \le C_f|x|^2 + C_1, \tag{1}$$

for some constants $C_f > 0,\ C_1 > 0$ and for all $x \in \mathbb{R}^n$.

The *quadratic lower compensated convex transform* for f (lower transform for short) is defined by

$$\mathcal{C}^l_\lambda(f)(x) = \mathbf{Co}[\lambda|\cdot|^2 + f(\cdot)](x) - \lambda|x|^2, \quad x \in \mathbb{R}^n,\ \text{for } \lambda > C_f, \tag{2}$$

where the part $\mathbf{Co}[\lambda|\cdot|^2 + f(\cdot)](x)$ is the value of the convex envelope of the function $y \mapsto f(y) + \lambda|y|^2$ at $x \in \mathbb{R}^n$ [10, 14, 15].

The *quadratic upper compensated convex transform* for f (upper transform for short) is defined by

$$\mathcal{C}^u_\lambda(f)(x) := -\mathbf{Co}[\lambda|\cdot|^2 - f(\cdot)](x) + \lambda|x|^2, \quad x \in \mathbb{R}^n,\ \text{for } \lambda > C_f. \tag{3}$$

The two *quadratic mixed compensated convex transforms* for f (mixed transforms for short) are defined, respectively, by $\mathcal{C}^u_\lambda(\mathcal{C}^l_\tau(f))$, and $\mathcal{C}^l_\lambda(\mathcal{C}^u_\tau(f))$ when $\lambda,\ \tau > C_f$. It is easy to see under our growth condition for f that $\mathcal{C}^u_\lambda(f)(x) = -\mathcal{C}^l_\lambda(-f)(x)$.

In this paper we will consider semiconvex and semiconcave functions, which are defined as follows [5, 21, 16].

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ be a non-empty open convex domain.

(i) We say that a function $f : \Omega \to \mathbb{R}$ is *semiconvex* if there is exsist a nondecrasing upper semicontinous function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim\limits_{t\to 0^+} \omega(t) = 0$, and

$$sf(x) + (1-s)f(y) - f(sx + (1-s)y) \ge -s(1-s)|x-y|\omega(|x-y|) \tag{4}$$

for all $x, y \in \Omega$ and for all $0 \le s \le 1$ where ω is the modulus of semiconvexity for f in Ω.

(ii) A function $f : \Omega \to \mathbb{R}$ is *semiconcave* with modulus ω in Ω if $-f$ is semiconvex with modulus ω.

(iii) When there is some constant $\lambda_0 \geq 0$ such that $\omega(r) = \lambda_0 r$ for all $r \geq 0$, we say that $f : \Omega \to \mathbb{R}$ is $2\lambda_0$-*semiconvex* with linear modulus. This is equivalent to that there is a convex function $g : \Omega \to \mathbb{R}$ such that $f(x) = g(x) - \lambda_0 |x|^2$ for all $x \in \Omega$ [5, 16].
Also, a function $f : \Omega \to \mathbb{R}$ is $2\lambda_0$-semiconcave with linear modulus if $-f$ is $2\lambda_0$-semiconvex with linear modulus. In this case, there is a concave function $g : \Omega \to \mathbb{R}$ such that $f(x) = g(x) + \lambda_0 |x|^2$ for all $x \in \Omega$ [5, 16].

(iv) A function $f : \Omega \to \mathbb{R}$ is called *locally semiconvex* (respectively, *locally semiconcave*) in Ω if, on every compact convex subset $K \subset \Omega$, f is semiconvex (respectively, semiconcave) with a modulus ω_K depending on K.

(v) A function $f : \Omega \to \mathbb{R}$ is called *locally linearly semiconvex* (respectively, *locally linearly semiconcave*) with linear modulus if for every convex compact subset $K \subset \Omega$, there is a constant $\lambda_K \geq 0$ and a convex function (respectively, concave function) $g_K : K \to \mathbb{R}$ such that we have $f(x) = g_K(x) - \lambda_K |x|^2$ (respectively, $f(x) = g_K(x) + \lambda_K |x|^2$) for all $x \in K$.

We will need also the following result on the minimal bounding sphere for a compact set in $\mathbb{R}^n$ [18, Lemma 1.2, p. 4].

Lemma 1.2. *Let $K \subset \mathbb{R}^n$ be a compact set. The following hold.*

(i) *There is a unique closed ball $\overline{B}(y_0, r)$ containing K in the sense that $\overline{B}(y_0, r)$ is the closed ball containing K with the smallest radius. The sphere $S_r(y_0) := \partial \overline{B}(y_0, r)$ is called the minimal bounding sharer of K.*

(ii) *Let d be the diameter of K , then $r \leq \sqrt{\dfrac{n}{2(n+1)}}\, d$.*

(iii) *The center y_0 of the ball satisfies $y_0 \in \boldsymbol{co}[K \cap S_r(y_0)]$, the convex hull of $K \cap S_r(y_0)$.*

The following theorem is our main result on approximations and geometric singular extraction of semiconvex functions by the upper transform. The result is related to the Fréchet subdifferential $\partial_- f(x)$ of semiconvex functions. For its definition, we refer to Definition 2.10 and to its characterization Proposition 2.16.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a non empty open convex domain. Assume that $f : \Omega \to \mathbb{R}$ is a bounded and semiconvex function with modulus ω and let $x_0 \in \Omega$ be a singular (non-differentiable) point of f. Then*

$$\lim_{\lambda \to +\infty} \nabla \mathcal{C}^u_\lambda(f)(x_0) = y_0, \tag{5}$$

where $y_0 \in \partial_- f(x_0)$ is the center of the minimal bounding sphere of $\partial_- f(x_0)$.

The minimal bounding sphere of $\partial_- f(x_0)$ is the boundary of the smallest ball containing the set of $\partial_- f(x_0)$ so the ball has the center y_0. A similar result holds for semiconcave functions with the difference that we have

$$\lim_{\lambda \to +\infty} \nabla \mathcal{C}^l_\lambda(f)(x_0) = y_0,$$

with $y_0 \in \partial_+ f(x_0)$ the center of the minimal bounding sphere of Fréchet superdifferential of the semiconcave function f at x_0.

We also have the following C^{1,ω_λ} regularity result for the upper transform of semiconvex functions with general modulus.

Theorem 1.4. *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded function, satisfying $|f(x)| \leq M$, for some constant $M > 0$, and is semiconvex with modulus ω. Then for $\lambda > 0$ sufficiently large, $\mathcal{C}^u_\lambda(f) \in C^{1,\omega_\lambda}(\mathbb{R}^n)$ in the sense that*

$$|\nabla \mathcal{C}^u_\lambda(f)(y) - \nabla \mathcal{C}^u_\lambda(f)(x)| \leq \omega_\lambda(|y-x|), \forall x, y \in \mathbb{R}^n, \tag{6}$$

where $\omega_\lambda(t) = 16\lambda t + 4\omega(4t), t \geq 0$.

As an immediate consequence of Theorem 1.4, we obtain the following local C^{1,ω_λ} regularity result.

Corollary 1.5. *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and is a semiconvex function with modulus ω in $\overline{B}(0; r)$ for some $r > 0$. Then for $\lambda > 0$ sufficiently large, $\mathcal{C}^u_\lambda(f) \in C^{1,\omega_\lambda}(\overline{B}(0; r/2))$ in the sense of (6) with the restriction that $x, y \in \overline{B}(0; r/2)$.*

We can derive some precise estimates if we take a simple explicit modulus. If we consider the special case $\omega(t) = \lambda_0 t^\alpha$ for $\lambda_0 > 0$, $0 < \alpha < 1$, then $\mathcal{C}^u_\lambda(f) \in C^{1,\alpha}(\mathbb{R}^n)$ for large $\lambda > 0$.

Corollary 1.6. *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and is a semiconvex function in $\overline{B}(0; r)$ for some $r > 0$ with modulus $\omega(t) = \lambda_0 t^\alpha$ for some $\lambda_0 > 0$, $0 < \alpha < 1$. Then we have for $\lambda > 0$ sufficiently large, $\mathcal{C}^u_\lambda(f) \in C^{1,\alpha}(\overline{B}(0; r/2))$ in the sense that*

$$|\nabla \mathcal{C}^u_\lambda(f)(y) - \nabla \mathcal{C}^u_\lambda(f)(x)| \leq L_{\lambda,\alpha}|y-x|^\alpha, \ x, y \in \overline{B}(0; r/2), \tag{7}$$

with $L_{\lambda,\alpha} = 16\left(\lambda r^{1-\alpha} + \lambda_0\right)$.

We need the following preparations for the proof of our main results.

Lemma 1.7. [10, Lemma 2.1.1] *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a semiconvex function and $\sigma_0(x) = \max\{p \cdot x, p \in S\}$ is the support function of $S = \partial_- f(0)$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $|x| \leq \delta$ implies*

$$|f(x) - f(0) - \sigma_0(x)| \leq \epsilon |x|. \tag{8}$$

Remark 1.8. We can define the support function

$$\sigma : x \in \mathbb{R}^n \to \sigma_x(h) = \max\{p \cdot h, p \in \partial_- f(x)\},$$

for semiconvex functions [6], where the subdifferential $\partial_- f(x)$ is a non-empty, compact, convex subset of $\mathbb{R}^n$ [3]. By a similar argument as in the proof of [10, Lemma 2.1.1], we can show that

$$\lim_{t \to 0^+, h' \to h} \frac{f(x + th') - f(x)}{t} = \sigma_x(h),$$

where the left member defines the directional Hadamard derivative of f at x in the direction $h \in \mathbb{R}^n$, and $|h| = 1$ [6].

Lemma 1.9. *Assume that Ω is a non-empty open cconvex set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is a bounded function. If $g \le f \le h$, in $\overline{B}(x_0, r) \subset \Omega$ where g and h are globally Lipschitz functions, then for $x \in \overline{B}(x_0, \frac{r}{2})$,*

$$\mathcal{C}^u_\lambda(g)(x) \le \mathcal{C}^u_\lambda(f)(x) \le \mathcal{C}^u_\lambda(h)(x), \tag{9}$$

whenever
$$\lambda > \left(\frac{L(2+\sqrt{2})}{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}} \right)^2 \quad \textit{and} \quad \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{\sqrt{\lambda}} < \frac{r}{2}, \tag{10}$$

where $\overline{f}$ is the upper semicontinuous envelope of f and $L \ge 0$ is an upper bound of the Lipschitz constants for both g and h.

Remark 1.10. If $\operatorname{osc}(f) = 0$ in $\overline{B}(x_0, r)$, then f is a constant in $\overline{B}(x_0, r)$ so $f = \overline{f}$ in $\overline{B}(x_0, r)$, where $\overline{f}$ is the upper semicontinuous envelope of f. So we can prove the result easily. For semiconcave functions, we have a similar result for the lower transform. The only change is that $\overline{f}$ is to be replaced by $\underline{f}$ where $\underline{f}$ is the lower semicontinuous envelope of f.

Lemma 1.11. *Assume $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a bounded and semiconvex function with modulus ω and let $x_0 \in \Omega$ be a singular point of f. Then the gradient of the upper transform of f at x_0 is bounded.*

The plan of the rest of the paper is as follows. In Section 2, we introduce some further preliminary results which are needed for the proofs of our main results. We prove our results in Section 3.

2. Preliminary results

We first collect some basic results from convex analysis which will be used in the following, and refer to [10, 14, 15] for further details and proof.

Definition 2.1. [7, 15] For a function $f : \mathbb{R}^n \to \mathbb{R}$ bounded below, we define its *convex envelope* at $x \in \mathbb{R}^n$, denoted by $\mathbf{Co}[f]$, by

$$\mathbf{Co}[f](x) = \sup\{g(x),\ g(y) \le f(y),\ \text{for all } y \in \mathbb{R}^n,\ g \text{ is convex}\}.$$

We often use the following characterization of convex envelope in the proofs.

Proposition 2.2. [19, 15] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded below and $x_0 \in \mathbb{R}^n$. Then*

(i) *The convex envelope $\boldsymbol{Co}[f](\cdot)$ of f at $x_0 \in \mathbb{R}^n$ is given by*

$$\boldsymbol{Co}[f](x_0) = \inf\left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) : \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i x_i = x_0,\ \lambda_i \ge 0,\ x_i \in \mathbb{R}^n \right\}. \tag{11}$$

In addition, if the function f is lower semicontinuous and coercive in the sense that $f(x)/|x| \to +\infty$ as $|x| \to +\infty$, then the infimum can be attained by some

(λ_i^*, x_i^*) *for* $0 < \lambda_i^* \le 1$, $x_i^* \in \mathbb{R}^n$, *for* $i = 1, \ldots, k$ *with* $1 \le i \le k \le n+1$, *satisfy* $\sum_{i=1}^k \lambda_i^* = 1$, *and* $\sum_{i=1}^k \lambda_i^* x_i^* = x_0$, *such that*

$$\boldsymbol{Co}[f](x_0) = \sum_{i=1}^k \lambda_i^* f(x_i^*).$$

(ii) *We can replace the general convex functions g in Definition* 2.1 *by affine functions* (*see* [17, 19]):

$$\boldsymbol{Co}[f](x_0) = \sup\left\{\ell(x_0) : \ \ell \text{ affine and } \ell(y) \le f(y) \text{ for all } y \in \mathbb{R}^n\right\}.$$

In particular if f is lower semicontinuous and coercive, then the sup *is attained by an affine function $\ell^* \in \mathbb{R}^n$.*

We recall the definition of modulus of continuity with some of its properties.

Definition 2.3. [8, 16] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function in $\mathbb{R}^n$. Then $\omega_f : [0, \infty) \to [0, \infty)$ defined by

$$\omega_f(t) = \sup\{|f(x) - f(y)| : x,\ y \in \mathbb{R}^n \text{ and } |x - y| \le t\}$$

is called the *modulus of continuity* of f. In particular, there is a concave continuous majorant $\tilde{\omega}_f$ of ω_f of f satisfying that $\tilde{\omega}_f(0) = 0$, $\omega_f(t) \le \tilde{\omega}_f(t)$, and $\tilde{\omega}_f \le a^* t + b^*$ for some constants $a^* > 0,\ b^* \ge 0$.

The modulus of continuity of f has the following properties.

Proposition 2.4. *[8, 16] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function in $\mathbb{R}^n$. Then the modulus of continuity ω_f of f satisfies the following properties:*

(i) $\omega_f(t) \longrightarrow \omega(0) = 0, \quad$ *as* $t \longrightarrow 0$.

(ii) ω_f *is non-negative, non-decreasing and continuous on* $[0, \infty)$.

(iii) ω_f *is subadditive* $\omega_f(t_1 + t_2) \le \omega_f(t_1) + \omega_f(t_2)$ *for all* $t_1, t_2 \ge 0$.

Next we list some basic properties of compensated convex transforms (see [17, 19, 20] for further details and proofs).

The lower and upper transforms are related to each other when $\lambda > 0$ by the following relation [17]

$$\mathcal{C}_\lambda^l(f)(x) = -\mathcal{C}_\lambda^u(-f)(x),$$

when both can be defined. Furthermore, the ordering property of compensated convex transform [17] holds for $x \in \mathbb{R}^n$, $0 < \lambda < \tau$,

$$\mathcal{C}_\lambda^l(f)(x) \le \mathcal{C}_\tau^l(f)(x) \le f(x) \le \mathcal{C}_\tau^u(f)(x) \le \mathcal{C}_\lambda^u(f)(x), \tag{12}$$

and for $f \le g$ in $\mathbb{R}^n$, we have that [17, 19]

$$\mathcal{C}_\lambda^l(f)(x) \le \mathcal{C}_\lambda^l(g)(x) \quad \text{and} \quad \mathcal{C}_\lambda^u(f)(x) \le \mathcal{C}_\lambda^u(g)(x),\ x \in \mathbb{R}^n. \tag{13}$$

In addition, the compensated convex transforms are affine invariant [19], that is,

$$\mathcal{C}^l_\lambda(f+\ell)(x) = \mathcal{C}^l_\lambda(f)(x) + \ell(x) \quad \text{and} \quad \mathcal{C}^u_\lambda(f+\ell)(x) = \mathcal{C}^u_\lambda(f)(x) + \ell(x), \tag{14}$$

where ℓ is any affine function and we also have [17, Theorem 2.1(iii)]

$$\mathcal{C}^u_\tau(\mathcal{C}^u_\lambda(f)) = \begin{cases} \mathcal{C}^u_\lambda(f), & \text{if } \tau \geq \lambda, \\ \mathcal{C}^u_\tau(f), & \text{if } \tau \leq \lambda; \end{cases} \quad \text{and} \quad \mathcal{C}^l_\tau(\mathcal{C}^l_\lambda(f)) = \begin{cases} \mathcal{C}^l_\lambda(f), & \text{if } \tau \geq \lambda, \\ \mathcal{C}^l_\tau(f), & \text{if } \tau \leq \lambda. \end{cases} \tag{15}$$

We also recall the following properties. If f is a continuous function with quadratic growth [17, Theroem 2.3(i)], then

$$\lim_{\lambda\to\infty} \mathcal{C}^l_\lambda(f)(x) = f(x) \quad \text{and} \quad \lim_{\lambda\to\infty} \mathcal{C}^u_\lambda(f)(x) = f(x), x \in \mathbb{R}^n.$$

Also, if f and g are both bounded real valued functions in $\mathbb{R}^n$ [19, Proposition 2.9], then for $\lambda > 0$, and $\tau > 0$, we have

$$\mathcal{C}^l_{\lambda+\tau}(f+g)(x) \geq \mathcal{C}^l_\lambda(f)(x) + \mathcal{C}^l_\tau(g)(x) \quad \text{and} \quad \mathcal{C}^u_{\lambda+\tau}(f+g)(x) \leq \mathcal{C}^l_\lambda(f)(x) + \mathcal{C}^l_\tau(g)(x).$$

The following property of compensated convex transforms in [19, Proposition 3.1] will also be used:

$$\mathcal{C}^u_\lambda(\overline{f})(x) = \mathcal{C}^u_\lambda(f)(x), \tag{16}$$

$$\mathcal{C}^l_\lambda(\underline{f})(x) = \mathcal{C}^l_\lambda(f)(x), \tag{17}$$

where $\overline{f}$ and $\underline{f}$ are the upper and lower semicontinuous envelopes of f [14, 10].

We will use the following simple translation-invariance property in our proofs in order to allow us to consider the point $x_0 = 0$ without loss of generality.

Proposition 2.5. [19, Proposition 2.10] (Translation-invariance property) *For any function $f : \mathbb{R}^n \to \mathbb{R}$ bounded below and for any affine function $\ell : \mathbb{R}^n \to \mathbb{R}$, we have $\boldsymbol{Co}[f+\ell] = \boldsymbol{Co}[f] + \ell$. Consequently, both $\mathcal{C}^l_\lambda(f)(x)$ and $\mathcal{C}^u_\lambda(f)(x)$ are translation invariant against the weight function, that is:*

$$\mathcal{C}^l_\lambda(f)(x) = \boldsymbol{Co}[\lambda|\cdot - x_0|^2 + f(\cdot)](x) - \lambda|x - x_0|^2,$$

$$\mathcal{C}^u_\lambda(f)(x) = -\boldsymbol{Co}[\lambda|\cdot - x_0|^2 - f(\cdot)](x) + \lambda|x - x_0|^2,$$

for all $x \in \mathbb{R}^n$ and for every fixed $x_0 \in \mathbb{R}^n$. In particular, at x_0

$$\mathcal{C}^l_\lambda(f)(x_0) = \boldsymbol{Co}[\lambda|\cdot - x_0|^2 + f(\cdot)](x_0), \quad \mathcal{C}^u_\lambda(f)(x_0) = -\boldsymbol{Co}[\lambda|\cdot - x_0|^2 - f(\cdot)](x_0).$$

Above and below the upper/lower semicontinous envelope of f, say $\overline{f}$ and $\underline{f}$, are defined as

$$\overline{f}(x) = \limsup_{y\to x} f(y) \text{ and } \underline{f}(x) = \liminf_{y\to x} f(y).$$

We give the definitions of general global $C^\alpha, C^{1,\beta}$, and Lipschitz functions as follows:

Definition 2.6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$.

(i) We say that f is a C^α-function (*Hölder continuous*) in $\mathbb{R}^n$ for some $0 < \alpha < 1$ if there is a constant $L > 0$ such that for all $x,\ y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \le L|x - y|^\alpha.$$

(ii) We say that f is a *globally Lipschitz function* in $\mathbb{R}^n$ if there is a constant $L > 0$ such that for all $x,\ y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \le L|x - y|.$$

(iii) We say that f is a $C^{1,\beta}$*-function* in $\mathbb{R}^n$ for some $0 < \beta < 1$ if there is a constant $L > 0$ such that for all $x,\ y \in \mathbb{R}^n$,

$$|Df(x) - Df(y)| \le L|x - y|^\beta.$$

(iv) We say that f is $C^{1,1}$*-function* in $\mathbb{R}^n$ if there is a constant $L > 0$ such that for all $x, y \in \mathbb{R}^n$, $|Df(x) - Df(y)| \le L|x - y|$.

We need the following definition from [19].

Definition 2.7. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and let $x_0 \in \mathbb{R}^n$.

(i) We say that f is *locally* C^α at x_0 for some $0 < \alpha \le 1$ if there are constants $\delta > 0$ and $L > 0$ such that

$$|f(x) - f(x_0)| \le L|x - x_0|^\alpha \quad \text{whenever } |x - x_0| \le \delta.$$

(ii) If $\alpha = 1$ in the inequality above, we say that f is *locally Lipschitz* at x_0.

(iii) We say f is *locally* C^α at x_0 for some $1 < \alpha < 2$ if f is differentiable in $\overline{B}(x_0; \delta)$and there are constants $\delta > 0$ and $L > 0$ such that

$$|Df(x) - Df(x_0)| \le L|x - x_0|^{\alpha-1} \quad \text{whenever } |x - x_0| \le \delta.$$

In fact if we let $\beta = \alpha - 1$, our definition gives the local $C^{1,\beta}$ property of f at x_0.

The following estimates for lower transform can be found in [19, Theorem 2.12], similar results hold for the upper transform.

Theorem 2.8. (i) *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous, f maps bounded sets to bounded sets and satisfies $f(x) \ge -A(1 + |x|^2)$ for some constant $A \ge 0$. If at some $x_0 \in \mathbb{R}^n$, f is locally C^α at x_0 for $0 < \alpha < 1$ with constant $L \ge 0$, or $C^{1,\beta}$ with $\alpha = 1 + \beta$ and $1 < \alpha < 2$ then for $\lambda > 0$ sufficiently large,*

$$\mathcal{C}^l_\lambda(f)(x_0) \le f(x_0) \le \mathcal{C}^l_\lambda(f)(x_0) + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{18}$$

(ii) *Suppose the $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous , f maps bounded sets to bounded sets and satisfies $f(x) \ge -A(1 + |x|^2)$ for some constant $A \ge 0$. If at some $x_0 \in \mathbb{R}^n$, f is differentiable, then for any $\epsilon > 0$, there is a $\Lambda > 0$, such that when $\lambda \ge \Lambda$,*

$$\mathcal{C}^l_\lambda(f)(x_0) \le f(x_0) \le \mathcal{C}^l_\lambda(f)(x_0) + \frac{\epsilon^2}{4\lambda}. \tag{19}$$

(iii) *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a globally Lipschitz function with Lipschitz constant $L \geq 0$. Then then for $\lambda > 0$ and for $x \in \mathbb{R}^n$,*

$$\mathcal{C}^l_\lambda(f)(x) \leq f(x) \leq \mathcal{C}^l_\lambda(f)(x) + \frac{L^2}{4\lambda}. \tag{20}$$

We also need the following locality result under various global continuity /smoothness assumptions obtained in [13].

Theorem 2.9. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the growth condition* (1) *and is a function under the continuity/smoothness assumptions below. Let $\lambda > 0$. Then for every $x_0 \in \mathbb{R}^n$, the lower transform of f at x_0 is given by*

$$\mathcal{C}^l_\lambda(f)(x_0) = \boldsymbol{Co}_{\overline{B}(x_0;\boldsymbol{R}_\lambda)}[f(\cdot) + \lambda|\cdot - x_0|^2](x_0), \tag{21}$$

where the radius $\boldsymbol{R}_\lambda \geq 0$ depends on the following continuity/smoothness conditions respectively.

(i) [21, Proposition 2.3] *If f is a Lipschitz function with Lipschitz constant $L > 0$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^L_\lambda = \frac{L(2+\sqrt{2})}{\lambda},$$

(ii) *If f is a C^α-function for $0 < \alpha < 1$ with a constant $L > 0$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^\alpha_\lambda = 2^{\left(\frac{5-2\alpha}{2-\alpha}\right)} \left(\frac{L}{\lambda}\right)^{\left(\frac{1}{2-\alpha}\right)},$$

(iii) *If f is a $C^{1,\beta}$-function for $0 < \beta < 1$ with $\alpha = 1 + \beta$ and a constant $L > 0$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^{1,\beta}_\lambda = (M(\alpha))^{1/(2-\alpha)} \left(\frac{L}{\lambda}\right)^{1/(2-\alpha)},$$

where $M(\alpha) > 0$ is a constant depending on $1 < \alpha < 2$.

(iv) *If f is a bounded lower semicontinuous function with its oscillation in $\mathbb{R}^n$ defined by* $\operatorname{osc}(f) = \sup(f) - \inf(f)$, *then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^B_\lambda = \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(f)}}{\sqrt{\lambda}},$$

(v) *If f is a bounded uniformly continuous function in $\mathbb{R}^n$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^U_\lambda = \frac{3\sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}(1+\sqrt{2})\right)} + 2\sqrt{\tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)}}{\sqrt{\lambda}},$$

where $\tilde{\omega}_f(\cdot)$, a^*, b^* are given by Definition 2.3.

(vi) *If f is a differentiable function, and its gradient is bounded and uniformly continuous, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^D_\lambda = \frac{4\ \omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)}{\lambda},$$

where ω_{Df} is the modulus of continuity of Df.

(vii) [17, Theorem (2.3,v)] *If f is $C^{1,1}$ with Lipschitz constant L for Df such that*

$$|Df(x) - Df(y)| \leq L|x - y|,$$

then $\mathcal{C}^l_\lambda(f)(x) = f(x)$ for all $x \in \mathbb{R}^n$ whenever $\lambda \geq L$, that is, $\boldsymbol{R}_\lambda = 0$ for $\lambda \geq L$.

Next we recall from [21, Definition 2.1] and [4] the following definition.

Definition 2.10. We say that $f : \Omega \to \mathbb{R}$ is *(Fréchet) upper semi-differentiable* (respectively, *lower semi-differentiable*) at $x \in \mathbb{R}^n$ if there exists $u \in \mathbb{R}^n$, such that

$$\limsup_{y \to x} \frac{f(x) - f(y) - \langle u, y - x\rangle}{|y - x|} \leq 0 \tag{22}$$

resp.:

$$\liminf_{y \to x} \frac{f(x) - f(y) - \langle u, y - x\rangle}{|y - x|} \leq 0. \tag{23}$$

The *(Fréchet) superdifferential* of f at x, denoted by, $\partial_+ f(x)$, is the set of vectors $u \in \mathbb{R}^n$ satisfying (22) and the *(Fréchet) subdifferential* of f at x, denoted by $\partial_- f(x)$, is the set of vectors $u \in \mathbb{R}^n$ satisfying (23).

We recall from [10, Defintion 1.1.1, Chapter D] the following definition

Definition 2.11. The *directional derivative* of f at x in the direction h (when it exists) is

$$f'(x, h) = \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}$$

Definition 2.12. [14, 16] Given a non empty compact set in $\mathbb{R}^n$, its *support function* is defined by $\sigma(x) = \max\{x \cdot p,\ p \in K\}$ for all $x \in \mathbb{R}^n$. A support function is also called *sublinear function* [10].

We conclude this section by recalling some propertie of semiconvex functions we need in our proofs.

We first recall from [5, 16] the following theorem.

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^n$ be a non-empty open convex domain. Any semiconvex/semiconcave function $f : \Omega \to \mathbb{R}$ is locally Lipschitz continuous in Ω.*

The following theorem [5, Theorem 3.3.7, p. 60] is valid for any modulus. More precisely, this theorem gives an estimate of the modulus of continuity of Df.

Theorem 2.14. *If $f : \Omega \to \mathbb{R}$ is both semiconcave and semiconvex function in Ω, then $f \in C^1(\Omega)$. In addition, on each compact subset of Ω, the modulus of continuity of Df is in the form $\omega_1(t) = c_1\omega(c_2 t)$, where ω is a modulus of semiconcvaity and seniconvexity for f and $c_1, c_2 > 0$ are constants.*

We also need the following result from [5, Proposition 3.3.4, p. 57] and [16, Corollary 10.31, p. 1038].

Proposition 2.15. *Let Ω be a nonempty open convex set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a semiconcave function with a modulus ω and let $x \in \Omega$. Then the following property holds true $\partial_+ f(x) \neq \phi$.*

The following property of subdifferential of semiconvex function is taken from [3, Proposition 2.1, p. 4] and [16, Theorem 10.29, p. 1037].

Proposition 2.16. *Let Ω be an open convex set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a semiconvex function in Ω. Then f is locally Lipschitz continuous in Ω, and a vector $p \in \mathbb{R}^n$ belongs to $\partial_- f(x)$ if and only if we have for any point y in Ω*

$$f(x) - f(y) - \langle p, y - x\rangle \geq -|y - x|\omega(|x - y|). \tag{24}$$

The following differentiability property is useful in our proofs [4, Corollary 2.5] and more generally [12, p. 726] and [16, Corollary 10.31].

Lemma 2.17. *If g is convex, f is upper semidifferentiable, $g \leq f$ and $g(x) = f(x)$, then g, f are both differentiable at x and $Df(x) = Dg(x)$.*

We recall from [16, Proposition 10.27] the following result.

Proposition 2.18. *Let X be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function which is semiconvex near x at which f is finite. Then the directional derivative*

$$f'(x, h) = \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}$$

exists in $\mathbb{R} \cup \{-\infty, +\infty\}$ for any directions $h \in X$ and the function $f'(x, .)$ is convex.

We need the following result from [6, Proposition 1.4, p. 248] and [16, Theorem 10.29, p. 1037].

Proposition 2.19. *Let Ω be a non-empty convex set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a semiconvex function. Then for any $h \in \mathbb{R}^n$*

$$f'(x, h) = \max\{h \cdot p,\ p \in \partial_- f(x)\}, \tag{25}$$

that is, $f'(x, \cdot)$ is the support function of $\partial_- f(x)$.

The following lemma is taken from [21, Lemma 2.5, pag 11].

Lemma 2.20. *Let $S \subset \mathbb{R}^n$ be a non-empty compact convex set, containing more than one element, and denote by $S_r(-a)$ the minimal bounding sphere of S with radius $r > 0$ and centre $-a \in \mathbb{R}^n$. Consider the support function*

$$\sigma : x \in \mathbb{R}^n \to \sigma(x) = \max\{p \cdot x, p \in S\}.$$

Then for a fixed $0 < \epsilon \leq \min\{1, r\}$ and for $\lambda > 0$, we have

$$\mathcal{C}^u_\lambda(\sigma - \epsilon|\cdot|)(0) = \frac{(r - \epsilon)^2}{4\lambda}, \tag{26}$$

$$\nabla \mathcal{C}^u_\lambda(\sigma)(0) = -a; \tag{27}$$

and for $0 < \epsilon \leq \min\{1, r\}$

$$\mathcal{C}^u_\lambda(\sigma + \epsilon|\cdot|)(0) \leq \mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma)(0) + \mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(0) = \frac{r^2}{4(1 - \epsilon)\lambda} + \frac{\epsilon}{4\lambda}, \tag{28}$$

where
$$\mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x) = \begin{cases} \epsilon\lambda|x|^2 + \frac{\epsilon}{4\lambda}, & |x| \leq \frac{1}{2\lambda}, \\ \epsilon|x|, & |x| \geq \frac{1}{2\lambda}. \end{cases} \tag{29}$$

We recall the following approximations theorem for locally semiconvex functions with linear modulus [21].

Theorem 2.21. *Assume $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a locally semiconvex function in Ω with linear modulus and let $x_0 \in \Omega$ be a singular point of f. Then*

$$\lim_{\lambda\to+\infty} \nabla \mathcal{C}^u_\lambda(f)(x_0) = y_0,$$

where $y_0 \in \partial_- f(x_0)$ is the center of the minimal bounding sphere of $\partial_- f(x_0)$.

A similar result holds for semiconcave functions with linear modulus. We also recall the following $C^{1,1}$ regularity result for the upper transform of semiconvex functions with linear modulus [21].

Theorem 2.22. *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant some $L > 0$. Assume that for some $r > 0$, f is $2\lambda_0-$semiconvex with linear modulus in the closed ball $\overline{B}(0; 2r)$, that is, $f(x) = g(x) - \lambda_0|x|^2$ for $x \in \overline{B}(0; 2r)$, where $\lambda_0 \geq 0$ is a constant and $g : \overline{B}(0; 2r) \to \mathbb{R}$ is convex. Then for $\lambda \geq \lambda_0$ sufficiently large, $\mathcal{C}^u_\lambda(f) \in C^{1,1}(\overline{B}(0; r))$ in the sense that*

$$|\nabla \mathcal{C}^u_\lambda(f)(y) - \nabla \mathcal{C}^u_\lambda(f)(x)| \leq 2\ \lambda(|y - x|),\ x, y \in \overline{B}(0; r).$$

3. Proofs of the main results

We postpone the proofs of Lemmas 1.7, 1.9 and 1.11 to the end, and with those lemmas we start with the proof of Theorem 1.4. Some features within the proof of Theorem 1.4, say the property (30), will be utilized to demonstrate Theorem 1.3.

Proof of Theorem 1.4: We show that when $\lambda > 0$ is sufficiently large, $\mathcal{C}^u_\lambda(f)$ is differentiable at x_0 and

$$-|y_0-x_0|\omega(|y_0-x_0|) \leq \mathcal{C}^u_\lambda(f)(y_0)-\mathcal{C}^u_\lambda(f)(x_0)-\nabla \mathcal{C}^u_\lambda(f)(x_0)\cdot(y_0-x_0) \leq \lambda|y_0-x_0|^2,$$

for $x_0, y_0 \in \mathbb{R}^n$. If we consider $\omega^*_\lambda(t) = \lambda t + \omega(t), t \geq 0$, then we obtain for all $x_0, y_0 \in \mathbb{R}^n$

$$|\mathcal{C}^u_\lambda(f)(y_0) - \mathcal{C}^u_\lambda(f)(x_0) - \nabla \mathcal{C}^u_\lambda(f)(x_0) \cdot (y_0 - x_0)| \leq |y_0 - x_0|\omega^*_\lambda(|y_0 - x_0|). \tag{30}$$

From (30) we see that $\mathcal{C}^u_\lambda(f)$ is both semiconvex and semiconcave with modulus ω^*_λ. Therefore by [5, Theorem 3.3.7, p. 60] (see also Theorem 2.14) we see that $\mathcal{C}^u_\lambda(f) \in C^{1,\omega^*_\lambda}(\mathbb{R}^n)$ satisfying

$$|\nabla \mathcal{C}^u_\lambda(f)(y) - \nabla \mathcal{C}^u_\lambda(f)(x)| \leq 4\ \omega^*_\lambda(4|y - x|)\ , \forall x, y \in \mathbb{R}^n. \tag{31}$$

Since f is bounded, by the locality property (Theorem 2.9 (iv)) for a bounded lower semicontinous function, when $\lambda > 0$ is sufficiently large, we have for $x_0 \in \overline{B}(0; r)$ with $|x_i| \leq \dfrac{(1 + \sqrt{2})\sqrt{\mathrm{osc}(f)}}{\sqrt{\lambda}} < r$,

$$\mathcal{C}^u_\lambda(f)(x_0) = -\mathbf{Co}[\lambda| \cdot -x_0|^2 - f(\cdot)](x_0) = -\sum_{i=1}^{k} \lambda_i[\lambda|x_i - x_0|^2 - f(x_i)], \tag{32}$$

with $2 \leq k \leq n + 1,\ \lambda_i \geq 0,\ |x_i - x_0| \leq r$ for $i = 1, \ldots, k$.

We define $g_\lambda(y) = \lambda|y - x_0|^2 - f(y)$ for $y \in \mathbb{R}^n$, and $\lambda > 0$. By [19, Proposition 2.1] (see also Proposition 2.2), there is an affine function $\ell(y) = a \cdot (y - x_0) + b$ for $y \in \mathbb{R}^n$, with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

(i) $\ell(y) \le g_\lambda(y)$ for all $y \in \mathbb{R}^n$,

(ii) $\ell(x_i) = g_\lambda(x_i)$ for $x_i \in \overline{B}(0; r)$.

Let
$$\Delta_{x_0} = \left\{ \sum_{i=1}^{k} \mu_i x_i,\ \mu_i \ge 0,\ i = 1, \dots, k, \sum_{i=1}^{k} \mu_i = 1, \right\},$$
be the simplex defined by $\{x_1, ...x_k\}$, then we see that $\mathbf{Co}[g_\lambda](y) = a \cdot (y - x_0) + b$ for $y \in \Delta_{x_0}$ as the set $U = \{(y, a \cdot y + b), y \in \Delta_0\}$ is contained in a face of convex hull of the epigraph $\mathbf{Co}[epi(g_\lambda)]$ of g_λ and
$$\{(x_1, g_\lambda(x_1)), \dots, (x_m, g_\lambda(x_m))\} \subset U \cap epi(g_\lambda).$$
Now we have $\mathbf{Co}[g_\lambda](y) \le g_\lambda(y)$ for $y \in \mathbb{R}^n$, and
$$\mathbf{Co}[g_\lambda](x_i) = g_\lambda(x_i) = a \cdot (x_i - x_0) + b,$$
for $x_i \in \overline{B}(0; r)$, $i = 1, \dots, k$ with
$$|x_i| \le \frac{(1 + \sqrt{2})\sqrt{\mathrm{osc}(f)}}{\sqrt{\lambda}} < r.$$
Since f is semiconvex with modulus ω, then by [5, Proposition 3.3.4, p. 57] (also see Proposition 2.15), f is lower semi-differentiable. Then $-f$ is upper semi-differentiable and the quadratic function automatically differentiable. Moreover,
$$g_\lambda(y) = \lambda|y - x_0|^2 - f(y)$$
is upper semi-differentiable. Thus by [4, Corollary 2.5] and [12, p. 726] (also see Lemma 2.17), we see that both $\mathbf{Co}[g_\lambda](y)$ and $g_\lambda(y)$ are differentiable at x_i and
$$\nabla\mathbf{Co}[g_\lambda](x_i) = \nabla g_\lambda(x_i) = 2\lambda(x_i - x_0) - \nabla f(x_i),$$
hence $\nabla f(x_i)$ exists for $i = 1, \dots, k$.. If we apply Lemma 2.17 again to the affine function $\ell(y)$ and the upper semi-differentiable $g_\lambda(y)$, we also have $\nabla g_\lambda(x_i) = a$, for $i = 1, \dots, k$.

Now we show that $\mathcal{C}^u_\lambda(f)$ is differentiable at x_0 and $\nabla\mathcal{C}^u_\lambda(f)(x_0) = -a$.

We know that $\mathbf{Co}[g_\lambda](x_0) = \sum_{i=1}^k \lambda_i g_\lambda(x_i)$ with $2 \le k \le n + 1$, and without loss of generality we can assume that $\lambda_1 \ge \cdots \ge \lambda_k > 0$, $|x_i - x_0| \le r$ (by the locality property), satisfying $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i x_i = x_0$. We then have
$$\lambda_1 > \frac{1}{k} \ge \frac{1}{n+1}\,.$$
Now for $y \in \mathbb{R}^n$, we have
$$x_0 + y = \lambda_1 \left(x_1 + \frac{y}{\lambda_1}\right) + \sum_{i=2}^{k} \lambda_i x_i.$$
By the convexity of $\mathbf{Co}[g_\lambda]$, we have
$$\mathbf{Co}[g_\lambda](x_0 + y) \le \lambda_1\ g_\lambda \left(x_1 + \frac{y}{\lambda_1}\right) + \sum_{i=2}^{k} \lambda_i g_\lambda(x_i).$$
Then by subtracting $\mathbf{Co}[g_\lambda](x_0) = \sum_{i=1}^k \lambda_i\ g_\lambda(x_i)$ from both sides of the inequality above we obtain for $y \in \mathbb{R}^n$

$$\begin{aligned}
&\mathbf{Co}[g_\lambda](x_0+y)-\mathbf{Co}[g_\lambda](x_0)\\
&\le \lambda_1\Big(g_\lambda\left(x_1+\frac{y}{\lambda_1}\right)-g_\lambda(x_1)\Big)+\Big(\sum_{i=2}^{k}\lambda_i[g_\lambda(x_i)-g_\lambda(x_i)]\Big)\\
&=\lambda_1\left(g_\lambda\left(x_1+\frac{y}{\lambda_1}\right)-g_\lambda(x_1)\right),
\end{aligned}$$

Since the left hand side of the above inequality is convex in y and the right hand side is upper semi-differentiable at $y=0$ and the two functions are equal at $y=0$, by Lemma 2.17, we see that $\nabla\mathbf{Co}[g_\lambda](x_0)=\nabla g_\lambda(x_1)$. Thus $\mathbf{Co}[g_\lambda]$ is differentiable at x_0.

Furthermore, since $\ell(y)\le g_\lambda(y)$ for $y\in\mathbb{R}^n$, by the definition of convex envelope we see that $\ell(y)\le\mathbf{Co}[g_\lambda](y)$ for $y\in\mathbb{R}^n$. We also have $\ell(x_0)=b=\mathbf{Co}[g_\lambda](x_0)$. Also, since $\mathbf{Co}[g_\lambda]$ is differentiable at x_0, again by Lemma 2.17, we have $\nabla\mathbf{Co}[g_\lambda](x_0)=a$. Thus $\nabla\mathcal{C}_\lambda^u(f)(x_0)=-a$. Therefore $\mathcal{C}_\lambda^u(f)$ is differentiable at x_0.

Now we prove that for all $x_0,y_0\in\mathbb{R}^n$, we have

$$\mathcal{C}_\lambda^u(f)(y_0)-\mathcal{C}_\lambda^u(f)(x_0)-\nabla\mathcal{C}_\lambda^u(f)(x_0)\cdot(y_0-x_0)\ge-|y_0-x_0|\omega(|y_0-x_0|), \tag{33}$$

so that $\mathcal{C}_\lambda^u(f)$ is semiconvex with modulus ω in $\mathbb{R}^n$. We see that (33) is equivalent to

$$\lambda|y_0-x_0|^2-\mathbf{Co}[g_\lambda](y_0)+\mathbf{Co}[g_\lambda](x_0)+\nabla\mathbf{Co}[g_\lambda](x_0)\cdot(y_0-x_0)\ge-|y_0-x_0|\omega(|y_0-x_0|),$$

which again is equivalent to

$$\begin{aligned}
&\mathbf{Co}[g_\lambda](y_0)-\mathbf{Co}[g_\lambda](x_0)-\nabla\mathbf{Co}[g_\lambda](x_0)\cdot(y_0-x_0)\\
&\le\lambda|y_0-x_0|^2+|y_0-x_0|\omega(|y_0-x_0|).
\end{aligned}\tag{34}$$

We will prove (34) above which is equivalent to (33). Note that

$$\mathbf{Co}[g_\lambda](x_0)=\sum_{i=1}^{k}\lambda_i\ g_\lambda(x_i),\ \nabla\mathbf{Co}[g_\lambda](x_0)=a,\ \nabla\mathbf{Co}[g_\lambda](x_i)=\ \nabla g_\lambda(x_i)=a.$$

Since $\forall y_0\in\mathbb{R}^n$, and $|x_i-x_0|\le r$, we see that $y_0=\displaystyle\sum_{i=1}^{k}\lambda_i\,(y_0+(x_i-x_0))$.

Thus $$\mathbf{Co}[g_\lambda](y_0)\le\sum_{i=1}^{k}\lambda_i\ g_\lambda\left(y_0+(x_i-x_0)\right).$$

We also have $$\mathbf{Co}[g_\lambda](x_0)=\sum_{i=1}^{k}\lambda_i\ g_\lambda\left(x_0+(x_i-x_0)\right),$$

and $$\begin{aligned}\nabla\mathbf{Co}[g_\lambda](x_0)\cdot(y_0-x_0)=a\cdot(y_0-x_0)&=\sum_{i=1}^{k}\lambda_i\ a\cdot(y_0-x_0)\\
&=\sum_{i=1}^{k}\lambda_i\nabla g_\lambda\left(x_0+(x_i-x_0)\right)\cdot(y_0-x_0).\end{aligned}$$

We notice that in $\mathbb{R}^n$, f is semiconvex with modulus ω and $g_\lambda(y)=\lambda|y-x_0|^2-f(y)$.

Thus $\mathbf{Co}[g_\lambda](y_0) - \mathbf{Co}[g_\lambda](x_0) - \nabla \mathbf{Co}[g_\lambda](x_0)\cdot(y_0-x_0)$

$$\le \sum_{i=1}^{k} \lambda_i \left(g_\lambda\left(y_0+(x_i-x_0)\right)-g_\lambda\left(x_0+(x_i-x_0)\right)-\nabla g_\lambda\left(x_0+(x_i-x_0)\right)\cdot(y_0-x_0)\right)$$

$$= \sum_{i=1}^{k} \lambda_i\ \lambda\left(|(y_0-x_0)+(x_i-x_0)|^2-|x_i-x_0|^2-2(y_0-x_0)\cdot(x_i-x_0)\right)$$

$$- \sum_{i=1}^{k} \lambda_i\left(f(y_0+(x_i-x_0))-f(x_0+(x_i-x_0))-\nabla f(x_0+(x_i-x_0))\cdot(y_0-x_0)\right)$$

$$\le \lambda|y_0-x_0|^2+|y_0-x_0|\omega(|y_0-x_0|).$$

Here we have used the facts that $\sum_{i=1}^{k}\lambda_i(x_i-x_0)=0$ and that f is a semiconvex function with modulus ω hence is lower semi-differentiable,

$$f(x)-f(y)-\langle p, y-x\rangle \ge -|y-x|\omega(|x-y|).$$

Hence $\mathcal{C}^u_\lambda(f)$ is semiconvex with modulus ω. Also, by the definition of the upper transform, we have, $\mathcal{C}^u_\lambda(f)$ is semiconcave with linear modulus, that is,

$$\mathcal{C}^u_\lambda(f)(y_0)-\mathcal{C}^u_\lambda(f)(x_0)-\nabla\mathcal{C}^u_\lambda(f)(x_0)\cdot(y_0-x_0)\le\lambda|y_0-x_0|^2. \tag{35}$$

Combining (33) and (35), then we have

$$-|y_0-x_0|\omega(|y_0-x_0|)\le\mathcal{C}^u_\lambda(f)(y_0)-\mathcal{C}^u_\lambda(f)(x_0)-\nabla\mathcal{C}^u_\lambda(f)(x_0)\cdot(y_0-x_0)\le\lambda|y_0-x_0|^2.$$

Furthermore, we conclude that (by adding the two bounds together)

$$\begin{aligned}-\lambda|y_0-x_0|^2-|y_0-x_0|\omega(|y_0-x_0|)&\le\mathcal{C}^u_\lambda(f)(y_0)-\mathcal{C}^u_\lambda(f)(x_0)-\nabla\mathcal{C}^u_\lambda(f)(x_0)\cdot(y_0-x_0)\\&\le\lambda|y_0-x_0|^2+|y_0-x_0|\omega(|y_0-x_0|).\end{aligned}$$

Thus, if we consider where $\omega^*_\lambda(t)=\lambda t+\omega(t), t\ge 0$, then we have

$$\begin{aligned}-|y_0-x_0|\omega^*_\lambda(|y_0-x_0|)&\le\mathcal{C}^u_\lambda(f)(y_0)-\mathcal{C}^u_\lambda(f)(x_0)-\nabla\mathcal{C}^u_\lambda(f)(x_0)\cdot(y_0-x_0)\\&\le|y_0-x_0|\omega^*_\lambda(|y_0-x_0|).\end{aligned}$$

Therefore

$$|\mathcal{C}^u_\lambda(f)(y_0)-\mathcal{C}^u_\lambda(f)(x_0)-\nabla\mathcal{C}^u_\lambda(f)(x_0)\cdot(y_0-x_0)|\le|y_0-x_0|\omega^*_\lambda(|y_0-x_0|). \tag{36}$$

We see that $\mathcal{C}^u_\lambda(f)$ is both semiconvex and semiconcave with modulus ω^*_λ. Hence, by [5, Theorem 3.3.7, p. 60] (also see Theorem 2.14), we obtain that $\mathcal{C}^u_\lambda(f)\in C^{1,\omega^*_\lambda}(\mathbb{R}^n)$ satisfying

$$|\nabla\mathcal{C}^u_\lambda(f)(y)-\nabla\mathcal{C}^u_\lambda(f)(x)|\le 4\ \omega^*_\lambda(4|y-x|)\,,\ \forall x,y\in\mathbb{R}^n.$$

We define $\omega_\lambda(|y-x|)=4\ \omega^*_\lambda(4|y-x|)=16\lambda|y-x|+4\omega(4|y-x|)$

where $\omega^*_\lambda(t)=\lambda t+\omega(t),\ t\ge 0.$

This completes the proof. □

Proof of Corollary 1.6: This is a direct consequence of Theorem 1.4, where $\omega(t) = \lambda_0 t^\alpha$ for $\lambda_0 > 0$, $0 < \alpha < 1$. In fact for some $r > 0$, if $x, y \in \overline{B}(0; r)$, then we have $|x - y| \leq r$. Then for $\lambda > 0$ sufficiently large, $\mathcal{C}^u_\lambda(f) \in C^{1,\alpha}(\overline{B}(0; r/2))$, and for $x, y \in \overline{B}(0; r/2)$, we have

$$\begin{aligned}
&|\nabla \mathcal{C}^u_\lambda(f)(y) - \nabla \mathcal{C}^u_\lambda(f)(x)| \leq \omega_\lambda(|y - x|) \leq 16\lambda|y - x| + 4\omega(4|y - x|) \\
&\leq 16\lambda|y - x| + 4\lambda_0(4|y - x|)^\alpha \leq 16\lambda|y - x| + 16\lambda_0|y - x|^\alpha \\
&\leq 16\lambda|y - x|^{1-\alpha+\alpha} + 16\lambda_0|y - x|^\alpha \leq 16\left(\lambda|y - x|^{1-\alpha} + \lambda_0\right)|y - x|^\alpha.
\end{aligned}$$

Then by using the triangle inequality, we have

$$|y - x|^{1-\alpha} \leq ||x| + |y||^{1-\alpha} \leq |r|^{1-\alpha} \leq r^{1-\alpha}.$$

Here we have used the fact that $x, y \in \overline{B}(0; r/2)$. Therefore

$$|\nabla \mathcal{C}^u_\lambda(f)(y) - \nabla \mathcal{C}^u_\lambda(f)(x)| \leq 16\left(\lambda r^{1-\alpha} + \lambda_0\right)|y - x|^\alpha, \quad x, y \in \overline{B}(0; r/2),$$

with $L_{\lambda,\alpha} = 16\left(\lambda r^{1-\alpha} + \lambda_0\right)$. The proof is finished. □

The next step is the proof of Theorem 1.3.

Proof of Theorem 1.3: Let $x_0 \in \Omega$ be a singular point of f. By the translation invariance property [19, Proposition 2.10] (also see Proposition 2.5), we may assume without loss of generality that $x_0 = 0$.

Since f is semiconvex on Ω, f is a locally Lipschitz continuous function [3, Proposition 2.1, p. 4] (also see Proposition 2.16). Following [16, Proposition 10.27] (also see Proposition 2.18, we conclude that the directional derivative of f exists and is finite.

Furthermore, $\partial_- f(0)$ is a non empty, compact set [3] (Proposition 2.16).

Let $\sigma_0(x) = \max\{p\cdot x, p \in \partial_- f(0)\}$ be the support function of $\partial_- f(0)$. By [6, Proposition 1.4, p. 248] (also see Proposition 2.19), we have that the directional derivative of f at 0 is the support function of $\partial_- f(0)$.

Now we may assume that $f(0) = 0$ in Lemma 1.7, so for every fixed $\epsilon > 0$, there is some δ with $0 < \delta \leq 1$, such that

$$|f(x) - \sigma_0(x)| \leq \epsilon|x|,$$

whenever $x \in \overline{B}(0, \delta)$. Therefore we have

$$\sigma_0(x) - \epsilon|x| \leq f(x) \leq \sigma_0(x) + \epsilon|x|,$$

for $x \in \overline{B}(0, \delta)$. Since f is bounded and $\sigma_0(x) \pm \epsilon|x|$ are globally Lipschitz functions, then for $x \in \overline{B}(0, \delta/2)$ and for sufficiently large $\lambda > 0$ satisfying the requirement of the proof of Lemma 1.9 we have by Lemma 1.9

$$\mathcal{C}^u_\lambda\left(\sigma_0 - \epsilon|\cdot|\right)(x) \leq \mathcal{C}^u_\lambda(f)(x) \leq \mathcal{C}^u_\lambda\left(\sigma_0 + \epsilon|\cdot|\right)(x). \tag{37}$$

Now we may apply Corollary 1.5 to obtain for large $\lambda > 0$, $\mathcal{C}^u_\lambda(f) \in C^{1,\omega_\lambda}(\overline{B}(0; \delta/2))$.

Let $p_\lambda = \nabla \mathcal{C}^u_\lambda(f)(0)$, by Lemma 1.11, we then have $|p_\lambda| \le c_0$ and by using (30), we obtain

$$|\mathcal{C}^u_\lambda(f)(x) - \mathcal{C}^u_\lambda(f)(0) - p_\lambda \cdot x| \le |x|\omega^*_\lambda(|x|), \tag{38}$$

for $x \in \overline{B}(0,\delta/2)$, where $\omega^*_\lambda(\cdot) = \lambda|\cdot| + \omega(\cdot)$ is the modulus of semiconvexity and semiconcavity of $\mathcal{C}^u_\lambda(f)$.

Now we show p_λ tends to $-a$ when $\lambda \to +\infty$ as follows, where $-a \in \mathbb{R}^n$ is the centre of the minimal bounding sphere of the subdifferential $\partial_- f(x_0)$ of f,

$$\begin{aligned} p_\lambda \cdot x &\le \mathcal{C}^u_\lambda(f)(x) - \mathcal{C}^u_\lambda(f)(0) + |x|\omega^*_\lambda(|x|) \\ &\le \mathcal{C}^u_\lambda\left(\sigma_0 + \epsilon|\cdot|\right)(x) - \mathcal{C}^u_\lambda\left(\sigma_0 - \epsilon|\cdot|\right)(0) + |x|\omega^*_\lambda(|x|) \\ &:= I - \frac{(r-\epsilon)^2}{4\lambda} + |x|\omega^*_\lambda(|x|), \end{aligned}$$

where $I = \mathcal{C}^u_\lambda\left(\sigma_0 + \epsilon|\cdot|\right)(x)$. In the above inequality, we have used the fact that

$$\mathcal{C}^u_\lambda(\sigma_0 - \epsilon|\cdot|)(0) = \frac{(r-\epsilon)^2}{4\lambda},$$

given by [21, Lemma 2.5 (2.5)] (also see Lemma 2.20 (26). By a similar argument to that used to show [21, Lemma 2.5 (2.7)] (see Lemma 2.20 (28)), we also have

$$I = \mathcal{C}^u_\lambda\left(\sigma_0 + \epsilon|\cdot|\right)(x) \le \mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma_0)(x) + \mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x) = J_1 + J_2.$$

Now for $a \in \mathbb{R}^n$, we conclude that

$$\begin{aligned} J_1 &= \mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma_0)(x) \\ &= \left(\mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma_0)(x) - \mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma_0)(0) + a\cdot x\right) + \mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma_0)(0) - a\cdot x \\ &\le |x|\omega^*_{(1-\epsilon)\lambda}(|x|) + \frac{r^2}{4(1-\epsilon)\lambda} - a\cdot x \\ &\le \lambda(1-\epsilon)|x|^2 + |x|\omega(|x|) + \frac{r^2}{4(1-\epsilon)\lambda} - a\cdot x. \end{aligned}$$

Here we have used (38) and the fact given by [21, Lemma 2.5 (2.7)](also see Lemma 2.20 (28)) that

$$\mathcal{C}^u_{(1-\epsilon)\lambda}(\sigma)(0) = \frac{r^2}{4(1-\epsilon)\lambda}. \tag{39}$$

Therefore when $\lambda > \max\left\{2\beta, \left((1+\sqrt{2})\sqrt{\operatorname{osc}(f)}\right)^2, \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(f)}}{\delta}\right\}$ and for large $\lambda > 0$ satisfying the requirement of the proof of Lemma 1.9, we have

$$\begin{aligned} p_\lambda \cdot x &\le \lambda(1-\epsilon)|x|^2 + |x|\omega(|x|) + \frac{r^2}{4(1-\epsilon)\lambda} - a\cdot x + \mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x) \\ &\quad - \frac{(r-\epsilon)^2}{4\lambda} + |x|\omega^*_\lambda(|x|) \\ &\le \lambda(1-\epsilon)|x|^2 + |x|\omega(|x|) + \frac{r^2}{4(1-\epsilon)\lambda} - a\cdot x + \mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x) \\ &\quad - \frac{(r-\epsilon)^2}{4\lambda} + \lambda|x|^2 + |x|\omega(|x|) \\ &\le \frac{\epsilon r^2}{4(1-\epsilon)\lambda} - a\cdot x + \mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x) + 2\lambda|x|^2 + 2|x|\omega(|x|). \end{aligned}$$

So that $$(p_\lambda + a)\cdot x \le \frac{\epsilon r^2}{4(1-\epsilon)\lambda} + \mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x) + 2\lambda|x|^2 + 2|x|\omega(|x|). \tag{40}$$

Since p_λ is bounded by c_0 (Lemma 1.11), by taking

$$x_\lambda = \frac{p_\lambda + a}{2^5(1+|a|+c_0)\lambda},$$

we have $|x_\lambda| \le \frac{1}{2^4\lambda} < \frac{\delta}{2}$ if $\lambda > \max\left\{2\beta, \left((1+\sqrt{2})\sqrt{\operatorname{osc}(f)}\right)^2, \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(f)}}{\delta}\right\}$.

Also for large $\lambda > 0$ satisfying the requirement of the proof of Lemma 1.9, furthermore, we have $|x_\lambda| < \frac{1}{2\lambda}$. So that by using explicit formula [20, Lemma 2.5 (2.8)](also see Lemma 2.20 (29)), we have

$$\mathcal{C}^u_{\epsilon\lambda}(\epsilon|\cdot|)(x_\lambda) = \epsilon\lambda|x_\lambda|^2 + \frac{\epsilon}{4\lambda}.$$

Thus, if we substitute x_λ into (40), we obtain

$$\begin{aligned}
\frac{|p_\lambda + a|^2}{2^5(1+|a|+c_0)\lambda} &\le \frac{\epsilon r^2}{4(1-\epsilon)\lambda} + \frac{|p_\lambda + a|^2}{2^9(1+|a|+c_0)^2\lambda} + \frac{\epsilon}{4\lambda} \\
&\quad + \frac{\epsilon|p_\lambda + a|^2}{2^{10}(1+|a|+c_0)^2\lambda} + \frac{|p_\lambda + a|}{2^4(1+|a|+c_0)\lambda}\omega\left(\frac{|p_\lambda + a|}{2^5(1+|a|+c_0)\lambda}\right) \\
&\le \frac{\epsilon r^2}{4(1-\epsilon)\lambda} + \frac{|p_\lambda + a|^2}{2^9(1+|a|+c_0)\lambda} + \frac{\epsilon}{4\lambda} \\
&\quad + \frac{\epsilon|p_\lambda + a|^2}{2^{10}(1+|a|+c_0)\lambda} + \frac{|p_\lambda + a|}{2^4(1+|a|+c_0)\lambda}\omega\left(\frac{|p_\lambda + a|}{2^5(1+|a|+c_0)\lambda}\right).
\end{aligned}$$

Furthermore, as $0 < \epsilon < 1$, by multiplying both sides by $2^5(1+|a|+c_0)\lambda$, we obtain

$$\begin{aligned}
|p_\lambda + a|^2 &\le 2^5(1+|a|+c_0)\left(\frac{\epsilon r^2}{4(1-\epsilon)} + \frac{\epsilon}{4}\right) + \frac{|p_\lambda + a|^2}{2^4} + \frac{|p_\lambda + a|^2}{2^5} \\
&\quad + 2|p_\lambda + a|\ \omega\left(\frac{|p_\lambda + a|}{2^5(1+|a|+c_0)\lambda}\right).
\end{aligned}$$

Thus

$$|p_\lambda + a|^2 \le 2^6(1+|a|+c_0)\left(\frac{\epsilon r^2}{4(1-\epsilon)} + \frac{\epsilon}{4}\right) + 4|p_\lambda + a|\ \omega\left(\frac{|p_\lambda + a|}{2^5(1+|a|+c_0)\lambda}\right).$$

Now we take the upper limit first as $\lambda \to +\infty$, due to the original property of ω that is, ($\omega(|t|) \to 0$ as $t \to 0$), this ω will disappear as x_λ goes to zero when $\lambda \to +\infty$, then we obtain

$$\limsup_{\lambda\to+\infty} |p_\lambda + a|^2 \le 2^6(1+|a|+c_0)\left(\frac{\epsilon r^2}{4(1-\epsilon)} + \frac{\epsilon}{4}\right).$$

Finally, let $\epsilon \to 0+$ we deduce that $p_\lambda \to -a$ as $\lambda \to +\infty$. Thus

$$\lim_{\lambda\to+\infty} \nabla\mathcal{C}^u_\lambda(f)(x_0) = -a,$$

with $-a$ is the centre of the minimal bounding sphere of $\partial_- f(0)$, which completes the proof. □

We pass now to the proofs of Lemmas 1.7, 1.9 and 1.11.

Proof of Lemma 1.7: Suppose for contradiction that there is $\epsilon > 0$ and a sequence (x_k) with $|x_k| := t_k \leq \frac{1}{k}$ such that

$$|f(x_k) - f(0) - \sigma_0(x_k)| > \epsilon\, t_k,$$

for $k = 1, 2, \ldots$. Extracting a subsequence if necessary, assume that $\frac{x_k}{t_k} \to d$ for some d of norm of 1. Then take a local lipschitz constant of f and we expand

$$\begin{aligned}
\epsilon\, t_k &< |f(x_k) - f(0) - \sigma_0(x_k)| \\
&\leq |f(x_k) - f(t_k d)| + |f(t_k d) - f(0) - \sigma_0(t_k d)| + |\sigma_0(t_k d) - \sigma_0(x_k)| \\
&\leq L\,|x_k - t_k d| + |f(t_k d) - f(0) - \sigma_0(t_k d)| + |\sigma_0(t_k d) - \sigma_0(x_k)|\,.
\end{aligned}$$

Then divide by $t_k > 0$,

$$\epsilon \leq L\left|\frac{x_k}{t_k} - d\right| + \frac{|f(t_k d) - f(0) - \sigma_0(t_k d)|}{t_k} + \frac{|\sigma_0(t_k d) - \sigma_0(x_k)|}{t_k}.$$

We pass to the limit to obtain the contradiction with $\epsilon \leq 0$. More precisely, every term will goes to zero. First term is to use the fact f is locally Lipschitz function so if we divide by t_k then when $\frac{x_k}{t_k} \to d$, this term goes to zero. The second term is the definition of directional derivative it goes to zero. For the third term, since σ_0 is continuous if we divide by t_k, then when $\frac{x_k}{t_k} \to d$, so that $\sigma_0(\frac{x_k}{t_k})$ goes to $\sigma_0(d)$ as σ_0 is 1-homogeneous. □

Proof of Lemma 1.9: By the translation invariance property [19, Proposition 2.10] (also see Proposition 2.5), we may assume without loss of generality that $x_0 = 0$.

By the locality property (Theorem 2.9 (iv)) for a bounded lower semicontinuous function when $\lambda > 0$ is large enough, we have

$$|x_i| \leq \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{\sqrt{\lambda}} = \mathbf{R}_\lambda^B < \frac{r}{2}.$$

Also, we have g and h are globally Lipschitz functions then by the locality property (Theorem 2.9 (i)) for the Lipschitz case, we have

$$|x_i| \leq \frac{L(2+\sqrt{2})}{\lambda} = \mathbf{R}_\lambda^L < \frac{r}{2}.$$

Furthermore, we can show that $\mathbf{R}_\lambda^B \geq \mathbf{R}_\lambda^L$ for large $\lambda > 0$ as follows:

$$r > \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{\sqrt{\lambda}} \Leftrightarrow \sqrt{\lambda} > \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{r} \Leftrightarrow \lambda > \left(\frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{r}\right)^2.$$

Also,

$$\frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{\sqrt{\lambda}} < \frac{r}{2} \Leftrightarrow \sqrt{\lambda} > \frac{2(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{r} \Leftrightarrow \lambda > \left(\frac{2(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{r}\right)^2,$$

and
$$\frac{L(2+\sqrt{2})}{\lambda} < \frac{r}{2} \Leftrightarrow \lambda > \left(\frac{2L(2+\sqrt{2})}{r}\right)^2.$$

Recall that $\mathbf{R}_\lambda^B \geq \mathbf{R}_\lambda^L$. So if (10) holds then

$$\frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{\sqrt{\lambda}} \geq \frac{L(2+\sqrt{2})}{\lambda} \Leftrightarrow \sqrt{\lambda} \geq \frac{L(2+\sqrt{2})}{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}} \Leftrightarrow \lambda \geq \left(\frac{L(2+\sqrt{2})}{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}\right)^2.$$

Hence, when λ is sufficiently large,

$$\lambda > \max\left\{\begin{array}{l} \left(\frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{r}\right)^2, \left(\frac{2(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}{r}\right)^2, \\ \left(\frac{2L(2+\sqrt{2})}{r}\right)^2, \ \left(\frac{L(2+\sqrt{2})}{(1+\sqrt{2})\sqrt{\operatorname{osc}(\overline{f})}}\right)^2 \end{array}\right\},$$

by the local ordered property we have

$$\mathcal{C}_\lambda^u(g)(x) \leq \mathcal{C}_\lambda^u(\overline{f})(x) \leq \mathcal{C}_\lambda^u(h)(x), \tag{41}$$

holds for $x \in \overline{B}(0, \frac{r}{2})$.

The proof of the local ordered property is just to use one of the equivalent definitions of the convex envelope and the property of the infimum. To prove the first part $\mathcal{C}_\lambda^u(g)(x) \leq \mathcal{C}_\lambda^u(\overline{f})(x)$, we have

$$\begin{aligned} -\mathcal{C}_\lambda^u(g)(x) &= \inf\left\{\sum_{i=1}^{n+1}\lambda_i[\lambda|y_i-x|^2 - g(y_i)], y_i \in \overline{B}(x, \mathbf{R}_\lambda^L), \lambda_i \geq 0, \sum_{i=1}^{n+1}\lambda_i = 1\right\} \\ &= \sum_{i=1}^{k}\lambda_i[\lambda|x_i-x|^2 - g(x_i)] \\ &\geq \sum_{i=1}^{k}\lambda_i[\lambda|x_i-x|^2 - \overline{f}(x_i)] \\ &\geq \inf\left\{\sum_{i=1}^{n+1}\lambda_i[\lambda|y_i-x|^2 - \overline{f}(y_i)], y_i \in \overline{B}(x, \mathbf{R}_\lambda^L), \lambda_i \geq 0, \sum_{i=1}^{n+1}\lambda_i = 1\right\} \\ &= -\mathcal{C}_\lambda^u(\overline{f})(x) \quad \text{as } \mathbf{R}_\lambda^L \leq \mathbf{R}_\lambda^B. \end{aligned}$$

Similarly, we can prove the second part $\mathcal{C}_\lambda^u(\overline{f})(x) \leq \mathcal{C}_\lambda^u(h)(x)$,

$$\begin{aligned} -\mathcal{C}_\lambda^u(\overline{f})(x) &= \inf\left\{\sum_{i=1}^{n+1}\lambda_i[\lambda|y_i-x|^2 - \overline{f}(y_i)], y_i \in \overline{B}(x, \mathbf{R}_\lambda^B), \lambda_i \geq 0, \sum_{i=1}^{n+1}\lambda_i = 1\right\} \\ &\geq \inf\left\{\sum_{i=1}^{n+1}\lambda_i[\lambda|y_i-x|^2 - h(y_i)], y_i \in \overline{B}(x, \mathbf{R}_\lambda^B), \lambda_i \geq 0, \sum_{i=1}^{n+1}\lambda_i = 1\right\} \\ &= \inf\left\{\sum_{i=1}^{n+1}\lambda_i[\lambda|y_i-x|^2 - h(y_i)], y_i \in \overline{B}(x, \mathbf{R}_\lambda^L), \lambda_i \geq 0, \sum_{i=1}^{n+1}\lambda_i = 1\right\} \\ &= -\mathcal{C}_\lambda^u(h)(x). \end{aligned}$$

The last equality is due to the locality property as the infimum for h over $\mathbb{R}^n$ is the same as that over $\overline{B}(x, \mathbf{R}_\lambda^L)$, which completes the proof. $\square$

Proof of Lemma 1.11: Let $x_0 \in \Omega$ be a singular point of f. By the translation invariance property [19, Proposition 2.10] (also see Proposition 2.5), we may assume without loss of generality that $x_0 = 0$. By the locality property for a bounded lower semicontinuous function (Theorem 2.9 (iv)), we define $g_\lambda(y) = \lambda|y|^2 - f(y)$ for $y \in \mathbb{R}^n$, and $\lambda > 0$. Then by [19, Proposition 2.1](see also Proposition 2.2), there is an affine function $\ell(y) = a \cdot y + b$ for $y \in \mathbb{R}^n$, with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

(i) $\ell(y) \le g_\lambda(y)$ for all $y \in \mathbb{R}^n$,

(ii) $\ell(x_i) = g_\lambda(x_i)$ for $i = 1, \cdots, k$.

Since f is bounded, again by the locality property (Theorem 2.9 (iv)) when $\lambda > 0$ is sufficiently large, we have

$$|x_i| \le \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(f)}}{\sqrt{\lambda}}.$$

Moreover, if $\sqrt{\lambda} \ge (1+\sqrt{2})\sqrt{\operatorname{osc}(f)}$, then $|x_i| \le 1$. By [5, Theorem 2.1.7, p. 33] (also see Theorem 2.13), we conclude f is locally Lipschitz in the unit ball $\overline{B}(0;1)$. By (ii) and $\mathcal{C}_\lambda^u(f)(0) = -\mathbf{Co}[\lambda|\cdot|^2 - f(\cdot)](0)$, we conclude that

$$\ell(0) = b = \mathbf{Co}[g_\lambda](0) = -\mathcal{C}_\lambda^u(f)(0).$$

Let $p_\lambda = \nabla\mathcal{C}_\lambda^u(f)(0)$, to derive the bound of p_λ, we evaluate (i) at $y = \frac{a}{2\lambda}$ to obtain the bound of $|a| = |p_\lambda|$ as follows.

$$\frac{a \cdot a}{2\lambda} + b \le \lambda\left|\frac{a}{2\lambda}\right|^2 - f\left(\frac{a}{2\lambda}\right),$$

so that
$$\frac{|a|^2}{4\lambda} \le -f\left(\frac{a}{2\lambda}\right) - b = -f\left(\frac{a}{2\lambda}\right) + \mathcal{C}_\lambda^u(f)(0)$$

Following [19, Theorem 2.12 (iii)] (also see Theorem 2.8 (iii) for the Lipschitz function f with Lipschitz constant $L > 0$, we have

$$\mathcal{C}_\lambda^u(f)(0) \le f(0) + \frac{L^2}{4\lambda}.$$

Then
$$\frac{|a|^2}{4\lambda} \le -f\left(\frac{a}{2\lambda}\right) + f(0) + \frac{L^2}{4\lambda}.$$

By the semiconvexity of f [3, Proposition 2.1] (also see Proposition 2.16), we have

$$f\left(\frac{a}{2\lambda}\right) - f(0) \ge \langle p, \frac{a}{2\lambda}\rangle - \left|\frac{a}{2\lambda}\right|\omega\left(\frac{|a|}{2\lambda}\right).$$

Thus
$$\frac{|a|^2}{4\lambda} \le -\langle p, \frac{a}{2\lambda}\rangle + \left|\frac{a}{2\lambda}\right|\omega\left(\frac{|a|}{2\lambda}\right) + \frac{L^2}{4\lambda}.$$

By using the Cauchy-Schwarz inequality, we conclude

$$\frac{|a|^2}{4\lambda} \le |p|\left|\frac{a}{2\lambda}\right| + \left|\frac{a}{2\lambda}\right|\omega\left(\frac{|a|}{2\lambda}\right) + \frac{L^2}{4\lambda}.$$

By using the sublinearity property of $\omega(\cdot)$, we have

$$\frac{|a|^2}{4\lambda} \leq |p| \left|\frac{a}{2\lambda}\right| + \left|\frac{a}{2\lambda}\right| \left(\alpha + \beta\frac{|a|}{2\lambda}\right) + \frac{L^2}{4\lambda}.$$

Hence
$$|a|^2 \leq L^2 + 2(|p| + \alpha)|a| + \frac{\beta}{\lambda}|a|^2.$$

Then
$$\left(1 - \frac{\beta}{\lambda}\right) |a|^2 - 2(|p| + \alpha)|a| - L^2 \leq 0.$$

Thus
$$|a| \leq \frac{2(|p| + \alpha) + \sqrt{4(|p| + \alpha)^2 + 4L^2 \left(1 - \frac{\beta}{\lambda}\right)}}{2\left(1 - \frac{\beta}{\lambda}\right)}.$$

For $\lambda > 2\beta$, we have

$$|a| \leq 2(|p| + \alpha) + 2\sqrt{(|p| + \alpha)^2 + L^2} = c_0. \tag{42}$$

Hence p_λ is bounded by c_0 when

$$\lambda > \max\left\{2\beta, \left((1 + \sqrt{2})\sqrt{\operatorname{osc}(f)}\right)^2\right\},$$

which completes the proof. □

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