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On Some Relative Operator Entropies by Convex Inequalities

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A considerable amount of literature on the theory of inequality is devoted to studying Jensen's and Young's inequality. This article presents a number of new inequalities involving the log-convex functions and the geometrically convex functions. As their consequences, we derive the refinements for Young's and Jensen's inequality. In addition, the operator Jensen's type inequality is also developed for conditioned two functions. Utilizing these new inequalities, we investigate the operator inequalities related to the relative operator entropy.

Keywords: Convexity, log-convex, geometrically convex, Jensen's inequality, relative operator entropy, and operator inequality.

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1. Introduction

Throughout this article $I \subset \mathbb{R}$ and $J \subset (0,\infty)$ denote two intervals. A function $f : I \to \mathbb{R}$ is said to be *convex* if it satisfies the well-known Jensen's inequality, namely

$$f\left((1-t)\,a + tb\right) \le (1-t)\, f\left(a\right) + tf\left(b\right), \tag{1}$$

for all $a, b \in I$ and all $0 \le t \le 1$. In general, we can express it as

$$f\left(\sum_{i=1}^{n} t_i a_i\right) \le \sum_{i=1}^{n} t_i f(a_i), \tag{2}$$

where $a_1, a_2, \ldots, a_n \in I$ and $t_1, t_2, \ldots, t_n \ge 0$ with $\sum\limits_{i=1}^{n} t_i = 1$. Young's inequality [5] is a consequence of Jensen's inequality, which can be stated as

$$a^{1-t} b^t \le (1-t)\, a + tb,$$

where $a, b > 0$ and $0 \le t \le 1$. The inequality is generated from inequality (1) applying $f\left(x\right) = -\log x$.

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A function $f : I \to (0, \infty)$ is called *log-convex*, if $f((1-t)a + tb) \le f^{1-t}(a)f^t(b)$ for all $a, b \in I$ and all $0 \le t \le 1$. In addition, we mention that a function $f : J \to (0, \infty)$ is called *geometrically convex* if $f(a^{1-t}b^t) \le f^{1-t}(a)f^t(b)$ for all $a, b \in I$ and all $0 \le t \le 1$.

A self-adjoint operator A is said to be *positive* if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space with inner product $\langle ., .\rangle$. We denote the identity operator in $\mathcal{H}$ by $\mathbf{1}_{\mathcal{H}}$. Given two self-adjoint operators A and B, we denote $A \ge B$ ($A > B$) when $A - B$ is a positive (strictly positive) operator. Therefore, $A \ge 0$ ($A > 0$) indicates A is a positive (strictly positive) operator.

This article is organized as follows. Section 2 illustrates our results on log-convex functions and the geometrically convex functions. Proposition 2.8 and its corollaries present a refinement of Jensen inequality. Theorem 2.14 discusses monotonicity and concavity. Section 3 presents the applications of the results developed in Section 2. Here, we consider the deformed logarithmic function and operator relative entropy. In this section, we construct several operator inequalities applicable to entropy theory. Remark 3.9 presents a refinement of Young's inequality.

2. Inequalities on convex functions

In this section, we prepare some inequalities for convex and geometrically convex functions as lemmas for Section 3. We denote the weighted arithmetic mean and the geometric mean of two positive real numbers a and b by $a\nabla_v b := (1-v)a + vb$ and $a\sharp_v b := a^{1-v}b^v$, respectively. The first result of this article describes a bound on the ratio of these two quantities, which is as follows:

Proposition 2.1. *Given two positive real numbers a and b we have*

$$\exp\left(n\left(1-\left(\frac{a\nabla_v b}{a\sharp_v b}\right)^{-1/n}\right)\right) \le \frac{a\nabla_v b}{a\sharp_v b} \le \exp\left(n\left(\left(\frac{a\nabla_v b}{a\sharp_v b}\right)^{1/n}-1\right)\right), \tag{3}$$

where $0 \le v \le 1$, and $n \in \mathbb{N}$.

Proof. Setting $t := x^{1/n} > 0$ in the well-known inequality

$$1 - \frac{1}{t} \le \log t \le t - 1, \quad (t > 0) \tag{4}$$

we obtain
$$n\left(1 - x^{-1/n}\right) \le \log x \le n\left(x^{1/n} - 1\right), \quad (x > 0). \tag{5}$$

Thus, we have the desired result by replacing $x := \dfrac{a\nabla_v b}{a\sharp_v b}$ in the above inequality. □

Remark 2.2. Consider two sequences of functions $\{a_n(x)\}$ and $\{b_n(x)\}$ defined by $a_n(x) := n\left(x^{1/n} - 1\right)$ and $b_n(x) := n\left(1 - x^{-1/n}\right)$, respectively. We observe that $\{a_n\}$ is a monotone decreasing sequence for $x \ge 1$, since the function

$$f(y) := ny^{n+1} - (n+1)y^n + 1 \quad \text{where} \quad y = x^{\frac{1}{n(n+1)}}$$

is monotone increasing for $y \ge 1$. Similarly, defining $g(z) := nz^{-n-1} - (n+1)z^{-n} + 1$ where $z := x^{\frac{1}{n(n+1)}} > 0$ we can prove that $\{b_n\}$ is a monotone increasing sequence when $x \ge 1$.

Moreover,
$$\lim_{n\to\infty} a_n(x) = \lim_{n\to\infty} b_n(x) = \log x.$$

Thus, both sides in (3) converge to $\dfrac{a\nabla_v b}{a\sharp_v b}$ in the limit $n \to \infty$.

Proposition 2.3. *Given a function $f : I \to \mathbb{R}$ and two real numbers s and $t \in I$ with $t > s$ the following inequality holds:*

$$\min\left\{\frac{(f(t)-f(s))^2}{t-s}, t-s\right\} \le |f(t)-f(s)| \le \max\left\{\frac{(f(t)-f(s))^2}{t-s}, t-s\right\}.$$

Proof. Given any real number $\alpha \le 1$ we have $\alpha^2 \le \alpha \le 1$, that is
$$\min\{\alpha^2, 1\} = \alpha^2 \le \alpha \le 1 = \max\{\alpha^2, 1\}.$$
Similarly, considering $\alpha \ge 1$, we observe $\alpha^2 \ge \alpha \ge 1$, which leads us to
$$\min\{\alpha^2, 1\} = 1 \le \alpha \le \alpha^2 = \max\{\alpha^2, 1\}.$$
Combining the inequalities we get
$$\min\{\alpha^2, 1\} \le \alpha \le \max\{\alpha^2, 1\}$$
for all $\alpha \in \mathbb{R}$. Replacing $\alpha := \dfrac{|f(t)-f(s)|}{t-s}$ in the above inequality we find

$$\min\left\{\left(\frac{f(t)-f(s)}{t-s}\right)^2, 1\right\} \le \frac{|f(t)-f(s)|}{t-s} \le \max\left\{\left(\frac{f(t)-f(s)}{t-s}\right)^2, 1\right\}.$$

Multiplying all terms in the above inequality with $t-s > 0$ completes the proof. □

Remark 2.4. (i) Consider a monotonically increasing function f, that is, $f(t) > f(s)$ for $t > s$. Hence, Proposition 2.3 takes the form

$$\min\left\{\frac{(f(t)-f(s))^2}{t-s}, t-s\right\} \le f(t)-f(s) \le \max\left\{\frac{(f(t)-f(s))^2}{t-s}, t-s\right\}.$$

(ii) Moreover, Proposition 2.3 leads us to

$$\frac{\min\left\{(f(t)-f(0))^2, t^2\right\}}{t} \le f(t)-f(0) \le \frac{\max\left\{(f(t)-f(0))^2, t^2\right\}}{t},$$

when $s = 0 \in I$ and $f : I \to \mathbb{R}$ is a monotone increasing function.

(iii) For $\alpha > 0$, $p \ge 1$ and $q \le 1$, we have $\min\{\alpha^p, \alpha^q\} \le \alpha \le \max\{\alpha^p, \alpha^q\}$. Now, given a monotone increasing function $f : I \to \mathbb{R}$ as well as two points $s, t \in I$ with $t > s$, we have

$$\begin{aligned}\min\left\{\frac{(f(t)-f(s))^p}{(t-s)^{p-1}}, \frac{(f(t)-f(s))^q}{(t-s)^{q-1}}\right\} &\le f(t)-f(s)\\ &\le \max\left\{\frac{(f(t)-f(s))^p}{(t-s)^{p-1}}, \frac{(f(t)-f(s))^q}{(t-s)^{q-1}}\right\}.\end{aligned}$$

The proof is similar to Theorem 2.3. The positivity of α can be dropped for $p = 2$, $q = 0$.

Corollary 2.5. *Consider a convex function $f : J \to \mathbb{R}$ and $0 \leq t \leq 1$. For $x_1, x_2, ..., x_n \in J$ and positive real numbers $w_1, w_2, ..., w_n$ with $\sum_{i=1}^{n} w_i = 1$ we have*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) + \frac{\psi(t)}{t} \leq \sum_{i=1}^{n} w_i f(x_i), \quad \textit{where}$$

$$\psi(t) = \min\left\{\left(\sum_{i=1}^{n} w_i f\left(t x_i + (1-t)\sum_{j=1}^{n} w_j x_j\right) - f\left(\sum_{i=1}^{n} w_i x_i\right)\right)^2, t^2\right\}.$$

Proof. We can prove that the function

$$g(t) = \sum_{i=1}^{n} w_i f\left(t x_i + (1-t)\sum_{j=1}^{n} w_j x_j\right)$$

is a monotonically increasing function on the interval $[0, 1]$ [1, Theorem 2.1].

Moreover, $$g(0) = f\left(\sum_{i=1}^{n} w_i x_i\right) \text{ and } g(1) = \sum_{i=1}^{n} w_i f(x_i).$$

Therefore, we get the desired result by applying Remark 2.4 (ii). □

Remark 2.6. Corollary 2.5 leads us to an improvement of Jensen's inequality. It also assists us in revising the arithmetic-geometric mean inequality. Applying $f(x) := -\log x$ in Corollary 2.5 we obtain

$$0 \leq \min\left\{\left(\log \frac{A(w_i, x_i)}{G(w_i, x_i \nabla_{1-t} A(w_i, x_i))}\right)^2, t^2\right\} \times \frac{1}{t} \leq A(w_i, x_i) - G(w_i, x_i),$$

where A and G are the weighted arithmetic mean and geometric mean, respectively, defined by

$$A(w_i, x_i) := \sum_{i=1}^{n} w_i x_i, \quad G(w_i, x_i) := \prod_{i=1}^{n} x_i^{w_i}.$$

Lemma 2.7. *If $f : J \to (0, \infty)$ is a differentiable log-convex function, then we have for any $s, t \in J$,*

$$\exp\left(\frac{f'(s)}{f(s)}(t-s)\right) \leq \frac{f(t)}{f(s)} \leq \exp\left(\frac{f'(t)}{f(t)}(t-s)\right). \tag{6}$$

If $f : J \to (0, \infty)$ is a differentiable log-concave function, then for any $s, t \in J$,

$$\exp\left(\frac{f'(t)}{f(t)}(s-t)\right) \leq \frac{f(t)}{f(s)} \leq \exp\left(\frac{f'(s)}{f(s)}(s-t)\right). \tag{7}$$

Proof. A function $f : J \to \mathbb{R}$ is log-convex if $\log f$ is convex. Inequality (8) indicates

$$(\log f(s))'(t-s) \leq \log f(t) - \log f(s) \leq (\log f(t))'(t-s),$$

that is $$\frac{f'(s)}{f(s)}(t-s) \leq \log \frac{f(t)}{f(s)} \leq \frac{f'(t)}{f(t)}(t-s).$$

Applying the exponential function, we get

$$\exp\left(\frac{f'(s)}{f(s)}(t-s)\right)f(s) \le f(t) \le \exp\left(\frac{f'(t)}{f(t)}(t-s)\right)f(s).$$

The function f is called *log-concave* if $-\log f$ is convex [9, Definition 1.3.1]. We obtain the second statement by applying the inequality (8) with $-\log f$ and exchanging s and t. □

Proposition 2.8. *If $f : J \to (0,\infty)$ be a differentiable geometrically convex function, then for any $s,t \in J$,*

$$\left(\frac{t}{s}\right)^{\frac{sf'(s)}{f(s)}} \le \frac{f(t)}{f(s)} \le \left(\frac{t}{s}\right)^{\frac{tf'(t)}{f(t)}}.$$

Proof. The function $f : J \to \mathbb{R}$ is geometrically convex if and only if $\log f(\exp(x))$ is convex on $\mathbb{R}$. Recall that, for any differentiable convex function f on the interval J is also convex so that we can write

$$f'(s)(t-s) \le f(t) - f(s) \le f'(t)(t-s), \tag{8}$$

for any $t \ge s > 0$. Hence, from (8), we infer that

$$(\log f(\exp s))'(t-s) \le \log\frac{f(\exp t)}{f(\exp s)} \le (\log f(\exp t))'(t-s).$$

We see that

$$\frac{(\exp s)f'(\exp s)}{f(\exp s)}(t-s) \le \log\frac{f(\exp t)}{f(\exp s)} \le \frac{(\exp t)f'(\exp t)}{f(\exp t)}(t-s).$$

We get the required result by replacing s and t by $\log s$ and $\log t$, respectively. □

Proposition 2.9. *Let $g : J \to (0,\infty)$ be a differentiable geometrically convex function and $a,b \in J$. Then for any $0 \le t \le 1$,*

$$\exp\left(G_0(g;a,b)t\right) \le \frac{g\left(a^{1-t}b^t\right)}{g^{1-t}(a)\,g^t(b)} \le \exp\left(G_t(g;a,b)t\right), \tag{9}$$

where $$G_t(g;a,b) := \log\frac{g(a)}{g(b)} - a^{1-t}b^t\left(\log\frac{a}{b}\right)\frac{g'(a^{1-t}b^t)}{g(a^{1-t}b^t)}.$$

Proof. If g is geometrically convex, then $f(t) = \dfrac{g\left(a^{1-t}b^t\right)}{g^{1-t}(a)\,g^t(b)}$ is a log-convex function. Indeed we calculate for $0 \le s,t \le 1$,

$$\begin{aligned} f\left(\frac{s+t}{2}\right) &= \frac{g\left(a^{1-\frac{s+t}{2}}b^{\frac{s+t}{2}}\right)}{g^{1-\frac{s+t}{2}}(a)g^{\frac{s+t}{2}}(b)} = \frac{g\left(\sqrt{a^{1-t}b^t}\sqrt{a^{1-s}b^s}\right)}{\sqrt{g^{1-t}(a)g^t(b)}\sqrt{g^{1-s}(a)g^s(b)}} \\ &\le \frac{\sqrt{g\left(a^{1-t}b^t\right)}\sqrt{g\left(a^{1-s}b^s\right)}}{\sqrt{g^{1-t}(a)g^t(b)}\sqrt{g^{1-s}(a)g^s(b)}} = \sqrt{f(t)f(s)}. \end{aligned}$$

Also, $\dfrac{f'(t)}{f(t)} = G_t(g;a,b)$. Taking $s=0$ as $f(0)=1$, we obtain the result. □

Remark 2.10. If the function $g : I \to (0, \infty)$ is differentiable log-convex and monotone increasing, we can also prove the inequality (9).

Corollary 2.11. *Let $a_1, a_2, \ldots, a_n \in J$, $t_1, t_2, \ldots, t_n$ be non-negative real numbers, such that, $\sum_{i=1}^n t_i = 1$. Denote $A(t_i, a_i) := \sum_{i=1}^n t_i a_i$. Now, we have the following inequalities:*

(i) *If $f : J \to (0, \infty)$ is a differentiable log-convex function, then*

$$L_n f\left(\sum_{i=1}^n t_i a_i\right) \le \sum_{i=1}^n t_i f(a_i) \le R_n f\left(\sum_{i=1}^n t_i a_i\right), \tag{10}$$

where
$$L_n := \sum_{i=1}^n t_i \exp\left(\frac{f'\left(A(t_i, a_i)\right)\left(a_i - A(t_i, a_i)\right)}{f\left(A(t_i, a_i)\right)}\right)$$

and
$$R_n := \sum_{i=1}^n t_i \exp\left(\frac{f'(a_i)\left(a_i - A(t_i, a_i)\right)}{f(a_i)}\right).$$

(ii) *If $f : J \to (0, \infty)$ is a differentiable geometrically convex function, then we also have*

$$\widehat{L_n} f\left(\sum_{i=1}^n t_i a_i\right) \le \sum_{i=1}^n t_i f(a_i) \le \widehat{R_n} f\left(\sum_{i=1}^n t_i a_i\right), \tag{11}$$

where

$$\widehat{L_n} := \sum_{i=1}^n t_i \left(\frac{a_i}{A(t_i, a_i)}\right)^{\frac{A(t_i,a_i) f'(A(t_i,a_i))}{f(A(t_i,a_i))}} \quad \text{and} \quad \widehat{R_n} := \sum_{i=1}^n t_i \left(\frac{a_i}{A(t_i, a_i)}\right)^{\frac{a_i f'(a_i)}{f(a_i)}}.$$

Proof. Replacing $t = a_i$ $(i = 1, 2, \ldots, n)$ in Theorem 2.7, we write

$$\exp\left(\frac{f'(s)}{f(s)}(a_i - s)\right) \le \frac{f(a_i)}{f(s)} \le \exp\left(\frac{f'(a_i)}{f(a_i)}(a_i - s)\right) \tag{12}$$

for any $i = 1, 2, \ldots, n$. Multiplying t_i to both sides in (12) and adding over i from 1 to n we deduce

$$\sum_{i=1}^n t_i \exp\left(\frac{f'(s)}{f(s)}(a_i - s)\right) \le \frac{\sum_{i=1}^n t_i f(a_i)}{f(s)} \le \sum_{i=1}^n t_i \exp\left(\frac{f'(a_i)}{f(a_i)}(a_i - s)\right). \tag{13}$$

Putting $s = A(t_i, a_i)$ in (13), then we infer the result (i). Similarly, we can prove result (ii) by substituting $s = A(t_i, a_i)$ and $t = a_i$ in Theorem 2.7. $\square$

Remark 2.12. A log-convex function $f : I \to (0, \infty)$ is always a convex function that satisfies the Jensen's inequality mentioned in (2). Now, we have the following observations:

(i) Since $e^\alpha > 0$ for all $\alpha \in \mathbb{R}$, we observe $L_n > 0$ and $R_n > 0$ in Corollary 2.11. Note that, $e^\alpha \ge 1$ if and only if $\alpha \ge 0$. Thus, the first inequality in (10) is a refined Jensen's inequality when $L_n \ge 1$. The condition $f'(A(t_i, a_i))(a_i - A(t_i, a_i)) \ge 0$ for all $i = 1, 2, \cdots, n$ is a strong sufficient condition for $L_n \ge 1$. The sign of $a_i - A(t_i, a_i)$ for each i is not determinate, since $A(t_i, a_i)$ is an arithmetic mean.

There are examples of the log-convex function f such that $f'(x) \geq 0$ or $f'(x) \leq 0$. The function $f(x) = \exp(x^p)$ for $x > 0$ and $p \geq 1$ is log-convex and $f'(x) > 0$. Also, the function $f(x) = \frac{1}{x^p}$ for $x > 0$ and $p > 0$ is another log-convex with $f'(x) < 0$. The second inequality in (10) is a reverse of Jensen's inequality.

(ii) Note that a differentiable monotonically decreasing geometrically convex function $f : J \to (0, \infty)$ is a convex function, since

$$f\left((1-t)x + ty\right) \leq f(x^{1-t}y^t) \leq f^{1-t}(x)f^t(y) \leq (1-t)f(x) + tf(y).$$

Therefore the first inequality in (11) provides a refinement of the Jensen's inequality for this class of functions. A typical example is $f(x) = \dfrac{1}{\sin x}$ in $(0, \pi/2)$. Clearly,

$$f'(x) = -\frac{\cos x}{\sin^2 x} \leq 0 \quad \text{and} \quad \frac{d^2}{dx^2}\log\left(f(e^x)\right) = \frac{e^x\left(e^x - \frac{1}{2}\sin 2x\right)}{\sin^2 x} > 0$$

in $(0, \pi/2)$, since $e^x > 1$ and $\sin 2x \leq 1$ in $(0, \pi/2)$. □

Here, we review the definition of $f(A)$ for a self–adjoint operator and continuous function f. Let A be a self–adjoint operator on a Hilbert space and $f(t)$ be a real valued continuous function defined on $[m, M]$, where $m := \inf_{\|x\|=1}\langle Ax, x\rangle$ and $M := \sup_{\|x\|=1}\langle Ax, x\rangle$. Since a self–adjoint operator A has a spectral decomposition $A := \int \lambda dE_\lambda$, where $\{E_\lambda : \lambda \in \mathbb{R}\}$ is a family of projections, an operator $f(A)$ has also spectral representation $f(A) = \int_m^M f(\lambda)dE_\lambda$ by the functional calculus. Thus an operator $f(A)$ makes a sense. In the lemma below, $f(A)$ is understood as the above.

Before closing this section, we state an interesting theorem that illustrates an inequality by monotonicity and convexity/concavity. We need the following well-known lemma as Jensen's operator inequality for a convex function.

Lemma 2.13. ([8],[10, Theorem 1.2]) *Let f be a concave function on $[a, b]$ where $a \geq 0$ and A be a self-adjoint operator with its spectrum in $[a, b]$. Then we have*

$$\langle f(A)x, x\rangle \leq f\left(\langle Ax, x\rangle\right)$$

for every unit vector $x \in \mathcal{H}$. If f is a convex function, then

$$\langle f(A)x, x\rangle \geq f\left(\langle Ax, x\rangle\right).$$

Theorem 2.14. *Let $a \geq 0$, $f : [a, b] \to [0, \infty)$ and $g : [a, b] \to [0, \infty)$ be differentiable functions satisfying the following conditions (i)-(iv):*

(i) f is monotone increasing and concave;

(ii) g is convex;

(iii) $f(x) \geq g(x)$ for all $x \in [a, b]$; and

(iv) $f(b) - f(a) \geq \mathrm{M}_{\mathrm{ratio}}(f, g)(g(b) - g(a)) \geq 0$, where

$$\mathrm{M}_{\mathrm{ratio}}(f, g) := \max_{x\in[a,b]}\{f(x)\} \Big/ \min_{x\in[a,b]}\{g(x)\}.$$

Then, for any positive operator A with $a\mathbf{1}_{\mathcal{H}} \leq A \leq b\mathbf{1}_{\mathcal{H}}$, and a unit vector $h \in \mathcal{H}$ we have $(g(b) - g(a))f(\langle Ah, h\rangle) \leq (f(b) - f(a))\langle g(A)h, h\rangle$.

Proof. Define a function $F : [a, b] \to \mathbb{R}$ by

$$F(t) = (f(b) - f(a))g(t) - (g(b) - g(a))f(t).$$

From the conditions (i),(ii), and (iv), we see that $F(t)$ is a convex function. As $f(t)$ and $g(t)$ are differentiable functions,

$$F'(t) = (f(b) - f(a))g'(t) - (g(b) - g(a))f'(t).$$

The Cauchy mean value theorem states that there exists $\xi \in (a, b)$ such that

$$\frac{g(b) - g(a)}{f(b) - f(a)} = \frac{g'(\xi)}{f'(\xi)}$$

or
$$(g(b) - g(a))f'(\xi) = (f(b) - f(a))g'(\xi)$$

or
$$(f(b) - f(a))g'(\xi) - (g(b) - g(a))f'(\xi) = 0, \text{ or } F'(\xi) = 0.$$

As $F(t)$ is a convex function, ξ is the minimum of $F(t)$. We have also

$$f(b) - f(a) \geq \mathrm{M}_{\mathrm{ratio}}(f, g)(g(b) - g(a)) \geq 0.$$

Hence,
$$f(b) - f(a) \geq (g(b) - g(a))\frac{f(\xi)}{g(\xi)}.$$

Thus we have $F(\xi) = (f(b)-f(a))g(\xi)-(g(b)-g(a))f(\xi) > 0$. Therefore, $F(t) \geq 0$ for all $t \in [a, b]$, that is

$$(g(b) - g(a))f(t) \leq (f(b) - f(a))g(t). \tag{14}$$

Consider $t = \langle Ah, h\rangle$. As there exists a and b such that $a\mathbf{1}_{\mathcal{H}} \leq A \leq b\mathbf{1}_{\mathcal{H}}$, we have $a \leq t \leq b$. Replacing it in equation (14) we have

$$(g(b) - g(a))f(\langle Ah, h\rangle) \leq (f(b) - f(a))g(\langle Ah, h\rangle).$$

As g is a convex function applying Lemma 2.13 we have

$$(g(b) - g(a))f(\langle Ah, h\rangle) \leq (f(b) - f(a))g(\langle Ah, h\rangle) \leq (f(b) - f(a))(\langle g(A)h, h\rangle),$$

which concludes this theorem. □

Corollary 2.15. *Let f and g satisfy the assumptions in Theorem 2.14. Then, for any positive operator A, B with $aA \leq B \leq bA$, we have*

$$A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \leq \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) A^{1/2} g\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Proof. From the convexity of f and $a\mathbf{1}_{\mathcal{H}} \leq A^{-1/2}BA^{-1/2} \leq b\mathbf{1}_{\mathcal{H}}$, we have with Lemma 2.13 and Theorem 2.14,

$$\left\langle f\left(A^{-1/2}BA^{-1/2}\right)h, h\right\rangle \leq \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)\left\langle g\left(A^{-1/2}BA^{-1/2}\right)h, h\right\rangle,$$

which implies the result. □

Corollary 2.16. *Let f and g satisfy the assumptions in Theorem* 2.14. *Given two positive operators A and B with $B \leq A$ we have*

$$f(B) \leq \frac{f(b)-f(a)}{g(b)-g(a)} g(A).$$

Proof. As we have shown in Theorem 2.14

$$f(\langle Ah, h\rangle) \leq \frac{f(b)-f(a)}{g(b)-g(a)} \langle g(A) h, h\rangle$$

for any unit vector $h \in \mathcal{H}$. Since $B \leq A$, then $\langle Bh, h\rangle \leq \langle Ah, h\rangle$ for any unit vector h. On the other hand, f is an increasing function, thus $f(\langle Bh, h\rangle) \leq f(\langle Ah, h\rangle)$. Therefore,

$$f(\langle Bh, h\rangle) \leq \frac{f(b)-f(a)}{g(b)-g(a)} \langle g(A) h, h\rangle.$$

Now, by Lemma 2.13,

$$\langle f(B) h, h\rangle \leq \frac{f(b)-f(a)}{g(b)-g(a)} \langle g(A) h, h\rangle,$$

for any unit vector $h \in \mathcal{H}$. By replacing $h = x/\|x\|$, we reach the desired inequality. □

3. Inequalities on relative operator entropies

We apply the results discussed in section 2 in the theory of relative operator entropy. Recall that, the t-logarithmic function $\ln_t$ is defined by

$$\ln_t(x) := \frac{x^t - 1}{t} \quad \text{for } x > 0 \text{ and } t \neq 0.$$

The deformed logarithm and their generalizations are widely utilized in mathematical inequality and information theory [5, 2, 3]. The function $\ln_t(x)$ is an inverse of $\exp_t(x) := (1+tx)^{1/t}$, which is defined for $1+tx > 0$. We easily find that

$$\lim_{t\to 0} \exp_t(x) = \exp(x) \quad \text{and} \quad \lim_{t\to 0} \ln_t(x) = \log(x).$$

Given positive operators A, and B the relative operator entropy $S(A|B)$ [4], the Tsallis relative operator entropy $T_t(A|B)$ [11], and the generalized relative operator entropy $S_t(A|B)$ [6] are respectively defined by

$$S(A|B) := A^{1/2} \log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2},$$

$$T_t(A|B) := A^{1/2} \ln_t\left(A^{-1/2} B A^{-1/2}\right) A^{1/2},$$

and

$$S_t(A|B) := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^t \log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

It is easy to observe that $\lim_{t\to 0} T_t(A|B) = S(A|B)$ and $\lim_{t\to 0} S_t(A|B) = S(A|B)$ by $\lim_{t\to 0} \ln_t x = \lim_{t\to 0} \ln_{-t} x = \log x$. In addition, the following inequalities are known [12],

$$A - AB^{-1}A \leq T_{-t}(A|B) \leq S(A|B) \leq T_t(A|B) \leq B - A, \tag{15}$$

where $t \in (0,1]$.

Note that, inequality (5) is equivalent to $\ln_{-t} x \le \log x \le \ln_t x$ when $t := \frac{1}{n}$.
The above inequality implies

$$T_{-t}(A|B) \le S(A|B) \le T_t(A|B) \tag{16}$$

which is covered by (15). See [5, Section 7.3] for these details.

Lemma 3.1. *The t-logarithmic function $\ln_t x$, is convex in t for $x \ge 1$, and concave in t for $0 < x \le 1$. Furthermore, this function is log-convex on $(-\infty, \infty)$. Moreover, $\ln_t x$ is a monotone increasing function on $t \in (-\infty, \infty)$.*

Proof. We define the function $f(t) := t(\log t)^2 - 2t \log t + 2t - 2$ for $t > 0$. Since $f(1) = 0$ and $f'(t) = (\log t)^2 \ge 0$, we have $f(t) \le 0$ for $0 < t \le 1$ and $f(t) \ge 0$ for $t \ge 1$. Note that

$$\frac{d^2}{dt^2}(\ln_t x) = \frac{1}{t^3}\left\{x^t(\log x^t)^2 - 2x^t \log x^t + 2x^t - 2\right\} = \frac{f(x^t)}{t^3}.$$

Hence, $\quad \frac{d^2}{dt^2}(\ln_t x) \ge 0$ if $x \ge 1$ and $\frac{d^2}{dt^2}(\ln_t x) \le 0$ if $0 < x \le 1$.

We also calculate $\frac{d^2}{dt^2}(\log(\ln_t x)) = \frac{(x^t - 1)^2 - x^t(\log x^t)^2}{t^2(x^t - 1)^2} \ge 0.$

This inequality holds by the relation $\sqrt{a} \le \frac{a-1}{\log a}$ for $a > 0$. Thus, the function $\ln_t x$ is log-convex for all $t \in \mathbb{R}$.

Finally, we calculate $\frac{d}{dt}(\ln_t x) = \frac{x^t \log x^t - (x^t - 1)}{t^2} \ge 0.$

This inequality can easily be proven using the fundamental inequality $\log a \le a - 1$ for $a > 0$. $\square$

Employing Lemma 3.1 and Remark 2.4 (i), we have:

Lemma 3.2. *Given $x > 0$ and $0 \le t \le 1$. Then we have $\log x + \frac{\theta(t,x)}{t} \le \ln_t x$, where $\theta(t,x) = \min\left\{(\ln_t x - \log x)^2, t^2\right\} \ge 0$.*

Note that $\lim_{t \to 0} \frac{(\ln_t x - \log x)^2}{t} = 0$. Therefore, $\lim_{t \to 0} \frac{\theta(t,x)}{t} = 0$.

Thus $\theta(t,x)/t$ is defined for $t = 0$.

We have the following consequence of Lemma 3.2:

Proposition 3.3. *Let A, and B be positive operators, such that, $mA \le B \le MA$ with $0 < m \le M$, and let $0 \le t \le 1$. Then we have the following results:*

(i) *If $M < 1$, then $S(A|B) + \frac{\theta(t,M)}{t} A \le T_t(A|B)$.*

(ii) *If $m \le 1 \le M$, then $S(A|B) \le T_t(A|B)$.*

(iii) *If $1 < m$, then $S(A|B) + \frac{\theta(t,m)}{t} A \le T_t(A|B)$.*

Proof. Note that the function $f_t(x) := \ln_t x - \log x$ defined for $x > 0$ and $0 \le t \le 1$ is monotone decreasing in $0 < x \le 1$ and monotone increasing in $x \ge 1$. From Lemma 3.2, for the case (i) we have

$$\log x + \min_{(0<)m \le x \le M(<1)} \frac{\theta(t,x)}{t} \le \ln_t x.$$

Thus we have the inequality in (i) by setting $x := A^{-1/2}BA^{-1/2}$ and multiplying $A^{1/2}$ to both sides. The inequalities in (ii) and (iii) similarly follow with $\theta(t,1) = 0$. $\square$

Proposition 3.3 includes a refinement of the second inequality in (16).

Lemma 3.4. *For $s, t > 0$ and $x \ge 1$, we have*

$$\exp\left(\frac{(x^s \log x - \ln_s x)(t-s)}{s \ln_s x}\right) \ln_s x \le \ln_t x \le \exp\left(\frac{(x^t \log x - \ln_t x)(t-s)}{t \ln_t x}\right) \ln_s x.$$

Proof. Observe that $\ln_t x \ge 0$ if $x \ge 1$ and $t > 0$. By Lemma 3.1, $\ln_t x$ is log-convex in $t > 0$. Then we can apply the inequalities (6) in Lemma 2.7 with $f(t) := \ln_t x$. $\square$

The following result gives the relation of two Tsallis relative operator entropies.

Proposition 3.5. *Let A, B be positive operators such that $mA \le B \le MA$ with $1 \le m \le M$, and let $s, t > 0$. Then we have the following results:*

(i) *If $t \ge s \ge 0$, then $0 \le e^{\eta(m,s)(t-s)} T_s(A|B) \le T_t(A|B) \le e^{\eta(M,t)(t-s)} T_s(A|B)$.*

(ii) *If $s \ge t \ge 0$, then $0 \le e^{\eta(M,s)(t-s)} T_s(A|B) \le T_t(A|B) \le e^{\eta(m,t)(t-s)} T_s(A|B)$.*

Here $\eta(x,a) := \dfrac{x^a \log x - \ln_a x}{a \ln_a x}$ *is defined for $a \ge 0$ and $x > 0$.*

Proof. Consider the function

$$\eta(x,a) = \frac{x^a \log x^a - x^a + 1}{a(x^a - 1)} \quad \text{for } a \ge 0 \text{ and } x > 0.$$

Then we have

$$\frac{d\eta(x,a)}{dx} = \frac{x^{a-1}(x^a - 1 - \log x^a)}{(x^a-1)^2} \ge 0 \quad \text{with} \quad \lim_{x \to 1} \eta(x,a) = 0.$$

Therefore the function $\eta(x,a)$ is monotone increasing in $x > 0$ for all $a \ge 0$, and $\eta(x,a) \le 0$ if $0 < x \le 1$, and $\eta(x,a) \ge 0$ if $x \ge 1$. For the case $t \ge s \ge 0$ and $m \ge 1$, we have

$$0 \le \exp(\eta(m,s)(t-s)) \ln_s x \le \ln_t x \le \exp(\eta(M,s)(t-s)) \ln_s x$$

thanks to Lemma 3.4. Putting $x := A^{-1/2}BA^{-1/2}$ in these inequalities and multiplying $A^{1/2}$ to both sides, then we have the operator inequalities given in (i). Taking into respect that for $a \ge 0$ we have $\ln_a x \ge 0$ if $x \ge 1$, (ii) can be proven similarly using Lemma 3.4. $\square$

We give another application of Lemma 2.7 for estimating the bounds of the relative operator entropy.

Theorem 3.6. *Let A, B be positive operators such that $mA \leq B \leq MA$ with $0 < m \leq M$.*

(i) *If $M \leq 1/e$, then we have*

$$-\exp\left(\frac{em-1}{em\log m}\right) A \leq S(A|B) \leq -\exp\left(1-eM\right) A \leq 0. \tag{17}$$

(ii) *If $1 \leq m \leq M \leq e$, then we have*

$$0 \leq \exp\left(\frac{e-M}{M\log M}\right) A \leq S(A|B) \leq \exp\left(\frac{e-m}{e}\right) A.$$

Proof. (i) Since $\frac{d^2}{dx^2}\left(\log\left(-\log x\right)\right) = \frac{-1-\log x}{x^2\left(\log x\right)^2} \geq 0$ for $x \in (0, 1/e]$, the function $f(x) := -\log x$ is log-convex. Also we observe that $f(x) \geq 0$ for $x \in (0, 1/e]$. Applying the inequalities (6) in Lemma 2.7 with $f(x) := -\log x$, $J := (0, 1/e]$ and $s := 1/e$ and manipulating its inequalities, we obtain

$$\exp(1-et) \leq -\log t \leq \exp\left(\frac{t-1/e}{t\log t}\right).$$

Putting $t := A^{-1/2}BA^{-1/2}$ in the above, from the condition $mA \leq B \leq MA$ with $0 < m \leq M$, we get

$$\min_{(0<)m\leq t\leq M(\leq 1/e)} \exp(1-et) \leq -\log A^{-1/2}BA^{-1/2} \leq \max_{(0<)m\leq t\leq M(\leq 1/e)} \exp\left(\frac{et-1}{et\log t}\right).$$

Multiplying $A^{1/2}$ to both sides, we have the inequalities (17), since the function $1-et$ is decrasing and the function $\frac{et-1}{et\log t}$ is decreasing on $(0, 1/e]$. Indeed, we calculate $\frac{d}{dt}\left(\frac{et-1}{et\log t}\right) = \frac{h(t)}{et^2(\log t)^2}$ with $h(t) := \log t - et + 1$. Since $h'(t) = 1/t - e \geq 0$ so that $h(t) \leq h(1/e) = -1$, we have $\frac{d}{dt}\left(\frac{et-1}{et\log t}\right) < 0$.

(ii) Observe that $\frac{d^2}{dx^2}\left(-\log\left(\log x\right)\right) = \frac{\log x + 1}{x^2(\log x)^2} \geq 0$ for $x \geq 1$. Also, it is trivial that $\log x \geq 0$ for $x \geq 1$. Applying the inequalities (7) in Lemma 2.7 with $f(x) := \log x$, $J := [1, \infty)$ and $s := e$ and manipulating its inequalities, we have

$$\exp\left(\frac{e-t}{t\log t}\right) \leq \log t \leq \exp\left(\frac{e-t}{e}\right).$$

In a similar way to (i), we obtain

$$\min_{(1\leq)m\leq t\leq M} \exp\left(\frac{e-t}{t\log t}\right) \leq \log A^{-1/2}BA^{-1/2} \leq \max_{(1\leq)m\leq t\leq M} \exp\left(\frac{e-t}{e}\right),$$

which implies the inequalities (17), since the function $\frac{e-t}{e}$ is decreasing and the function $\frac{e-t}{t\log t}$ is also decreasing for $t \in [1, e]$.

Indeed, we calculate $\frac{d}{dt}\left(\frac{e-t}{t\log t}\right) = \frac{k(t)}{t^2(\log t)^2}$ with $k(t) := t - e - e\log t$.

Since $k'(t) = -e/t + 1 \le 0$ for $1 \le t \le e$ so that $k(t) \le k(1) = 1 - e < 0$, we finally obtain $\frac{d}{dt}\left(\frac{e-t}{t\log t}\right) < 0$. $\square$

Lemma 3.7. *Let $f : \mathbb{R} \to (0,\infty)$ be a log-convex function. Then the functions*

$$g(t) := \frac{f(t)}{(1-t)f(0) + tf(1)} \quad \text{and} \quad h(t) := \frac{f(t)}{f(0)^{1-t}f(1)^t}, \quad t \in \mathbb{R}$$

are log-convex too.

Proof. Since f is log-convex, we have

$$\frac{d^2}{dt^2}\log f(t) = \frac{f''(t)f(t) - f'(t)^2}{f(t)^2} \ge 0,$$

and sinde $f''(t)f(t) - f'(t)^2 \ge 0$ we obtain

$$\frac{d^2}{dt^2}\left(\log\left(\frac{f(t)}{(1-t)f(0) + tf(1)}\right)\right)$$
$$= \frac{f(t)^2\{f(1) - f(0)\}^2 + \left\{f''(t)f(t) - f'(t)^2\right\}\{(1-t)f(0) + tf(1)\}^2}{f(t)^2\{(1-t)f(0) + tf(1)\}^2} \ge 0,$$

which shows the log-convexity of $g(t)$. We have further

$$h\left(\frac{t+s}{2}\right) = \frac{f\left(\frac{t+s}{2}\right)}{f(0)^{1-\frac{t+s}{2}}f(1)^{\frac{t+s}{2}}} \le \frac{\sqrt{f(t)f(s)}}{\sqrt{f(0)^{1-t}f(1)^t}\sqrt{f(0)^{1-s}f(1)^s}} = \sqrt{h(t)h(s)},$$

which shows the log-convexity of $h(t)$. $\square$

Corollary 3.8. *Let $a, b \ge 0$ and $0 \le t \le 1$. Then*

$$\exp\left(\left(\log\frac{b}{a} + 1 - \frac{b}{a}\right)t\right) \le \frac{a\sharp_t b}{a\nabla_t b} \le \exp\left(\left(\log\frac{b}{a} - \frac{b-a}{a\nabla_t b}\right)t\right). \tag{18}$$

Proof. In Lemma 3.7, we take $f(t) := a\sharp_t b$. Then the function

$$g(t) = \frac{f(t)}{(1-t)f(0) + tf(1)} = \frac{a\sharp_t b}{a\nabla_t b}$$

is log-convex. Since we have

$$g(0) = 1, \quad g'(t) = g(t)\left(\log\frac{b}{a} - \frac{(b-a)}{a\nabla_t b}\right) \quad \text{and} \quad g'(0) = \log\frac{b}{a} + 1 - \frac{b}{a},$$

we get the desired result by Lemma 2.7 with $s = 0$. $\square$

Remark 3.9. (i) Putting $a = 1$, $b = x$ and taking the logarithm of both sides of the first inequality in (18), we have $\log\{(1-t) + tx\} \le t(x-1)$. Now putting $x := A^{-1/2}BA^{-1/2}$ in this inequality and multiplying $A^{1/2}$ to both sides, we have

$$A^{1/2}\log\left((1-t)I + tA^{-1/2}BA^{-1/2}\right)A^{1/2} \le t(B - A),$$

which includes the known inequality (see e.g.,[5, Eq.(7.3.1)]), $S(A|B) \le B - A$ when $t = 1$.

(ii) The inequalities (18) is rewritten as

$$\exp\left(\left(\frac{b-a}{a\nabla_t b} - \log\frac{b}{a}\right)t\right) \leq \frac{a\nabla_t b}{a\sharp_t b} \leq \exp\left(\left(\frac{b}{a} - 1 - \log\frac{b}{a}\right)t\right). \tag{19}$$

Since we have $\log x \leq x-1$ for all $x>0$, $\exp\left(\left(\frac{b}{a}-1-\log\frac{b}{a}\right)t\right) \geq 1$ for $a,b\geq 0$ and $0\leq t\leq 1$, and the second inequality of (19) gives the reverse of the Young's inequality. On the other hand, the function

$$\varphi(t,x) := \frac{x-1}{(1-t)+tx} - \log x$$

takes both positive and negative values. For example we get $\varphi(1/4,4) \simeq 0.327991$, whereas $\varphi(1/2,4) \simeq -0.186294$. Therefore, the first inequality of (19) does not always give the refinement of Young's inequality. However, we say that the first inequality of (19) gives the refinement of Young's inequality for $a \geq b > 0$ and $0 \leq t \leq 1/2$. Indeed, the function $\varphi(t,x)$ is monotone decreasing in $t\in[0,1]$ by

$$\frac{d\varphi(t,x)}{dt} = -\frac{(x-1)^2}{\{(1-t)+tx\}^2} \leq 0.$$

If $0\leq t\leq 1/2$ and $0<x\leq 1$, then we have

$$\varphi(t,x) \geq \varphi(1/2,x) = \frac{2(x-1)}{x+1} - \log x \geq 0$$

since we have $\dfrac{x-1}{\log x} \leq \dfrac{x+1}{2}$ for all $x>0$. □

Proposition 3.10. ([7, Theorem 1]) *For positive operators A,B and $p>0$, we have $S(A|B) \leq T_p(A|B) \leq S_p(A|B)$ and their reverse inequalities hold when $p<0$.*

Proof. Putting $t=1$ in (9), we have

$$b\left(\log\frac{a}{b}\right)\frac{g'(b)}{g(b)} \leq \log\frac{g(a)}{g(b)} \leq a\left(\log\frac{a}{b}\right)\frac{g'(a)}{g(a)}.$$

Taking a geometrically convex function $g(x) := \exp(x^p)$, $(p\neq 0)$ in the above as an example, we have

$$b^p\left(\log\frac{a^p}{b^p}\right) \leq a^p - b^p \leq a^p\left(\log\frac{a^p}{b^p}\right), \quad (a,b>0,\ p\neq 0),$$

which implies $$b^p\left(\log\frac{a}{b}\right) \leq \ln_p a - \ln_p b \leq a^p\left(\log\frac{a}{b}\right), \quad (a,b>0,\ p>0)$$

and $$b^p\left(\log\frac{a}{b}\right) \geq \ln_p a - \ln_p b \geq a^p\left(\log\frac{a}{b}\right), \quad (a,b>0,\ p<0).$$

The above inequalities give the relations among relative operator entropies for $A,B\geq 0$:

$$S(A|B) \leq T_p(A|B) \leq S_p(A|B), \quad (p>0)$$

and $$S(A|B) \geq T_p(A|B) \geq S_p(A|B), \quad (p<0)$$

by taking $b=1$, $a=x=: A^{-1/2}BA^{-1/2}$ and multiplying $A^{1/2}$ to both sides. □

We have the following results on the bounds of the Tsallis relative operator entropy $T_t(A|B)$. The application below of Corollary 2.15 is a simple case such as $f(a) = g(a)$ and $f(b) = g(b)$, namely $\frac{f(b) - f(a)}{g(b) - g(a)} = 1$.

Proposition 3.11. *For two positive operators A, B such that $mA \le B \le MA$, where $1 \le m < M$, we have*

$$T_t(A|B) \le \left(\frac{\ln_t M - \ln_t m}{M - m}\right) B + \left(\frac{M \ln_t m - m \ln_t M}{M - m}\right) A, \quad (t \le 1,\ t \neq 0). \tag{20}$$

We also have the reverse inequality of (20) *for $t \ge 1$.*

Proof. If we take $f(x) = \ln_t x$, $(t \le 1,\ t \neq 0)$ and

$$g(x) = \left(\frac{\ln_t M - \ln_t m}{M - m}\right)(x - m) + \ln_t m \quad \text{on} \quad [m, M], \text{ where } m \ge 1,$$

then these functions satisfy the conditions (i)–(iv) in Theorem 2.14. Since we have $1 \le m \le A^{-1/2} B A^{-1/2} \le M$, we have inequality (20) by Corollary 2.15. Since the functions

$$f(x) = \left(\frac{\ln_t M - \ln_t m}{M - m}\right)(x - m) + \ln_t m \quad \text{and} \quad g(x) = \ln_t x,$$

$(t \ge 1)$ on $[m, M]$, where $m \ge 1$ also satisfy the conditions (i)–(iv) in Theorem 2.14, we then have the reverse inequality of (20) for $t \ge 1$, by Corollary 2.15. $\square$

Remark 3.12. (i) In (20), we take $m = 1$, then we have

$$T_t(A|B) \le \frac{\ln_t M}{M - 1}(B - A) \le B - A, \quad (t \le 1,\ t \neq 0,\ M \ge 1). \tag{21}$$

The last inequality holds since we have $\ln_t x \le x - 1$, $(x > 0,\ t \le 1,\ t \neq 0)$. Therefore, the inequality (21) gives a refinement for the upper bound of the Tsallis relative operator entropy $T_t(A|B)$ given in (15). We also have

$$T_t(A|B) \ge \frac{\ln_t M}{M - 1}(B - A) \ge B - A, \quad (t \ge 1,\ M \ge 1),$$

since we have the inequality $\ln_t x \ge x - 1$, $(x > 0,\ t \ge 1)$. Further, we put $M = e$ and take a limit $t \to 0$ in (21), we have the slightly refined upper bound for the relative operator entropy $S(A|B)$:

$$S(A|B) \le \frac{1}{e - 1}(B - A) \le B - A.$$

(ii) Since the function f is concave in Theorem 2.14, we have

$$(g(b) - g(a))(\langle f(A)h, h\rangle) \le (f(b) - f(a))\langle g(A)h, h\rangle$$

by Lemma 2.13. We take the functions $f(x) := \log x$ and $g(x) := \frac{x - 1}{e - 1}$ in the interval $[1, e]$ to satisfy the conditions in Theorem 2.14. Then we have

$$\langle (\log A)\, h, h\rangle \le \langle \left(\frac{A - I}{e - 1}\right) h, h\rangle$$

for any unit vector $h \in \mathcal{H}$.

Thus we have the operator inequality $\log A \leq \dfrac{A - \mathbf{1}_{\mathcal{H}}}{e-1}$ for $\mathbf{1}_{\mathcal{H}} \leq A \leq e\mathbf{1}_{\mathcal{H}}$. $\square$

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References

[1] Y. J. Cho, M. Matic, J. Pečarić: *Two mappings in connection to Jensen's inequality*, Panamerican Math. J. 12 (2002) 43–50.

[2] S. Dutta, S. Furuichi: *On log-sum inequalities*, Linear Multilinear Algebra 72/5 (2024) 812–827.

[3] S. Dutta, S. Furuichi, P. Guha: *A two-parameter entropy and its fundamental properties*, Rev. Math. Phys. 33 (2021), art. no. 2130003.

[4] J. I. Fujii, E. Kamei: *Relative operator entropy in noncommutative information theory*, Math. Japon. 34 (1989) 341–348.

[5] S. Furuichi, H. R. Moradi: *Advances in Mathematical Inequalities*, De Gruyter, Berlin (2020).

[6] T. Furuta: *Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators*, Linear Algebra Appl. 381 (2004) 219–235.

[7] H. Isa, M. Ito, E. Kamei, H. Tohyama, M. Watanabe: *Relative operator entropy, operator divergence, and Shannon inequality*, Sci. Math. Japon. Online e-2012(30) (2012) 353–362.

[8] B. Mond: *On Jensen's inequality for operator convex functions*, Houston J. Math. 21 (1995) 739–754.

[9] C. P. Niculescu, L. E. Persson: *Convex Functions and their Applications*, 2nd ed., Springer, Berlin (2018).

[10] J. Pečarić, T. Furuta, J. Mićić Hot, Y. Seo: *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb (2005).

[11] K. Yanagi, K. Kuriyama, S. Furuichi: *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Algebra Appl. 394 (2005) 109–118.

[12] L. Zou: *Operator inequalities associated with Tsallis relative operator entropy*, Math. Inequal. Appl. 18/2 (2015) 401–406.