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Second-Order Optimality Conditions for Efficiency in C^1-Smooth Robustly Quasiconvex Multiobjective Programming Problems

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We study some second-order optimality conditions in terms of second-order Mordukhovich's subdifferentials for the weak efficiency of C^1-smooth robustly quasiconvex multiobjective programming problem with set, inequality and equality constraints. We provide some second - order generalized constraint qualifications and two necessary conditions for the robust quasiconvexity of C^1-smooth/ $C^{1,1}-$smooth mappings. Using this result, weak and strong second-order Karush-Kuhn-Tucker-type necessary optimality conditions in terms of Mordukhovich's subdifferentials / Fréchet's subdifferentials are provided. Under some suitable assumptions, some second-order Karush-Kuhn-Tucker type sufficient optimality conditions are presented too. Examples demonstrate the applicability of the obtained results.

Keywords: C^1-smooth robustly quasiconvex multiobjective programming, constraints, weak efficiency, KKT-type second-order optimality conditions, second-order constraint qualifications, second-order Mordukhovich's subdifferential.

2000 Mathematics Subject Classification: 90C46, 90C29, 49K27, 49J52.

1. Introduction

The quasiconvex functions share many nice properties of convex functions and cover some models which are effective and adaptable to real-world situations. As far as we know, if the continuously differentiable function f of the minimization problem is quasiconvex and its derivative at the optimal point is 0, then the origin is the local minimum of that problem. Therefore, the class of quasiconvex functions play an important role in nonconvex optimization; for example see Mangasarian [23, 24]. More recently, Khanh and Phat [17, 18] have provided the second-order fundamental characterizations of C^1-smooth robustly quasiconvex functions, where the authors have achieved some necessary conditions in terms of Mordukhovich's subdifferentials. To the best of our knowledge, these characterizations are useful for establishing second-order optimality conditions for robustly quasiconvex multiobjective programming

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problems via the Mordukhovich subdifferentials with C^1-smooth even $C^{1,1}$-smooth data. However, this application to the class of quasiconvex optimization problems is not considered; for instance see Mordukhovich [25–27], An and Yen [1] and the references therein. So, it is important to find some second-order necessary and sufficient optimality conditions in terms of Mordukhovich's subdifferentials for a class of nonlinear programming problems with non-convex and C^1-smooth data.

C^1-smooth robustly quasiconvex multiobjective programming problems are multiobjective optimization problems whose concerned functions are quasiconvex under small linear perturbations (known as robustly quasiconvex functions). These problems lie among the class of quasiconvex and convex multiobjective optimization problems. Any well-known result in the literature which is true for quasiconvex problems will remain true for robustly quasiconvex problems. Further, the C^1-smooth robustly quasiconvex multiobjective programming problems also share a lot of nice properties of convex multiobjective optimization problems, more precisely, we refer the reader to the references [1, 3, 7, 8, 12, 13, 15, 28]. Second-order optimality conditions have attracted many outstanding mathematicians in the last years and the interest in this kind of problems has grown more and more because of their applications in optimal control, economic, games, etc, e.g., in [14, 19, 21] and the references therein. Again, to the best of our knowledge, the results on optimality of second-order via the second-order Mordukhovich/Fréchet subdifferentials for the class of these problems are not considered, see [2, 18, 20, 22–24, 29–31, 34] and the references therein.

As far as we know, the second-order Mordukhovich and/or limiting subdifferentials, which was introduced by Mordukhovich [25, 26], has been well recognized as a convenient tool to study many important issues in variational analysis and optimization, see [1, 4–6, 9–11, 16, 27] and some other related publications. Recently, as shown in [17], the authors obtained two necessary conditions for the the robust quasiconvexity of C^1-smooth functions via the second-order Fréchet and/or Mordukhovich subdifferentials. This results are of interest and can be used to study the C^1-smooth robustly quasiconvex multiobjective programming problems with set, inequality and equality constraints. This is a motivation for our present work.

In this work, we first analyze a C^1-smooth robustly quasiconvex multiobjective programming problem with set, inequality and equality constraints. Secondly, we introduce the cone of critical directions and subcritical directions at a given point and three constraint qualifications of the (GBCQ), (GBCQs) and (NCQ) types. We also obtain a necessary condition for the class of robustly quasiconvex functions with moduli $\alpha > 0$ via the second-order Mordukhovich subdifferentials. In additon, weak and strong Karush-Kuhn-Tucker necessary and/or sufficient second-order optimality conditions for weak efficiency to such problem in terms of the second-order Mordukhovich/Fréchet subdifferentials are derived.

The remainder of this paper is organized as follows. After some preliminaries and definitions, Section 3 is devoted to constructing strong and weak Karush-Kuhn-Tucker second-order necessary optimality conditions for the weakly efficient solution via the second-order Mordukhovich/ Fréchet subdifferentials for the considered problem. Section 4 presents Karush-Kuhn-Tucker second-order sufficient optimality conditions for local weakly efficient solution via the second-order Mordukhovich/ Fréchet subdifferentials for the considered problem. Illustrative examples are provided for our results.

2. Preliminaries and definitions

Throughout the paper we use standard notation of convex analysis, set-valued analysis, variational analysis and generalized differentiation. We refer the reader to books [2, 16, 23–28] for more details and discussions. For the sake of clarity, let $\mathbb{R}^n$ be the usual n-dimensional Euclidean space, $\mathscr{B}$ be the closed unit ball in $\mathbb{R}^n$, $\langle x, v\rangle$ the standard inner product of $x, v \in \mathbb{R}^n$. We will write $\langle A, x\rangle + \langle B, y\rangle \prec z$ with respect to a binary relation $\prec$ on $\mathbb{R}$, to mean that $\langle a, x\rangle + \langle b, y\rangle \prec z$ for all $a \in A, b \in B$, where $x, y, z \in \mathbb{R}^n$, $A, B \subset \mathbb{R}^n$. For each $A \subset \mathbb{R}^n$, the interior, the closure, the relative interior and the convex hull of A are denoted respectively by $\mathrm{int}A$, $\mathrm{cl}A$, $\mathrm{ri}A$ and $\mathrm{co}A$. We use the symbol $\mathrm{cone}A$ to denote the *cone* generated by A, and $|A|$ signifies the cardinality of A, i.e., the number of elements of A. The *positive polar cone* of A is denoted by A^+ and given as

$$A^+ := \big\{x^* \in \mathbb{R}^n \,\big|\, \langle x^*, a\rangle \geq 0,\ \forall\, a \in A\big\}.$$

The *strictly positive polar cone* of A is denoted by $A^{s,+}$ and given as

$$A^{s,+} := \big\{x^* \in \mathbb{R}^n \,\big|\, \langle x^*, a\rangle \,>\, 0, \forall\, a \in A \setminus \{0\}\big\}.$$

The *contingent cone* to A at $\overline{x} \in \mathrm{cl}A$ is denoted by $T(A; \overline{x})$ and given as

$$T(A;\overline{x}) = \Big\{v \in \mathbb{R}^n \,|\, \exists\, t_k \to 0^+,\ \exists\, v_k \to v \text{ such that } \overline{x} + t_k v_k \in A\ \forall\, k \geq 1\Big\}.$$

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be a set-valued mapping. The *effective domain* and the *graph* of F are defined as

$$\mathrm{dom}F := \{x \in \mathbb{R}^n \,|\, F(x) \neq \emptyset\}, \ \text{ and } \ \mathrm{graph}F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, y \in F(x)\}.$$

We recall from [25–27, 32, 33] that the sequential Painlevé-Kuratowski upper/outer limit of F as $x \to \overline{x}$ is

$$\operatorname*{Limsup}_{x \longrightarrow \overline{x}} F(x) = \left\{ x^* \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists \text{ sequences } x_k \longrightarrow \overline{x} \text{ and } x_k^* \longrightarrow x^* \\ \text{with } x_k^* \in F(x_k) \ \text{ for every } \ k \in \mathbb{N} \end{array} \right\}. \tag{1}$$

Given $\Omega \subset X$ and $\overline{x} \in \Omega$. Then the Fréchet normal cone to Ω at $\overline{x}$ is defined as

$$\hat{N}(\overline{x}; \Omega) := \Big\{x^* \in \mathbb{R}^n \,\Big|\, \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \leq 0\Big\}, \tag{2}$$

where the symbol $x \xrightarrow{\Omega} \overline{x}$ signifies that $x \to \overline{x}$ with $x \in \Omega$. Letting for the sake of convenience $\hat{N}(\overline{x}; \Omega) = \emptyset$ if $\overline{x} \in \mathbb{R}^n \setminus \Omega$ and employing the outer limit (1) to $\hat{N}(. , \Omega)$, we define the Mordukhovich normal cone to $\Omega \subset \mathbb{R}^n$ at $\overline{x}$ as follows:

$$N(\overline{x}; \Omega) := \operatorname*{Limsup}_{x \xrightarrow{\Omega} \overline{x}} \hat{N}(x; \Omega). \tag{3}$$

It should be mentioned that $N(\overline{x}; \Omega)$ is often nonconvex, in contrast to $\hat{N}(\overline{x}; \Omega)$, which may be null at boundary points. When Ω is a convex set, both constructions

(2)–(3) are reduced to the classical normal cone. Following Mordukhovich [25–27], the Mordukhovich (resp. Fréchet) subdifferential of $\varphi:\mathbb{R}^n\to\mathbb{R}$ at $\overline{x}\in\mathbb{R}$ is defined by

$$\partial\varphi(\overline{x}) := \left\{x^* \in \mathbb{R}^n \,\middle|\, (x^*, -1) \in N((\overline{x}, \varphi(\overline{x})); \mathrm{epi}\varphi)\right\}$$
$$\left(\hat{\partial}\varphi(\overline{x}) := \left\{x^* \in \mathbb{R}^n \,\middle|\, (x^*, -1) \in \hat{N}((\overline{x}, \varphi(\overline{x})); \mathrm{epi}\varphi)\right\}\right),$$

where $\mathrm{epi}\varphi := \{(x,r)\in\mathbb{R}^n\times\mathbb{R} : r \geq \varphi(x)\}$. From the notion of the graph of F, for all $y^*\in\mathbb{R}$, define the Fréchet coderivative $\hat{D}^*F(\overline{x},\overline{y}) : \mathbb{R} \rightrightarrows \mathbb{R}^n$ at $(\overline{x},\overline{y})\in\mathrm{graph}F$ by

$$\hat{D}^*F(\overline{x},\overline{y})(y^*) := \left\{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in \hat{N}((\overline{x},\overline{y}); \mathrm{graph}F)\right\},$$

and the Mordukhovich coderivative $D^*F(\overline{x},\overline{y}) : \mathbb{R} \rightrightarrows \mathbb{R}^n$ at $(\overline{x},\overline{y}) \in \mathrm{graph}F$ by

$$D^*F(\overline{x},\overline{y})(y^*) := \left\{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N((\overline{x},\overline{y}); \mathrm{graph}F)\right\}, \;\; y^* \in \mathbb{R},$$

or equivalently,

$$D^*F(\overline{x},\overline{y})(y^*) := \left\{x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{l}\exists\,(x_n,y_n) \to (\overline{x},\overline{y}),\ (x_n^*,y_n^*) \to (x^*,y^*) \text{ with}\\ (x_n^*,-y_n^*) \in N((x_n,y_n);\, \mathrm{graph}F) \text{ as } n \to +\infty\end{array}\right\},$$

where $\overline{y}$ is omitted when $F = f : \mathbb{R}^n \to \mathbb{R}$ is real-valued function.

Definition 2.1. [25–27] Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Fréchet differentiable function at $\overline{x} \in \mathbb{R}^n$.

(i) The *second-order Fréchet subdifferential* of φ at $\overline{x}$ relative to $\overline{y} = \nabla\varphi(\overline{x})$ is the set-valued mapping $\hat{\partial}^2\varphi(\overline{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\hat{\partial}^2\varphi(\overline{x},\overline{y})(u) := (\hat{D}^*\partial\varphi)(\overline{x},\overline{y})(u),\;\; \forall\, u \in \mathbb{R}^n.$$

Each x^* in the second-order Fréchet subdifferential of the function φ at $\overline{x}$ relative to $\overline{y} = \nabla\varphi(\overline{x})$ is called a *second-order Fréchet subgradient* of φ at $\overline{x}$ relative to $\overline{y} = \nabla\varphi(\overline{x})$.

(ii) The *second-order Mordukhovich subdifferential* of φ at $\overline{x}$ relative to $\overline{y} = \nabla\varphi(\overline{x})$ is a set-valued mapping $\partial^2\varphi(\overline{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\partial^2\varphi(\overline{x},\overline{y})(u) := (D^*\partial\varphi)(\overline{x},\overline{y})(u),\;\; \forall\, u \in \mathbb{R}^n.$$

Each x^* in the second-order Mordukhovich subdifferential of φ at $\overline{x}$ relative to $\overline{y} = \nabla\varphi(\overline{x})$ is called *second-order Mordukhovich subgradient* of φ at $\overline{x}$ relative to $\overline{y} = \nabla\varphi(\overline{x})$. □

According to [14,20,21], φ is $C^{1,1}$ around $\overline{x}$, which is denoted by $\varphi \in C^{1,1}$ around $\overline{x}$ iff it is Fréchet differentiable around $\overline{x}$ with its gradient $\nabla\varphi$ being locally Lipschitzian around $\overline{x}$; φ is C^1 around $\overline{x}$, which is denoted by $\varphi \in C^1$ around $\overline{x}$ iff it is continuously Fréchet differentiable in a neighborhood of $\overline{x}$. Similarly, $\varphi \in C^2$ around $\overline{x}$ iff it is twice continuously Fréchet differentiable in a neighborhood of $\overline{x}$. We mention that when we say that $\nabla\varphi$ is continuous at $\overline{x}$ we assume that $\nabla\varphi(x)$ exists for all x close $\overline{x}$. Due to Khan, Tammer and Zalinescu [16], one says that φ is Lipschitz on U iff,

$|\varphi(x) - \varphi(y)| \leq L\|x - y\|$ for any $x, y \in U$ and for some $L > 0$. If φ is Lipschitz on some neighborhood of $\overline{x}$, we will say that φ is Lipschitz around $\overline{x}$. We have the following implications:

$$\varphi \in C^{1,1} \text{ around } \overline{x} \implies \nabla\varphi \text{ continuous around } \overline{x} \implies \varphi \text{ Lipschitz around } \overline{x}.$$

In general, the Fréchet second-order subdifferential and the one of Mordukhovich are incomparable. However, if $\varphi \in C^1$ around $\overline{x}$, i.e., continuously Fréchet differentiable in a neighborhood of $\overline{x}$, then

$$\hat{\partial}^2\varphi(\overline{x}, \overline{y})(u) \subset \partial^2\varphi(\overline{x}, \overline{y})(u), \ \ \forall\, u \in \mathbb{R}^n. \tag{4}$$

From now on, if not otherwise specified, for $\varphi \in C^{1,1}$ around $\overline{x}$ and for $u \in \mathbb{R}^n$, by φ_u we will denote the function $\varphi_u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi_u(x) = \langle \nabla\varphi(x), u\rangle\,, \forall\, x \in \mathbb{R}^n.$$

We remark that by virtue of the $C^{1,1}$-smoothness of φ, φ_u is locally Lipschitz continuous on $\mathbb{R}^n$. For this case, we omit the following scalarization formulas

$$\partial^2\varphi(\overline{x}, \overline{y})(u) := \partial^2\varphi(\overline{x})(u) := \partial\varphi_u(\overline{x}), \ \ \forall\, u \in \mathbb{R}^n. \tag{5}$$

If $\varphi \in C^2$ around $\overline{x}$, it holds that

$$\partial^2\varphi(\overline{x})(u) = \hat{\partial}^2\varphi(\overline{x})(u) = \{\nabla^2\varphi(\overline{x})u\}, \ \ \forall\, u \in \mathbb{R}^n. \tag{6}$$

The following generalized convexity plays a key role in the literature.

Definition 2.2. [28] A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be

(i) *convex* if for any $x, y \in \mathbb{R}^n$ and for any $t \in [0, 1]$,

$$\varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y).$$

(ii) *quasiconvex* if for any $x, y \in \mathbb{R}^n$ and for any $t \in [0, 1]$,

$$\varphi(tx + (1 - t)y) \leq \max\{\varphi(x), \ \varphi(y)\}.$$

(iii) *pseudoconvex* if for any $x, y \in \mathbb{R}^n$,

$$\varphi(x) < \varphi(y) \Longrightarrow \limsup_{t \to 0^+} \frac{\varphi(y + t(x - y)) - \varphi(y)}{t} < 0.$$

(iv) *robustly quasiconvex* with modulus $\alpha > 0$ if the function $\varphi_v : x \mapsto \varphi(x) + \langle v, x\rangle$ is quasiconvex for all $v \in \alpha\mathscr{B}$.

(v) *robustly pseudoconvex* with modulus $\alpha > 0$ if the function $\varphi_v : x \mapsto \varphi(x) + \langle v, x\rangle$ is pseudoconvex for all $v \in \alpha\mathscr{B}$. □

It follows, immediately, from the above definitions, that pseudoconvexity implies quasiconvexity, φ is robustly quasiconvex with modulus $\alpha > 0$ iff φ_v is quasiconvex for all $v \in \mathbb{R}^n$ such that $\|v\| < \alpha$, φ is convex iff φ is robustly quasiconvex for any modulus $\alpha > 0$, and robust quasiconvexity is equivalent to robust pseudoconvexity for C^1-smooth functions.

In the case when φ is a robustly quasiconvex function with modulus $\alpha > 0$, we write

$$L_{(\varphi,\alpha)} := \Big\{(x, u) \in \mathbb{R}^{2n} \,\Big|\, |\langle \nabla\varphi(x), u\rangle| < \alpha\|u\|\Big\}.$$

Lemma 2.3. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a robustly quasiconvex function with modulus $\alpha > 0$. If φ is a $C^{1,1}$-smooth function then for all $(x, u) \in L_{(\varphi,\alpha)}$, then*

$$\Big\langle \hat{\partial}^2\varphi(x)(u),\ u\Big\rangle \geq 0.$$

Proof. Follows from Khanh and Phat [17], which completes the proof. □

Lemma 2.4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a robustly quasiconvex function with modulus $\alpha > 0$. If φ is a C^1-smooth function then for all $(x, u) \in L_{(\varphi,\alpha)}$, then*

$$\big\langle \partial^2\varphi(x)(u),\ u\big\rangle \geq 0.$$

Proof. Analogous to the proof of Lemma 2.3, we get the desired conclusion. □

In the present paper, we deal with the following C^1-smooth robustly quasiconvex multiobjective programming problem with set, inequality and equality constraints:

$$\text{(RQMP)} \qquad \begin{array}{l} \text{minimize } \ f(x) \\ \text{subject to } \ x \in K := \big\{x \in \Omega \,|\, g(x) \leq 0, \ \ h(x) = 0\big\}, \end{array}$$

where f, g and h are functions from $\mathbb{R}^n$ into $\mathbb{R}^m$, $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, Ω a nonempty convex subset of $\mathbb{R}^n$ and K is the feasible set for the problem (RQMP), $f = (f_1, \ldots, f_m)$, $g = (g_1, \ldots, g_p)$ and $h = (h_1, \ldots, h_q)$ with f_i ($\forall\, i \in I := \{1, \ldots, m\}$), g_j ($\forall\, j \in J := \{1, \ldots, p\}$) are robustly quasiconvex functions with moduli $\alpha_i > 0$ ($\forall\, i \in I$), $\beta_j > 0$ ($\forall\, j \in J$), respectively, and h_k ($\forall\, k \in T := \{1, \ldots, q\}$) are Fréchet differentiable functions at the point under consideration.

Definition 2.5. A vector $\overline{x} \in K$ is said to be a *weakly efficient solution* of problem (RQMP) iff, there is no $x \in K$ such that $f(x) \leq f(\overline{x})$ and $f(x) \neq f(\overline{x})$. If there exists $r > 0$ such that there is no $x \in K \cap B(\overline{x}, r)$ such that $f(x) \leq f(\overline{x})$ and $f(x) \neq f(\overline{x})$, then $\overline{x}$ is called a *local weakly efficient solution* of problem (RQMP).

For the sake of brevity, for each $\overline{x} \in K$, consider the active index set $J(\overline{x})$, defined by $J(\overline{x}) := \{j \in J \,|\, g_j(\overline{x}) = 0\}$. We set

$$K(H) = \ker\nabla h(\overline{x}) := \{v \in \mathbb{R}^n \,|\, \langle \nabla h(\overline{x}), v\rangle = 0\},$$

$H := \{x \in \mathbb{R}^n \,|\, h(x) = 0\}$, $\alpha := \min_{i\in I} \alpha_i > 0$, $\beta := \min_{i\in J(\overline{x})} \beta_j > 0$, and for each $\overline{x} \in \mathrm{cl}\Omega$ and $v \in C(\overline{x})$, set $M := \prod_{i\in I} \partial^2 f_i(\overline{x})(v) \times \prod_{j\in J(\overline{x})} \partial^2 g_j(\overline{x})(v)$,

$$\nabla f_i(\overline{x})(T(\Omega; \overline{x})) := \bigcup_{w\in T(\Omega;\overline{x})} \langle \nabla f_i(\overline{x}),\ w\rangle\,,\ \ i \in I,$$

$$\nabla g_j(\overline{x})(T(\Omega; \overline{x})) := \bigcup_{w\in T(\Omega;\overline{x})} \langle \nabla g_j(\overline{x}),\ w\rangle\,,\ \ j \in J,$$

$$\nabla h_k(\overline{x})(T(\Omega; \overline{x})) := \bigcup_{w\in T(\Omega;\overline{x})} \langle \nabla h_k(\overline{x}),\ w\rangle\,,\ \ k \in T.$$

We define the following sets

$$C(K,\overline{x}) := \Big\{ v \in \mathbb{R}^n \,\Big|\, \big|\,\langle \nabla g_j(\overline{x}), v\rangle\,\big| < \beta\|v\|,\;\; j \in J(\overline{x}) \Big\} \cup \{0\},$$

$$C(f,\overline{x}) := \Big\{ v \in \mathbb{R}^n \,\Big|\, \big|\,\langle \nabla f_i(\overline{x}), v\rangle\,\big| < \alpha\|v\|,\;\; i \in I \Big\} \cup \{0\}.$$

It is easy to verify that

$$C(K,\overline{x}) \subset \bigcap_{j\in J(\overline{x})} \big\{ v \in \mathbb{R}^n \,\big|\, \big|\,\langle \nabla g_j(\overline{x}), v\rangle\,\big| < \beta_j\|v\| \big\} \cup \{0\}, \tag{7}$$

$$C(f,\overline{x}) \subset \bigcap_{i\in I} \big\{ v \in \mathbb{R}^n \,\big|\, \big|\,\langle \nabla f_i(\overline{x}), v\rangle\,\big| < \alpha_i\|v\| \big\} \cup \{0\}. \tag{8}$$

At the end of this section, in order to establish optimality via the second-order Mordukhovich subdifferentials for the weakly efficient solution of problem (RQMP), we introduce the critical/subcritical directions cone and the constraint qualifications (GBCQ) and (GBCQs) (for some $s \in I$), which will be useful in the present paper.

Definition 2.6. Let a feasible vector $\overline{x} \in K$ to the problem (RQMP).

(i) The *critical directions cone* is denoted as $C(\overline{x})$, defined by

$$C(\overline{x}) := C(K,\overline{x}) \cap C(f,\overline{x}).$$

(ii) The *subcritical directions cone* is denoted as $C_T(\overline{x})$, defined by

$$C_T(\overline{x}) := C(\overline{x}) \cap T(\Omega;\overline{x}). \qquad \square$$

For simplicity, for each $\overline{x} \in \Omega$ and some $s \in I$, we define the following sets

$$C^1_{\overline{x}} := \Big\{ w \in \mathbb{R}^n \mid \langle \nabla f_i(\overline{x}), w\rangle < 0,\;\; \langle \nabla g_j(\overline{x}), w\rangle < 0,\;\; \forall\, i \in I,\;\; \forall\, j \in J(\overline{x}) \Big\},$$

$$D(g,h) := \left\{ (v,w) \in \mathbb{R}^{2n} \;\middle|\; \begin{array}{l} \langle \nabla g_j(\overline{x}), w\rangle + \big\langle \partial^2 g_j(\overline{x})(v), v\big\rangle \\ \in -\big[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\big] \setminus \{0\},\;\; \forall\, j \in J(\overline{x}), \\ \langle \nabla h_k(\overline{x}), w\rangle \in -\big[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\big] \;\forall\, k \in T \end{array} \right\},$$

$$D_s(f,g,h) := \left\{ (v,w) \in D(g,h) \;\middle|\; \begin{array}{l} \langle \nabla f_i(\overline{x}), w\rangle + \big\langle \partial^2 f_i(\overline{x})(v), v\big\rangle \\ \in -\big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big] \setminus \{0\}\, \forall\, i \in I \setminus \{s\} \end{array} \right\}.$$

Definition 2.7. The following constraint qualifications at $\overline{x} \in K$ are considered:

(GBCQ) $\qquad C(\overline{x}) \times T(\Omega;\overline{x}) \cap D(g,h) \neq \emptyset;$

(GBCQs) $\qquad C(\overline{x}) \times T(\Omega;\overline{x}) \cap D_s(f,g,h) \neq \emptyset$ for some $s \in I$.

We mention that the constraint qualifications (GBCQ) and (GBCQs) are called the second-order generalized Ben-Tal constraint qualifications in the sense of second-order Mordukhovich subdifferentials. The constraint qualifications above are used to establish weak and strong Krush-Kuhn-Tucker second-order optimality conditions for the problem (RQMP) in terms of the Mordukhovich second-order subdifferentials, which are different from the existing one in the literature, e.g., in Mordukhovich [25–27] and the references therein.

3. Primal and dual second-order necessary optimality conditions

In this section, we always consider problem (RQMP) and we suppose that all of the following assumptions hold:

(A1) $\overline{x} \in \Omega \cap H$;

(A2) h is continuous in a neighborhood of $\overline{x}$;

(A3) the constraint qualification of the (CQ) type holds, where

$$\text{(CQ)} \quad 0 \in \sum_{k\in T} \gamma_k \nabla h_k(\overline{x}) - [T(\Omega;\overline{x})]^+ \implies \gamma_1 = \ldots = \gamma_q = 0.$$

It is well-known that if h is Fréchet differentiable at $\overline{x}$, then

$$T(\Omega;\overline{x}) \cap K(H) = T(\Omega \cap H;\overline{x}) = \mathrm{cl}\big[K(H) \cap \mathrm{cone}(\Omega - \overline{x})\big]. \tag{9}$$

In order to obtain primal and dual second-order necessary optimality conditions for the weak efficiency of problem (RQMP), we establish the following fundamental lemma.

Lemma 3.1. *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Then*

$$C^1_{\overline{x}} \cap T(\Omega;\overline{x}) \cap K(H) = \emptyset. \tag{10}$$

Proof. Let $\overline{x} \in K$ be a feasible solution to the problem (RQMP). By the initial assumptions, it follows from (9) that the equality $T(\Omega;\overline{x}) \cap K(H) = T(\Omega \cap H;\overline{x})$ holds, which, combined with the equality $C^1_{\overline{x}} \cap T(\Omega;\overline{x}) \cap K(H) = \emptyset$, guarantees that

$$C^1_{\overline{x}} \cap T(\Omega \cap H;\overline{x}) = \emptyset.$$

In consequence, there is no $w \in \mathbb{R}^n$ which solves the system

$$\begin{cases} \langle \nabla f_i(\overline{x}), w\rangle < 0, \ \ \forall\, i \in I, \\ \langle \nabla g_j(\overline{x}), w\rangle < 0, \ \ \forall\, j \in J(\overline{x}), \\ w \in T(\Omega \cap H;\overline{x}). \end{cases}$$

Taking into account Lemma 2.3 in [31], we get the desired conclusion. □

Theorem 3.2. (Primal second-order optimality condition) *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that $v \in C(\overline{x})$, the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^1-smooth and h satisfies* (A2) *and* (A3). *Then the following system is incompatible in $w \in \mathbb{R}^n$:*

$$\begin{cases} \langle \nabla f_i(\overline{x}), w\rangle + \langle \partial^2 f_i(\overline{x})(v), v\rangle < 0, \ \ \forall\, i \in I, \\ \langle \nabla g_j(\overline{x}), w\rangle + \langle \partial^2 g_j(\overline{x})(v), v\rangle < 0, \ \ \forall\, j \in J(\overline{x}), \\ \langle \nabla h_k(\overline{x}), w\rangle = 0, \ \ \forall\, k \in T, \\ w \in T(\Omega;\overline{x}). \end{cases} \tag{11}$$

Proof. Suppose the contrary, that is, there exists $w \in \mathbb{R}^n$, which solves the system (11). It is evident that $w \in T(\Omega;\overline{x}) \cap K(H)$. Taking into account Lemma 3.1, it follows that

$$w \notin C^1_{\overline{x}}. \tag{12}$$

On one side we have $v \in C(\overline{x})$, which is equivalent to $v \in C(K, \overline{x})$ and $v \in C(f, \overline{x})$. As $v \in C(K, \overline{x})$, it follows from the inclusion (7) that $(\overline{x}, v) \in \bigcap_{j \in J(\overline{x})} L_{(g_j, \beta_j)}$.
In view of Lemma 2.4 we obtain $\langle \partial^2 g_j(\overline{x})(v), v \rangle \geq 0, \ \forall\, j \in J(\overline{x})$.
Consequently, $\langle \nabla g_j(\overline{x}), w \rangle < 0$ for all $j \in J(\overline{x})$. On the other side, in a similar way to the previous proof, it results that $\langle \nabla f_i(\overline{x}), w \rangle < 0$ for all $i \in I$. In consequence, $w \in C^1_{\overline{x}}$, which conflicts with (12), what we had to show. □

Corollary 3.3. (Primal second-order optimality condition) *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP)*. Suppose that $v \in C(\overline{x})$, the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are, either $C^{1,1}$-smooth, or C^1-smooth, and h satisfies* (A2) *and* (A3)*. Then the following system is incompatible in $w \in \mathbb{R}^n$:*

$$\begin{cases} \langle \nabla f_i(\overline{x}), w \rangle + \left\langle \hat{\partial}^2 f_i(\overline{x})(v), v \right\rangle < 0, \ \ \forall\, i \in I, \\ \langle \nabla g_j(\overline{x}), w \rangle + \left\langle \hat{\partial}^2 g_j(\overline{x})(v), v \right\rangle < 0, \ \ \forall\, j \in J(\overline{x}), \\ \langle \nabla h_k(\overline{x}), w \rangle = 0, \ \ \forall\, k \in T, \\ w \in T(\Omega; \overline{x}). \end{cases} \tag{13}$$

Proof. By an argument analogous to that used for the proof of Theorem 3.2, if f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^1-smooth, then one obtains $\langle \partial^2 g_j(\overline{x})(v), v \rangle \geq 0$ $\forall\, j \in J(\overline{x})$. It should be pointed out that

$$\left\langle \hat{\partial}^2 g_j(\overline{x})(v), v \right\rangle \geq 0, \ \ \forall\, j \in J(\overline{x}). \tag{14}$$

If $\hat{\partial}^2 g_j(\overline{x})(v) = \emptyset$ ($j \in J(\overline{x})$), there is nothing to prove. Conversely, under (4), it entails that $\langle \hat{\partial}^2 g_j(\overline{x})(v), v \rangle \geq 0$, and so, inequality (14) is satisfied. In case the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are $C^{1,1}$-smooth, then by arguing similarly as for proving Theorem 3.2 with observing Lemma 2.3 replacing Lemma 2.4, it ensures that (14) is satisfied too. The remaining part of the proof is similar as mentioned in the proof of Theorem 3.2. □

Remark 3.4. It should be mentioned here that (CQ) is equivalent to the following regularity condition (RC):

$$\text{(RC)} \qquad \sum_{k \in T} \gamma_k \nabla h_k(\overline{x})(T(\Omega; \overline{x})) \geq 0 \Longrightarrow \gamma_1 = \ldots = \gamma_q = 0.$$

In fact, by the definition of the positive polar cone of the cone $T(\Omega; \overline{x})$, the inequality $\sum_{k \in T} \gamma_k \nabla h_k(\overline{x})(T(\Omega; \overline{x})) \geq 0$ is equivalent to $0 \in \sum_{k \in T} \gamma_k \nabla h_k(\overline{x}) - [T(\Omega; \overline{x})]^+$, which proves that $\gamma_1 = \ldots = \gamma_q = 0$, as required. □

Theorem 3.5. (Dual second-order optimality condition) *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP)*. Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^1-smooth and h satisfies* (A2) *and* (A3)*. Then for all $v \in C(\overline{x})$, $z^*_{i,f} \in \partial^2 f_i(\overline{x})(v)$ ($\forall\, i \in I$) and $z^*_{j,g} \in \partial^2 g_j(\overline{x})(v)$ ($\forall\, j \in J(\overline{x})$), there exist*

$$\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega; \overline{x}))\right]^+ \ (\forall\, i \in I), \qquad \mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega; \overline{x}))\right]^+ \ (\forall\, j \in J)$$

not all equal to zero and $\gamma_k \in \left[\nabla h_k(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, k \in T$) such that

$$\sum_{i\in I}\lambda_i\left\langle\nabla f_i(\overline{x}),w\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\nabla g_j(\overline{x}),w\right\rangle+\sum_{k\in T}\gamma_k\left\langle\nabla h_k(\overline{x}),w\right\rangle\geq 0\ \forall\, w\in T(\Omega;\overline{x}), \tag{15}$$

$$\sum_{i\in I}\lambda_i\left\langle z^*_{i,f},v\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle z^*_{j,g},v\right\rangle\geq 0, \tag{16}$$

$$\mu_j g_j(\overline{x})=0,\ \ j\in J. \tag{17}$$

Proof. By assumptions, f_i ($\forall\, i\in I$) and g_j ($\forall\, j\in J(\overline{x})$) are C^1-smooth, h is Fréchet differentiable at $\overline{x}$ and satisfies (A2) and (A3). We consider the cone N defined by

$$N=\prod_{i\in I}\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\times\prod_{j\in J(\overline{x})}\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\times\prod_{k\in T}\nabla h_k(\overline{x})(T(\Omega;\overline{x})).$$

Since Ω is convex, the cone $T(\Omega;\overline{x})=\operatorname{cl}\operatorname{cone}(\Omega-\overline{x})$ is convex too, which ensures the convexity of N. In view of Theorem 3.2, we deduce that $\langle -a,v\rangle\notin N$, where $a:=\big((z^*_{i,f})_{i\in I},(z^*_{j,g})_{j\in J(\overline{x})}\big)\in M$. It can be easily seen that $\langle -a,v\rangle\in\mathbb{R}^m\times\mathbb{R}^{|J(\overline{x})|}$, $\mathrm{ri}N\neq\emptyset$ and $\langle -a,v\rangle\notin\mathrm{ri}N$. By using two seperation theorems in Rockafellar's book (see [28], for instance), there would exist $\lambda_i\in\mathbb{R}$ ($i\in I$), $\mu_j\in\mathbb{R}$ ($j\in J(\overline{x})$) and $\gamma_k\in\mathbb{R}$ ($k\in T$), not all equal to zero, such that for every $w_1,w_2,w_3\in T(\Omega;\overline{x})$,

$$\sum_{i\in I}\lambda_i\left\langle\nabla f_i(\overline{x}),w_1\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\nabla g_j(\overline{x}),w_2\right\rangle+\sum_{k\in T}\gamma_k\left\langle\nabla h_k(\overline{x}),w_3\right\rangle\geq 0,$$

$$0>-\sum_{i\in I}\lambda_i\left\langle z^*_{i,f},v\right\rangle-\sum_{j\in J(\overline{x})}\mu_j\left\langle z^*_{j,g},v\right\rangle.$$

We need to point out that

$$\lambda_i\in\big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big]^+\ (\forall\, i\in I),\quad \mu_j\in\big[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\big]^+\ (\forall\, j\in J(\overline{x}))$$

and $\gamma_k\in\big[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\big]^+$ ($\forall\, k\in T$). In fact, let us take $w_1,w_2,w_3\in T(\Omega;\overline{x})$ be arbitrary. For any $t,s,r>0$, it results that $tw_1,\ sw_2,\ rw_3\in T(\Omega;\overline{x})$ since $T(\Omega;\overline{x})$ is cone, and further, the following inequality holds true

$$\sum_{i\in I}\lambda_i\left\langle\nabla f_i(\overline{x}),tw_1\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\nabla g_j(\overline{x}),sw_2\right\rangle+\sum_{k\in T}\gamma_k\left\langle\nabla h_k(\overline{x}),rw_3\right\rangle\geq 0.$$

If there exists $i_0\in I$ such that $\lambda_{i_0}\notin\big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big]^+$, which means that there exists $w_0\in T(\Omega;\overline{x})$ such that for t be sufficiently large,

$$\sum_{i\in I}\lambda_i\left\langle\nabla f_i(\overline{x}),tw_0\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\nabla g_j(\overline{x}),sw_2\right\rangle+\sum_{k\in T}\gamma_k\left\langle\nabla h_k(\overline{x}),rw_3\right\rangle< 0.$$

This is a contradiction. This allows us to conclude that $\lambda_i\in\big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big]^+$ for every $i\in I$. Similarly, we also have a conclusion that $\mu_j\in\big[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\big]^+$ for all $j\in J(\overline{x})$ and $\gamma_k\in\big[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\big]^+$ for all $k\in T$. Thus, conditions (15)–(16) hold. By taking $\mu_j=0$ for any $j\notin J(\overline{x})$, equality (17) holds.

Under (CQ), it should be pointed out that $(\lambda,\mu)\neq(0,0)$. Indeed, if it was not so, then $\gamma\neq 0$.

In this case, for some $w \in T(\Omega;\overline{x})$, we obtain $\sum_{k\in T} \gamma_k \langle \nabla h_k(\overline{x}), w\rangle < 0$, or equivalently,

$$\sum_{i\in I} \lambda_i \langle \nabla f_i(\overline{x}), w\rangle + \sum_{j\in J(\overline{x})} \mu_j \langle \nabla g_j(\overline{x}), w\rangle + \sum_{k\in T} \gamma_k \langle \nabla h_k(\overline{x}), w\rangle < 0,$$

which contradicts with (15), what we had to show. □

Corollary 3.6. *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are, either C^1-smooth, or $C^{1,1}$-smooth, and h satisfies* (A2) *and* (A3). *Then, for all $v \in C(\overline{x})$, $z^*_{i,f} \in \hat{\partial}^2 f_i(\overline{x})(v)$ ($\forall\, i \in I$) and $z^*_{j,g} \in \hat{\partial}^2 g_j(\overline{x})(v)$ ($\forall\, j \in J(\overline{x})$), there exist*

$$\lambda_i \in \big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big]^+ \ (\forall\, i \in I), \quad \mu_j \in \big[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\big]^+ \ (\forall\, j \in J)$$

not all equal to zero and $\gamma_k \in \big[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\big]^+$ ($\forall\, k \in T$) such that conditions (15), (16) *and* (17) *are fulfilled.*

Proof. Arguing as in the proof of Theorem 3.5 by using Corollary 3.3 in place of Theorem 3.2, we get the desired conclusion. □

Corollary 3.7. *Let $\Omega = \mathbb{R}$ and $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are, either $C^{1,1}$-smooth, or C^1-smooth, and h satisfies* (A2) *and* (A3). *Then for all $v \in C(\overline{x})$, $z^*_{i,f} \in \hat{\partial}^2 f_i(\overline{x})(v)$ ($\forall\, i \in I$) and $z^*_{j,g} \in \hat{\partial}^2 g_j(\overline{x})(v)$ ($\forall\, j \in J(\overline{x})$), there exist*

$$\lambda_i \in \big[\nabla f_i(\overline{x})(\mathbb{R}^n)\big]^+ \ (\forall\, i \in I), \quad \mu_j \in \big[\nabla g_j(\overline{x})(\mathbb{R}^n)\big]^+ \ (\forall\, j \in J)$$

not all equal to zero and $\gamma_k \in \big[\nabla h_k(\overline{x})(\mathbb{R}^n)\big]^+$ ($\forall\, k \in T$) satisfying (16), (17) *and*

$$\sum_{i\in I} \lambda_i \nabla f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \nabla g_j(\overline{x}) + \sum_{k\in T} \gamma_k \nabla h_k(\overline{x}) = 0.$$

Proof. Follows immediately from the proof of Theorem 3.5 by considering the exact equalities

$$T(\Omega;\overline{x}) = \operatorname{cl}\operatorname{cone}(\Omega - \overline{x}) = \operatorname{cl}\operatorname{cone}(\mathbb{R}^n - \overline{x}) = \mathbb{R}^n.$$ □

Theorem 3.5 is illustrated by the following example.

Example 3.8. Let the real-valued functions f, g and h defined on $\mathbb{R}$ by

$$f(x) := 3\int_0^{|x|} \Phi(t)dt - vx \ \ \forall v \in]-2, 2[, \quad \Phi(t) := \begin{cases} 2t^2 + t^2 \sin(\frac{1}{t}) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

$$g(x) = -2 - vx \ \ \forall v \in]-1, 1[, \ \forall x \in \mathbb{R}, \quad h(x) = \sin(\pi x) \ \ \forall x \in \mathbb{R}.$$

Consider the problem (RQMP) being given above in which $\Omega = \mathbb{R}$, $T(\Omega;\overline{x}) = \mathbb{R}$, $K = \mathbb{R}$, $C(\overline{x}) = \mathbb{R}$, and $\overline{x} = 0$ is a weakly efficient solution of problem (RQMP).

Observe that f_v is a pseudoconvex $C^{1,1}$-smooth function for all $v \in [-2, 2]$, therefore f_v is also quasiconvex for all $v \in [-2, 2]$. Consequently, f is robustly quasiconvex with modulus 2. In other words, h is Fréchet differentiable at $\overline{x}$ with $\nabla h(\overline{x}) = \pi$ and moreover h is continuous on a neighborhood of $\overline{x}$. If $\gamma \nabla h(\overline{x})(\mathbb{R}) \geq 0$, i.e., $\pi\gamma a \geq 0$ for all $a \in \mathbb{R}$, it entails that $\gamma = 0$. So, the constraint qualification of the (RC) type holds. We compute that $I = \{1\}$, $J = \{1\}$, $T = \{1\}$, $J(\overline{x}) = \emptyset$ and the Fréchet derivative of f_1 at x is given by

$$\nabla f_1(x) = \begin{cases} 3\Phi(x) - v & \text{if } x \geq 0, \\ -3\Phi(-x) - v & \text{if } x < 0. \end{cases}$$

Consequently $\partial^2 f_1(\overline{x})(u) = [-3|u|,\ 3|u|]$ ($u \in \mathbb{R}$), and $\nabla f_1(\overline{x})(T(\Omega; \overline{x})) = \{0\}$ if $v = 0$, $\nabla f_1(\overline{x})(T(\Omega; \overline{x})) = \mathbb{R}$ if $v \neq 0$. Therefore, $[\nabla f_1(\overline{x})(T(\Omega; \overline{x}))]^+ = \mathbb{R}$ if $v = 0$, $[\nabla f_1(\overline{x})(T(\Omega; \overline{x}))]^+ = \{0\}$ if $v \neq 0$. All these results allow us to conclude that for all $u \in \mathbb{R}$ and $z^*_{1,f} \in [-3|u|,\ 3|u|]$, there exist $\lambda_1 \in \mathbb{R} \setminus \{0\}$ and $\mu_1 = \gamma_1 = 0$ satisfying (15), (16) and (17). In fact, in this setting, conditions (15) and (17) are obtained obviously. In order to check (16), by an easy computation one may pick $\lambda_1 \neq 0$ if $u.z^*_{1,f} = 0$, $\lambda_1 > 0$ if $u.z^*_{1,f} > 0$, $\lambda_1 < 0$ if $u.z^*_{1,f} < 0$, as we have to check. □

The next result provides a weak Karush-Kuhn-Tucker type second-order necessary optimality condition for the weakly efficient solution of problem (RQMP).

Theorem 3.9. (Fritz John second-order optimality condition) *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^1-smooth and h satisfies* (A2) *and* (A3). *Suppose also that the set M is a closed convex and the constraint qualification of the* (GBCQ) *type holds. Then for every $v \in C(\overline{x})$, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, i \in I$) with $\lambda := (\lambda_i)_{i \in I} \neq 0$, $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, j \in J$) and $\gamma_k \in \left[\nabla h_k(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, k \in T$) such that*

$$\sum_{i \in I} \lambda_i \langle \nabla f_i(\overline{x}), w\rangle + \sum_{j \in J(\overline{x})} \mu_j \langle \nabla g_j(\overline{x}), w\rangle + \sum_{k \in T} \gamma_k \langle \nabla h_k(\overline{x}), w\rangle \geq 0 \ \forall w \in T(\Omega; \overline{x}), \quad (18)$$

$$\sum_{i \in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v), v\right\rangle + \sum_{j \in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \geq 0, \quad (19)$$

$$\mu_j g_j(\overline{x}) = 0, \ \ j \in J. \quad (20)$$

Proof. Due to the result obtained in Theorem 3.2, we deduce that $\langle -M, v\rangle \cap N = \emptyset$. By an argument analogous to that used for the proof of Theorem 3.5, there exist reals $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, i \in I$), $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, j \in J$) not all equal to zero and $\gamma_k \in \left[\nabla h_k(\overline{x})(T(\Omega; \overline{x}))\right]^+$ ($\forall\, k \in T$) such that conditions (18)–(20) are fulfilled. We want to prove that $\lambda \neq 0$ under (GBCQ). In fact, if it was not so, then, we have $\lambda := (\lambda_i)_{i \in I} = 0$, and so, $\mu := (\mu_j)_{j \in J(\overline{x})} \neq 0$. Adding up both sides of the inequalities (18) - (19), we arrive at the following inequality

$$\sum_{j \in J(\overline{x})} \mu_j \Big[\langle \nabla g_j(\overline{x}), T(\Omega; \overline{x})\rangle + \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \Big] + \sum_{k \in T} \gamma_k \langle \nabla h_k(\overline{x}), T(\Omega; \overline{x})\rangle \geq 0. \quad (21)$$

Under (GBCQ), it means that there is $(v_0, w_0) \in C(\overline{x}) \times T(\Omega; \overline{x})$ such that

$$\langle \nabla g_j(\overline{x}), w_0 \rangle + \left\langle \partial^2 g_j(\overline{x})(v_0), v_0 \right\rangle \in -\left[\nabla g_j(\overline{x})(T(\Omega; \overline{x}))\right] \setminus \{0\}, \ \ \forall\, j \in J(\overline{x}),$$
$$\langle \nabla h_k(\overline{x}), w_0 \rangle \in -\left[\nabla h_k(\overline{x})(T(\Omega; \overline{x}))\right], \ \ \forall\, k \in T.$$

Since $\mu := (\mu_j)_{j \in J} \neq 0$, due to the proof of Theorem 3.5 it infers that $\mu_j = 0$ for any $j \notin J(\overline{x})$, and thus, there exists $j_0 \in J(\overline{x})$ such that $\mu_{j_0} \in \left[\nabla g_{j_0}(\overline{x})(T(\Omega; \overline{x}))\right]^{s,+}$. By invoking the concept of the strict positive polar cone $\left[\nabla g_{j_0}(\overline{x})(T(\Omega; \overline{x}))\right]^{s,+}$ gives us

$$\mu_{j_0} \Big[\langle \nabla g_{j_0}(\overline{x}), w_0 \rangle + \left\langle \partial^2 g_{j_0}(\overline{x})(v_0), v_0 \right\rangle \Big] < 0.$$

Observe that for all $j \in J(\overline{x})$, $j \neq j_0$, $k \in T$, it holds that

$$\mu_j \Big[\langle \nabla g_j(\overline{x}), w_0 \rangle + \left\langle \partial^2 g_j(\overline{x})(v_0), v_0 \right\rangle \Big] + \gamma_k \, \langle \nabla h_k(\overline{x}), w_0 \rangle \leq 0.$$

Consequently,

$$\sum_{j \in J(\overline{x})} \mu_j \Big[\langle \partial g_j(\overline{x}), w_0 \rangle + \left\langle \partial^2 g_j(\overline{x})(v_0), v_0 \right\rangle \Big] + \sum_{k \in T} \gamma_k \, \langle \nabla h_k(\overline{x}), w_0 \rangle < 0,$$

which conflicts with (21), and completes the proof. □

We state a direct consequence from Theorem 3.9, which its easy proof can be omitted.

Corollary 3.10. *Let $\Omega = \mathbb{R}^n$ and $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i $(\forall\, i \in I)$ and g_j $(\forall\, j \in J(\overline{x}))$ are C^1-smooth, and h satisfies* (A2) *and* (A3). *Suppose also that the set M is a closed convex and the constraint qualification of the* (GBCQ) *type holds. Then for every $v \in C(\overline{x})$, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(\mathbb{R}^n)\right]^+$ $(\forall\, i \in I)$ with $\lambda := (\lambda_i)_{i \in I} \neq 0$, $\mu_j \in \left[\nabla g_j(\overline{x})(\mathbb{R}^n)\right]^+$ $(\forall\, j \in J)$ and $\gamma_k \in \left[\nabla h_k(\overline{x})(\mathbb{R}^n)\right]^+$ $(\forall\, k \in T)$ such that*

$$\sum_{i \in I} \lambda_i \nabla f_i(\overline{x}) + \sum_{j \in J(\overline{x})} \mu_j \nabla g_j(\overline{x}) + \sum_{k \in T} \gamma_k \nabla h_k(\overline{x}) = 0,$$
$$\sum_{i \in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v), v \right\rangle + \sum_{j \in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v), v \right\rangle \geq 0,$$
$$\mu_j g_j(\overline{x}) = 0, \ \ j \in J.$$

The next result considers the second-order generalized Ben-Tal constraint qualification (GBCQs) for some $s \in I$ holds.

Theorem 3.11. (Weak KKT second-order optimality condition) *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i $(\forall\, i \in I)$ and g_j $(\forall\, j \in J(\overline{x}))$ are C^1-smooth, and h satisfies conditions* (A2) *and* (A3). *Suppose also that M is a closed convex set and the constraint qualification of the* (GBCQs) *type for some $s \in I$ holds. Then for every $v \in C(\overline{x})$, there exist $\lambda_s \in \left[\nabla f_s(\overline{x})(T(\Omega; \overline{x}))\right]^{s,+}$, $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega; \overline{x}))\right]^+$ $(\forall\, i \in I \setminus \{s\})$, and $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega; \overline{x}))\right]^+$ $(\forall\, j \in J)$ and $\gamma_k \in \left[\nabla h_k(\overline{x})(T(\Omega; \overline{x}))\right]^+$ $(\forall\, k \in T)$ such that conditions* (18)–(20) *are fulfilled.*

Proof. Under (GBCQs) for some $s \in I$, it is guaranteed that the constraint qualification (GBCQ) is true. According to Theorem 3.9, for every $v \in C(\overline{x})$, there exist $\lambda_i \in \big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big]^+$ $(\forall\, i \in I)$, $\lambda := (\lambda_i)_{i\in I} \neq 0$, $\mu_j \in \big[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\big]^+$ $(\forall\, j \in J)$ and $\gamma_k \in \big[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\big]^+$ $(\forall\, k \in T)$ satisfying (18)–(20). We need to point out that $\lambda_s \in \big[\nabla f_s(\overline{x})(T(\Omega;\overline{x}))\big]^{s,+}$. Indeed, from $\lambda_s \in \big[\nabla f_s(\overline{x})(T(\Omega;\overline{x}))\big]^+$ it follows that $\langle \lambda_s, \nabla f_s(\overline{x})(T(\Omega;\overline{x}))\rangle \geq 0$, i.e., for all $x_s \in \nabla f_s(\overline{x})(T(\Omega;\overline{x}))$, $\langle \lambda_s, x_s\rangle \geq 0$. If $\lambda_s \notin \big[\nabla f_s(\overline{x})(T(\Omega;\overline{x}))\big]^{s,+}$, then there would exists $x_s \in \nabla f_s(\overline{x})(T(\Omega;\overline{x})) \setminus \{0\}$ such that $\langle \lambda_s, x_s\rangle \leq 0$. So, we deduce that $\langle \lambda_s, x_s\rangle = 0$ implies $\lambda_s = 0$ due to $x_s \in \mathbb{R} \setminus \{0\}$. It is evident that

$$\sum_{i\in I} \lambda_i \Big[\langle \nabla f_i(\overline{x}), T(\Omega;\overline{x})\rangle + \big\langle \partial^2 f_i(\overline{x})(v), v\big\rangle \Big] + \sum_{k\in T} \gamma_k \,\langle \nabla h_k(\overline{x}), T(\Omega;\overline{x})\rangle$$
$$+ \sum_{j\in J(\overline{x})} \mu_j \Big[\langle \nabla g_j(\overline{x}), T(\Omega;\overline{x})\rangle + \big\langle \partial^2 g_j(\overline{x})(v), v\big\rangle \Big] \geq 0$$

is equivalent to

$$\sum_{i\in I\setminus\{s\}} \lambda_i \Big[\langle \nabla f_i(\overline{x}), T(\Omega;\overline{x})\rangle + \big\langle \partial^2 f_i(\overline{x})(v), v\big\rangle \Big] + \sum_{k\in T} \gamma_k \,\langle \nabla h_k(\overline{x}), T(\Omega;\overline{x})\rangle$$
$$+ \sum_{j\in J(\overline{x})} \mu_j \Big[\langle \nabla g_j(\overline{x}), T(\Omega;\overline{x})\rangle + \big\langle \partial^2 g_j(\overline{x})(v), v\big\rangle \Big] \geq 0. \tag{22}$$

Under the constraint qualification (GBCQs), there exist $(v_0, w_0) \in C(\overline{x}) \times T(\Omega;\overline{x})$ such that $\langle \nabla f_i(\overline{x}), w_0\rangle + \langle \partial^2 f_i(\overline{x})(v_0), v_0\rangle \in -\big[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\big] \setminus \{0\}$, $\forall\, i \in I \setminus \{s\}$, $\langle \nabla g_j(\overline{x}), w_0\rangle + \langle \partial^2 g_j(\overline{x})(v_0), v_0\rangle \in -\big[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\big] \setminus \{0\}$, $\forall\, j \in J(\overline{x})$, and furthermore $\langle \nabla h_k(\overline{x}), w_0\rangle \in -\big[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\big]$, $\forall\, k \in T$. Since $\lambda \neq 0$, one finds $i_0 \in I \setminus \{s\}$ with $\lambda_{i_0} \in \big[\nabla f_{i_0}(\overline{x})(T(\Omega;\overline{x}))\big]^{s,+}$. Using the definitions we obtain the following inequalities

$$\begin{aligned}
&\lambda_{i_0}\Big[\langle \nabla f_{i_0}(\overline{x}), w_0\rangle + \big\langle \partial^2 f_{i_0}(\overline{x})(v_0), v_0\big\rangle \Big] < 0,\\
&\lambda_i\Big[\langle \nabla f_i(\overline{x}), w_0\rangle + \big\langle \partial^2 f_i(\overline{x})(v_0), v_0\big\rangle \Big] \leq 0 \ \ \forall\, i \in I \setminus \{i_0, s\},\\
&\mu_j\Big[\langle \nabla g_j(\overline{x}), w_0\rangle + \big\langle \partial^2 g_j(\overline{x})(v_0), v_0\big\rangle \Big] \leq 0 \ \ \forall\, j \in J(\overline{x}),\\
&\gamma_k\Big[\langle \nabla h_k(\overline{x}), w_0\rangle \leq 0 \ \ \forall\, k \in T.
\end{aligned}$$

Adding up both sides of the inequalities above, it results that

$$\sum_{i\in I\setminus\{s\}} \lambda_i \Big[\langle \nabla f_i(\overline{x}), w_0\rangle + \big\langle \partial^2 f_i(\overline{x})(v_0), v_0\big\rangle \Big] + \sum_{k\in T} \gamma_k \,\langle \nabla h_k(\overline{x}), w_0\rangle$$
$$+ \sum_{j\in J(\overline{x})} \mu_j \Big[\langle \nabla g_j(\overline{x}), w_0\rangle + \big\langle \partial^2 g_j(\overline{x})(v_0), v_0\big\rangle \Big] < 0,$$

which contradicts inequality (22), what we had to show. $\square$

Theorem 3.12. (Strong KKT second-order optimality condition) *Let $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^1-smooth, and h satisfies* (A2) *and* (A3). *Suppose also that M is a closed convex set and the constraint qualification of the* (GBCQs) *type for all $s \in I$ holds. Then for every $v \in C(\overline{x})$, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\right]^{s,+}$ ($\forall\, i \in I$), $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, j \in J$), and $\gamma_k \in \left[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, k \in T$) satisfying* (18), (19) *and* (20).

Proof. We fix $s \in I$. This implies following Theorem 3.11 that for all $v \in C(\overline{x})$, there exist $\lambda_s^{[s]} \in \left[\nabla f_s(\overline{x})(T(\Omega;\overline{x}))\right]^{s,+}$, $\lambda_i^{[s]} \in \left[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, i \in I \setminus \{s\}$), $\mu_j^{[s]} \in \left[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, j \in J$) and $\gamma_k^{[s]} \in \left[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, k \in T$) such that

$$\sum_{i\in I} \lambda_i^{[s]} \langle \nabla f_i(\overline{x}), T(\Omega;\overline{x})\rangle + \sum_{j\in J(\overline{x})} \mu_j^{[s]} \langle \nabla g_j(\overline{x}), T(\Omega;\overline{x})\rangle + \sum_{k\in T} \gamma_k^{[s]} \langle \nabla h_k(\overline{x}), T(\Omega;\overline{x})\rangle \geq 0, \tag{23}$$

$$\sum_{i\in I} \lambda_i^{[s]} \left\langle \partial^2 f_i(\overline{x})(v), v\right\rangle + \sum_{j\in J(\overline{x})} \mu_j^{[s]} \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \geq 0, \tag{24}$$

$$\mu_j^{[s]} g_j(\overline{x}) = 0, \;\; j \in J. \tag{25}$$

Taking $\lambda_i = \sum_{s\in I} \lambda_i^{[s]} \in \left[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\right]^{s,+}$ ($\forall i \in I$), $\mu_j = \sum_{s\in I} \mu_j^{[s]} \in \left[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, j \in J(\overline{x})$), $\gamma_k = \sum_{s\in I} \gamma_k^{[s]} \in \left[\nabla h_k(\overline{x})(T(\Omega;\overline{x}))\right]^{+}$ ($\forall\, k \in T$), and then, by adding up both sides of the inequalities with respect to $s = 1, \ldots, m$ from (23) to (25), it results that $\mu_j g_j(\overline{x}) = 0$, $j \in J$,

$$\sum_{i\in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v), v\right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \geq 0,$$

$$\sum_{i\in I} \lambda_i \langle \nabla f_i(\overline{x}), T(\Omega;\overline{x})\rangle + \sum_{j\in J(\overline{x})} \mu_j \langle \nabla g_j(\overline{x}), T(\Omega;\overline{x})\rangle + \sum_{k\in T} \gamma_k \langle \nabla h_k(\overline{x}), T(\Omega;\overline{x})\rangle \geq 0,$$

which completes the proof of the theorem. □

Corollary 3.13. *Let $\Omega = \mathbb{R}^n$ and $\overline{x} \in K$ be a weakly efficient solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^1-smooth, and h satisfies* (A2) *and* (A3). *Suppose also that the set M is a closed convex and the constraint qualification of the* (GBCQs) *type for all $s \in I$ holds. Then for every $v \in C(\overline{x})$, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(\mathbb{R}^n)\right]^{s,+}$ ($\forall\, i \in I$), $\mu_j \in \left[\nabla g_j(\overline{x})(\mathbb{R}^n)\right]^{+}$ ($\forall\, j \in J$) and $\gamma_k \in \left[\nabla h_k(\overline{x})(\mathbb{R}^n)\right]^{+}$ ($\forall\, k \in T$) such that $\mu_j g_j(\overline{x}) = 0$, $j \in J$ and*

$$\sum_{i\in I} \lambda_i \nabla f_i(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j \nabla g_j(\overline{x}) + \sum_{k\in T} \gamma_k \nabla h_k(\overline{x}) = 0,$$

$$\sum_{i\in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v), v\right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \geq 0.$$

Proof. Follows immediately from Theorem 3.12. □

Remark 3.14. We remark that the statements in Theorem 3.9, Theorem 3.11, Theorem 3.12, Corollary 3.10 and Corollary 3.13 are still true if the inequality

$$\sum_{i\in I}\lambda_i\left\langle\partial^2 f_i(\overline{x})(v),v\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\partial^2 g_j(\overline{x})(v),v\right\rangle\geq 0$$

is removed and replaced by

$$\sum_{i\in I}\lambda_i\left\langle\hat{\partial}^2 f_i(\overline{x})(v),v\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\hat{\partial}^2 g_j(\overline{x})(v),v\right\rangle\geq 0.$$

Especially, this remark still holds if f_i ($\forall\, i\in I$) and g_j ($\forall\, j\in J(\overline{x})$) are $C^{1,1}$-smooth mappings. Furthermore, all the preceding obtained results are still true if the critical cone $C(\overline{x})$ is removed and replaced by the subcritical cone $C_T(\overline{x})$. We also remark that all these results have not been considered before (see Mordukhovich [25–27], for instance). □

4. Second-order sufficient optimality conditions

In this section, we establish second-order Karush-Kuhn-Tucker sufficient optimality conditions for a local weakly efficient solution of problem (RQMP) with set and inequality constraints (without equality constraints) being given as in Section 2. The conditions are given in terms of the second-order Mordukhovich subdifferentials/ the second-order Fréchet subdifferentials.

Regarding this issue, for every $\overline{x}\in \mathrm{cl}\Omega$, we denote by

$$C_-(\overline{x}):=\left\{v\in T(\Omega;\overline{x})\,\middle|\,\langle\nabla f_i(\overline{x}),v\rangle\leq 0,\ \langle\nabla g_j(\overline{x}),v\rangle\leq 0,\ \forall\, i\in I,\ \forall\, j\in J(\overline{x})\right\},$$

$$C_0(\overline{x}):=\left\{v\in T(\Omega;\overline{x})\,\middle|\,\langle\nabla f_i(\overline{x}),v\rangle= 0,\ \langle\nabla g_j(\overline{x}),v\rangle= 0,\ \forall\, i\in I,\ \forall\, j\in J(\overline{x})\right\}.$$

It can be easily seen that
$$C_0(\overline{x})\subset C_-(\overline{x}). \tag{26}$$

In order to obtain second-order sufficient optimality conditions, we say that the constraint qualification of the (NCQ) type is true for $\overline{x}\in \mathrm{cl}\Omega$ iff,

$$\text{(NCQ)}\qquad C_-(\overline{x})\subset C_0(\overline{x}).$$

Under the constraint qualification (NCQ), it follows from (26) that $C_0(\overline{x})=C_-(\overline{x})$.

To begin with, we provide a second-order sufficient optimality condition for efficiency in terms of second-order Mordukhovich's subdifferentials as follows.

Theorem 4.1. *Let $\overline{x}$ be a feasible solution of problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i\in I$) and g_j ($\forall\, j\in J(\overline{x})$) are C^1-smooth, the set Ω is nonempty convex and the constraint qualification of the* (NCQ) *type holds.*
Then, if for every $v\in C(\overline{x})\setminus\{0\}$, there exist $\lambda_i\in\left[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\right]^+$ ($\forall\, i\in I$) and $\mu_j\in\left[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\right]^+$ ($\forall\, j\in J(\overline{x})$) such that

$$\sum_{i\in I}\lambda_i\left\langle\partial^2 f_i(\overline{x})(v),v\right\rangle+\sum_{j\in J(\overline{x})}\mu_j\left\langle\partial^2 g_j(\overline{x})(v),v\right\rangle\ >\ 0, \tag{27}$$

then $\overline{x}$ is a local weakly efficient solution for the problem (RQMP).

Proof. Suppose by contradiction, that the feasible solution $\overline{x}$ is not a local weakly efficient solution for the problem (RQMP). By the definition of local weak efficiency to the (RQMP), there exists sequence $(x_n)_{n\geq 1}$ with $x_n \in K \cap B(\overline{x}, \frac{1}{n}) \setminus \{\overline{x}\}$ such that

$$f(x_n) \leq f(\overline{x}),\ \ \forall\, n \geq 1.$$

Consider the boundary sequence $(v_n)_{n\geq 1} \subset \mathbb{R}^n$, given by

$$v_n := \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|},\ \ \forall\, n \in \mathbb{N}.$$

We may assume without loss of generality that v_n converges to v_0 with $\|v_0\| = 1$ due to $\|v_n\| = 1$ $(\forall\, n \geq 1)$. Consider the positive real numbers sequence $(t_n)_{n\geq 1}$, defined by

$$t_n := \|x_n - \overline{x}\|,\ \ \forall\, n \in \mathbb{N}.$$

Since $x_n = \overline{x} + t_n v_n \in K$ $(\forall\, n \geq 1)$ and $t_n \to 0^+$, which proves the following relation

$$v_0 \in T(\Omega; \overline{x}) \setminus \{0\}. \tag{28}$$

Because f_i $(\forall\, i \in I)$ and g_j $(\forall\, j \in J(\overline{x}))$ are continuously Fréchet differentiable at $\overline{x}$, we have on the one hand that

$$\langle \nabla f_i(\overline{x}), v_0 \rangle \leq 0,\ \ \forall\, i \in I, \tag{29}$$

$$\langle \nabla g_j(\overline{x}), v_0 \rangle \leq 0,\ \forall\, j \in J(\overline{x}). \tag{30}$$

Since the constraint qualification (NCQ) holds true for $\overline{x} \in \mathrm{cl}\Omega$, which combined with (28), (29) and (30) gives us $v_0 \in C_0(\overline{x}) \setminus \{0\}$. It is not hard to see that $v_0 \in C(\overline{x}) \setminus \{0\}$. On the other hand, by the initial assumptions, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega; \overline{x}))\right]^+$ $(\forall\, i \in I)$ and $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega; \overline{x}))\right]^+$ $(\forall\, j \in J(\overline{x}))$ such that

$$\sum_{i\in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v_0), v_0 \right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v_0), v_0 \right\rangle \, > \, 0. \tag{31}$$

It must be mentioned that for every $i \in I$ and $j \in J(\overline{x})$, the sets $\partial^2 f_i(\overline{x})(v_0)$ and $\partial^2 g_j(\overline{x})(v_0)$ are nonempty due to Mordukhovich [25, 26]. We have $v_0 \in T(\Omega; \overline{x})$, which guarantees that

$$\nabla f_i(\overline{x})(v_0) \in \nabla f_i(\overline{x})(T(\Omega; \overline{x}))\ (\forall\, i \in I),\ \ \nabla g_j(\overline{x})(v_0) \in \nabla g_j(\overline{x})(T(\Omega; \overline{x}))\ (\forall\, j \in J(\overline{x})).$$

Consequently, one obtains the following inequalities

$$\lambda_i \left\langle \nabla f_i(\overline{x}), v_0 \right\rangle \geq 0,\ \ \forall\, i \in I, \tag{32}$$

$$\eta_j \left\langle \nabla g_j(\overline{x}), v_0 \right\rangle \geq 0,\ \ \forall\, j \in J(\overline{x}). \tag{33}$$

Combining (29) with (32) as well as (30) with (33) yields that $\lambda_i \leq 0$ $(\forall\, i \in I)$ and $\mu_j \leq 0$ $(\forall\, j \in J(\overline{x}))$. In view of the definition of $C(\overline{x})$ and $L_{(\varphi,\alpha)}$ in Section 2, it is evident that $(\overline{x}, v_0) \in L_{(f_i,\alpha_i)}$ for all $i \in I$ and $(\overline{x}, v_0) \in L_{(g_j,\beta_j)}$ for all $j \in J(\overline{x})$. In other words, the mappings f_i $(\forall\, i \in I)$ and g_j $(\forall\, j \in J(\overline{x}))$ are C^1-smooth, and taking into account Lemma 2.4, this implies that $\langle \partial^2 f_i(\overline{x})(v_0), v_0 \rangle \geq 0$ $\forall\, i \in I$ and $\langle \partial^2 g_j(\overline{x})(v_0), v_0 \rangle \geq 0$ $\ \forall\, j \in J(\overline{x})$.

These inequalities combined with $\lambda_i \leq 0$ $(\forall i \in I)$ and $\mu_j \leq 0$ $(\forall j \in J(\overline{x}))$, ensure that $\lambda_i \langle \partial^2 f_i(\overline{x})(v_0), v_0\rangle \leq 0 \;\; \forall i \in I$ and $\mu_j \langle \partial^2 g_j(\overline{x})(v_0), v_0\rangle \leq 0 \; \forall j \in J(\overline{x})$. Adding up both sides of the above inequalities, we obtain the following one

$$\sum_{i\in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v_0), v_0 \right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v_0), v_0 \right\rangle \leq 0,$$

which contradicts (31), what we had to show. $\square$

The next result provides a second-order sufficient optimality condition for local weak efficiency based on the second-order Fréchet subdifferentials.

Theorem 4.2. *Let $\overline{x}$ be a feasible solution of problem* (RQMP). *Suppose that the mappings f_i $(\forall i \in I)$ and g_j $(\forall j \in J(\overline{x}))$ are $C^{1,1}$-smooth, the set Ω is nonempty convex and the constraint qualification of the* (NCQ) *type holds. Suppose also that the second-order Fréchet subdifferentials $\hat{\partial}^2 f_i(\overline{x})(v)$ $(\forall i \in I)$ and $\hat{\partial}^2 g_j(\overline{x})(v)$ $(\forall j \in J(\overline{x}))$ are nonempty for every $v \in C(\overline{x})$. Then, if $\forall v \in C(\overline{x}) \setminus \{0\}$, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\right]^+$ $(\forall i \in I)$ and $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\right]^+$ $(\forall j \in J(\overline{x}))$ such that*

$$\sum_{i\in I} \lambda_i \left\langle \hat{\partial}^2 f_i(\overline{x})(v), v \right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \hat{\partial}^2 g_j(\overline{x})(v), v \right\rangle > 0, \tag{34}$$

then $\overline{x}$ is a local weakly efficient solution for the problem (RQMP).

Proof. Arguing as in the proof of Theorem 4.1, with the new situation where all the objective and constraint functions are $C^{1,1}$-smooth, and utilizing Lemma 2.3 in place of Lemma 2.4, we also get the desired conclusion. $\square$

Corollary 4.3. *Let $\overline{x}$ be a feasible solution of problem* (RQMP). *Suppose that the mappings f_i $(\forall i \in I)$ and g_j $(\forall j \in J(\overline{x}))$ are C^2-smooth, the set Ω is nonempty convex and the constraint qualification of the* (NCQ) *type holds.*
Then, if for every $v \in C(\overline{x}) \setminus \{0\}$, there exist $\lambda_i \in \left[\nabla f_i(\overline{x})(T(\Omega;\overline{x}))\right]^+$ $(\forall i \in I)$ and $\mu_j \in \left[\nabla g_j(\overline{x})(T(\Omega;\overline{x}))\right]^+$ $(\forall j \in J(\overline{x}))$ such that

$$\sum_{i\in I} \lambda_i \left\langle \nabla^2 f_i(\overline{x})v, v \right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \nabla^2 g_j(\overline{x})v, v \right\rangle > 0, \tag{35}$$

then $\overline{x}$ is a local weakly efficient solution for the problem (RQMP).

Proof. By the initial assumptions, we always have a conclusion for every $v \in C(\overline{x})$ that $\hat{\partial}^2 f_i(\overline{x})(v)$ $(i \in I)$ and $\hat{\partial}^2 g_j(\overline{x})(v)$ $(j \in J(\overline{x}))$ are nonempty, where

$$\begin{aligned}\partial^2 f_i(\overline{x})(v) &= \hat{\partial}^2 f_i(\overline{x})(v) = \{\nabla^2 f_i(\overline{x})v\},\\ \partial^2 g_j(\overline{x})(v) &= \hat{\partial}^2 g_j(\overline{x})(v) = \{\nabla^2 g_j(\overline{x})v\}.\end{aligned}$$

By repeating the proof of Theorem 4.1 even the proof of Theorem 4.2, we arrive at the desired conclusion. $\square$

Example 4.4. Consider the real-valued functions f and g defined on $\mathbb{R}$, where $f := (f_1, f_2)$ and $g := g_1$, defined by

$$f(x) := \Big(x^2, \ \ -x\sin x\Big), \ \ \forall\, x \in \mathbb{R},$$

$$g(x) := \int_0^{|x|} \phi(\tau)d\tau - \alpha x, \ \ \forall\, x \in \mathbb{R}, \ \ \forall\, \alpha \in]-1, 1[,$$

where
$$\phi(\tau) := \begin{cases} 2\tau^2 + \tau^2 \sin(\frac{1}{\tau}) & \text{if } \ \tau > 0, \\ 0 & \text{if } \ \tau = 0. \end{cases}$$

Consider the C^1-smooth robustly quasiconvex multiobjective programming problem (RQMP) given above for which $\Omega := [0, \frac{\pi}{2}]$ and $\overline{x} = 0$. Then the functions f_1, f_2 are C^2-smooth around $\overline{x} = 0$, and g_1 is robustly pseudoconvex $C^{1,1}$ smooth function with modulus 1, so it is also robustly quasiconvex $C^{1,1}$-smooth function with modulus 1. By an easy computation shows that $I := \{1, 2\}$, $J = J(\overline{x}) = \{1\}$, $T(\Omega; \overline{x}) = \mathbb{R}_+$, $\nabla f_i(\overline{x}) = 0$, $(i = 1, 2)$, $\nabla^2 f_i(\overline{x}) = 2(-1)^{i-1}$ $(i = 1, 2)$, and moreover, $\nabla g_1(\overline{x}) = -\alpha$ for all $\alpha \in \mathbb{R}$ such that $|\alpha| < 1$ and $\partial^2 g_1(\overline{x})(v) = [-|v|, \ |v|]$ for all $v \in \mathbb{R}$. We thus obtain the result $\nabla f_1(\overline{x})(T(\Omega; \overline{x})) = \nabla f_2(\overline{x})(T(\Omega; \overline{x})) = \{0\}$ or equivalently,

$$\big[\nabla f_1(\overline{x})(T(\Omega; \overline{x}))\big]^+ = \big[\nabla f_2(\overline{x})(T(\Omega; \overline{x}))\big]^+ = \mathbb{R}.$$

We see that $\nabla g_1(\overline{x})(T(\Omega; \overline{x})) = \{0\}$ is equivalent to $\big[\nabla g_1(\overline{x})(T(\Omega; \overline{x}))\big]^+ = \mathbb{R}$ if $\alpha = 0$, and $\nabla g_1(\overline{x})(T(\Omega; \overline{x})) = \mathbb{R}$ is equivalent to $\big[\nabla g_1(\overline{x})(T(\Omega; \overline{x}))\big]^+ = \{0\}$ if $-1 < \alpha < 1$ and $\alpha \neq 0$. We remark that $C_0(\overline{x}) = C_-(\overline{x}) = \mathbb{R}_+$ if $\alpha = 0$, which means that (NCQ) holds. In this case, for all $v \in \mathbb{R}_+ \setminus \{0\} \iff v > 0$, using the precedingresults, we assert that $\partial^2 f_1(\overline{x})(v)$, $\partial^2 f_2(\overline{x})(v)$ and $\partial^2 g_1(\overline{x})(v)$ are nonempty. By taking $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$ one obtains the following inequality

$$\sum_{i\in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v), v\right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \geq v^2 \ > \ 0.$$

By applying the results obtained in Section 4, we can conclude that $\overline{x} = 0$ is a local weakly efficient solution for the problem (RQMP), as we had to check. □

We state now a series of direct corollaries, whose easy proofs are omitted.

Corollary 4.5. *Let $\Omega = \mathbb{R}^n$ and $\overline{x}$ be a feasible solution for the problem* (RQMP)*. Suppose that the mappings f_i $(\forall\, i \in I)$ and g_j $(\forall\, j \in J(\overline{x}))$ are C^1-smooth and the constraint qualification of the* (NCQ) *type holds. Then, if for every $v \in C(\overline{x}) \setminus \{0\}$, there exist $\lambda_i \in \big[\nabla f_i(\overline{x})(\mathbb{R}^n)\big]^+$ $(\forall\, i \in I)$ and $\mu_j \in \big[\nabla g_j(\overline{x})(\mathbb{R}^n)\big]^+$ $(\forall\, j \in J(\overline{x}))$ such that*

$$\sum_{i\in I} \lambda_i \left\langle \partial^2 f_i(\overline{x})(v), v\right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \partial^2 g_j(\overline{x})(v), v\right\rangle \ > \ 0,$$

then $\overline{x}$ is a local weakly efficient solution for the problem (RQMP)*.*

Corollary 4.6. *Let $\Omega = \mathbb{R}^n$ and $\overline{x}$ be a feasible solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are $C^{1,1}$-smooth and the constraint qualification of the* (NCQ) *type holds. Suppose also that the second-order Fréchet subdifferentials $\hat{\partial}^2 f_i(\overline{x})(v)$ ($\forall\, i \in I$) and $\hat{\partial}^2 g_j(\overline{x})(v)$ ($\forall\, j \in J(\overline{x})$) are nonempty for any $v \in C(\overline{x})$. Then, if for every $v \in C(\overline{x}) \setminus \{0\}$, there exist $\lambda_i \in \big[\nabla f_i(\overline{x})(\mathbb{R}^n)\big]^+$ ($\forall\, i \in I$) and $\mu_j \in \big[\nabla g_j(\overline{x})(\mathbb{R}^n)\big]^+$ ($\forall\, j \in J(\overline{x})$) such that*

$$\sum_{i\in I} \lambda_i \left\langle \hat{\partial}^2 f_i(\overline{x})(v), v \right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \hat{\partial}^2 g_j(\overline{x})(v), v \right\rangle \,>\, 0,$$

then $\overline{x}$ is a local weakly efficient solution for the problem (RQMP).

Corollary 4.7. *Let $\Omega = \mathbb{R}^n$ and $\overline{x}$ be a feasible solution for the problem* (RQMP). *Suppose that the mappings f_i ($\forall\, i \in I$) and g_j ($\forall\, j \in J(\overline{x})$) are C^2-smooth and the constraint qualification of the* (NCQ) *type holds. Then, if for every $v \in C(\overline{x}) \setminus \{0\}$, there exist $\lambda_i \in \big[\nabla f_i(\overline{x})(\mathbb{R}^n)\big]^+$ ($\forall\, i \in I$) and $\mu_j \in \big[\nabla g_j(\overline{x})(\mathbb{R}^n)\big]^+$ ($\forall\, j \in J(\overline{x})$) such that*

$$\sum_{i\in I} \lambda_i \left\langle \nabla^2 f_i(\overline{x})v, v \right\rangle + \sum_{j\in J(\overline{x})} \mu_j \left\langle \nabla^2 g_j(\overline{x})v, v \right\rangle \,>\, 0,$$

then $\overline{x}$ is a local weakly efficient solution for the problem (RQMP).

Remark 4.8. We remark that when $\Omega = \mathbb{R}^n$, it can be seen that $T(\Omega; \overline{x}) = \mathbb{R}^n$. Moreover, the strict inequality (27) implies the strict inequality (34). Hence, the result obtained in Corollary 4.6 is the same if the strict inequality (34) is removed and replaced by the strict inequality (27). The strict inequalities (27), (34) and (35) coincide in the case the objective and constraint functions are C^2-smooth. Furthermore, all the results obtained in Section 4 have not been considered before (see Mordukhovich [25–27] for more details). □

5. Conclusions

Based on the notions of the second-order Mordukhovich subdifferentials/ Fréchet subdifferentials, constraint qualifications of the (GBCQ) and (GBCQs) types are provided. Using these constraint qualifications, we have established the primal and dual Karush-Kuhn-Tucker type second-order necessary optimality conditions for (local) weak efficiency of problem (RQMP). We also propose the constraint qualification of the (NCQ) type. Under (NCQ), the second-order Karush-Kuhn-Tucker type sufficient optimality conditions are presented.

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Compliance with ethical standards

Conflicts of interest: The authors declare that they have no conflict of interest.

Ethical approval: This manuscript does not contain any studies with human participants or animals performed by any of the authors.

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References

[1] D. T. V. An, N. D. Yen: *Optimality conditions based on the Fréchet second-order subdifferential*, J. Global Optim. 81 (2021) 351–365.

[2] J.-P. Aubin, H. Frankowska: *Set-Valued Analysis*, Birkhäuser, Boston (1990).

[3] M. Avriel: *r-Convex functions*, Math. Programming 2 (1972) 309–323.

[4] T. Q. Bao, B. S. Mordukhovich: *Necessary conditions in multiobjective optimization with equilibrium constraints*, J. Optim. Theory Appl. 135/2 (2007) 179–-203.

[5] T. Q. Bao, B. S. Mordukhovich: *Necessary conditions for super minimizers in constrained multiobjective optimization*, J. Global Optim. 43/4 (2009) 533–552.

[6] T. Q. Bao, B. S. Mordukhovich: *Relative Pareto minimizers for multiobjective problems: existence and optimality conditions*, Math. Programming Ser. A 122/2 (2010) 301–347.

[7] R. I. Bot, S.-M. Grad, G. Wanka: *Duality in Vector Optimization*, Springer, Berlin (2009).

[8] A. Cambina, L. Martein: *Generalized Convexity and Optimization. Theory and Applications*, Springer, Berlin (2009).

[9] T. D. Chuong: *Optimality and duality for robust multiobjective optimization problems*, Nonlinear Analysis 134 (2016) 127–143.

[10] T. D. Chuong: *Robust optimality and duality in multiobjective optimization problems under data uncertainty*, SIAM J. Optim. 30/2 (2020) 1501–1526.

[11] E. Constantin: *Necessary conditions for weak minima and for strict minima of order two in nonsmooth constrained multiobjective optimization*, J. Global Optim. 80 (2021) 177–193.

[12] J.-P. Crouzeix: *On second order conditions for quasiconvexity*, Math. Programming 18 (1980) 349–352.

[13] J.-P. Crouzeix: *Characterizations of generalized convexity and monotonicity, a survey*, in: *Generalized Convexity, Generalized Monotonicity: Recent Results*, J.-P. Crouzeix et al. (eds.), Nonconvex Optimization and its Applications Vol. 27, Kluwer, Dordrecht (1998) 237–256.

[14] C. Gutiérrez, B. Jiménez, V. Novo: *On second-order Fritz John type optimality conditions in nonsmooth multiobjective programming*, Math. Programming Ser. B 123 (2010) 199–223.

[15] B. Jiménez, V. Novo: *First order optimality conditions in vector optimization involving stable functions*, Optimization 57/3 (2008) 449–471.

[16] A. Khan, C. Tammer, C. Zalinescu: *Set-Valued Optimization*, Springer, Berlin (2015).

[17] P. D. Khanh, V. T. Phat: *Second-order characterizations of C^1-smooth robustly quasiconvex functions*, Oper. Res. Letters 46/6 (2018) 568–572.

[18] P. D. Khanh, V. T. Phat: *Second-order characterizations of quasiconvexity and pseudoconvexity for differentiable functions with Lipschitzian derivatives*, Optimization Letters 14 (2020) 2413–2427.

[19] L. P. Liu: *The second-order conditions of nondominated solutions for $C^{1,1}$ generalized multiobjective mathematical programming*, J. Systems Sci. Math. Sci. 4 (1991) 128–131.

[20] D. T. Luc: *Characterizations of quasiconvex function*, Bull. Aust. Math. Soc. 48 (1993) 393–405.

[21] D. T. Luc: *Taylor's formula for $C^{k,1}$-functions*, SIAM J. Optimization 5 (1995) 659–669.

[22] D. V. Luu: *Second-order necessary efficiency conditions for nonsmooth vector equilibrium problems*, J. Global Optim. 70 (2018) 437–453.

[23] O. L. Mangasarian: *Pseudo-convex functions*, J. SIAM, Ser. A Control 3 (1965) 281–290.

[24] O. L. Mangasarian: *Nonlinear Programming*, McGraw-Hill, New York (1969).

[25] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. I: Basic Theory*, Springer, Berlin (2006).

[26] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. II: Applications*, Springer, Berlin (2006).

[27] B. S. Mordukhovich; *Variational Analysis and Applications*, Springer, Berlin (2018).

[28] R. T. Rockafellar: *Convex Analysis*, Princeton Mathematical Series Vol. 28, Princeton University Press, Princeton (1970).

[29] T. V. Su: *New second-order optimality conditions for vector equilibrium problems with constraints in terms of contingent derivatives*, Bull. Braz. Math. Soc. New Ser. 51/2 (2020) 371–395.

[30] T. V. Su, D. H. Dinh: *Duality results for interval-valued pseudoconvex optimization problem with equilibrium constraints with applications*, Comp. Appl. Math. 39/2 (2020), art. no. 127, 24 pp.

[31] T. V. Su, N. D. Hien: *Strong Karush-Kuhn-Tucker optimality conditions for weak efficiency in constrained multiobjective programming problems in terms of Mordukhovich subdifferentials*, Optimization Letters 15 (2021) 1175–1194.

[32] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. Part I: General Theory*, World Scientific, Singapore (2023).

[33] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. Part II: Special Classes of Functions and Sets*, World Scientific, Singapore (2023).

[34] P. N. Tinh: *Optimality conditions for nonsmooth vector problems in normed spaces*, Optimization 69/6 (2020) 1151–1186.