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Dislocation Hyperbolic Kernel Function

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A new kernel function is introduced in Mathematical Optimization. This function is called the dislocation hyperbolic kernel function. It is based on the dislocation hyperbolic function. Finally we present some applications of this new function.

Keywords: Dislocation hyperbolic function, Fenchel conjugate, kernel function, self-concordant.

2020 Mathematics Subject Classification: 46N10.

1. Introduction

In nonlinear programming a problem that is very studied is the following

$$(\mathrm{P}) \qquad min\left\{f(x) \mid x \in S\right\}, \tag{1}$$

with
$$S = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \;\; i = 1, ..., m\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, ..., m.$ The problem (1) can be solved by a wide variety of algorithms, some of them are: the penalty and augmented Lagrangian algorithms.

In 1967, Zangwill ([31]) studied the exact penalty function (EPF) in his penalty algorithm. This function is as follows:

$$\beta \in \mathbb{R} \quad \mapsto \quad (\beta)_+ = -\,min\left\{\beta, 0\right\}, \tag{2}$$

see Figure 1.1. Later EPF is smoothed and used in [27], [11], [22] and [13].

1.1. Hyperbolic penalty function

In the hyperbolic penalty algorithm (HPA), the solution to problem (P) is obtained by means of solving a sequence of subproblems, $k = 1, 2, ...$, defined by minimization of the modified objective function

$$F(x, \lambda^k, \tau^k) = f(x) + \sum_{i=1}^{m} P\left(g_i(x), \lambda_i^k, \tau_i^k\right), \tag{3}$$

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where $P(\cdot,\cdot,\cdot)$ is the hyperbolic penalty function (HPF), defined as follows

$$P(y,\lambda,\tau) = -\lambda y + \sqrt{(\lambda y)^2 + \tau^2}, \tag{4}$$

where $P:(-\infty,+\infty)\times\mathbb{R}_+\times\mathbb{R}_{++}\to\mathbb{R}$, see Figure 1.2. In HPA, the parameters λ and τ are updated. The HPF was originally proposed in [27] and studied in [28] and [29]. The HPF is equivalent to a smoothing of the penalty studied by Zangwill [31].

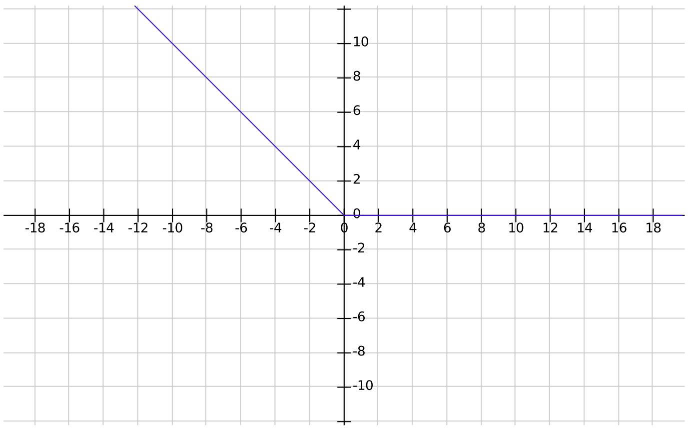

Figure 1.1: Function $(\beta)_+ = -\,min\{\beta, 0\}$

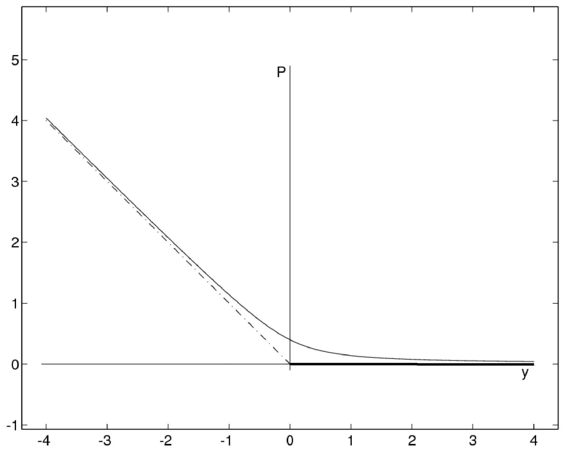

Figure 1.2: Function P

1.2. Hyperbolic augmented Lagrangian function

In the hyperbolic augmented Lagrangian algorithm (HALA), the solution to problem (P) is obtained by means of solving a subproblem, defined by the minimization of the hyperbolic augmented Lagrangian function (HALF) $L_H:\mathbb{R}^n\times\mathbb{R}^m_{++}\times\mathbb{R}_{++}\to\mathbb{R}$,

$$\begin{aligned} L_H(x,\lambda,\tau) &= f(x) + \sum_{i=1}^{m} P(g_i(x),\lambda_i,\tau) \\ &= f(x) + \sum_{i=1}^{m}\left(-\lambda_i g_i(x) + \sqrt{(\lambda_i g_i(x))^2 + \tau^2}\right), \end{aligned} \tag{5}$$

where τ is the penalty parameter that is fixed and λ is the Lagrange multiplier which is updated by means of a formula. The HALA was proposed in [28], and recently its convergence was guaranteed in [18]. Note that this function L_H belongs to class C^∞ if the involved functions $f(x)$ and $g_i(x),\ i = 1, ..., m$, are too.

1.3. Dislocation hyperbolic augmented Lagrangian function

We define the function $l_H : \mathbb{R}^n \times \mathbb{R}^m_{++} \times \mathbb{R}_{++} \to \mathbb{R}$, as,

$$\begin{aligned} l_H(x, \lambda, \tau) &= f(x) + \sum_{i=1}^{m} p(g_i(x), \lambda_i, \tau) \\ &= f(x) + \sum_{i=1}^{m} \left(-\lambda_i g_i(x) + \sqrt{(\lambda_i g_i(x))^2 + \tau^2} - \tau \right), \end{aligned} \tag{6}$$

where τ is the penalty parameter that is fixed and λ is the Lagrange multiplier. In [28], the dislocation hyperbolic function is mentioned as follows:

$$p(y, \lambda, \tau) = -\lambda y + \sqrt{(\lambda y)^2 + \tau^2} - \tau. \tag{7}$$

1.4. Hyperbolic smoothing function

In [30] the hyperbolic smoothing function (HSF) was introduced, in [3] this function is studied. This function is defined as

$$\Phi(x, \tau) = \frac{x + \sqrt{x^2 + \tau^2}}{2}. \tag{8}$$

The paper is organized as follows: In Section 2, we present some basic definitions. In Section 3, we introduce the dislocation hyperbolic kernel function and some of its properties are studied. In Section 4, we show some applications of the dislocation hyperbolic function. In Section 5, we show some applications of the dislocation hyperbolic kernel function. We also propose a new self-concordant function. In Section 6, we discuss some ideas for future work and conclusions from our work.

2. Preliminaries

Definition 2.1. (See [19] and [20]) A function $f \in C^3 : (0, \infty) \to (0, \infty)$ is *self-concordant* (SC) if f is convex, three times continuously differentiable, and satisfies for some constant $\alpha > 0$ the condition

$$|f'''(t)| \leq \alpha \left(f''(t)\right)^{\frac{3}{2}}, \quad \forall t \in (0, \infty).$$

Definition 2.2. (See [19] and [20]) A univariate function $f \in C^3$ is said to be a *self-concordant barrier* for the domain $\mathbb{R}_+$ if it satisfies the following two conditions:

(i) The function $f(t)$ is self-concordant.

(ii) There exists a constant α such that $|f'(t)| \leq \alpha \left(f''(t)\right)^{\frac{1}{2}}$.

The kernel function (KF) is any univariate strictly convex function $k(t)$ that is minimal at $t = 1$ and such that $k(1) = 0$. The KF is widely studied in primal-dual path-following method ([5] and [1]), also in stochastic programming [9].

Here are some examples:

Examples 2.3. (1) $k(t) = t\log(t) - t + 1$, $t \geq 0$, see, Kullback-Leible [16].
(2) $k(t) = (1 - \sqrt{t})^2$, $t \geq 0$, see, Hellinger Distance [10].

3. The dislocation hyperbolic kernel function

In this section we introduce the Dislocation Hyperbolic Kernel Function (DHKF).

3.1. Dislocation hyperbolic function

A new approach of the hyperbolic functions is studied in this section. We are going to write the expression (6) as follows

$$l_H(x, \lambda, \tau) = f(x) + \sum_{i=1}^{m} p\left(g_i(x), \lambda_i, \tau\right) \tag{9}$$

$$= f(x) - \sum_{i=1}^{m} \left(\lambda_i g_i(x) - \sqrt{\left(\lambda_i g_i(x)\right)^2 + \tau^2} + \tau \right)$$

$$= f(x) - \sum_{i=1}^{m} \tau \left(\frac{\lambda_i g_i(x)}{\tau} - \sqrt{\left(\frac{\lambda_i g_i(x)}{\tau}\right)^2 + 1} + 1 \right). \tag{10}$$

We rewrite (10) now as

$$l_H(x, \lambda, \tau) = f(x) - \sum_{i=1}^{m} \tau\, h\left(\frac{\lambda_i g_i(x)}{\tau}\right), \tag{11}$$

where the function h will be called by the dislocation hyperbolic function (DHF), and is defined as

$$h(t) = t - \sqrt{t^2 + 1} + 1,$$

with $h \in C^\infty$ and is defined $h : (-\infty, \infty) \to \mathbb{R}$, see Figure 3.1.

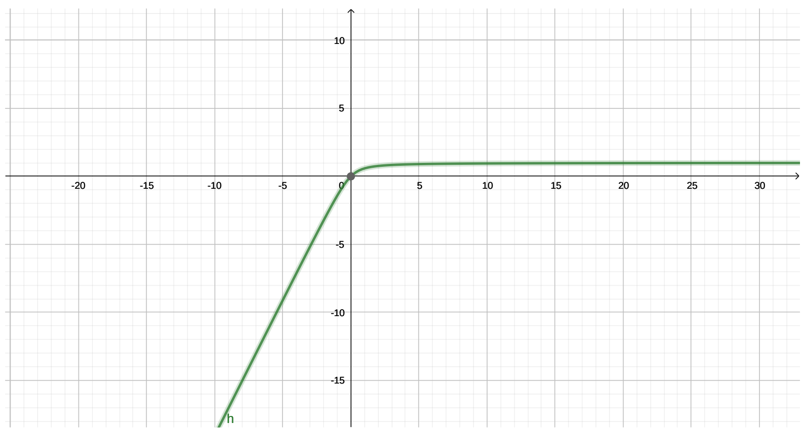

Figure 3.1: Function $h(t)$

The function h satisfies the following properties:

(H1) (i) $h(0) = 0$.

(ii) $\lim_{t\to+\infty} h(t) = 1$.

(iii) $\lim_{t\to-\infty} h(t) = -\infty$.

(H2) (i) h is increasing, i.e., $h'(t) = 1 - \dfrac{t}{\sqrt{t^2+1}} > 0, \ \ \forall t \in (-\infty, \infty)$.

(ii) $h'(0) = 1$.

(iii) $\lim_{t\to+\infty} h'(t) = \lim_{t\to+\infty} \left(1 - \frac{t}{|t|\sqrt{1+\frac{1}{t^2}}}\right) = 0, \ \ t > 0$.

(iv) $\lim_{t\to-\infty} h'(t) = \lim_{t\to-\infty} \left(1 - \frac{t}{|t|\sqrt{1+\frac{1}{t^2}}}\right) = 2, \ \ t < 0$.

(v) $0 < h'(t) < 2$.

See Figure 3.2.

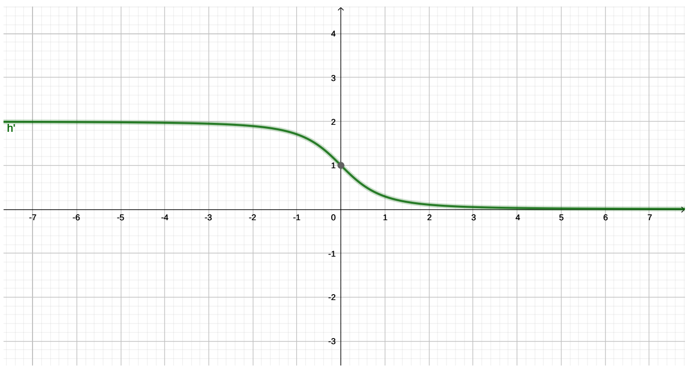

Figure 3.2: Function $h'(t)$

(H3) (i) The h function is strictly concave, i.e.,

$$h''(t) = \frac{-1}{(t^2+1)^{\frac{3}{2}}} < 0, \quad \forall t \in (-\infty, \infty).$$

(ii) $\lim_{t\to+\infty} h''(t) = 0$.

(iii) $\lim_{t\to-\infty} h''(t) = 0$.

See Figure 3.3.

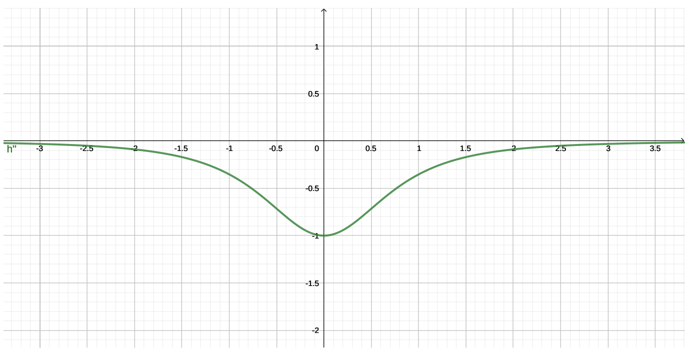

Figure 3.3: Function $h''(t)$

(H4) $\lim_{t\to0^+} th\left(\frac{z}{t}\right) = \lim_{t\to0^+} t\left(\frac{z}{t} - \sqrt{\left(\frac{z}{t}\right)^2+1}+1\right) = \lim_{t\to0^+}\left(z-\sqrt{z^2+t^2}+t\right)=0$, for all $z \geq 0$.

We denote by Ψ_H this class of functions.

3.2. The Fenchel conjugate for DHF

The Fenchel conjugate of h is given by

$$h^*(s) = \inf\left\{st - h(t) \ \mid \ t \in \mathbb{R}\right\} = st(s) - h\left(t(s)\right),$$

due to (H3) for any $0 < s < 2$ and the equation $(st - h(t))'_t = s - h'(t) = 0$. It follows that $h'(t) = s$. This can be solved for t as follows

$$1 - \frac{t}{\sqrt{t^2+1}} = s.$$

In this last equality we will perform some operations, in this way we will obtain the following equalities

$$(1-s)^2 = \frac{t^2}{t^2+1}, \quad \text{hence} \quad (t^2+1)(1-s)^2 = t^2,$$

so,
$$(1-s)^2 = t^2\left(1-(1-s)^2\right) \quad \text{and} \quad t^2 = \frac{(1-s)^2}{1-(1-s)^2}.$$

Finally from the equality above we obtain

$$t = \frac{1-s}{\sqrt{1-(1-s)^2}}.$$

Therefore there is an inverse function h'^{-1} as follows:

$$h'^{-1}(s) = t(s) = \frac{(1-s)}{\sqrt{1-(1-s)^2}}\,. \tag{12}$$

Thus, the conjugate of the dislocation hyperbolic function is

$$\begin{aligned}
h^*(s) &= \inf_{t\in\mathbb{R}}\left\{st(s) - h(t(s))\right\} = \inf_{t\in dom(h)}\left\{st(s) - h(t(s))\right\}\\
&= \inf\left\{ s\left(\frac{(1-s)}{\sqrt{1-(1-s)^2}}\right) - \left(\frac{(1-s)}{\sqrt{1-(1-s)^2}} - \sqrt{\frac{(1-s)^2}{1-(1-s)^2}+1}+1\right)\right\}\\
&= \inf\left\{\frac{s(1-s)}{\sqrt{1-(1-s)^2}} - \frac{(1-s)}{\sqrt{1-(1-s)^2}} + \sqrt{\frac{(1-s)^2}{1-(1-s)^2}+1} - 1\right\}\\
&= \inf\left\{\frac{s(1-s)-(1-s)}{\sqrt{1-(1-s)^2}} + \sqrt{\frac{(1-s)^2+1-(1-s)^2}{1-(1-s)^2}} - 1\right\}\\
&= \inf\left\{\frac{(s-1)(1-s)}{\sqrt{1-(1-s)^2}} + \frac{1}{\sqrt{1-(1-s)^2}} - 1\right\}\\
&= \inf\left\{\frac{-(1-s)^2}{\sqrt{1-(1-s)^2}} + \frac{1}{\sqrt{1-(1-s)^2}} - 1\right\}\\
&= \inf\left\{\frac{1-(1-s)^2}{\sqrt{1-(1-s)^2}} - 1\right\}.
\end{aligned} \tag{13}$$

On the other hand, we know

$$\sqrt{1-(1-s)^2} \leq \frac{1-(1-s)^2}{\sqrt{1-(1-s)^2}}, \quad \forall s \in (0,2). \tag{14}$$

Finally, let us consider (14) in (13), so we get

$$h^*(s) = \sqrt{1-(1-s)^2} - 1, \qquad \forall s \in (0,2). \tag{15}$$

We obtain, $-1 \leq h^*(s) \leq 0$ for $0 \leq s \leq 1$.
Now, since $h^*(1) = 0$, we obtain $max\,\{h^*(s) : 0 \leq s \leq 2\} = h^*(1) = 0$ and

$$(0,2) = ri(dom\ h^*) \subset range\ h' \subset dom\ h^* = [0,2]. \tag{16}$$

To obtain expression (16) we follow an analysis similar to the Section 24 of [26], Proposition 3.1 of [24] and Section 3 of [23].

3.3. The kernel for the dislocation hyperbolic function

In this section we are going to define the dislocation hyperbolic kernel function (DHKF). Now, given $h \in \Psi_H$, we introduce the function $\varphi : [0,2] \to \mathbb{R}$ as

$$\varphi(s) = -h^*(s) = 1 - \sqrt{1-(1-s)^2}, \tag{17}$$

The function h^* is concave and hence $\varphi = -h^*$ is convex, see Figure 3.4.

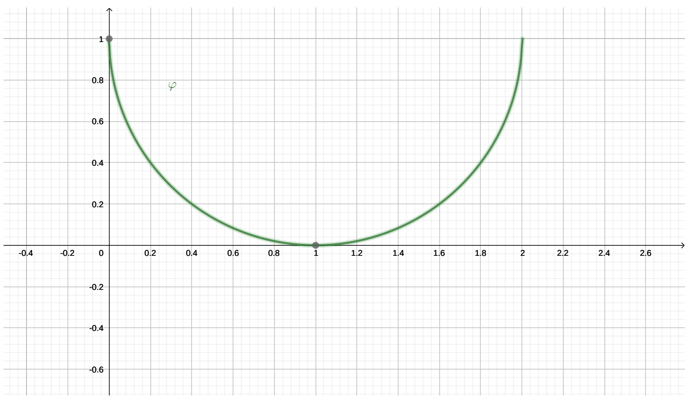

Figure 3.4: Function $\varphi(s)$

The first, second and third derivative of φ with respect to s are, respectively:

$$\varphi'(s) = \frac{-(1-s)}{\sqrt{1-(1-s)^2}}, \quad \varphi''(s) = \frac{1}{(1-(1-s)^2)^{\frac{3}{2}}}, \quad \varphi'''(s) = \frac{-3(1-s)}{(1-(1-s)^2)^{\frac{5}{2}}},$$

for all $s \in (0,2)$, see Figures 3.5, 3.6 and 3.7 respectively.

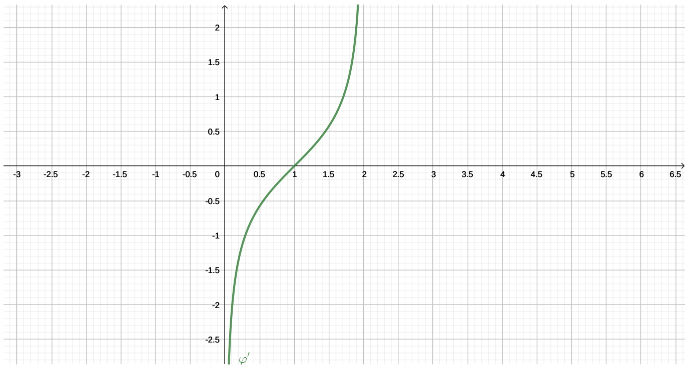

Figure 3.5: Function $\varphi'(s)$

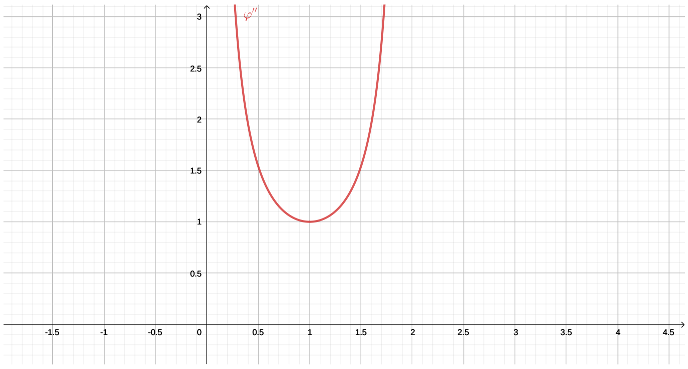

Figure 3.6: Function $\varphi''(s)$

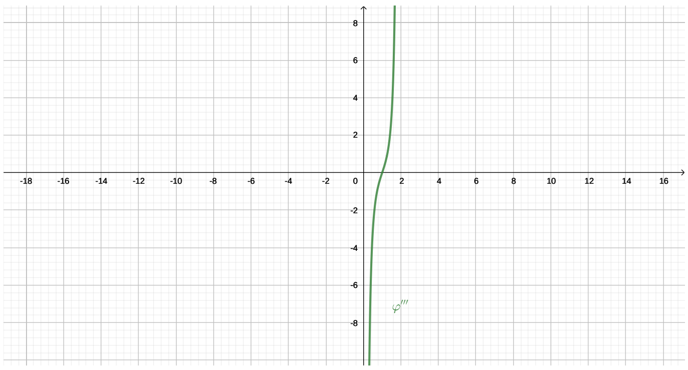

Figure 3.7: Function $\varphi'''(s)$

The function φ has the following properties:

Proposition 3.1. *Let* $h \in \Psi_H$. *Then the function* φ *is a strictly convex differentiable function in the set* $(0,2)$ *which satisfies*

$$\varphi(1) = \varphi'(1) = 0 \quad \textit{and} \quad \lim_{s \to 0^+} \varphi'(s) = -\infty. \tag{18}$$

Proof. The function φ is strictly convex, i.e.,

$$\varphi''(s) = \frac{1}{(1-(1-s)^2)^{\frac{3}{2}}} > 0, \quad \forall s \in (0,2).$$

The proof of (18) is similar to the proof of Proposition 3.1 of [24]. $\square$

Proposition 3.2.

(K1) (i) $\varphi'(s) < 0, \quad \forall\, s \in (0,1)$.
(ii) $\varphi'(s) > 0, \quad \forall\, s \in (1,2)$.

(K2) $\lim_{s\to 2^-} \varphi'(s) = +\infty$.

(K3) (i) $\varphi'''(s) < 0, \quad \forall\, s \in (0,1)$.
(ii) $\varphi'''(1) = 0$,
(iii) $\varphi'''(s) > 0, \quad \forall\, s \in (1,2)$.

(K4) (i) $\lim_{s\to 0^+} \varphi(s) = 1$.
(ii) $\lim_{s\to 2^-} \varphi(s) = 1$.

Proof. The demonstrations are direct. □

After all of the above, we now propose the following definition.

Definition 3.3. A univariate function $\varphi : (0,2) \to \mathbb{R}$ is called the *dislocation hyperbolic kernel function* if $\varphi(s)$ is continuously differentiable and the following conditions hold:

(i) $\varphi(1) = \varphi'(1) = 0$;

(ii) $\varphi''(s) > 0$;

(iii) $\lim_{s\to 0^+} \varphi(s) = 1$ and $\lim_{s\to 2^-} \varphi(s) = 1$.

We can see that our function kernel φ does not have the right or left coercivity property. Also, due to the condition (iii) in Definition 3.3, we see that the function $\varphi(s)$ does not possess the general property of the barrier function. Something similar to our Definition 3.3 can be seen in [5] and [12]. In these works, the kernel functions have only right-hand coercivity.

4. Applications of DHF

4.1. Information measures

This section is based on the work [8]. We propose a function based on information measures function studied in [8]. For any $h : \mathbb{R}_+ \to \mathbb{R}$ which is concave and normalized, i.e.,

$$h(1) = 0, \tag{19}$$

and the adjoint function w, defined by,

$$w(t) = t\, h\left(\frac{1}{t}\right), \quad \forall t \in \mathbb{R}_+. \tag{20}$$

4.2. Dislocation hyperbolic quadratic function

In the work of Kort and Bertsekas [15] a class of functions is proposed, with this class of functions an augmented Lagrangian type algorithm is proposed and its convergence is ensured. In this section, we propose the function ϕ_1 which belongs to the class of functions of Kort and Bertsekas. This new function is defined as,

$$\phi_1 : \mathbb{R} \to \mathbb{R}, \quad \phi_1(t) = \begin{cases} t + \sqrt{t^2+1} - 1, & t < 0 \\ t + t^2, & t \geq 0 \end{cases}$$

see Figure 4.1.

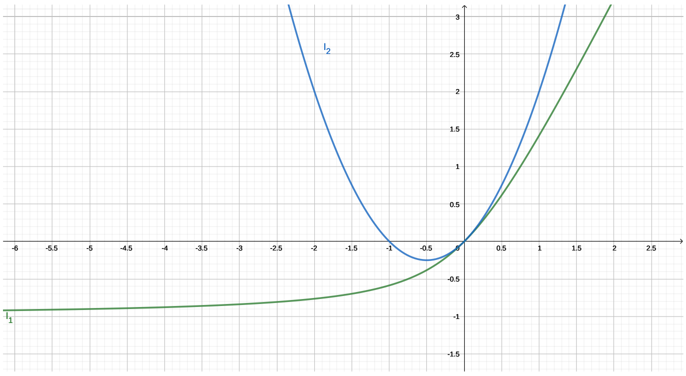

Figure 4.1: Function $\phi_1(t)$

We note the following properties of $\phi_1(x)$:

(a1) $\phi_1''(t) > 0,\ \forall t \in \mathbb{R}$, (a2) $\phi_1(0) = 0$, (a3) $\phi_1'(0) = 1$

(a4) $\lim_{t\to-\infty} \phi_1(t) = -1$, (a5) $\lim_{t\to\infty} \phi_1'(t) = \infty$.

5. Applications of DHKF

5.1. Hyperbolic distance

This section is based on Subsection 1.2 of [14]. Let Φ_1 be the set of functions $\varphi_1 : (0,\infty) \to [0,\infty)$ which satisfy the following properties:

(P1) φ_1 is strictly convex and thrice continuously differentiable.

(P2) $\varphi_1(1) = 0$.

(P3) $\varphi_1'(1) = 0$.

(P4) $\varphi_1''(1) > 0$.

(P5) $\lim_{t\to 0} \varphi_1'(t) = -\infty$.

Define $d_{\varphi_1} : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}_+$ as $d_{\varphi_1}(x,y) = \sum_{j=1}^{n} y_j \varphi_1\left(\frac{x_j}{y_j}\right)$.

When $\varphi_1 \in \Phi_1$, d_{φ_1} is said to be a φ_1-divergence. Now, in our case, since φ has the properties (K1)–(K4) and the Proposition 3.1 then check the conditions (P1)–(P5). Then $\varphi \in \Phi_1$ and the dislocation hyperbolic kernel distance function (DHKDF) is defined by $d_\varphi : \mathbb{R}^n_+ \times \mathbb{R}^n_{++} \to \mathbb{R}$ as

$$d_\varphi(x, x^k) = \sum_{i=1}^{p} x_i^k \varphi\left(\frac{x_i}{x_i^k}\right) = \sum_{i=1}^{p} x_i^k \left(1 - \sqrt{1 - \left(1 - \frac{x_i}{x_i^k}\right)^2}\right),$$

i.e., d_φ is said to a $\varphi-$divergence.

Lemma 5.1. *Let $\varphi \in \Phi_1$.*

(a) *$d_\varphi(x,y) \geq 0$ and equality holds iff $x = y$;*

(b) *$d_\varphi(x,y)$ is jointly convex in (x,y).*

Proof. (a) Since φ is convex, we have

$$\varphi\left(\frac{x_j}{y_j}\right) \geq \varphi(1) + \varphi'(1)\left(\frac{x_j}{y_j} - 1\right), \quad j = 1, ..., n,$$

since $\varphi \in \Phi_1$, and by (P2) and (P3), it follows from the inequality above that

$$\varphi\left(\frac{x_j}{y_j}\right) \geq 0, \quad j = 1, ..., n,$$

so,

$$d_\varphi(x, y) = \sum_{j=1}^{n} y_j \varphi\left(\frac{x_j}{y_j}\right) \geq 0.$$

If $x = y$, then by (P2) it follows that

$$d_\varphi(x, x) = \sum_{j=1}^{n} x_j \varphi\left(\frac{x_j}{x_j}\right) = \sum_{j=1}^{n} x_j \varphi(1) = 0, \quad \text{i.e.} \quad d_\varphi(x, x) = 0.$$

If $d_\varphi(x, y) = 0$, then

$$d_\varphi(x, y) = \sum_{j=1}^{n} y_j \varphi\left(\frac{x_j}{y_j}\right) = \sum_{j=1}^{n} y_j \left(1 - \sqrt{1 - \left(1 - \frac{x_j}{y_j}\right)^2}\right) = 0,$$

so,

$$x_j = y_j, \quad j = 1, ..., n.$$

(b) We show that $l(x, t) = t\varphi\left(\frac{x}{t}\right)$ is jointly convex in (x, t). From the convexity of φ, and $\forall\ (x_1, t_1), (x_2, t_2) \in \mathbb{R}^2_{++}$ and $c \in (0, 1)$, from all this it follows that

$$\varphi\left(\frac{cx_1 + (1-c)x_2}{ct_1 + (1-c)t_2}\right) = \varphi\left(\frac{ct_1}{ct_1 + (1-c)t_2}\frac{x_1}{t_1} + \frac{(1-c)t_2}{ct_1 + (1-c)t_2}\frac{x_2}{t_2}\right)$$

$$\leq \frac{c}{ct_1 + (1-c)t_2} t_1 \varphi\left(\frac{x_1}{t_1}\right) + \frac{(1-c)}{ct_1 + (1-c)t_2} t_2 \varphi\left(\frac{x_2}{t_2}\right),$$

then it follows that

$$l(cx_1 + (1-c)x_2, ct_1 + (1-c)t_2) \leq cl(x_1, t_1) + (1-c)l(x_2, t_2),$$

what had to be shown. $\square$

5.2. New self-concordant functions

The $\left(\left(\frac{3}{4}\right)^{\frac{3}{4}}\right)$-self-concordant dislocation hyperbolic kernel function

The function φ is a $\left(\left(\frac{3}{4}\right)^{\frac{3}{4}}\right)$-self-concordant dislocation hyperbolic kernel function for $\frac{1}{2} < s < 1$.

We have

$$\frac{|\varphi'''(s)|}{2\left(\varphi''(s)\right)^{\frac{3}{2}}} = \frac{\left|\frac{-3(1-s)}{(1-(1-s)^2)^{\frac{5}{2}}}\right|}{2\left(\frac{1}{(1-(1-s)^2)^{\frac{3}{2}}}\right)^{\frac{3}{2}}} = \frac{3(1-s)\left(1-(1-s)^2\right)^{\frac{9}{4}}}{2\left(1-(1-s)^2\right)^{\frac{5}{2}}}$$

$$= \frac{3\,(1-s)}{2\left(1-(1-s)^2\right)^{\frac{1}{4}}} \tag{21}$$

On the other hand, we know that,

$$0 < \frac{3(1-s)}{2\left(1-(1-s)^2\right)^{\frac{1}{4}}} < \frac{3}{4\left(1-(1-s)^2\right)^{\frac{1}{4}}} \tag{22}$$

and

$$\frac{1}{\left(1-(1-s)^2\right)^{\frac{1}{4}}} < \frac{1}{\left(\frac{3}{4}\right)^{\frac{1}{4}}}. \tag{23}$$

From (21), (22) and (23), we get $\dfrac{|\varphi'''(s)|}{2\left(\varphi''(s)\right)^{\frac{3}{2}}} < \left(\dfrac{3}{4}\right)^{\frac{3}{4}}$.

The $\left(2^{\frac{1}{4}}\right)$-self-concordant dislocation hyperbolic kernel barrier function

The function φ is a $\left(2^{\frac{1}{4}}\right)$-self-concordant dislocation hyperbolic kernel barrier function for $0 < s < 1$. We have

$$\frac{|\varphi'(s)|}{\left(\varphi''(s)\right)^{\frac{1}{2}}} = \frac{\left|\frac{-(1-s)}{(1-(1-s)^2)^{\frac{1}{2}}}\right|}{\left(\frac{1}{(1-(1-s)^2)^{\frac{3}{2}}}\right)^{\frac{1}{2}}} = \frac{\frac{(1-s)}{(1-(1-s)^2)^{\frac{1}{2}}}}{\frac{1}{(1-(1-s)^2)^{\frac{3}{4}}}}$$

$$= \frac{(1-s)\left(1-(1-s)^2\right)^{\frac{3}{4}}}{\left(1-(1-s)^2\right)^{\frac{1}{2}}} = (1-s)\left(1-(1-s)^2\right)^{\frac{1}{4}}$$

$$= (1-s)s^{\frac{1}{4}}(2-s)^{\frac{1}{4}} < 2^{\frac{1}{4}}.$$

6. Conclusions and future work

We present a new kernel function and a new proximal type distance. Using our Proposition 3.1 and properties K1-K4 our future work is to find new suitable properties to propose a proximal point algorithm and subsequently ensure a convergence theory for this proximal point algorithm based on our new kernel function.

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