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# Correction to "Analysis of the Implicit Euler Time-Discretization of a Class of Descriptor-Variable Linear Cone Complementarity Systems", J. Convex Analysis 29/2 (2022) 481–517

**Quang-Hung Pham, Bernard Brogliato**
*Univ. Grenoble Alpes, INRIA, CNRS, Grenoble INP, LJK, Grenoble, France*
*quang-hung.pham@inria.fr, bernard.brogliato@inria.fr*

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This note corrects a mistake in a recent paper of B. Brogliatio [*Analysis of the implicit Euler time-discretization of a class of descriptor-variable linear cone complementarity systems*, J. Convex Analysis 29/2 (2022) 481–517] concerning the characterization of the domain of an operator of the form $\Phi(t,\cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$, $D = D^\top$ a constant positive semi-definite matrix, $\Gamma(t)$ a nonempty closed convex set for each $t$, $\sigma_{\Gamma(t)}(\cdot)$ its support function.

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The results in [2, Theorems 3, 4] are wrongly used in [3] to conclude that

$$\mathrm{dom}(\Phi(t,\cdot)) = \Gamma(t) + \mathrm{Im}(D),$$

while it can only be inferred that $\mathrm{dom}(\Phi(t,\cdot)) \simeq \Gamma(t) + \mathrm{Im}(D)$, where $\simeq$ has the meaning as stated in [2]. Actually, the equality holds if in addition $\Gamma(t)$ is polyhedral. The set $\Gamma(t)$ is defined as $\Gamma(t) = K^\star - F(t)$ in [3], where $F : \mathbb{R}_+ \to \mathbb{R}^m$ is a function of time. Assume that $K \subseteq \mathbb{R}^m$ is a closed convex nonempty polyhedral cone defined as

$$K := \left\{ v \in \mathbb{R}^m \mid v = G^\top \xi,\ \xi \in \mathbb{R}^j_+ \right\}, G \in \mathbb{R}^{j\times m}.$$

Thus $K^\star = \{z \in \mathbb{R}^m \mid Gz \geq 0\} \subseteq \mathbb{R}^m$ (this is assumed in the subsequent developments in [3] hence it has no consequence on the results which are stated after section 3.2.1 "Calculation of sets" in that article). We will also use the following. If $S := \{\gamma \in \mathbb{R}^m : A\gamma \geq b\}$ for some $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$, then for any $x \in S$, the normal cone to $S$ at $x$ is given by:

$$\mathcal{N}_S(x) = \left\{ p \in \mathbb{R}^m : p = -A^\top \lambda, 0 \leq \lambda \perp Ax - b \geq 0, \lambda \in \mathbb{R}^n \right\}. \tag{1}$$

Also $\partial\sigma_S(\cdot) = \mathcal{N}_S^{-1}(\cdot)$, where $\partial$ denotes the subdifferential of convex analysis. Let $D \succcurlyeq 0$ (not necessarily symmetric), then $\mathrm{SOL}\left(0, GDG^\top\right)$ is a polyhedral convex cone [5, p.1054], where $\mathrm{SOL}\left(0, GDG^\top\right)$ is the set of solutions to the homogeneous Linear Complementarity Problem (LCP): $0 \leq \lambda \perp GDG^\top\lambda \geq 0$.

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Therefore, there exists a positive number $k$ and a matrix $\hat{G} \in \mathbb{R}^{k\times j}$ such that $\mathrm{SOL}\left(0,\, GDG^\top\right) = \left\{v \in \mathbb{R}^j \mid v = \hat{G}^\top\xi,\ \xi \in \mathbb{R}^k_+\right\}$ (for instance, if $GDG^\top \succ 0$, then $k$ can be chosen as any positive number, and $\hat{G} \in \mathbb{R}^{k\times j}$ is a zero matrix since the only solution is null, while if $GDG^\top = 0$ then $k = j$ can be chosen and $\hat{G}$ is an identity matrix since all vectors in $\mathbb{R}^j_+$ are solutions). Let us remind that $\Phi(t,\cdot) := (\partial\sigma_{\Gamma(t)} + D)^{-1}(\cdot)$. The following result holds:

**Lemma 1.1.** *Let* $\mathcal{H} := \left\{h \in \mathbb{R}^m : \hat{G}Gh \geq 0\right\}$.
*Then we have* $\mathrm{dom}(\Phi(t,\cdot)) = \mathcal{H} - F(t)$ *for all* $t \geq 0$.

**Proof:** For each $t \geq 0$, we have $\mathrm{dom}(\Phi(t,\cdot)) = \mathrm{Im}\big(\mathcal{N}^{-1}_{K^\star - F(t)} + D\big)$. We prove

$$\mathrm{Im}\left(\mathcal{N}^{-1}_{K^\star - F(t)} + D\right) = \mathcal{H} - F(t).$$

Take any $p \in \mathrm{Im}\big(\mathcal{N}^{-1}_{K^\star - F(t)} + D\big)$, then there exists $x \in \mathbb{R}^m$ such that

$$p \in \big(\mathcal{N}^{-1}_{K^\star - F(t)} + D\big)(x), \quad \text{i.e.,} \quad x \in \mathcal{N}_{K^\star - F(t)}(p - Dx).$$

Since $K^\star - F(t) = \{z \in \mathbb{R}^m : Gz + GF(t) \geq 0\}$, using (1) there exists $\lambda \in \mathbb{R}^j$ such that

$$\begin{cases} x = -G^\top\lambda \\ w = G(p - Dx) + GF(t) \\ 0 \leq \lambda \perp w \geq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} w = GDG^\top\lambda + Gp + GF(t) \\ 0 \leq \lambda \perp w \geq 0. \end{cases}$$

This implies that $\mathrm{LCP}\left(Gp + GF(t), GDG^\top\right)$ is solvable, where $\mathrm{LCP}(q, M)$ denotes the LCP of the form: $0 \leq \lambda \perp M\lambda + q \geq 0$, for some matrix $M$ and vector $q$. Since $GDG^\top \succcurlyeq 0$ due to $D \succcurlyeq 0$, it is copositive-plus [6, Exercise 3.12.1]. Then using [6, Corollary 3.8.10], we get

$$Gp + GF(t) \in \left(\mathrm{SOL}\left(0, GDG^\top\right)\right)^\star = \left\{z \in \mathbb{R}^j : \hat{G}z \geq 0\right\}.$$

Therefore, $p \in \mathcal{H} - F(t)$. Reversing the above process, we also obtain

$$p \in \mathrm{Im}\big(\mathcal{N}^{-1}_{K^\star - F(t)} + D\big) \quad \text{for any} \quad p \in \mathcal{H} - F(t).$$

The proof is complete. □

Let us now prove the second part of the statement, i.e., $\mathrm{dom}(\Phi(t,\cdot)) = \Gamma(t) + \mathrm{Im}(D)$ if $D = D^\top \succcurlyeq 0$. Firstly, since $\mathrm{Im}(D) = \left(\ker D^\top\right)^\perp$ [1, equation (2.4.14)], using the symmetry of $D$ and [7, Example 6.23], we get $\mathrm{Im}(D) = (\ker D)^\star$. Therefore, $K^\star + \mathrm{Im}(D) = (K \cap \ker D)^\star$ [7, Corollary 11.25].

We now prove that $\mathcal{H} = (K \cap \ker D)^\star$.

[⊆] Let any $y \in \mathcal{H}$, then $\hat{G}Gy \geq 0$, that is, $Gy \in \left(\mathrm{SOL}\left(0, GDG^\top\right)\right)^\star$. This implies that

$$\langle Gy, z\rangle \geq 0, \quad \forall z \in \mathrm{SOL}\big(0, GDG^\top\big). \tag{2}$$

Take any $\alpha \in K \cap \ker D$; then there exists $\xi \in \mathbb{R}^j_+$ such that $\alpha = G^\top\xi$ and $D\alpha = 0$. Therefore, we obtain $GDG^\top\xi = 0$, and thus $\xi \in \mathrm{SOL}\left(0, GDG^\top\right)$. It follows from

(2) that $\langle y, \alpha\rangle = \langle y, G^\top \xi\rangle = \langle Gy, \xi\rangle \geq 0$. Because $\alpha$ is chosen arbitrarily, $\langle y, \alpha\rangle \geq 0$ for all $\alpha \in K \cap \ker D$. This means that $y \in (K \cap \ker D)^\star$.

[$\supseteq$] Let any $h \in (K \cap \ker D)^\star$, then $\langle h, z\rangle \geq 0$ for all $z \in K \cap \ker D$. To obtain $h \in \mathcal{H}$, we prove that $Gh \in \big(\mathrm{SOL}\,(0, GDG^\top)\big)^\star$. Indeed, take any $\alpha \in \mathrm{SOL}\,(0, GDG^\top)$, then $\alpha \in \mathbb{R}^j_+$ and $\langle \alpha, GDG^\top \alpha\rangle = 0$. Since $D = D^\top \succcurlyeq 0$, there exists $\tilde{D} = \tilde{D}^\top \succcurlyeq 0$ such that $\tilde{D}^2 = D$. Moreover, it follows from the symmetry of $\tilde{D}$ that $\ker D = \ker \tilde{D}$. Therefore, we get

$$0 = \langle \alpha, GDG^\top \alpha\rangle = \langle \tilde{D}G^\top \alpha, \tilde{D}G^\top \alpha\rangle = \|\tilde{D}G^\top \alpha\|^2.$$

Hence, $G^\top \alpha \in \ker \tilde{D} = \ker D$. Alternatively, $G^\top \alpha \in K$. Therefore, $G^\top \alpha \in K \cap \ker D$ and thus $\langle h, G^\top \alpha\rangle \geq 0$, i.e., $\langle Gh, \alpha\rangle \geq 0$.

Therefore it is proved that

$$\mathrm{dom}(\Phi(t, \cdot)) = \mathcal{H} - F(t) = K^\star + \mathrm{Im}(D) - F(t) = \Gamma(t) + \mathrm{Im}(D).$$

**Remark 1.2.** If $D \succ 0$ (not necessarily symmetric) then

$$\mathrm{rint}(\mathrm{dom}(\Phi(t, \cdot))) = \mathrm{rint}(\Gamma(t)) + \mathrm{Im}(D) \quad \text{for all} \quad t \geq 0$$

[8, Proposition 3]. Moreover, in this case, $\mathrm{Im}(D) = \mathbb{R}^m$ and thus

$$\mathrm{rint}(\mathrm{dom}(\Phi(t, \cdot))) = \mathrm{dom}(\Phi(t, \cdot)) = \mathcal{H} = \mathbb{R}^m \quad \text{for all} \quad t \geq 0.$$

Another characterization is in [4, Theorem 4.5].

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