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Fixed Points of Generalized Pseudo-Contractions: Existence and Stability

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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We present an extension of Nadler's fixed point theorem and its local version for a class of multivalued maps that contains the pseudo-contractions. Relatively to a modulus, we propose conditions to ensure existence of fixed points with a quantitative estimate (depending on the modulus) about the distance of a reference point to the fixed point set. Some stability properties are established and are applied to the study of convergence of fixed point sets.

Keywords: Fixed points, multivalued maps, pseudo-contractions, iterative sequences, stability, convergence.

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1. Introduction

In the metric branch of fixed point theory, Banach's Contraction Principle and its version for multivalued contractions proved by Nadler [20] are fundamental and are presented in many books (see, e.g., [2], [12], [13], [17], [21]). Among numerous extensions (some are presented in [16], [18]), the case of pseudo-contractions is important for applications. Existence is proved in [11]; see also [5] where some stability properties are also presented.

The aim of this paper is to extend these results for a class of multivalued maps defined by some metric conditions based on a given function $\varphi : [0, +\infty) \to [0, +\infty)$ for which some conditions have to be required. In the case of single-valued maps, Boyd and Wong [7] proved such results with general conditions. In particular, if φ is continuous and $\varphi(t) < t$ for all $t > 0$: if f is a self-mapping of a complete metric space satisfying $d(f(x), f(y)) \leq \varphi(d(x, y))$ then it admits a unique fixed point. Another possible condition on φ is: φ is nondecreasing and the compositions $\varphi^{\langle n \rangle}(t) \to 0$ for all $t > 0$ (see for instance [13] and [16] for a proof of this result of Matkowski). In these cases each iterative sequence of f converges to the unique fixed point. But no quantitative estimates are obtained. Our objective is to impose some conditions on such a function φ in order to obtain some error estimates for the convergent iterative sequences and also to consider multivalued maps in this study.

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In § 2 we define Banach-φ-iterative sequences of a multivalued map T and we establish existence of fixed points with error estimates. In § 3 we define the property for T to be a Banach-φ-contraction w.r.t. a pair of its graph and we apply the results of the previous section to get generalizations of fixed point theorems for pseudo-contractions. In § 4 we get stability results for the general class of multivalued maps defined in § 3 and we deduce various convergence properties of the fixed point sets under similar convergence of the multivalued maps.

In this paper (X, d) is a metric space. We denote by $B(x_0, r)$ the closed ball with center $x_0 \in X$ and radius $r > 0$. For subsets of X, the excess from A to B is defined by $\mathbf{e}(A, B) = \sup\{d(a, B) : a \in A\}$ where the distance from a point a to B is defined by $d(a, B) = \inf\{d(a, b) : b \in B\}$. By convention: $d(a, \emptyset) = \infty$, $\mathbf{e}(\emptyset, B) = 0$ for $B \subset X$ and $\mathbf{e}(A, \emptyset) = \infty$ for $\emptyset \neq A \subset X$. The Hausdorff-Pompeiu distance between nonempty subsets A and B is defined by $\mathbf{h}(A, B) = \max(\mathbf{e}(A, B), \mathbf{e}(B, A))$.

By a multivalued map $T : X \rightrightarrows X$ we mean a function T from X into the family $Cl(X)$ of *closed subsets* of X. For $x \in X$, we denote by Tx its value from T. The domain of T is the subset $\mathrm{dom}(T)$ of points with nonempty values. The graph of T is defined as $\mathrm{Gr}(T) = \{(x, y) \in X \times X : y \in Tx\}$. A fixed point of T is a point $\overline{x}$ such that $\overline{x} \in T\overline{x}$; we will denote by $\mathrm{Fix}(T)$ the subset of all the fixed points of T.

We will use the following basic notions.

- For a function $\varphi : [0, +\infty) \to [0, +\infty)$ we define $\widehat{\varphi} : [0, +\infty) \to [0, +\infty]$ by $\widehat{\varphi}(t) = \sum_{k=0}^{+\infty} \varphi^{\langle k \rangle}(t)$ with $\varphi^{\langle k \rangle}$ the composition $\varphi \circ \cdots \circ \varphi$ with k factors (where $\varphi^{\langle 0 \rangle} : t \mapsto t$). We also denote $\varphi_n = \sum_{k=0}^{n} \varphi^{\langle k \rangle}$ and $\widehat{\varphi}_n = \sum_{k=n}^{+\infty} \varphi^{\langle k \rangle}$ for all n. We remark that $\widehat{\varphi}_n = \widehat{\varphi} \circ \varphi^{\langle n \rangle}$.
- $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be a *Banach modulus* iff φ is nondecreasing with $\varphi(0) = 0$ and $\widehat{\varphi}(\tau) < \infty$ for some $\tau > 0$.

Such a modulus (with $\widehat{\varphi}(t) < \infty$ for all $t > 0$) has been considered in [6] to define a generalization of a single-valued contraction $T : X \to X$ that admits a unique fixed point in a complete metric space.

The classical modulus $t \mapsto \theta t$ is a Banach modulus iff $0 \le \theta < 1$ and in this case $\widehat{\varphi}(t) = \alpha t$ (for all $t \ge 0$) with $\alpha = \sum_{k=0}^{+\infty} \theta^k = (1 - \theta)^{-1}$. Let us remark that Banach moduli are not restricted to these moduli $t \mapsto \theta t$ with $0 \le \theta < 1$. Consider for instance the function defined by $\varphi(t) = t/2$ for $0 \le t \le 2$ and $\varphi(t) = t - 1$ for $2 < t$.

A Banach modulus φ is a particular modulus of continuity as defined in [8]. This is a consequence of some elementary properties of a Banach modulus φ such that $\widehat{\varphi}(\tau) < \infty$ (with $\tau > 0$):

(i) $\widehat{\varphi}$ is a nondecreasing real-valued function on $[0, \tau]$,

(ii) $\varphi(t) < t$ for all $t \in (0, \tau]$ and φ is continuous at 0,

(iii) if φ is continuous on $[0, \tau]$ then $\widehat{\varphi}$ is continuous on $[0, \tau]$.

Indeed, (i) is obvious since all $\varphi^{\langle k \rangle} : [0, +\infty) \to [0, +\infty)$ and thus $\widehat{\varphi}$ are nondecreasing functions and then $\widehat{\varphi}(t) \le \widehat{\varphi}(\tau)$ on $(0, \tau]$. Property (ii) holds since if $\varphi(t) \ge t$ then $\varphi^{\langle k+1 \rangle}(t) \ge \varphi^{\langle k \rangle}(t) \ge \cdots \ge t$ which contradicts the convergence $\varphi^{\langle k \rangle}(t) \to 0$ for any $t \in (0, \tau]$. We note that $\varphi([0, \tau]) \subset [0, \tau]$. Finally (iii) is proved by using the continuity of all $\varphi^{\langle k\rangle}$ and thus all φ_n on $[0,\tau]$ and by using the inequalities

$$0 \leq \widehat{\varphi}(t) - \varphi_n(t) = \widehat{\varphi}_{n+1}(t) \leq \widehat{\varphi}_{n+1}(\tau)$$

for $t \in [0,\tau]$ which imply that the sequence of functions (φ_n) is uniformly convergent to $\widehat{\varphi}$ on $[0,\tau]$.

2. Existence of fixed points from Banach-φ-iterative sequences

In this section, we propose some sufficient conditions, relative to a given Banach modulus, for the existence of a fixed point of a multivalued map $T : X \rightrightarrows X$ which will be associated with an a priori estimate from an iterative construction.

We introduce two definitions concerning an *iterative sequence of* T which is a sequence (x_n) such that for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$:

- (x_n) is said to be a *φ-iterative sequence of T* iff (x_n) is an iterative sequence of T such that, for all n, $d(x_{n+2}, x_{n+1}) \leq \varphi\left(d(x_{n+1}, x_n)\right)$ where φ is a Banach modulus;
- (x_n) is said to be a *Banach-φ-iterative sequence of T* iff (x_n) is a φ-iterative sequence of T and $\widehat{\varphi}(d(x_0, x_1)) < \infty$.

We now prove that a Banach-φ-iterative sequence is necessarily a Cauchy sequence and that some estimates can be obtained in case of convergence.

Theorem 2.1. *If (x_n) is a Banach-φ-iterative sequence of T, then (x_n) is a Cauchy sequence in $B\left(x_0, \widehat{\varphi}(d(x_0, x_1))\right)$. If moreover (x_n) converges to x_∞, then*

$$d(x_\infty, x_n) \leq \widehat{\varphi}_n\left(d(x_0, x_1)\right) \quad (n \geq 0)\,, \tag{1}$$

$$d(x_\infty, Tx_\infty) \leq \liminf\ \mathbf{e}\left(Tx_n \cap B\left(x_0, \widehat{\varphi}(d(x_0, x_1))\right), Tx_\infty\right). \tag{2}$$

Proof. We denote $R = \widehat{\varphi}(d(x_0, x_1)) \in [0,\infty)$. The property

$$d(x_{n+1}, x_n) \leq \varphi^{\langle n\rangle}\left(d(x_0, x_1)\right)$$

follows from the facts that (x_n) is a φ-iterative sequence and φ is nondecreasing. Using $d(x_{n+p}, x_n) \leq \sum_{k=n}^{n+p-1} d(x_{k+1}, x_k)$ we get

$$d(x_{n+p}, x_n) \leq \sum_{k=n}^{\infty} \varphi^{\langle k\rangle}\left(d(x_0, x_1)\right) = \widehat{\varphi}_n\left(d(x_0, x_1)\right).$$

Thus (x_n) is a Cauchy sequence and $d(x_p, x_0) \leq \widehat{\varphi}_0(d(x_0, x_1)) = R$. Assume now that (x_n) converges to x_∞. We immediately deduce $d(x_\infty, x_n) \leq \widehat{\varphi}_n\left(d(x_0, x_1)\right)$. To prove (2) we can assume that Tx_∞ is nonempty and in consequence of the inequality $d(x_\infty, Tx_\infty) \leq d(x_\infty, x_{n+1}) + d(x_{n+1}, Tx_\infty)$ and by the fact that $x_{n+1} \in Tx_n \cap B(x_0, R)$ we obtain

$$d(x_\infty, Tx_\infty) \leq d(x_\infty, x_{n+1}) + \mathbf{e}\left(Tx_n \cap B(x_0, R), Tx_\infty\right),$$

which allows us to conclude that (2) holds. $\square$

With some properties formulated on the multivalued map T we will obtain that the limit x_∞ of a convergent Banach-φ-iterative sequence (x_n) of T satisfies: $x_\infty \in Tx_\infty$ and also, by Theorem 2.1, the a priori estimate $d(x_\infty, x_n) \leq \widehat{\varphi}_n\left(d(x_0, x_1)\right)$ for all n.

Theorem 2.2. *Let (x_n) be a Banach-φ-iterative sequence of T which converges to x_∞. Then x_∞ is a fixed point of T in $B\left(x_0, \widehat{\varphi}(d(x_0,x_1))\right)$ if anyone of the following assumptions holds:*

(i) *the function $x \mapsto d(x,Tx)$ is lower semicontinuous (at x_∞),*

(ii) *the graph of T is closed (or more generally T is closed at x_∞),*

(iii) *$Tx_\infty \neq \emptyset$ and $\liminf \mathbf{e}\left(Tx_n \cap B\left(x_0, \widehat{\varphi}(d(x_0,x_1))\right), Tx_\infty\right) \leq \theta d(x_\infty, Tx_\infty)$ for some $0 \leq \theta < 1$,*

(iv) *$Tx_\infty \neq \emptyset$ and there exist $0 \leq \theta < 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $R \geq \widehat{\varphi}\left(d(x_0,x_1)\right)$ such that for all $x, y \in B(x_0,R)$,*

$$\begin{aligned} &\mathbf{e}\left(Tx \cap B(x_0,R), Ty\right) \leq \theta \max\left(d(x,Ty), d(y,Ty)\right) + \alpha d(x,y) \\ &\quad + \beta \max\left(d(x, Tx \cap B(x_0,R)), d(y, Tx \cap B(x_0,R))\right) + \gamma\varphi(d(x,y)) \end{aligned} \tag{3}$$

Proof. In the cases (i) and (iii) we will prove that $d(x_\infty, Tx_\infty) = 0$ in order to conclude that x_∞ belongs to the closed subset Tx_∞. We can use the obvious facts $(x_n, x_{n+1}) \in \mathrm{Gr}(T)$ and $(x_n, x_{n+1}) \to (x_\infty, x_\infty)$. By (i) we get

$$d(x_\infty, Tx_\infty) \leq \liminf\, d(x_n, Tx_n) \leq \lim\, d\left(x_n, x_{n+1}\right) = 0.$$

By (ii) we can assert that $\lim(x_n, x_{n+1}) = (x_\infty, x_\infty) \in \mathrm{Gr}(T)$ and then $x_\infty \in Tx_\infty$. By the assumption (iii) and the inequality (2) we get $d(x_\infty, Tx_\infty) \leq \theta d(x_\infty, Tx_\infty)$ and thus $d(x_\infty, Tx_\infty) = 0$. Now, let us prove that (iv) is a sufficient condition for (iii) to be satisfied. By Theorem 2.1 it is known that $x_n, x_\infty \in B(x_0,R)$ which allows us to use the above property (3) for $x = x_n$, $y = x_\infty$. Using also $x_{n+1} \in Tx_n \cap B(x_0,R)$ we obtain that $\mathbf{e}\left(Tx_n \cap B(x_0,R), Tx_\infty\right)$ is majorized by the sum of the four terms $\theta \max\left(d(x_n, Tx_\infty), d(x_\infty, Tx_\infty)\right)$, $\alpha d(x_n, x_\infty)$, $\beta \max(d(x_n, x_{n+1}), d(x_\infty, x_{n+1}))$ and $\gamma\varphi(d(x_n, x_\infty))$. Hence, letting $n \to \infty$ we can conclude by continuity of φ at 0 that (iii) holds. □

Of course, the assumption $Tx_\infty \neq \emptyset$ in (iii) and (iv) is satisfied when the domain of T is the whole metric space or simply contains $B\left(x_0, \widehat{\varphi}(d(x_0,x_1))\right)$. It is also important to note that Tx_∞ is nonempty when the inequalities in (iv) are valid with $\theta = 0$ (consider $x = x_0$ and $y = x_\infty$).

We remark that conditions (i) and (ii) are independent of φ. The technical condition (iii), closely relative to (2), is satisfied in particular when

$$x \mapsto \mathbf{e}\left(Tx \cap B\left(x_0, \widehat{\varphi}(d(x_0,x_1))\right), Tx_\infty\right)$$

is continuous at x_∞ (when $Tx_\infty \neq \emptyset$). Obviously, (iii) holds when

$$\liminf\, \mathbf{e}\left(Tx_n, Tx_\infty\right) \leq \theta d(x_\infty, Tx_\infty) < \infty$$

for some $0 \leq \theta < 1$ (a condition independent of φ) and thus when $x \mapsto \mathbf{e}\left(Tx, Tx_\infty\right)$ is continuous at x_∞ (when $Tx_\infty \neq \emptyset$). The condition (iv) involves a relationship between six natural distances; many generalizations of the classical notion of contraction are constructed with these terms in order to get a generalization of Banach's theorem. We can remark that a quasi-contraction $T : X \to X$ in the sense of Ciric [9] implies (iv) with $R = \infty$ (see also [24] for a survey of such generalizations of contractions for single-valued maps).

We now apply these last two theorems for classical maps: Kannan maps (that can be discontinuous) and contractions.

For that purpose, consider a multivalued map $T : X \rightrightarrows X$ with nonempty values which satisfies for $K, \theta \geq 0$ with $2K + \theta < 1$:

$$\forall x, y \in X, \forall x' \in Tx, \exists y' \in Ty, d(x', y') \leq Kd(x, Tx) + Kd(y, Ty) + \theta d(x, y). \quad (4)$$

We remark that the case $0 < K < \frac{1}{2}$ and $\theta = 0$ corresponds to a Kannan multivalued map and that the case $0 < \theta < 1$ and $K = 0$ corresponds to a Banach contraction in the strong sense

$$\forall x, y \in X, \forall x' \in Tx, \exists y' \in Ty, d(x', y') \leq \theta d(x, y),$$

which in particular is satisfied for a Banach contraction with constant $\tilde{\theta} \in (0, 1)$ in the usual sense

$$\forall x, y \in X, \mathbf{e}(Tx, Ty) \leq \tilde{\theta} d(x, y),$$

by taking θ such that $0 < \tilde{\theta} < \theta < 1$. With $\lambda = (K + \theta)(1 - K)^{-1}$ the function $\varphi(t) = \lambda t$ is a Banach modulus (since $0 \leq \lambda < 1$) and we easily prove that T satisfies

$$\forall x \in X, \forall x' \in Tx, \exists x'' \in Tx', d(x', x'') \leq \lambda d(x, x') = \varphi\left(d(x, x')\right).$$

Starting with $x_1 \in Tx_0$ we can construct a sequence such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$ for $n \geq 1$. Thus (x_n) is a Banach-φ-iterative sequence of T (we can also apply Proposition 3.1 below). Moreover, the property (iv) of Theorem 2.2 is satisfied. We can obtain the existence of fixed point (see [15], [23]) with an error estimate.

Corollary 2.3. *Let $T : X \rightrightarrows X$ be a map with nonempty values which satisfies (4) for $K, \theta \geq 0$ with $2K + \theta < 1$. Consider $x_1 \in Tx_0$ and $\varphi(t) = \lambda t$ such that $\lambda = (K + \theta)(1 - K)^{-1}$. Then there exists a Banach-φ-iterative sequence (x_n) of T. Moreover, if the space X is complete then (x_n) converges to a fixed point x_∞ of T and $d(x_\infty, x_n) \leq \lambda^n (1 - \lambda)^{-1} d(x_0, x_1)$.*

3. Fixed points of Banach-φ-contractions

The concept of pseudo-Lipschitz maps introduced by Aubin [3] plays an important role in set-valued analysis (see, e.g., [4], [21], [26], [27]). Especially the concept of pseudo-contractions that have been used for some extensions of the inverse mapping theorem as in [11] where Dontchev and Hager first proved a fixed point theorem for a pseudo-contraction which is a generalization of Nadler's theorem (see [20] and [10]) and also a generalization of a local version of Banach's principle [13, Cor. (1.2)] (see [25], [14] for multivalued contractions).

We will generalize the classical definition of a pseudo-contraction with constant $\theta \in (0, 1)$ in the following. Our definitions will depend on a Banach modulus φ and a pair $(x_0, R) \in X \times [0, \infty]$ or a pair $(x_0, x_1) \in \mathrm{Gr}(T)$.

- $T : X \rightrightarrows X$ is said to be a *pseudo-φ-contraction w.r.t. $x_0 \in X$ and $R \geq 0$* (also said w.r.t. the ball $B(x_0, R)$) iff φ is a Banach modulus and

$$\forall x, y \in B(x_0, R), \forall x' \in Tx \cap B(x_0, R), \exists y' \in Ty, d(x', y') \leq \varphi\left(d(x, y)\right). \quad (5)$$

- $T : X \rightrightarrows X$ is said to be a *Banach-φ-contraction w.r.t. the points* (x_0, x_1) iff $(x_0, x_1) \in \mathrm{Gr}(T)$ with $\widehat{\varphi}(d(x_0, x_1)) < \infty$ and T is a pseudo-φ-contraction w.r.t. $x_0 \in X$ and $R = \widehat{\varphi}(d(x_0, x_1))$.

The definition in (5) is a slightly stronger form than the classical form:

$$\forall x, y \in B(x_0, R), \forall x' \in Tx \cap B(x_0, R), d(x', Ty) \leq \varphi\left(d(x, y)\right)$$

also written with the excess as:

$$\forall x, y \in B(x_0, R), \mathbf{e}(Tx \cap B(x_0, R), Ty) \leq \varphi\left(d(x, y)\right). \tag{6}$$

This version is used for the existence of fixed points in [5] by Azé and Penot with an open ball $U(x_0, R)$ and the classical modulus $t \mapsto \theta t$ with $\theta \in (0,1)$ and under the condition $(1-\theta)R > d(x_0, Tx_0)$. By choosing $x_1 \in Tx_0$ such that $d(x_0, x_1) < (1-\theta)R$ and $\theta' \in (\theta, 1)$, $R' \in (0, R)$ such that $(1-\theta')R' > d(x_0, x_1)$, we can assert that T is then a pseudo-φ'-contraction w.r.t. (x_0, x_1) and also w.r.t. $B(x_0, R')$ by denoting $\varphi'(t) = \theta' t$.

The following proposition proves that if T is a Banach-φ-contraction w.r.t. (x_0, x_1), then there exists a Banach-φ-iterative sequence (x_n) of T starting from $x_0, x_1, ...$ and contained in $B(x_0, \widehat{\varphi}(d(x_0, x_1)))$. In fact we will use a weaker form of (5) where formally $y = x'$ for each $(x, x') \in \mathrm{Gr}(T) \cap B(x_0, R) \times B(x_0, R)$.

Proposition 3.1. *Let $x_0 \in X$, $x_1 \in Tx_0$ and $R \in (0, \infty)$. Assume that we have $R \geq \widehat{\varphi}\left(d(x_0, x_1)\right)$ with φ a Banach modulus and*

$$\forall x \in B(x_0, R), \forall x' \in Tx \cap B(x_0, R), \exists x'' \in Tx', d(x', x'') \leq \varphi\left(d(x, x')\right). \tag{7}$$

Then there exists a Banach-φ-iterative sequence (x_n) of T in $B(x_0, R)$.

Proof. We will prove the existence of $x_2, \cdots, x_n$ such that, for $1 \leq k \leq n$, we have (i) $x_k \in Tx_{k-1} \cap B(x_0, R)$, (ii) $d(x_{k-1}, x_k) \leq \varphi^{\langle k-1 \rangle}\left(d(x_0, x_1)\right)$ and, for $2 \leq k \leq n$, we have (iii) $d(x_{k-1}, x_k) \leq \varphi\left(d(x_{k-2}, x_{k-1})\right)$.

These properties are satisfied by assumption for $n = 1$. Assume (i)-(ii)-(iii) for n and then apply (7) with $x = x_{n-1}$ and $x' = x_n$ (from (i)). Given by x'' we can consider $x_{n+1} \in Tx_n$ with (iii)' $d(x_n, x_{n+1}) \leq \varphi\left(d(x_{n-1}, x_n)\right)$. From (ii) for $k = n$ with φ nondecreasing and from (iii)' we deduce (ii)' $d(x_n, x_{n+1}) \leq \varphi^{\langle n \rangle}\left(d(x_0, x_1)\right)$. From (ii)-(ii)' we get $d(x_0, x_{n+1}) \leq \sum_{k=1}^{n+1} \varphi^{\langle k-1 \rangle}\left(d(x_0, x_1)\right) \leq \widehat{\varphi}\left(d(x_0, x_1)\right)$. Thus (i)' $x_{n+1} \in Tx_n \cap B(x_0, R)$ and we can conclude. □

The following theorem is a generalization of a theorem of Bianchini and Grandolfi stated in the case of single-valued maps [6, Teorema 1]. It points out that our definition of Banach-φ-contraction w.r.t. (x_0, x_1) is important for the existence of a fixed point in a complete metric space.

Theorem 3.2. *If $T : X \rightrightarrows X$ is a Banach-φ-contraction w.r.t. (x_0, x_1) and if $B(x_0, \widehat{\varphi}\left(d(x_0, x_1)\right))$ is complete, then T admits a fixed point in this ball. Moreover any φ-iterative sequence (x_n) of T starting from (x_0, x_1) is convergent and its limit x_∞ is a fixed point of T satisfying $d(x_\infty, x_n) \leq \widehat{\varphi}_n\left(d(x_0, x_1)\right)$.*

Proof. From Proposition 3.1, we can consider a Banach-φ-iterative sequence (x_n) of T in $B(x_0,R)$ by denoting $R=\widehat{\varphi}\left(d(x_0,x_1)\right)$. Assuming that this subset is complete we can assert from Theorem 2.1 that (x_n) is convergent to some $\overline{x}\in B(x_0,R)$. By (6) and by Theorem 2.2 (iv) with $\theta=\alpha=\beta=0,\gamma=1$ we conclude that $\overline{x}\in T\overline{x}\cap B(x_0,R)$. The fact that any φ-iterative sequence (x_n) of T converges to a fixed point x_∞ satisfying (1) is consequence of these two theorems. □

Let us give an elementary example of application of Theorem 3.2.

Example 3.3. Consider the space $X=[0,1]\cup\{2,3,4,..\}$ and the metric d defined on X by $d(x,y)=|x-y|$ if $x,y\in[0,1]$ and $d(x,y)=x+y$ otherwise; this metric space is complete (see [7]). We now define a single-valued map $T:X\to X$ by $Tx=x/2$ if $x\in[0,1]$ and $Tx=x-1$ if $x\in\{2,3,4,..\}$. It is straightforward to establish that $d(Tx,Ty)\le\varphi\left(d(x,y)\right)$ for $x,y\in X$ where φ is defined by $\varphi(t)=t/2$ for $0\le t\le 2$ and $\varphi(t)=t-1$ for $2<t$. We can remark that T is not a contraction since $d(Tn,T0)/d(n,0)\to 1$ but T is contractive from $\varphi(t)<t$ (and so T has at most one fixed point). Since φ is a Banach modulus such that $\widehat{\varphi}(\tau)<\infty$ for all $\tau>0$ we can apply Theorem 3.2 for all (x_0,x_1) in the graph of T: each iterative sequence $(T^{\langle k\rangle}(x_0))$ converge to the unique fixed point $x_\infty=0$ and the error estimate $d(x_\infty,x_n)\le\widehat{\varphi}_n\left(d(x_0,Tx_0)\right)$ holds. Multivalued versions of this map T can be considered to apply Theorem 3.2 (with, for instance, $x\mapsto\{Tx,1\}$). □

When the modulus φ is continuous we can get an estimate relative to the distance from the point x_0 to the set Tx_0 and the function $\widehat{\varphi}$. We thus introduce the definition:

$T:X\rightrightarrows X$ is a *Banach-φ-contraction at* $x_0\in\mathrm{dom}(T)$ iff T is a Banach-φ-contraction w.r.t. (x_0,x_1) for some $x_1\in Tx_0$.

Corollary 3.4. *Let X be a complete space and let φ be a continuous Banach modulus. If $T:X\rightrightarrows X$ is a Banach-φ-contraction at x_0, then* $\mathrm{Fix}(T)\neq\emptyset$ *and*

$$d(x_0,\mathrm{Fix}(T))\le\widehat{\varphi}(d(x_0,Tx_0)).$$

This direct consequence is based on classical arguments and we leave the proof to the reader.

For a pseudo-contraction T around $(x_0,x_1)\in\mathrm{Gr}(T)$ with constant $\tilde{\theta}$ and relative to a closed ball centered at x_0 with radius $R>(1-\tilde{\theta})^{-1}d(x_0,x_1)$ which is complete we obtain the existence of a fixed point (see [11]). We also obtain some convergence properties to a fixed point x_∞ for some θ-Lipschitz-iterative (in the sense φ-iterative with $\varphi(t)=\theta t$) sequence (x_n) of T with an a priori estimate (8). In particular $d(x_\infty,x_0)\le(1-\theta)^{-1}d(x_0,x_1)$ (see [5]).

Corollary 3.5. *Let $x_0\in X$, $x_1\in Tx_0$, $R>0$ and $0<\tilde{\theta}<1$. Assume that $R>(1-\tilde{\theta})^{-1}d(x_0,x_1)$ and $B(x_0,R)$ is complete. If T satisfies*

$$\forall x,y\in B(x_0,R),\mathbf{e}(Tx\cap B(x_0,R),Ty)\le\tilde{\theta}d(x,y),$$

then T admits a fixed point in $B(x_0,R)$.

Moreover, for $\tilde{\theta}\le\theta<1-R^{-1}d(x_0,x_1)$, if (x_n) is a *θ-Lipschitz-iterative sequence of T, then (x_n) is convergent and its limit x_∞ is a fixed point of T satisfying*

$$d(x_\infty,x_n)\le\theta^n(1-\theta)^{-1}d(x_0,x_1). \tag{8}$$

Proof. We can consider $\theta > \tilde{\theta}$ such that $R > (1-\theta)^{-1} d(x_0, x_1)$. We can apply Theorem 3.2 with the modulus $\varphi(t) = \theta t$ since

$$\forall x \in B(x_0, R), \forall y \in B(x_0, R) \setminus \{x\}, \forall x' \in Tx \cap B(x_0, R), d(x', Ty) < \theta d(x, y)$$

and then (8) holds. □

Remark 3.6. For the general class of multivalued maps defined in Proposition 3.1, it is possible to get fixed points by a direct application of Theorem 2.2. We can assert that under the assumptions of Proposition 3.1 where $B\left(x_0, \widehat{\varphi}(d(x_0, x_1))\right)$ is assumed to be complete: if the graph of T is closed or if the function $x \mapsto d(x, Tx)$ is lower semicontinuous, then T admits a fixed point in $B\left(x_0, \widehat{\varphi}(d(x_0, x_1))\right)$. We can also give estimates by Theorem 2.1. Such results have been obtained by Petrusel et al. in [22, Thm. 5.3] for the classical modulus $\varphi(t) = \theta t$.

4. Stability and convergence of fixed points

Stability of fixed points of multivalued maps plays an important role in the applications. Stability for fixed points of contractions was investigated in [20] and [19] (see also [1]). Generalizations for pseudo-contractions are given in [5] (see also [22] for other generalizations). We will present some stability properties for the fixed points of pseudo-φ-contractions where some local excess between two fixed point sets will be controlled by the value for the function $\widehat{\varphi}$ of a uniform excess between the two maps.

From now on, in order to simplify the notations we denote

$$\mathbf{e}_{x_0,r}\left(A, B\right) = \mathbf{e}\left(A \cap B(x_0, r), B\right),$$

$$\mathbf{e}_{x_0,r,R}\left(A, B\right) = \mathbf{e}\left(A \cap B(x_0, r), B \cap B(x_0, R)\right),$$

and similar notations for the Hausdorff-Pompeiu distance.

We first give an easy condition for a pseudo-φ-contraction to be a Banach-φ-contraction.

Proposition 4.1. *Let φ be a Banach modulus and $0 < r < R$. Let $T : X \rightrightarrows X$ be a pseudo-φ-contraction w.r.t. $B(x_0, R)$. If $y_0 \in B\left(x_0, r\right)$ and $y_0' \in Ty_0$ with $r + \widehat{\varphi}(d(y_0, y_0')) \leq R$, then T is a Banach-φ-contraction w.r.t. (y_0, y_0').*

Proof. Denote $\delta = \widehat{\varphi}(d(y_0, y_0'))$. The inclusion $B(y_0, \delta) \subset B(x_0, R)$ follows from $d(y, x_0) \leq d(y, y_0) + d(y_0, x_0)$ and $\delta + r \leq R$. Thus, T is a pseudo-φ-contraction w.r.t. $B(y_0, \widehat{\varphi}(d(y_0, y_0')))$ and we conclude from $(y_0, y_0') \in \mathrm{Gr}(T)$. □

We first state a stability property in a version relative to a given fixed point.

Theorem 4.2. *Let φ be a Banach modulus and let $R, r, \tau > 0$ with $r + \widehat{\varphi}(\tau) \leq R$. Let $T : X \rightrightarrows X$ be a pseudo-φ-contraction w.r.t. the ball $B(x_0, R)$ assumed to be complete. Let $S : X \rightrightarrows X$ be a multivalued map satisfying $\mathbf{e}_{x_0,r}\left(Sy_0, Ty_0\right) < \tau$ for some $y_0 \in B\left(x_0, r\right)$. If y_0 is a fixed point of S, then T admits a fixed point in $B\left(x_0, R\right)$. If moreover φ is continuous on $[0, \tau]$, then*

$$d(y_0, \mathrm{Fix}(T) \cap B(x_0, R)) \leq \widehat{\varphi}\left(\mathbf{e}_{x_0,r}\left(Sy_0, Ty_0\right)\right) \leq R - r.$$

Proof. By denoting $\tau_0 = \mathbf{e}_{x_0,r}(Sy_0, Ty_0)$ where $y_0 \in \mathrm{Fix}(S) \cap B(x_0, r)$, we consider arbitrarily $\tau_1 \in (\tau_0, \tau)$. Since $d(y_0, Ty_0) \leq \tau_0$ there exists $y_0' \in Ty_0$ such that $d(y_0, y_0') \leq \tau_1$. We have $r + \widehat{\varphi}(d(y_0, y_0')) \leq R$ and thus T is a Banach-φ-contraction w.r.t. (y_0, y_0') by Proposition 4.1. Moreover, the closed ball $B(y_0, \widehat{\varphi}(d(y_0, y_0')))$ contained in $B(x_0, R)$ is also complete. Applying now Theorem 3.2 we conclude that T admits a fixed point $\overline{x}$ in $B(y_0, \widehat{\varphi}(d(y_0, y_0')))$ and thus in $B(x_0, R)$. We can assert that

$$d(y_0, \mathrm{Fix}(T) \cap B(x_0, R)) \leq d(y_0, \overline{x}) \leq \widehat{\varphi}(d(y_0, y_0')) \leq \widehat{\varphi}(\tau_1).$$

In consequence of the continuity of φ and thus of $\widehat{\varphi}$ on $[0, \tau]$ and letting $\tau_1 \to \tau_0$ we get the conclusion. □

We can now deduce the uniform version concerning the two fixed point sets.

Corollary 4.3. *Let φ be a continuous Banach modulus and let $R, r, \tau > 0$ with $r + \widehat{\varphi}(\tau) \leq R$. Let $T : X \rightrightarrows X$ be a pseudo-φ-contraction w.r.t. the closed ball $B(x_0, R)$ which is assumed to be complete. Let $S : X \rightrightarrows X$ be a multivalued map satisfying the τ-approximation condition with T:*

$$\sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(Sy, Ty) < \tau.$$

Then
$$\mathbf{e}_{x_0,r,R}(\mathrm{Fix}(S), \mathrm{Fix}(T)) \leq \widehat{\varphi}\Big(\sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(Sy, Ty)\Big).$$

Proof. We can assume that $\mathrm{Fix}(S) \cap B(x_0, r)$ is nonempty (for otherwise the conclusion is obvious). We apply Theorem 4.2 for any fixed point y_0 of S in $B(x_0, r)$. We first remark the obvious inequalities $\widehat{\varphi}(\mathbf{e}_{x_0,r}(Sy_0, Ty_0)) \leq \widehat{\varphi}(e_{x_0,r}(S, T)) \leq \widehat{\varphi}(\tau)$ where $e_{x_0,r}(S, T)$ denotes the supremum of $\mathbf{e}_{x_0,r}(Sy, Ty)$ where $y \in B(x_0, r)$. We get

$$d(y_0, \mathrm{Fix}(T) \cap B(x_0, R)) \leq \widehat{\varphi}(e_{x_0,r}(S, T)) \leq R - r$$

and the conclusion is immediate since this holds for all $y_0 \in \mathrm{Fix}(S) \cap B(x_0, r)$. □

We remark that when the assumptions of this corollary are satisfied the pseudo-φ-contraction T admits a fixed point in $B(x_0, R)$ as soon as the perturbation S admits a fixed point in $B(x_0, r)$.

We conclude this paper by some applications of the previous stability properties for the study of convergence of a sequence of fixed point sets. For the sequel we consider a *continuous* Banach modulus φ and some real numbers $0 < r < R$ and a point $x_0 \in X$. We also consider some multivalued maps $T_n : X \rightrightarrows X$ for each $n \in \mathbb{N}$ and $T_\infty : X \rightrightarrows X$.

Proposition 4.4. *For each $n \in \mathbb{N}$, let $T_n : X \rightrightarrows X$ be a pseudo-φ-contraction w.r.t. the ball $B(x_0, R)$ assumed to be complete. Let $T_\infty : X \rightrightarrows X$ be a multivalued map satisfying the pointwise convergence condition with (T_n):*

$$\forall y \in B(x_0, r),\ \mathbf{e}_{x_0,r}(T_\infty y, T_n y) \to 0.$$

If T_∞ admits a fixed point y_0 in $B(x_0, r)$ then T_n admits a fixed point in $B(x_0, R)$ for n large enough and

$$d(y_0, \mathrm{Fix}(T_n) \cap B(x_0, R)) \to 0.$$

Proof. Let y_0 be a fixed point of T_∞ in $B(x_0, r)$. By continuity of the Banach modulus φ, the function $\widehat{\varphi}$ is continuous on $[0, \tau_0]$ for some $\tau_0 > 0$. Since $\widehat{\varphi}$ is continuous at 0 we can choose $\tau > 0$ such that $r + \widehat{\varphi}(\tau) \leq R$ and thus we apply Theorem 4.2 with $T = T_n$ and $S = T_\infty$ since $\mathbf{e}_{x_0,r}(T_\infty y_0, T_n y_0) < \tau$ for $n \geq N$ large enough. We have the existence of a fixed point in $B(x_0, R)$ for each T_n (with $n \geq N$) and

$$d(y_0, \mathrm{Fix}(T_n) \cap B(x_0, R)) \leq \widehat{\varphi}\left(\mathbf{e}_{x_0,r}(T_\infty y_0, T_n y_0)\right).$$

We conclude by continuity of $\widehat{\varphi}$ with the assumption $\mathbf{e}_{x_0,r}(T_\infty y_0, T_n y_0) \to 0$. □

Proposition 4.5. *For each $n \in \mathbb{N}$, let $T_n : X \rightrightarrows X$ be a pseudo-φ-contraction w.r.t. the ball $B(x_0, R)$ assumed to be complete. Let $T_\infty : X \rightrightarrows X$ be a multivalued map satisfying the uniform convergence condition with (T_n):*

$$\sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(T_\infty y, T_n y) \to 0.$$

If T_∞ admits a fixed point in $B(x_0, r)$ then, for n large enough, T_n admits a fixed point in $B(x_0, R)$ and

$$\mathbf{e}_{x_0,r,R}(\mathrm{Fix}(T_\infty), \mathrm{Fix}(T_n)) \to 0.$$

Proof. We assume that T_∞ admits a fixed point in $B(x_0, r)$. By continuity of φ, we can consider $\tau > 0$ such that $\widehat{\varphi}$ is continuous on $[0, \tau]$ and $r + \widehat{\varphi}(\tau) \leq R$. Since $\delta_n := \sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(T_\infty y, T_n y) \to 0$ we can fix an integer N such that $\delta_n < \tau$ for $n \geq N$. The conclusion is a direct consequence of Corollary 4.3: each T_n (with $n \geq N$) admits a fixed point in $B(x_0, R)$ and

$$\mathbf{e}_{x_0,r,R}(\mathrm{Fix}(T_\infty), \mathrm{Fix}(T_n)) \leq \widehat{\varphi}\Big(\sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(T_\infty y, T_n y)\Big). \tag{9}$$

We conclude from $\widehat{\varphi}(\delta_n) \to \widehat{\varphi}(0) = 0$. □

Proposition 4.6. *Let $T_\infty : X \rightrightarrows X$ be a pseudo-φ-contraction w.r.t. the ball $B(x_0, R)$ assumed to be complete. Let $T_n : X \rightrightarrows X$ ($n \in \mathbb{N}$) be a sequence of multivalued maps satisfying the uniform convergence condition with T_∞:*

$$\sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(T_n y, T_\infty y) \to 0.$$

If each T_n admits a fixed point in $B(x_0, r)$ then, T_∞ admits a fixed point in $B(x_0, R)$ and

$$\mathbf{e}_{x_0,r,R}(\mathrm{Fix}(T_n), \mathrm{Fix}(T_\infty)) \to 0.$$

Proof. We assume that each T_n admits a fixed point in $B(x_0, r)$. We can fix $\tau > 0$ such that $\widehat{\varphi}$ is continuous on $[0, \tau]$ and $r + \widehat{\varphi}(\tau) \leq R$. We can choose an integer N such that $\Delta_n := \sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(T_n y, T_\infty y) < \tau$ for $n \geq N$. We immediately conclude by Corollary 4.3 (with $T = T_\infty$ and $S = T_n$) which asserts that T_∞ admits a fixed point in $B(x_0, R)$ and (for $n \geq N$)

$$\mathbf{e}_{x_0,r,R}(\mathrm{Fix}(T_n), \mathrm{Fix}(T_\infty)) \leq \widehat{\varphi}\Big(\sup_{y \in B(x_0,r)} \mathbf{e}_{x_0,r}(T_n y, T_\infty y)\Big). \tag{10}$$

The proof is achieved from $\widehat{\varphi}(\Delta_n) \to 0$. □

Corollary 4.7. *Let T_∞ and T_n $(n \in \mathbb{N})$ be pseudo-φ-contractions w.r.t. $B(x_0, R)$ assumed to be complete. If T_∞ and T_n $(n \in \mathbb{N})$ admit fixed points in the ball $B(x_0, r)$ and if*

$$\sup_{y \in B(x_0,r)} \mathbf{h}_{x_0,r}\left(T_n y, T_\infty y\right) \to 0,$$

then
$$\mathbf{h}_{x_0,r,R}\left(\mathrm{Fix}(T_n), \mathrm{Fix}(T_\infty)\right) \to 0.$$

Proof. The corollary is a direct consequence of the two previous propositions; we remark that for n large enough

$$\mathbf{h}_{x_0,r,R}\left(\mathrm{Fix}(T_n), \mathrm{Fix}(T_\infty)\right) \leq \widehat{\varphi}\Big(\sup_{y \in B(x_0,r)} \mathbf{h}_{x_0,r}\left(T_n y, T_\infty y\right)\Big). \tag{11}$$

□

It is worth noticing that we also have some informations on the speed of convergence of the fixed point sets given respectively by (9), (10) and (11) for the last corresponding results.

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