Journal of Convex Analysis
Volume 31 (2024), No. 4, 1057–1138

Evolution Inclusions with and without Rough Signals

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

Received: August 29, 2023
Accepted: December 22, 2023

The paper studies diverse differential evolution inclusions involving rough signals and governed by time-dependent maximal monotone operators or subdifferentials evaluated at the velocity or the state. Their coupling with fractional order differential equations as well as their corresponding sweeping processes are also considered.

Keywords: Sweeping process, maximal monotone operator, rough signal, fractional differential inclusion, prox-regular set, Volterra-type integral.

2020 Mathematics Subject Classification: 34A60, 49I52, 49J53.

1. Introduction

Given an interval $I = [0, T]$ in the real line $\mathbb{R}$, a real Hilbert space E, mappings $f, g : I \times E \to E$, $h : I \times E \times E \to E$ and $F : I \times I \times E \to E$, and maximal monotone operators $A_t : E \rightrightarrows E$ for $t \in I$, we are mainly interested, in the present paper, on the one hand in perturbed evolution inclusions with rough signals as

$$\begin{cases} -\frac{du}{dt}(t) \in A_t u(t) + h\big(t, u(t), \int_0^t b(s, u(s))\, dz_s\big) \text{ a.e. } t \in I \\ u(0) = u_0 \end{cases} \tag{1.1}$$

and on the other hand in differential variational evolution problems with rough signals in the form

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

$$\begin{cases} \int_0^t b(s,u(s))dz_s + \int_0^t F(t,s,u(s))ds + f(t,u(t)) \\ \qquad\qquad\qquad -\Lambda_2 \frac{du}{dt}(t) \in A_t\left(\Lambda_1 \frac{du}{dt}(t) + g(t,u(t))\right) \\ u(0) = u_0. \end{cases} \tag{1.2}$$

Above $b(t,\cdot)$ is a continuous mapping from E into the space of linear mappings from E into E, and $\int_0^t b(s,u(s))\,dz_s$ is the Young integral along a mapping $z : I \to E$ which is continuous with bounded variation, see Section 4.2 for the concepts and details. In our development, $\Lambda_1, \Lambda_2 : E \to E$ will be two positive symmetric bijective linear operators.

Certain previous works must be mentioned. With closed convex sets $C(t)$ in E and the normal cone $N_{C(t)}(\cdot)$ the differential inclusion

$$\begin{cases} f_0(t) - L_1 \frac{du}{dt}(t) - L_0 u(t) \in N_{C(t)}\left(\frac{du}{dt}(t)\right) \text{ a.e. } t \in I \\ u(0) = u_0, \end{cases} \tag{1.3}$$

is dealt in [2], where $L_0, L_1 : E \to E$ are two symmetric linear operators which are merely semi-definite positive (non bijective); in [2] some assumptions allow to involve an implicit discretization via a surjectivity result of the subdifferential of a suitable convex function. An implicit discretization method is also utilized in [1] for the other problem

$$\begin{cases} -\frac{du}{dt}(t) \in N_{C(t)}\left(L_1 \frac{du}{dt}(t) + L_2 u(t)\right) \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $L_1, L_2 : E \to E$ are symmetric linear operators with L_1 coercive. This system (1.4) is illustrated in [1] as another model for certain quasistatic frictional contact problems considered in [72]. We also mention the previous work [14] with the particular case of (1.4) where $L_2 \equiv 0$ and L_1 is the identity mapping I_E on E. In [4] some other frictional contact problems are modeled with the differential inclusion

$$\begin{cases} -\frac{du}{dt}(t) \in N_{\Gamma\left(t,\mathcal{R}\frac{du}{dt}(t)\right)}\left(\Phi \frac{du}{dt}(t) + \Psi u(t) + \mathcal{S}\frac{du}{dt}(t)\right) \\ u(0) = u_0, \end{cases} \tag{1.5}$$

where $\Psi : E \to E$ is a Lipschiz continuous mapping, $\Phi : E \to E$ is a strongly monotone Lipschitz continuous operator, $\Gamma(t,v) = \ell(t) - \partial j(v,\cdot)(0)$ for some continuous mapping $\ell : I \to E$ and the subdifferential at zero $\partial j(v,\cdot)(0)$ of some specific sublinear function $j(v,\cdot) : E \to \mathbb{R} \cup \{+\infty\}$ with $v \in E$, and $\mathcal{R}, \mathcal{S} : \mathcal{C}_E(I) \to \mathcal{C}_E(I)$ are two almost history-dependent operators (see [73] for the definition), $\mathcal{C}_E(I)$ being the space of continuous mappings from I into E. To avoid the above particular structure of $\Gamma(t, \mathcal{R}\frac{du}{dt}(t))$, a closed convex moving set $K(t)$ (independent of the velocity $\frac{du}{dt}(t)$) is involved in [61] for (1.5) with $K(t)$ in place of $\Gamma(t, \mathcal{R}\frac{du}{dt}(t))$, and the unique solvability is shown under a Mosco convergence type condition of $K(t)$. A more general closed convex set $\Gamma(t, \mathcal{R}\frac{du}{dt}(t))$ is utilized in [62] to obtain a result encompassing both [4] and [61]. This result in [62] is proved under the continuity of the mapping $t \mapsto P_{\Gamma(v,t)}u$ for every (u,v) and the Lipschitz continuity of $v \mapsto P_{\Gamma(v,t)}u$ with a uniform Lipschitz constant independent of (t,u), where $P_{\Gamma(t,v)}u$ denotes the metric projection of u over the closed convex set $\Gamma(t,v)$.

The two differential inclusions (1.3), (1.4) as well as (1.5) and its aforementioned similar ones from [61] and [62], are variants of the well-known Moreau sweeping process [57–59]

$$-\frac{du}{dt}(t) \in N_{C(t)}u(t) \quad \text{with } u(0) \in C(0).$$

For several variants of the Moreau sweeping process we refer, e.g., to [8, 10–12, 17, 28, 36, 40, 41, 46, 47, 74, 81]. The case in (1.2) where $b(\cdot,\cdot) \equiv 0$, $F(\cdot,\cdot,\cdot) \equiv 0$ and $A_t(\cdot) = N_{C(t)}(\cdot)$, namely the differential inclusion

$$\begin{cases} f(t,u(t)) - \Lambda_2 \frac{du}{dt}(t) \in N_{C(t)}\left(\Lambda_1 \frac{du}{dt}(t) + g(t,u(t))\right) \\ u(0) = u_0, \end{cases} \tag{1.6}$$

is solved in [49] by a remarkable reduction to an ordinary differential equation; see also [78] for a certain extension of [49]. Inspired by the method of [49] we will first deal with inclusions similar to (1.6) with either a time-dependent subdifferential $\partial\varphi(t,\cdot)$ or a time-dependent maximal monotone operator A_t. Then, through some important convergence lemmas requiring non-trivial proofs, we show the existence of solutions for general systems of type (1.2). This is achieved on the one hand with subdifferentials of time-dependent convex functions $\varphi(t,\cdot)$ and on the other hand with time-dependent maximal monotone operators $A_t(\cdot)$; in both cases no continuity control in t is assumed but merely a measurability condition. Regarding systems of the other type (1.1), the existence of solutions is established via some other new lemmas of compactness and convergence. The situation where (1.2) is coupled with fractional differential equations is also studied. The coupling of other inclusions with fractional differential equations can be found, e.g., in [22,24–26] while diverse evolution problems coupled with either rough signals or stochastic differential equations are considered, e.g., in [29, 31, 32, 37, 42, 52, 68]. We also investigate the above problems in the setting where solutions are required to be of bounded variation (BV, for short). For other works with BV solutions of sweeping processes and of inclusions with maximal monotone operators, we refer, e.g., to [3, 21, 30, 40, 41, 55, 58–60, 63, 75].

Our paper is organized as follows. In Section 2 we recall some needed features on convex analysis and on measurable multimappings. Section 3.1 is devoted to differential variational inequalities with subdifferentials evaluated mainly at velocity arguments, more precisely the differential inclusion (1.2) with $b \equiv 0$ and $F \equiv 0$. We also examine in Section 3.1 the situation where in addition history-dependent operators are present in the mentioned case of (1.2) with $b \equiv 0$ and $F \equiv 0$ as well as the corresponding situation of variational inequalities with normal cones at velocities. The latter problems with time-dependent maximal operators instead of subdifferentials are the subject of Section 3.2. The equation (1.1) is dealt in Section 4.3. The general problem (1.2) with rough signals and Volterra-type integrals is studied with subdifferentials of time-dependent convex functions in Section 4.4, and with time-dependent maximal operators in Section 4.5. Section 5 is dedicated to evolution inclusions governed by operators/subdifferentials with velocity arguments $\frac{du}{dt}(t)$ and coupled with fractional differential equations related to Riemann-Liouville fractional derivatives, rough signals and Volterra-type integrals. Under a condition of absolute continuity of the multimapping $t \mapsto A_t$ with respect to a suitable pseudo-distance dis on the set of maximal monotone operators (see Section 3.2 for the definition), problems of the latter type with maximal monotone operators A_t evaluated merely at the state $u(t)$ are also studied in the same Section 5. The above problems in Section 5 with fractional differential equations related to Caputo fractional derivatives (instead of Riemann-Liouville fractional derivatives) are the subject of Section 6. In the last Section 7 the above absolute continuity condition for the mapping $t \mapsto A_t$ is relaxed to a condition of bounded variation with respect to the pseudo-distance dis.

2. Notation and terminology

In this section we consider a separable Hilbert space E with its inner product $\langle \cdot, \cdot \rangle$ and its associated norm $\| \cdot \|_E$ (also denoted $\| \cdot \|$ when there is no risk of confusion), and we collect diverse concepts and results which will be crucial for the development of the paper. The closed (resp. open) ball centered at $x \in E$ with radius $r > 0$ will be denoted by $B[x, r]$ (resp. $B(x, r)$), and $\mathbb{B}_E$ (or $\mathbb{B}$ if there is no ambiguity) will be the closed unit ball, that is, $\mathbb{B}_E := B[0, 1]$. The closure of a subset S in E will be denoted $\operatorname{cl} S$ or $\overline{S}$. The space of continuous mappings from a compact interval I of $\mathbb{R}$ into E will be denoted $\mathcal{C}(I, E)$ or $\mathcal{C}_E(I)$ and it will be equipped (unless otherwise stated) with the topology of uniform convergence associated with the supremum norm $\| \cdot \|_{\infty;I}$, also denoted $\| \cdot \|_\infty$ when there is no risk of ambiguity. When E is the Euclidean space $\mathbb{R}^d$ with $d \in \mathbb{N}$, its Euclidean norm will be denoted by $\| \cdot \|_{\mathbb{R}^d}$. For any subinterval J of I, by $\mathbf{1}_J$ we mean the function from J into $\mathbb{R}$ defined by $\mathbf{1}_J(t) = 1$ if $t \in J$ and $\mathbf{1}_J(t) = 0$ if $t \in I \setminus J$.

Given $p \in [1, +\infty]$ and a measure space $(\mathcal{I}, \mathfrak{T}, \nu)$ with a σ-field $\mathfrak{T}$ on $\mathcal{I}$ and a positive measure ν, we will use the standard notation $L^p_E(\mathcal{I}, \nu)$ for the space of $\mathfrak{T}$-measurable mappings $v : \mathcal{I} \to E$ from $\mathcal{I}$ into E such that $\int_{\mathcal{I}} \|v(t)\|^p \, d\nu(t) < +\infty$ if $p < +\infty$, and $v(\cdot)$ is bounded ν-almost everywhere (ν-a.e., for short) if $p = +\infty$. When $\mathcal{I}$ is a Lebesgue measurable subset S of $\mathbb{R}$ and ν is the Lebesgue measure dt, we will write $L^p_E(S)$ in place of $L^p_E(S, dt)$ for $p \in [1, +\infty]$.

Consider now a real $p \geq 1$, reals $0 \leq T_0 < T$ and the interval $I := [T_0, T]$. We denote as usual by $W^{1,p}_E(I)$ the Banach space of mappings $u : I \to E$ which are almost everywhere differentiable with $\frac{du}{dt}$ (also denoted $\dot{u}$) in $L^p_E(I)$ and such that

$$u(t) = u(T_0) + \int_{T_0}^{t} \frac{du}{d\tau}(\tau) \, d\tau \quad \text{for all } t \in I,$$

endowed with the norm $\|u\|_{W^{1,p}_E} := \left(\|u\|^p_{L^p_E} + \|\frac{du}{dt}\|^p_{L^p_E} \right)^{1/p}$. When $p = 1$, it is known that $W^{1,1}_E(I)$ coincides with the space of all *absolutely continuous mappings* from I into E (according to the reflexivity of E). The absolute continuity of $u : I \to E$ means that for every real $\varepsilon > 0$ there exists a real $\eta > 0$ such that for any

$$T_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1} \leq T \quad \text{with} \quad \sum_{i=1}^{n} (t_{i+1} - t_i) < \eta$$

one has
$$\sum_{i=1}^{n} \|u(t_{i+1}) - u(t_i)\| < \varepsilon.$$

The concept of *measurable multimappings* will also be used. Given a measurable space $(\mathcal{I}, \mathfrak{T})$ where $\mathfrak{T}$ is a σ-field on $\mathcal{I}$, a multimapping $M : \mathcal{I} \rightrightarrows E$ is said to be $\mathfrak{T}$-*measurable* if for every open set U in E the set

$$M^{-1}(U) := \{t \in \mathcal{I} : M(t) \cap U \neq \emptyset\}$$

belongs to the σ-field $\mathfrak{T}$. Denoting $\operatorname{Dom} M$ or $D(M)$ the (effective) *domain* of M, i.e.,

$$\operatorname{Dom} M = D(M) := \{t \in \mathcal{I} : M(t) \neq \emptyset\},$$

the $\mathfrak{T}$-measurability of M clearly entails the inclusion $D(M) \in \mathfrak{T}$. When the σ-field $\mathfrak{T}$ is *complete* with respect to some positive σ-finite measure, the $\mathfrak{T}$-measurability of M is equivalent (see [35, 71]) to the $\mathfrak{T} \otimes \mathcal{B}(E)$ measurability of the graph of M

$$\operatorname{gph} M := \{(t, x) \in \mathcal{I} \times E : x \in M(t)\}. \tag{2.1}$$

As usual $\mathcal{B}(E)$ is the Borel σ-field of E. A mapping $u : D(M) \to E$ is a *selection of* M if $u(t) \in M(t)$ for all $t \in D(M)$; if for a measure μ on $(\mathcal{I}, \mathfrak{T})$ one has $u(t) \in M(t)$ for μ-almost every $t \in D(M)$, one says that u is a selection of M μ-almost everywhere (a.e., for short). If M is a closed-valued multimapping which is $\mathfrak{T}$-measurable with $D(M) \neq \emptyset$, it is known (see, e.g., [35, 70, 71]) that the set S_M of $\mathfrak{T}$-measurable selections of M is nonempty and that there exists a sequence $(u_n)_n$ in S_M for which the following Castaing representation

$$M(t) = \operatorname{cl}_E \{u_n(t) : n \in \mathbb{N}\}, \tag{2.2}$$

holds for every $t \in D(M)$. If M is nonempty-valued but not necessarily closed-valued and if $\operatorname{gph} M$ belongs to $\mathfrak{T} \otimes \mathcal{B}(E)$ with $\mathfrak{T}$ complete with respect to a σ-finite positive measure, it is known (see, e. g., [35, Theorem III-22]) that there exists a sequence $(u_n)_n$ of $\mathfrak{T}$-measurable selections of M such that for each $t \in \mathcal{I}$

$$\text{the set } \{u_n(t) : n \in \mathbb{N}\} \text{ is dense in } M(t). \tag{2.3}$$

The situation of the Lebesgue σ-field $\mathcal{L}(I)$ with $I = [T_0, T]$ will be of great interest throughout the paper. For any real $p \in [1, +\infty]$, it will be convenient, under the Lebesgue measurability of M, to denote S^p_M the set of all selections of M which are in $L^p_E(D(M))$. If in addition to the Lebesgue measurability of the multimapping M and the closedness property there is a non-negative function $\gamma(\cdot) \in L^p_{\mathbb{R}}(I)$ such that $M(t) \subset \gamma(t)\mathbb{B}_E$ for all $t \in I$, then the sequence $(u_n)_n$ in the above Castaing representation as well as any Lebesgue selection of M are clearly included in S^p_M. When there is no ambiguity, we will keep S_M (resp. S^p_M) to denote the set of all selections of M Lebesgue a.e. on $D(M)$ (resp. which in addition are in $L^p_E(D(M))$).

The Hausdorff-Pompeiu distance will also be involved in several places of the text. Given two non-empty subsets S, S' of E, the excess of S over S' is the non-negative extended real (see, e.g., [64, 71, 76])

$$\operatorname{exc}(S, S') := \sup_{x \in S} d(x, S'),$$

where $d(x, S') := \inf_{x' \in S'} \|x - x'\|$ (also denoted $d_{S'}(x)$). The *Hausdorff-Pompeiu distance* between S and S' is then defined as

$$\operatorname{haus}(S, S') := \max\{\operatorname{exc}(S, S'), \operatorname{exc}(S', S)\}.$$

3. Existence results for evolution inclusions with time-dependent operators evaluated mainly at velocity arguments

3.1. Convex integrands and differential variational inequalities under time-dependent subdifferentials evaluated mainly at velocity arguments

Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The properness property of ϕ means that its (effective) *domain*

$$\operatorname{dom} \phi := \{x \in E : \phi(x) < +\infty\}$$

is nonempty, which is equivalent to say that its *epigraph*

$$\operatorname{epi} \phi := \{(x, r) \in E \times \mathbb{R} : \phi(x) \leq r\}$$

is nonempty. Notice also that the convexity (resp. lower semicontinuity) of ϕ is equivalent to the convexity (resp. closedness) of $\operatorname{epi} \phi$ in $E \times \mathbb{R}$. The *subdifferential* of ϕ at a point $x \in \operatorname{dom} \phi$ is given by (see, e.g., [35, 56, 76])

$$\partial\phi(x) := \{\zeta \in E : \langle \zeta, y - x\rangle \leq \phi(y) - \phi(x),\ \forall y \in E\},$$

and $\partial\phi(x) = \emptyset$ if $\phi(x) = +\infty$. When ϕ is the *indicator function* $\delta_S : E \to \mathbb{R} \cup \{+\infty\}$ of a nonempty closed convex set S in E, that is,

$$\delta_S(x) = 0 \quad \text{if } x \in S, \quad \text{and} \quad \delta_S(x) = +\infty \quad \text{otherwise},$$

the *normal cone* $N_S(x)$ of S at $x \in E$ is defined as the subdifferential of δ_S at x, that is, $N_S(x) = \partial\delta_S(x)$, which can also be written as

$$N_S(x) = \{\zeta \in E : \langle \zeta, y - x\rangle \leq 0,\ \forall y \in S\} \quad \text{if } x \in S$$

and $N_S(x) = \emptyset$ if $x \notin S$.

The *proximal mapping* $\operatorname{prox}_\phi : E \to E$ of the proper lower semicontinuous convex function ϕ is defined for any $x \in E$ by $\operatorname{prox}_\phi(x)$ as the unique minimizer on E of the function

$$E \ni y \mapsto \phi(y) + \frac{1}{2}\|x - y\|^2.$$

One knows (see, e.g., [56, 76]) for such a function ϕ that prox_ϕ is Lipschitz on E with 1 as Lipschitz constant, that is,

$$\left\|\operatorname{prox}_\phi(x_1) - \operatorname{prox}_\phi(x_2)\right\| \leq \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in E, \tag{3.1}$$

and one also knows that $\operatorname{prox}_\phi = (I_E + \partial\phi)^{-1}$, where I_E denotes here and in all the paper the identity mapping on E.

We pass now to convex normal integrands (see [35, 70, 71]).

Given the interval $I = [T_0, T]$, a function $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ is a *normal integrand* (resp. *convex normal integrand*) if the multimapping from I into $E \times \mathbb{R}$ defined with $\varphi_t := \varphi(t, \cdot)$ by $t \mapsto \operatorname{epi} \varphi_t$ is a Lebesgue measurable multimapping with closed (resp. closed convex) values.

According to the completeness of the σ-field $\mathcal{L}(I)$, this is known to be equivalent to saying (see [35, p. 195,196], [70, 71]) that φ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable and $\varphi(t, \cdot)$ is proper, lower semicontinuous (resp. and convex) for each $t \in I$. When φ is a convex normal integrand, it is known that the mapping $(t, x) \mapsto \mathrm{prox}_{\varphi_t}(x)$ from $I \times E$ into E is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable since it is measurable in t and (Lipschitz) continuous in x.

Certain symmetric linear operators will be employed below and in other places in the paper. Consider a linear mapping $L : E \to E$ from the Hilbert space E into itself which is symmetric in the sense that $\langle Ly, x\rangle = \langle y, Lx\rangle$ for all $x, y \in E$, so it is continuous (as well-known) since its graph is closed. If L is in addition positive, i.e. $\langle Lx, x\rangle \geq 0$ for all $x \in E$, one knows (see, e.g., [69, p. 265]) that its (unique) positive symmetric square root exists, i.e. there is a (unique) positive symmetric (continuous) linear mapping $L_0 : E \to E$ such that

$$L = L_0 L_0, \tag{3.2}$$

and clearly the linear mapping L_0 is bijective whenever L is itself bijective. It is worth mentioning that (by the Lax-Milgram theorem or by Theorem 2.20 in [16]) a (continuous) symmetric linear mapping $L : E \to E$ is bijective whenever it is *coercive* (called also strongly monotone) in the sense that there exists a real constant $c > 0$ such that $\langle Lx, x\rangle \geq c\|x\|^2$ for all $x \in E$.

Through the above features and inspired by A. Jourani and E. Vilches paper [49] we are now in a position to establish the existence result in Theorem 3.1. In fact, the proof of the theorem is based on an adaptation of diverse arguments of Jourani and Vilches [49] where the normal cone $N_{C(t)}$ of a moving set $C(t)$ in E is considered there instead of the subdifferential $\partial\varphi_t$ here.

Theorem 3.1. *Let $p \in [1, +\infty]$ and let $I := [0, T]$ and $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there are a mapping $\zeta \in L^p_E(I)$ such that $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ is in $L^p_{\mathbb{R}_+}(I)$ and two non-negative functions $\alpha, \beta : I \to \mathbb{R}$ with $\alpha \in L^p_{\mathbb{R}_+}(I)$ and $\sqrt{\beta} \in L^p_{\mathbb{R}_+}(I)$ such that*

$$-\alpha(t)\|x\| - \beta(t) \leq \varphi(t, x) \quad \textit{for all } (t, x) \in I \times E$$

(note that $\sqrt{\beta} \in L^p_{\mathbb{R}_+}(I)$ if $\beta \in L^p_{\mathbb{R}_+}(I)$). Let $f, g : I \times E \to E$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in E$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t, x) \in I \times E$

$$\|f(t, x)\| \leq \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t, x)\| \leq \alpha_g(t)\|x\| + \beta_g(t) \tag{3.3}$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_E$

$$\|f(t, x) - f(t, y)\| \leq \gamma_{f,r}(t)\|x - y\|, \ \|g(t, x) - g(t, y)\| \leq \gamma_{g,r}(t)\|x - y\|.$$

Let $\Lambda : E \to E$ be a positive symmetric bijective linear operator. Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t, u(t)) - \Lambda \frac{du}{dt}(t) \in \partial\varphi\left(t, \frac{du}{dt}(t) + g(t, u(t))\right) & \textit{a.e. } t \in I \\ u(0) = u_0 \end{cases} \tag{3.4}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I.

Further, $u(\cdot) \in W^{1,p}_E(I)$ *and more precisely there are a real constant* $\kappa > 0$ *and two non-negative functions* $a(\cdot)$ *and* $b(\cdot)$ *in* $L^p_{\mathbb{R}_+}(I)$ *such that*

$$\|u(t)\| \le \kappa \quad \textit{for all } t \in I \quad \textit{and} \quad \|\dot u(t)\| \le a(t)\|u(t)\| + b(t) \quad \textit{a.e. } t \in I,$$

where $a(\cdot)$ *and* $b(\cdot)$ *depend only on* Λ*, on the functions* α_f*,* β_f*,* α_g*,* β_g *satisfying* (3.3)*, and on the normal integrand* φ *(via the mappings* $\zeta(\cdot)$*,* $\alpha(\cdot)$*,* $\beta(\cdot)$*), and where the constant* κ *depends only on* $\|u_0\|$ *and* Λ *and on the same above functions* α_f*,* β_f*,* α_g*,* β_g *and the integrand* φ *(via the mappings* $\zeta(\cdot), \alpha(\cdot), \beta(\cdot)$*).*

Proof. Let $\Lambda_0 : E \to E$ be (see (3.2)) the positive symmetric bijective (continuous) linear operator such that $\Lambda = \Lambda_0\Lambda_0$.

(I) Consider the function $\psi : I \times E \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\psi(t,x) := \varphi(t, \Lambda_0^{-1}x) \quad \text{for all } (t,x) \in I \times E,$$

so ψ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable and $\psi(t,\cdot)$ is proper, lower semicontinuous and convex for every $t \in I$, hence ψ is a convex normal integrand. Note that for every $(t,x) \in I \times E$ we have $\partial\psi_t(x) = \Lambda_0^{-1}\partial\varphi_t(\Lambda_0^{-1}x)$, hence

$$\Lambda_0^{-1}\partial\varphi_t(x) = \partial\psi_t(\Lambda_0 x),$$

so we deduce for any $y \in E$ that

$$y - \Lambda_0\Lambda_0 x \in \partial\varphi_t(x) \Leftrightarrow {\Lambda_0}^{-1}y - \Lambda_0 x \in \partial\psi_t(\Lambda_0 x) \Leftrightarrow \Lambda_0 x = \mathrm{prox}_{\psi_t}(\Lambda_0^{-1}y).$$

Writing the inclusion in (3.4) as

$$f(t,u(t)) + \Lambda_0\Lambda_0 g(t,u(t)) - \Lambda_0\Lambda_0\big(\dot u(t) + g(t,u(t))\big) \in \partial\varphi_t\big(\dot u(t) + g(t,u(t))\big),$$

we see that it is equivalent to

$$\Lambda_0\dot u(t) + \Lambda_0 g(t,u(t)) = \mathrm{prox}_{\psi_t}\big(\Lambda_0^{-1}f(t,u(t)) + \Lambda_0 g(t,u(t))\big).$$

Therefore, defining $F : I \times E \to E$ by

$$F(t,x) := -g(t,x) + \Lambda_0^{-1}\mathrm{prox}_{\psi_t}\big(\Lambda_0^{-1}f(t,x) + \Lambda_0 g(t,x)\big) \text{ for all } (t,x) \in I \times E,$$

we see that the differential inclusion (3.4) is equivalent to the differential equation on I

$$\begin{cases} \dot u(t) = F(t,u(t)) & \text{a.e. } t \in I, \\ u(0) = u_0. \end{cases} \tag{3.5}$$

$F(\cdot,x)$ is Lebesgue measurable for each $x \in E$ since the mapping $(t,x) \mapsto \mathrm{prox}_{\psi_t}(x)$ is Lebesgue measurable in t and Lispchitz in x.

(II) We also observe that for each real $r > 0$ we can write for each $t \in I$ and all $x, y \in r\mathbb{B}$

$$\begin{aligned} &\|F(t,x) - F(t,y)\| \\ &\le \|-g(t,x) + g(t,y)\| + \|\Lambda_0^{-1}\|\,\|\Lambda_0^{-1}(f(t,x) - f(t,y)) + \Lambda_0(g(t,x) - g(t,y))\| \\ &\le \gamma_{g,r}(t)\|x-y\| + \|\Lambda_0^{-1}\|^2\gamma_{f,r}(t)\|x-y\| + \|\Lambda_0^{-1}\|\,\|\Lambda_0\|\gamma_{g,r}(t)\|x-y\|, \end{aligned}$$

so for $\gamma_{F,r}(\tau) := \|\Lambda_0^{-1}\|^2\gamma_{f,r}(\tau) + \big(1 + \|\Lambda_0\|\,\|\Lambda_0^{-1}\|\big)\,\gamma_{g,r}(\tau)$ with $\gamma_{F,r}(\cdot) \in L_{\mathbb{R}_+}(I)$

we have $$\|F(t,x) - F(t,y)\| \le \gamma_{F,r}(t)\|x - y\|.$$

(III) On the other hand, putting $y_t := \operatorname{prox}_{\psi_t}(0)$, we have $-\Lambda_0 y_t \in \partial\varphi_t(\Lambda_0^{-1}y_t)$, which by the assumption on $\zeta(\cdot)$ gives

$$\langle -\Lambda_0 y_t, \zeta(t) - \Lambda_0^{-1}y_t\rangle \le \varphi_t(\zeta(t)) - \varphi_t(y_t) \le \varphi_t(\zeta(t)) + \alpha(t)\|y_t\| + \beta(t),$$

hence $$\|y_t\|^2 - (\alpha(t) + \|\Lambda_0\|\,\|\zeta(t)\|)\|y_t\| - \big(\beta(t) + (\varphi_t(\zeta(t)))^+\big) \le 0.$$

Writing $$\Delta(t) := (\alpha(t) + \|\Lambda_0\|\,\|\zeta(t)\|)^2 + 4\,\big(\beta(t) + (\varphi_t(\zeta(t)))^+\big) \ge 0,$$

we see that $$\|y_t\| \le \frac{1}{2}\big(\alpha(t) + \|\Lambda_0\|\,\|\zeta(t)\| + \sqrt{\Delta(t)}\big) =: \sigma(t),$$

with $\sigma(\cdot) \in L^p_{\mathbb{R}_+}(I)$. Put $q(t,x) := \Lambda_0^{-1}f(t,x) + \Lambda_0 g(t,x)$. Then by the 1-Lipschitz property of $\operatorname{prox}_{\psi_t}$ (see (3.1))

$$\begin{aligned}
\|F(t,x)\| &\le \|g(t,x)\| + \|\Lambda_0^{-1}\|\,\|\operatorname{prox}_{\psi_t}(q(t,x))\| \\
&\le \|g(t,x)\| + \|\Lambda_0^{-1}\|\,\|q(t,x)\| + \|\Lambda_0^{-1}\|\,\|\operatorname{prox}_{\psi_t}(0)\| \\
&\le \|g(t,x)\| + \|\Lambda_0^{-1}\|\,\|q(t,x)\| + \sigma(t)\|\Lambda_0^{-1}\|,
\end{aligned}$$

and this clearly furnishes α_F, β_F in $L^p_{\mathbb{R}_+}(I)$ (hence also in $L^1_{\mathbb{R}_+}(I)$) such that

$$\|F(t,x)\| \le \alpha_F(t)\|x\| + \beta_F(t) \quad \text{for all } (t,x) \in I \times E.$$

It results that the differential equation (3.5) has one and only one absolutely continuous solution, which yields that the evolution inclusion (3.4) admits one and only one absolutely continuous solution $u(\cdot)$ on I.

It is clear from what precedes that the functions α_F and β_F depend only on Λ, on the functions α_f, $\beta_f, \alpha_g, \beta_g$ satisfying (3.3), and on the mappings ζ, α, β related to the integrand φ. Further we have $\|\dot u(t)\| \le \alpha_F(t)\|u(t)\| + \beta_F(t)$ for a.e. $t \in I$. Putting $\rho(t) := \|u(t)\|$ for all $t \in I$, the function ρ and the mapping u are both absolutely continuous, so both are differentiable on a set $P \subset]0,T[$ with $I \setminus P$ Lebesgue negligible. For any $t \in P$ with $u(t) \ne 0$ we deduce by the chain rule $\dot\rho(t) \le \|\dot u(t)\|$, and for any $t \in P$ with $u(t) = 0$ we see from the definition of derivate that $\dot\rho(t) = 0$. Consequently, $\dot\rho(t) \le \alpha_F(t)\rho(t) + \beta_F(t)$ for a.e. $t \in I$, so the Gronwall lemma yields for every $t \in I$

$$\rho(t) \le \|u_0\| \exp\left(\int_0^t \alpha_F(s)\,ds\right) + \int_0^t \beta_F(\tau) \exp\left(\int_\tau^t \alpha_F(s)\,ds\right) d\tau.$$

This furnishes a real constant $\kappa > 0$ depending on $\|u_0\|$, α_F and β_F such that

$$\|u(t)\| = \rho(t) \le \kappa \text{ for all } t \in I.$$

The proof of the theorem is complete with $a(t) := \alpha_F(t)$ and $b(t) := \beta_F(t)$ for all $t \in I$. □

Remark 3.2. Previously, S. A. Timoshin and A. A. Tolstonogov [78, Theorem 1.1] established with $I := [0,T]$ the existence of an absolutely continuous solution of the differential inclusion (with notation in [78])

$$\begin{cases} -\frac{du}{dt}(t) \in \partial\varphi\left(t, A\frac{du}{dt}(t) + B(t,u(t))\right) + U(t,u(t)) \text{ a.e. } t \in I \\ u(0) = u_0, \end{cases} \tag{3.6}$$

where $A : H \to H$ is a linear operator, $\varphi : I \times H \to \mathbb{R} \cup \{+\infty\}$ is a convex integrand as above, $B : I \times H \to H$ is a mapping and $U : I \times H \rightrightarrows H$ is a multimapping with nonconvex closed values. The linear operator A is assumed to satisfy $A = P^*P$ for some bijective linear operator $P : H \to H$, the mapping $B(\cdot, x)$ and multimapping $U(\cdot, x)$ are Lebesgue measurable and there exist ℓ_B, m_B, n_B, ℓ_U, m_U and n_U in $L^1(I, \mathbb{R}_+)$ such that for all $t \in I$ and $x \in H$

$$\|B(t,x)\| \le m_B(t)\|x\| + n_U(t), \quad d(0, U(t,x)) \le m_U(t)\|x\| + n_U(t)$$

along with the Lipschitz properties for all $t \in I$ and $x, y \in H$

$$\|B(t,x) - B(t,y)\| \le \ell_B(t)\|x-y\|, \quad \operatorname{haus}(U(t,x), U(t,y)) \le \ell_U(t)\|x-y\|.$$

In Theorem 3.1 above such a multimapping U is not involved and this allowed us to provide with our data a simpler proof than the one in [78] by applying directly an approach similar to the method of A. Jourani and E. Vilches [49] for the normal cone $N_{C(t)}(\cdot)$ of a convex moving set $C(t)$ instead of the subdifferential $\partial\varphi_t(\cdot)$ of the convex integrand φ. $\square$

A first variant of Theorem 3.1 is deduced as follows.

Corollary 3.3. *Let $p \in [1, +\infty]$ and let $I := [0,T]$ and $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there are a mapping $\zeta \in L^p_E(I)$ such that $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ is in $L^p_{\mathbb{R}_+}(I)$ and two non-negative functions $\alpha, \beta : I \to \mathbb{R}$ with $\alpha \in L^p_{\mathbb{R}_+}(I)$ and $\sqrt{\beta} \in L^p_{\mathbb{R}_+}(I)$ such that*

$$-\alpha(t)\|x\| - \beta(t) \le \varphi(t,x) \quad \textit{for all } (t,x) \in I \times E$$

(note that $\sqrt{\beta} \in L^p_{\mathbb{R}_+}(I)$ if $\beta \in L^p_{\mathbb{R}_+}(I)$). Let $f, g : I \times E \to E$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in E$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times E$

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_E$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x-y\|, \ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x-y\|.$$

Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators. Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t,u(t)) - \Lambda_2\frac{du}{dt}(t) \in \partial\varphi\left(t, \Lambda_1\frac{du}{dt}(t) + g(t,u(t))\right) \ \textit{a.e. } t \in I \\ u(0) = u_0 \end{cases} \tag{3.7}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I.

Further, there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that

$$\|u(t)\| \le \kappa \quad \textit{for all}\, t \in I \quad \textit{and} \quad \|\dot u(t)\| \le a(t)\|u(t)\| + b(t) \quad \textit{a.e.}\ t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ_1 and Λ_2, on the functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ satisfying (3.7)*, and on the normal integrand φ (via the mappings $\zeta(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$), and where the constant κ depends only on $\|u_0\|$ and Λ_1, Λ_2 and on the same above functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ and the integrand φ (via $\zeta(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$).*

Proof. Using the change of mappings $U(t) = \Lambda_1 u(t)$ and

$$f_1(t,x) = f\left(t, \Lambda_1^{-1}x\right), \quad g_1(t,x) = g\left(t, \Lambda_1^{-1}x\right), \quad \text{and} \quad \Lambda := \Lambda_2\Lambda_1^{-1},$$

we see that the differential inclusion (3.4) can be rewritten as

$$\begin{cases} f_1(t,U(t)) - \Lambda \frac{dU}{dt}(t) \in \partial\varphi\left(t, \frac{dU}{dt}(t) + g_1(t,U(t))\right) \ \text{a.e. } t \in I \\ U(0) = \Lambda_1 u_0. \end{cases} \tag{3.8}$$

It is clear that Λ is still a positive symmetric bijective linear operator from E into E and that the types of conditions satisfied by the mappings f and g are also enjoyed by f_1 and g_1. Then Theorem 3.1 provides a unique absolutely continuous solution for (3.8) with the additional properties therein. From this the unique absolutely continuous solution for the differential inclusion of the corollary is obtained with the additional stated properties. □

Taking $\varphi(t,\cdot)$ in Corollary 3.3 as the indicator function $\delta_{C(t)}(\cdot)$ furnishes the following second corollary related to sweeping processes; see also the previous result of Jourani and Vilches [49]. We must mention that the corollary below does not cover the sweeping process in Adly, Haddad and Thibault [2]

$$\begin{cases} f(t) - \Lambda_1 \frac{du}{dt}(t) - \Lambda_0 u(t) \in N_{C(t)}\left(\frac{du}{dt}(t)\right) \ \text{a.e. } t \in I \\ u(0) = u_0, \end{cases}$$

where $\Lambda_0, \Lambda_1 : E \to E$ are two symmetric (continuous) linear operators which are merely semi-definite positive (non bijective).

Corollary 3.4. *Let $p \in [1, +\infty]$ and let $I := [0,T]$ and $C : I \rightrightarrows E$ be a closed convex valued multimapping for which there is a selection $\zeta \in L^p_E(I)$ of $C(\cdot)$, i.e. $\zeta(t) \in C(t)$ for all $t \in I$. Let $f, g : I \times E \to E$ be two mappings such that $f(\cdot,x)$ and $g(\cdot,x)$ are Lebesgue measurable for all $x \in E$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times E$*

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_E$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x-y\|, \ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x-y\|.$$

Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators.

Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t,u(t)) - \Lambda_2 \frac{du}{dt}(t) \in N_{C(t)}\left(\Lambda_1 \frac{du}{dt}(t) + g(t,u(t))\right) \;\; a.e.\; t \in I \\ u(0) = u_0 \end{cases}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I.

Further, $u(\cdot) \in W^{1,p}_E(I)$ and there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that

$$\|u(t)\| \le \kappa \;\; \text{for all}\; t \in I \quad \text{and} \quad \|\dot{u}(t)\| \le a(t)\|u(t)\| + b(t) \;\; a.e.\; t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ_1 and Λ_2, on the functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ satisfying (3.7)*, and on the moving set $C(\cdot)$ (via $\zeta(\cdot)$), and where the constant κ depends only on $\|u_0\|$ and Λ_1, Λ_2 and on the same above functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ and the moving set $C(\cdot)$ (via $\zeta(\cdot)$).*

Another situation is concerned with the presence of operators with some history dependence. Let $I = [0,T]$, let X and Y be two nonempty closed subsets of two Banach spaces V and W respectively, and let $\mathcal{C}_X(I)$ be the set of all continuous mappings from I into X. A mapping $\mathcal{R} : \mathcal{C}_X(I) \to \mathcal{C}_Y(I)$ is called a *history-dependent operator* (see [72, p. 70-71]) provided there exists a real constant $L_{\mathcal{R}} \ge 0$ such that for all $u, v \in \mathcal{C}_X(I)$ and $t \in [0,T]$

$$\|(\mathcal{R}u)(t) - (\mathcal{R}v)(t)\| \le L_{\mathcal{R}} \int_0^t \|u(s) - v(s)\|\, ds. \tag{3.9}$$

If, instead, there are two real constants $0 \le \ell_{\mathcal{R}} < 1$ and $L_{\mathcal{R}} \ge 0$ such that

$$\|(\mathcal{R}u)(t) - (\mathcal{R}v)(t)\| \le \ell_{\mathcal{R}}\|u(t) - v(t)\| + L_{\mathcal{R}} \int_0^t \|u(s) - v(s)\|\, ds, \tag{3.10}$$

the operator $\mathcal{R}$ is called (see [73, p. 39]) *almost history-dependent.* We will have to deal with a slight extension of (3.9) already implicitly contained in the inequality (2.7) in [39] and in the inequality in [38, Remark p. 710] (with another terminology). We say that $\mathcal{R} : \mathcal{C}_X(I) \to \mathcal{C}_Y(I)$ is a *history-dependent operator with time-dependent rate* if there exists a non-negative function $Q_{\mathcal{R}}(\cdot) \in L^1_{\mathbb{R}_+}(I)$ such that we have for all $u, v \in \mathcal{C}_X(I)$ and all $t \in I$

$$\|(\mathcal{R}u)(t) - (\mathcal{R}v)(t)\| \le \int_0^t Q_{\mathcal{R}}(s)\|u - v\|_{\infty;s}\, ds, \tag{3.11}$$

where by convenience $\|\cdot\|_{\infty;s} := \|\cdot\|_{\infty;[0,s]}$, i.e., $\|\zeta\|_{\infty;s} = \sup_{\tau\in[0,s]} \|\zeta(\tau)\|$ for $\zeta \in \mathcal{C}_X(I)$ (note that the function $s \mapsto \|u - v\|_{\infty;s}$ is easily seen to be measurable and bounded on I). The latter function $\tau \mapsto Q_{\mathcal{R}}(\tau)$ can be seen as a time-dependent rate of the operator $\mathcal{R}$. We also say that $\mathcal{R} : \mathcal{C}_X(I) \to \mathcal{C}_Y(I)$ is an *almost history-dependent operator with time-dependent rates* if there exist two non-negative functions $q_{\mathcal{R}}(\cdot)$, $Q_{\mathcal{R}}(\cdot)$ in $L^1_{\mathbb{R}_+}(I)$ such that for all $u, v \in \mathcal{C}_X(I)$ and all $t \in I$

$$\|(\mathcal{R}u)(t) - (\mathcal{R}v)(t)\| \le q_{\mathcal{R}}(t)\|u - v\|_{\infty;t} + \int_0^t Q_{\mathcal{R}}(s)\|u - v\|_{\infty;s}\, ds. \tag{3.12}$$

The latter inequality is just an adaptation of the inequality (3.10) defining *almost history-dependent operators* where it is required that $\ell_{\mathcal{R}}$ and $L_{\mathcal{R}}$ are non-negative real constants with $\ell_{\mathcal{R}} < 1$. It will be convenient to write below $\mathcal{R}u(t)$ in place of $(\mathcal{R}u)(t)$ when there is no risk of ambiguity.

The theorem related to the presence of operators of some history dependence will utilize the next fixed point lemma. Before stating it, consider with $I := [0, T]$ a non-negative function $g \in L^1_{\mathbb{R}_+}(I)$ and put $g_1 := g$ and $g_{k+1}(s) = g(s) \int_0^s g_k(r)\,dr$.

Put also $G(s) = \int_0^s g(r)\,dr$ and note that G is absolutely continuous on I with $G(0) = 0$ and derivative $G'(s) = g(s)$ for almost every $s \in I$. Then the function $s \mapsto (G(s))^k$ is also absolutely continuous on I with derivative $kg(s)(G(s))^{k-1}$ a.e. $s \in I$. Then, for each $t \in I$ we have

$$\int_0^t g_2(s)\,ds = \int_0^t \left(g(s) \int_0^s g(r)\,dr \right) ds = \int_0^t G'(s)G(s)\,ds = \frac{(G(t))^2}{2} = \frac{(G(t))^2}{2!},$$

$$\int_0^t g_3(s)\,ds = \int_0^t \left(g(s) \int_0^s g_2(r)\,dr \right) ds = \int_0^t G'(s)\frac{(G(s))^2}{2!}ds = \frac{(G(t))^3}{3(2!)} = \frac{(G(t))^3}{3!},$$

and hence by induction $$\int_0^t g_k(s)\,ds = \frac{(G(t))^k}{k!}.$$

We deduce for each $t \in I$ that $\int_0^t g_k(s)\,ds \to 0$ as $k \to \infty$.

Lemma 3.5. *Let $I = [0, T]$ and let X be a nonempty closed subset of a Banach space V. Let $\mathcal{R} : \mathcal{C}_X(I) \to \mathcal{C}_X(I)$ be a history-dependent operator with time-dependent rate. Then $\mathcal{R}$ admits one and only one fixed point $u(\cdot) \in \mathcal{C}_X(I)$.*

Proof. Fix any $u, v \in \mathcal{C}_X(I)$ and let us follow the arguments in [39, p. 318]. Writing $Q(\cdot)$ in place of $Q_{\mathcal{R}}(\cdot)$ in the definition in (3.11), we easily see that

$$\|\mathcal{R}u - \mathcal{R}v\|_{\infty;t} \le \int_0^t Q(s)\|u - v\|_{\infty;s}\,ds,$$

so by induction we obtain with $\mathcal{R}^k := \mathcal{R} \circ \cdots \circ \mathcal{R}$ (k-times)

$$\|\mathcal{R}^k u - \mathcal{R}^k v\|_{\infty;t} \le \int_0^t Q_k(s)\|u - v\|_{\infty;s}\,ds,$$

where $Q_1(s) := Q(s)$ and $Q_k(s) := Q(s) \int_0^s Q_{k-1}(r)\,dr$ for $k \ge 2$. This yields

$$\|\mathcal{R}^k u - \mathcal{R}^k v\|_{\infty;T} \le \left(\int_0^T Q_k(s)\,ds \right) \|u - v\|_{\infty;T},$$

and we know that $\int_0^T Q_k(s)\,ds \to 0$ as $k \to \infty$ according to what precedes the lemma. Therefore, for k large enough $\mathcal{R}^k$ is a strict contraction of $\mathcal{C}_X(I)$ into itself, and hence $\mathcal{R}$ admits one and only one fixed point. $\square$

Theorem 3.6. *Let $I := [0,T]$ and $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there are a mapping $\zeta \in L^1_E(I)$ such that $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ is in $L^1_{\mathbb{R}_+}(I)$ and two non-negative functions $\alpha, \beta : I \to \mathbb{R}_+$ with $\alpha \in L^1_{\mathbb{R}_+}(I)$ and $\sqrt{\beta} \in L^1_{\mathbb{R}_+}(I)$ such that*

$$-\alpha(t)\|x\| - \beta(t) \leq \varphi(t,x) \quad \textit{for all } (t,x) \in I \times E$$

(note that $\sqrt{\beta} \in L^1_{\mathbb{R}_+}(I)$ if $\beta \in L^1_{\mathbb{R}_+}(I)$). Let $g : I \times E \to E$ be a mapping such that $g(\cdot,x)$ is Lebesgue measurable for each $x \in E$ and such that $g(\cdot,0) \in L^1_E(I)$ and there is a non-negative function $\gamma(\cdot) \in L^1_{\mathbb{R}+}$ satisfying

$$\|g(t,x) - g(t,y)\| \leq \gamma(t)\|x-y\| \quad \textit{for all } t \in I \textit{ and } x,y \in E.$$

Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators (hence continuous) and let $\mathcal{R} : \mathcal{C}_E(I) \to \mathcal{C}_E(I)$ be an almost history-dependent operator with time-dependent rates. Then given any $u_0 \in E$ the differential inclusion

$$\begin{cases} -\Lambda_2 \frac{du}{dt}(t) \in \partial\varphi\left(t, \Lambda_1 \frac{du}{dt}(t) + g(t,u(t)) + (\mathcal{R}u)(t)\right) & a.e.\ t \in I \\ u(0) = u_0 \end{cases} \tag{3.13}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I.

Proof. For each $h \in \mathcal{C}_E(I)$ the mapping $t \mapsto g(t,h(t)) + (\mathcal{R}h)(t)$ is in $L^1_E(I)$ (since $\|g(t,h(t))\| \leq \gamma(t)\|h\|_{\infty;T} + \|g(t,0)\|$), hence there exists by Corollary 3.3 a unique absolutely continuous solution u_h to the differential inclusion

$$\begin{cases} -\Lambda_2 \frac{du_h}{dt}(t) \in \partial\varphi\left(t, \Lambda_1 \frac{du_h}{dt}(t) + g(t,h(t)) + (\mathcal{R}h)(t)\right) & \text{a.e. } t \in I \\ u_h(0) = u_0. \end{cases}$$

We follow some ideas that can be found in [38, 39] and [73], and we consider the mapping $\mathcal{S} : \mathcal{C}_E(I) \to \mathcal{C}_E(I)$ defined by

$$\mathcal{S}h = u_h \quad \text{for every } h \in \mathcal{C}_E(I).$$

Let $q(\cdot) := q_{\mathcal{R}}(\cdot)$ and $Q(\cdot) := Q_{\mathcal{R}}(\cdot)$ in $L^1_{\mathbb{R}_+}(I)$ be given by the definition (3.12) of almost history-dependent operator with time-dependent rates. For any h_1, h_2 in $\mathcal{C}_E(I)$ we have by the monotonicity of $\partial\varphi(t,\cdot)$

$$\begin{aligned} &\langle \Lambda_2 (\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)), \Lambda_1 (\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)) \rangle \\ &\leq - \langle \Lambda_2 (\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)), (\mathcal{R}h_1)(s) - (\mathcal{R}h_2)(s) \rangle \\ &\qquad - \langle \Lambda_2 (\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)), g(s,h_1(s)) - g(s,h_2(s)) \rangle, \end{aligned}$$

and hence

$$\begin{aligned} &\langle \Lambda_2 (\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)), \Lambda_1 (\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)) \rangle \\ &\leq \|\Lambda_2\| \, \|\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)\| \left(\int_0^s Q(\tau)\|h_1 - h_2\|_{\infty;\tau} \, d\tau + q(s)\|h_1 - h_2\|_{\infty;s} \right) \\ &\qquad + \gamma(s)\|\Lambda_2\| \, \|\dot{u}_{h_1}(s) - \dot{u}_{h_2}(s)\| \, \|h_1(s) - h_2(s)\|. \end{aligned}$$

Choosing the positive symmetric bijective linear operator $P : E \to E$ as the square root of $\Lambda_1\Lambda_2$, i.e. $\Lambda_1\Lambda_2 = PP$, we note that for every $x \in E$

$$\langle \Lambda_2 x, \Lambda_1 x \rangle = \langle Px, Px \rangle = \|Px\|^2 \geq \frac{1}{\|P^{-1}\|^2}\|x\|^2.$$

It ensues that

$$\frac{1}{\|P^{-1}\|^2}\|\dot u_{h_1}(s)-\dot u_{h_2}(s)\|$$
$$\le \|\Lambda_2\|(\gamma(s)+q(s))\,\|h_1-h_2\|_{\infty;s}+\|\Lambda_2\|\int_0^s Q(\tau)\|h_1-h_2\|_{\infty;\tau}\,d\tau.$$

Writing

$$\int_0^t \Big(\int_0^s Q(\tau)\|h_1-h_2\|_{\infty;\tau}\,d\tau\Big)\,ds=\int_0^t \Big(\int_\tau^t Q(\tau)\|h_1-h_2\|_{\infty;\tau}\,ds\Big)\,d\tau$$
$$\le T\int_0^t Q(\tau)\|h_1-h_2\|_{\infty;\tau}\,d\tau = T\int_0^t Q(s)\|h_1-h_2\|_{\infty;s}\,ds$$

yields the inequality

$$\frac{1}{\|P^{-1}\|^2}\|u_{h_1}(t)-u_{h_2}(t)\|\le \|\Lambda_2\|\int_0^t(\gamma(s)+q(s)+TQ(s))\|h_1-h_2\|_{\infty;s}\,ds.$$

This tells us by (3.11) that $\mathcal{S}$ is a history-dependent operator with time-dependent rate, so by Lemma 3.5 it has a (unique) fixed point, say $h=\mathcal{S}h=u_h$. This means for $u:=u_h$ that $u(0)=u_0$ and

$$-\Lambda_2\dot u(t)\in\partial\varphi\big(t,\Lambda_1\dot u(t)+g(t,u(t))+(\mathcal{R}u)(t)\big)\ \text{a.e. } t\in I.$$

The proof is complete. □

The next corollary regarding corresponding sweeping processes directly follows.

Corollary 3.7. *Let $I:=[0,T]$ and $C:I\rightrightarrows E$ be a closed convex valued multimapping for which there exists a selection $\zeta\in L^1_E(I)$ of $C(\cdot)$, i.e., $\zeta(t)\in C(t)$ for all $t\in I$. Let $g:I\times E\to E$ be a mapping such that $g(\cdot,x)$ is Lebesgue measurable for each $x\in E$ and such that $g(\cdot,0)\in L^1_E(I)$ and there is a non-negative function $\gamma(\cdot)\in L^1_{\mathbb{R}_+}(I)$ satisfying*

$$\|g(t,x)-g(t,y)\|\le\gamma(t)\|x-y\|\quad \textit{for all}\, t\in I \textit{ and}\, x,y\in E.$$

Let $\Lambda_1,\Lambda_2:E\to E$ be two positive symmetric bijective linear operators (hence continuous) and let $\mathcal{R}:\mathcal{C}_E(I)\to\mathcal{C}_E(I)$ be an almost history-dependent operator with time-dependent rates. Then given any $u_0\in E$ the differential inclusion

$$\begin{cases}-\Lambda_2\frac{du}{dt}(t)\in N_{C(t)}\left(t,\Lambda_1\frac{du}{dt}(t)+g(t,u(t))+(\mathcal{R}u)(t)\right) & a.e.\ t\in I\\ u(0)=u_0\end{cases}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I.

Remark 3.8. We emphasize that Corollary 3.7 (under its hypotheses) is different from the results in [4, 61, 62] concerning the inclusion (1.5) in the introduction, according to certain basic assumptions in those papers recalled in the introduction. Neither the corollary covers those results nor the converse. □

In finite dimensional Euclidean spaces the Lispchitz conditions for $f(t,\cdot)$ and $g(t,\cdot)$ in the proof of Theorem 3.1 and in the statement of Corollary 3.4 can be relaxed to the mere continuity to furnish the following result.

Theorem 3.9. *Assume that E is a finite dimensional Euclidean space, $p \in [1, +\infty]$ and let $I := [0, T]$. Let $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there are a mapping $\zeta \in L^p_E(I)$ such that $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ is in $L^p_{\mathbb{R}_+}(I)$ and two non-negative functions $\alpha, \beta : I \to \mathbb{R}$ with $\alpha \in L^p_{\mathbb{R}_+}(I)$ and $\sqrt{\beta} \in L^p_{\mathbb{R}_+}(I)$ such that*

$$-\alpha(t)\|x\| - \beta(t) \leq \varphi(t, x) \quad \textit{for all } (t, x) \in I \times E$$

(note that $\sqrt{\beta} \in L^p_{\mathbb{R}_+}(I)$ if $\beta \in L^p_{\mathbb{R}_+}(I)$). Let $f, g : I \times E \to E$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in E$ and $f(t, \cdot)$ and $g(t, \cdot)$ are continuous for all $t \in I$, and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ such that for all $(t, x) \in I \times E$

$$\|f(t, x)\| \leq \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t, x)\| \leq \alpha_g(t)\|x\| + \beta_g(t).$$

Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators.
Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t, u(t)) - \Lambda_2 \frac{du}{dt}(t) \in \partial\varphi\left(t, \Lambda_1 \frac{du}{dt}(t) + g(t, u(t))\right) & \textit{a.e. } t \in I \\ u(0) = u_0 \end{cases} \tag{3.14}$$

admits at least one absolutely continuous solution $u(\cdot)$ on I.
Further, there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that for any solution $u(\cdot)$ of (3.14) one has

$$\|u(t)\| \leq \kappa \textit{ for all } t \in I \quad \textit{and} \quad \|\dot{u}(t)\| \leq a(t)\|u(t)\| + b(t) \quad \textit{a.e. } t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ, on the functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ satisfying (3.3), and on the normal integrand φ (via the mappings $\zeta(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$), and where the constant κ depends only on $\|u_0\|$ and Λ and on the same above functions α_f, β_f, α_g, β_g and the integrand φ (via the mappings $\zeta(\cdot), \alpha(\cdot), \beta(\cdot)$).

Proof. We will argue with the development of the proof of Theorem 3.1 and with the mapping $F : I \times E \to E$ as defined therein.

First, the part I) in that proof still holds and says that $u(\cdot)$ is a solution of the differential inclusion in the present theorem if and only if it is a solution of the differential equation

$$\begin{cases} \frac{du}{dt} = F(t, u(t)) \text{ a.e. } t \in [0, T] \\ u(0) = u_0. \end{cases} \tag{3.15}$$

The part I) also gives that $F(\cdot, x)$ is Lebesgue-measurable for each $x \in E$, and it is clear that $F(t, \cdot)$ is continuous on E for each $t \in I$. Further, the Part III) of the proof of the same Theorem 3.1 still holds and the functions $\alpha_F(\cdot)$ and $\beta_F(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ therein still satisfy the growth condition

$$\|F(t, x)\| \leq \alpha_F(t)\|x\| + \beta_F(t) \quad \text{for all } t \in I, x \in E.$$

Consequently, the differential equation (3.15) admits at least one absolutely continuous solution. Finally, part (III) of the proof of Theorem 3.1 also shows that any solution of (3.15) enjoys all the properties in the statement of the present theorem. □

Corollary 3.10. *Assume that E is a finite dimensional Euclidean space, $p\in[1,+\infty]$ and let $I := [0,T]$ and $C : I \rightrightarrows E$ be a closed convex convex valued multimapping for which there is a selection $\zeta \in L^p_E(I)$ of $C(\cdot)$, i.e. $\zeta(t) \in C(t)$ for all $t \in I$. Let $f, g : I \times E \to E$ be two mappings such that $f(\cdot,x)$ and $g(\cdot,x)$ are Lebesgue measurable for all $x \in E$ and $f(t,\cdot)$ and $g(t,\cdot)$ are continuous for all $t \in I$, and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ such that for all $(t,x) \in I \times E$*

$$\|f(t,x)\| \leq \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \leq \alpha_g(t)\|x\| + \beta_g(t)$$

and for each $t \in I$ the mappings $f(t,\cdot)$ and $g(t,\cdot)$ are continuous. Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators.

Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t,u(t)) - \Lambda_2 \frac{du}{dt}(t) \in N_{C(t)}\left(\Lambda_1 \frac{du}{dt}(t) + g(t,u(t))\right) & a.e.\ t \in I \\ u(0) = u_0 \end{cases} \tag{3.16}$$

admits at least one absolutely continuous solution $u(\cdot)$ on I.

Further, there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that for any solution $u(\cdot)$ of (3.16) one has

$$\|u(t)\| \leq \kappa \textit{ for all } t \in I \quad \textit{and} \quad \|\dot{u}(t)\| \leq a(t)\|u(t)\| + b(t) \quad a.e.\ t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ_1 and Λ_2, on the functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ satisfying (3.7), and on the moving set $C(\cdot)$ (via $\zeta(\cdot)$), and where the constant κ depends only on $\|u_0\|$ and Λ_1, Λ_2 and on the same above functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ and the moving set $C(\cdot)$ (via $\zeta(\cdot)$).

3.2. Evolution inclusions with time-dependent maximal monotone operators evaluated mainly at velocity arguments

Let $A : E \rightrightarrows E$ be a multivalued operator. The domain, the range and the graph of A are the following sets

$$D(A) = \{x \in E : \ Ax \neq \emptyset\},$$
$$R(A) = \{y \in E : \ \exists\, x \in D(A),\ y \in Ax\} = \bigcup_{x\in D(A)} Ax,$$
$$\mathrm{gph}\,(A) = \{(x,y) \in E \times E : \ x \in D(A),\ y \in Ax\}.$$

It can also be convenient to denote a multivalued operator A as $A : D(A) \subset E \rightrightarrows E$. One says that $A : D(A) \subset E \rightrightarrows E$ is monotone, if $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$ whenever $(x_i, y_i) \in \mathrm{gph}\,(A)$, $i = 1,2$. Given for each $t \in I := [0,T]$ a monotone multivalued operator $A_t : D(A_t) \subset E \rightrightarrows E$ it is remarkable that for any $u_0 \in E$ and any mapping $h : I \to E$, the monotonicity of A_t (as is easily seen) entails that the evolution inclusion

$$\begin{cases} -\frac{du}{dt}(t) \in A_t u(t) + h(t) & \text{a.e. } t \in I \\ u(0) = u_0 \end{cases}$$

admits **at most** one absolutely continuous solution, otherwise stated, if this evolution inclusion has an absolutely continuous solution, this solution is necessarily unique.

A monotone multivalued operator $A : E \rightrightarrows E$ is *maximal monotone*, if its graph is not strictly contained in the graph of any other monotone multivalued operator. In the sequel, when there is no ambiguity, we will just say monotone operator instead of monotone multivalued operator. Given a fixed real $\lambda > 0$, a monotone operator $A : D(A) \subset E \rightrightarrows E$ is known to be maximal monotone if and only if the basic surjectivity equality (see, e.g., [15])

$$R(I_E + \lambda A) = E \tag{3.17}$$

holds, where I_E stands for the identity mapping of E as said in the previous subsection.

Let A be a maximal monotone operator on E. Then, for every $x \in D(A)$, Ax is nonempty, closed and convex. So, the projection A^0x of the origin into Ax, exists, it is the element of minimal norm of Ax and it is unique. For any real $\lambda > 0$, one defines the *resolvent* and the *Yosida approximation* of A respectively by

$$J_\lambda^A = (I + \lambda A)^{-1} \quad \text{and} \quad A_\lambda = \frac{1}{\lambda}\left(I - J_\lambda^A\right).$$

These operators are both single valued and defined on the whole space E, and one has

$$J_\lambda^A(x) \in D(A) \quad \text{and} \quad A_\lambda(x) \in A\left(J_\lambda^A(x)\right), \text{ for every } x \in E,$$
$$\|A_\lambda(x)\| \le \|A^0x\| \quad \text{for every } x \in D(A).$$

Given two maximal monotone operators $A : D(A) \subset E \rightrightarrows E$ and $B : D(B) \subset E \rightrightarrows E$, we denote by $\operatorname{dis}(A, B)$ the *Vladimorov pseudo-distance* between A and B introduced in [82] and defined by

$$\operatorname{dis}(A, B) = \sup\left\{\frac{\langle y - y', x' - x\rangle}{1 + \|y\| + \|y'\|} : (x, y) \in \operatorname{gph}(A),\ (x', y') \in \operatorname{gph}(B)\right\}. \tag{3.18}$$

Clearly, $\operatorname{dis}(A, B) \in [0, +\infty]$, $\operatorname{dis}(A, B) = \operatorname{dis}(B, A)$ and $\operatorname{dis}(A, B) = 0$ if and only if $A = B$.

We summarize below some useful other features. Lemma 3.11 can be found, e.g., in [15, Theorem 2.2 and Proposition 2.7].

Lemma 3.11. *Let A be a maximal monotone operator of E. The following hold.*

(a) *The closure $\overline{D(A)}$ of the domain of A is convex.*

(b) *If $x \in \overline{D(A)}$ and $y \in E$ are such that $\langle A^0z - y, z - x\rangle \ge 0$ for all $z \in D(A)$, then $x \in D(A)$ and $y \in Ax$.*

The result in Lemma 3.12 is shown in [51, Lemma 3]. It states a property related to the convergence with respect to the pseudo-distance $\operatorname{dis}(\cdot, \cdot)$.

Lemma 3.12. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of E such that one has $\operatorname{dis}(A_n, A) \to 0$. Assume also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in E$. Then $x \in D(A)$ and $y \in Ax$.*

The property (b) in Lemma 3.13 is classical (see, e.g., [15, Proposition 2.2]), and the property (a) is taken from [51, Lemma 4].

Lemma 3.13. *Let A, B be maximal monotone operators of E. Then the following holds.*

(a) *For $\lambda > 0$ and $x \in D(A)$*

$$\|x - J_\lambda^B(x)\| \leq \lambda \|A^0 x\| + \mathrm{dis}\,(A, B) + \sqrt{\lambda\big(1 + \|A^0 x\|\big)\mathrm{dis}\,(A, B)}.$$

(b) *For $\lambda > 0$ and $x, x' \in E$ we have* $\|J_\lambda^A(x) - J_\lambda^A(x')\| \leq \|x - x'\|$.

The following other property of convergence with respect to the pseudo-distance $\mathrm{dis}\,(\cdot,\cdot)$ is also taken from [51, Lemma 5].

Lemma 3.14. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of E such that one has $\mathrm{dis}\,(A_n, A) \to 0$ and such that for some real constant $c > 0$ one has $\|A_n^0 x\| \leq c(1 + \|x\|)$ for all $n \in \mathbb{N}$ and $x \in D(A_n)$. Then for every $z \in D(A)$ there exists a sequence $(z_n)_n$ such that*

$$z_n \in D(A_n), \quad z_n \to z \ \text{ and } \ A_n^0 z_n \to A^0 z. \tag{3.19}$$

We recall now a useful existence result in [5] for differential inclusions of the type

$$-\dot{u}(t) \in A_t u(t) + h(t)$$

with time-dependent maximal monotone operators A_t evaluated at the state $u(t)$ instead of the velocity $\dot{u}(t)$. This existence result will be utilized in Subsection 4.1.

Theorem 3.15. *Assume that for every $t \in I = [0, T]$, $A_t : D(A_t) \subset E \rightrightarrows E$ is a maximal monotone operator satisfying:*

($\mathcal{A}_1$) *there exists a function $r \in W^{1,2}(I)$ which is non-negative on $[0, T[$ and non-decreasing with $r(T) < \infty$ and $r(0) = 0$ such that*

$$\mathrm{dis}\,(A_t, A_s) \leq |r(t) - r(s)|, \quad \text{for all}\, t, s \in I;$$

($\mathcal{A}_2$) *there exists a real constant $c \geq 0$ such that*

$$\|A_t^0 x\| \leq c(1 + \|x\|), \quad \text{for all}\, t \in I,\ x \in D(A_t).$$

Let $h : I \to E$ be a mapping in $L_E^2(I)$. Then, for each $u_0 \in D(A_0)$, the problem

$$\begin{cases} -\dot{u}(t) \in A_t u(t) + h(t) \ a.e.\ t \in I \\ u(0) = u_0 \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$. Further, this solution $u(\cdot) \in W_E^{1,2}(I)$ with $u(t) \in D(A_t)$ for every $t \in I$ and it satisfies

$$\|\dot{u}(t)\| \leq K\big(1 + \dot{r}(t)\big) + (1 + K)\|h(t)\| \ \ a.e.\ t \in I,$$

for some non-negative real constant K (independent of t) depending on $\|u_0\|, c, I, r(\cdot)$.

Following partly the arguments by Jourani and Vilches [49] as in Theorem 3.1 above, we also have the existence result in Theorem 3.16 below.

Before stating the theorem, let us notice the following. Let $\Gamma : E \rightrightarrows E$ be a multimapping and $L : E \to E$ be a continuous linear operator. Define $\Gamma_L : E \rightrightarrows E$ by $\Gamma_L(x) = L^*(\Gamma(Lx))$ for all $x \in E$, where L^* is the adjoint continuous linear operator of L. If Γ is monotone, it is easily seen that Γ_L is also monotone. Indeed, for any $y_i \in \Gamma_L x_i$, with $i = 1, 2$, there is $z_i \in \Gamma(Lx_i)$ with $y_i = L^* z_i$, so

$$\langle y_1 - y_2, x_1 - x_2 \rangle = \langle L^*(z_1 - z_2), x_1 - x_2 \rangle = \langle z_1 - z_2, Lx_1 - Lx_2 \rangle \geq 0,$$

implying that Γ_L is monotone. It is remarkable (see, e.g. [9, Theorem 24.5]) that

$$\big(L \text{ surjective and } \Gamma \text{ maximal monotone}\big) \Rightarrow \big(\Gamma_L \text{ maximal monotone}\big). \tag{3.20}$$

Theorem 3.16. *Let $p \in [1, +\infty]$ and let $I := [0, T]$ and $f, g : I \times E \to E$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in E$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t, x) \in I \times E$*

$$\|f(t, x)\| \leq \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t, x)\| \leq \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_E$

$$\|f(t, x) - f(t, y)\| \leq \gamma_{f,r}(t)\|x - y\|, \ \|g(t, x) - g(t, y)\| \leq \gamma_{g,r}(t)\|x - y\|.$$

Let $\Lambda : E \to E$ be a positive symmetric bijective linear operator. For each $t \in I$ let $A_t : D(A_t) \subset E \rightrightarrows E$ be a maximal monotone operator such that the multimapping $t \mapsto \operatorname{gph} A_t$ from I into $E \times E$ is Lebesgue measurable and there are $\zeta(\cdot) \in L^p_E(I)$ and $\xi(\cdot) \in L^p_E(I)$ with $\xi(t) \in A_t(\zeta(t))$ for all $t \in I$. Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t, u(t)) - \Lambda \frac{du}{dt}(t) \in A_t\left(\frac{du}{dt}(t) + g(t, u(t))\right) \ \ a.e.\ t \in I \\ u(0) = u_0 \end{cases} \tag{3.21}$$

admits one and only one absolutely continuous solution $u(\cdot)$ on I.

Further, there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that

$$\|u(t)\| \leq \kappa \quad \textit{for all } t \in I \quad \textit{and} \quad \|\dot{u}(t)\| \leq a(t)\|u(t)\| + b(t) \ a.e.\ t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ, on the functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ satisfying (3.3), *and on the maximal monotone operators $(A_t)_{t \in I}$ (via the mappings $\zeta(\cdot)$, $\xi(\cdot)$), and where the constant κ depends only on $\|u_0\|$ and Λ and on the same above functions α_f, β_f, α_g, β_g and the maximal monotone operators $(A_t)_{t \in I}$ (via $\zeta(\cdot)$, $\xi(\cdot)$).*

Proof. (I) As already seen (cf. (3.2)) there exists a (unique) positive symmetric bijective continuous linear operator $\Lambda_0 : E \to E$ such that $\Lambda = \Lambda_0 \Lambda_0$. For each $t \in I$ consider the multivalued operator $B_t : E \rightrightarrows E$ defined by

$$B_t x := \Lambda_0^{-1} A_t(\Lambda_0^{-1} x) \quad \text{for all } x \in E,$$

and note that B_t is maximal monotone according to (3.20).

Note also that the latter equality is equivalent to

$$\Lambda_0^{-1}A_t(x) = B_t(\Lambda_0 x) \text{ for all } x \in E,$$

so denoting $\Phi_t := (I_E + B_t)^{-1}$ (the λ-resolvent of B_t with $\lambda = 1$) it follows that for any $y \in E$

$$y - \Lambda_0\Lambda_0 x \in A_t(x) \Leftrightarrow {\Lambda_0}^{-1}y - \Lambda_0 x \in B_t(\Lambda_0 x) \Leftrightarrow \Lambda_0 x = \Phi_t(\Lambda_0^{-1}y).$$

Writing the inclusion in (3.21) in the form

$$f(t,u(t)) + \Lambda_0\Lambda_0 g(t,u(t)) - \Lambda_0\Lambda_0\big(\dot u(t) + g(t,u(t))\big) \in A_t\big(\dot u(t) + g(t,u(t))\big),$$

we see that it is equivalent to

$$\begin{aligned}\Lambda_0\dot u(t) + \Lambda_0 g(t,u(t)) &= (I_E + B_t)^{-1}\big(\Lambda_0^{-1}f(t,u(t)) + \Lambda_0 g(t,u(t))\big)\\ &= \Phi_t\big(\Lambda_0^{-1}f(t,u(t)) + \Lambda_0 g(t,u(t))\big).\end{aligned}$$

Consequently, defining $F : I \times E \to E$ by

$$F(t,x) := -g(t,x) + \Lambda_0^{-1}\Phi_t\big(\Lambda_0^{-1}f(t,x) + \Lambda_0 g(t,x)\big) \text{ for all } (t,x) \in I \times E,$$

we see that the differential inclusion (3.21) is equivalent to the differential equation on I

$$\begin{cases} \dot u(t) = F(t,u(t)) & \text{a.e. } t \in I,\\ u(0) = u_0. \end{cases} \tag{3.22}$$

Clearly, for $\qquad G = \{(t,x,y) \in I \times E \times E : (x,y) \in \operatorname{gph} A_t\}$

we have that $G \in \mathcal{L}(I) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$ by the assumption of Lebesgue measurability of the multimapping $t \mapsto \operatorname{gph} A_t$. We notice that

$$y = \Phi_t(x) \Leftrightarrow x \in y + \Lambda_0^{-1}A_t(\Lambda_0^{-1}y),$$

hence putting $p(t,x,y) := (t, \Lambda_0^{-1}y, \Lambda_0(x-y))$ yields

$$\{(t,x,y) \in I \times E \times E : y = \Phi_t(x)\} = p^{-1}(G),$$

with $p^{-1}(G) \in \mathcal{L}(I) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$. According to this and to the completeness of the σ-field $\mathcal{L}(I)$ we easily see by (2.1) that for each $x \in E$ the mappings $t \mapsto \Phi_t(x)$ and $F(\cdot,x)$ are Lebesgue measurable.

(II) Using the 1-Lipschitz property of Φ_t (see (b) in Lemma 3.13), we also observe that for each real $r > 0$ we can write for each $t \in I$ and all $x, y \in r\mathbb{B}$

$$\begin{aligned}&\|F(t,x) - F(t,y)\|\\ &\le \|-g(t,x) + g(t,y)\| + \|\Lambda_0^{-1}\|\,\|\Lambda_0^{-1}(f(t,x) - f(t,y)) + \Lambda_0(g(t,x) - g(t,y))\|\\ &\le \gamma_{g,r}(t)\|x-y\| + \|\Lambda_0^{-1}\|^2\gamma_{f,r}(t)\|x-y\| + \|\Lambda_0^{-1}\|\,\|\Lambda_0\|\gamma_{g,r}(t)\|x-y\|,\end{aligned}$$

so for $\gamma_{F,r}(\tau) := \|\Lambda_0^{-1}\|^2\gamma_{f,r}(\tau) + (1 + \|\Lambda_0\|\,\|\Lambda_0^{-1}\|)\gamma_{g,r}(\tau)$ with $\gamma_{F,r}(\cdot) \in L_{\mathbb{R}_+}(I)$ we have

$$\|F(t,x) - F(t,y)\| \le \gamma_{F,r}(t)\|x-y\|.$$

(III) On the other hand, putting $y_t := \Phi_t(0)$ we have $-y_t \in B_t y_t$ or equivalently $-\Lambda_0 y_t \in A_t(\Lambda_0^{-1} y_t)$, which by the monotonicity of A_t gives with the assumption on $\zeta(\cdot)$ and $\xi(\cdot)$

$$\langle \Lambda_0 y_t + \xi(t), \Lambda_0^{-1} y_t - \zeta(t) \rangle \le 0,$$

or equivalently

$$\langle \Lambda_0 y_t, \Lambda_0^{-1} y_t \rangle - \langle y_t, \Lambda_0 \zeta(t) - \Lambda_0^{-1} \xi(t) \rangle - \langle \zeta(t), \xi(t) \rangle \le 0,$$

hence
$$\|y_t\|^2 - \|\Lambda_0 \zeta(t) - \Lambda_0^{-1} \xi(t)\| \, \|y_t\| - \langle \zeta(t), \xi(t) \rangle \le 0.$$

Writing

$$\begin{aligned}
\Delta(t) : &= \|\Lambda_0 \zeta(t) - \Lambda_0^{-1} \xi(t)\|^2 + 4\langle \zeta(t), \xi(t) \rangle \\
&= \|\Lambda_0 \zeta(t)\|^2 + \|\Lambda_0^{-1} \xi(t)\|^2 - 2\langle \Lambda_0 \zeta(t), \Lambda_0^{-1} \xi(t) \rangle + 4\langle \Lambda_0 \zeta(t), \Lambda_0^{-1} \xi(t) \rangle \\
&= \|\Lambda_0 \zeta(t) + \Lambda_0^{-1} \xi(t)\|^2
\end{aligned}$$

we see that

$$\|y_t\| \le \frac{1}{2}\big(\|\Lambda_0 \zeta(t) - \Lambda_0^{-1} \xi(t)\| + \|\Lambda_0 \zeta(t) + \Lambda_0^{-1} \xi(t)\|\big) =: \sigma(t),$$

$\sigma(\cdot) \in L^p_{\mathbb{R}_+}(I)$. Put $q(t,x) := \Lambda_0^{-1} f(t,x) + \Lambda_0 g(t,x)$. Then by the 1-Lipschitz property of Φ_t again

$$\begin{aligned}
\|F(t,x)\| &\le \|g(t,x)\| + \|\Lambda_0^{-1}\| \, \|\Phi_t(q(t,x))\| \\
&\le \|g(t,x)\| + \|\Lambda_0^{-1}\| \, \|q(t,x)\| + \|\Lambda_0^{-1}\| \, \|\Phi_t(0)\| \\
&\le \|g(t,x)\| + \|\Lambda_0^{-1}\| \, \|q(t,x)\| + \sigma(t)\|\Lambda_0^{-1}\|,
\end{aligned}$$

and this clearly furnishes α_F, β_F in $L^p_{\mathbb{R}_+}(I)$ such that

$$\|F(t,x)\| \le \alpha_F(t)\|x\| + \beta_F(t) \quad \text{for all } (t,x) \in I \times E.$$

It ensues that the differential equation (3.22) has one and only one absolutely continuous solution, which in turn assures us that the differential inclusion (3.21) admits one and only one absolutely continuous solution $u(\cdot)$ on I.

Finally, regarding the real constant $\kappa > 0$ and the integrable functions $a(\cdot)$ and $b(\cdot)$ as in the statement, they are obtained with arguments similar to those in Theorem 3.1. □

Proceeding as for Corollary 3.3 furnishes from Theorem 3.16 the following.

Corollary 3.17. *Let $p \in [1, +\infty]$ and let $I := [0,T]$ and $f, g : I \times E \to E$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in E$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times E$*

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \text{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_E$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x - y\|, \ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x - y\|.$$

Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators. For each $t \in I$ let $A_t : D(A_t) \subset E \rightrightarrows E$ be a maximal monotone operator such that the multimapping $t \mapsto \operatorname{gph} A_t$ from I into $E \times E$ is Lebesgue measurable and there are $\zeta(\cdot) \in L^p_E(I)$ and $\xi(\cdot) \in L^p_E(I)$ with $\xi(t) \in A_t(\zeta(t))$ for all $t \in I$. Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t, u(t)) - \Lambda_2 \frac{du}{dt}(t) \in A_t\left(\Lambda_1 \frac{du}{dt}(t) + g(t, u(t))\right) & a.e.\ t \in I \\ u(0) = u_0 \end{cases}$$

admits one and only one absolutely continuous solution $u(\cdot)$ on I.

Further, there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that

$$\|u(t)\| \le \kappa \quad \text{for all } t \in I \quad \text{and} \quad \|\dot u(t)\| \le a(t)\|u(t)\| + b(t) \quad a.e.\ t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ_1 and Λ_2, on the functions α_f, β_f, α_g, β_g satisfying (3.3)*, and on the maximal monotone operators $(A_t)_{t\in I}$ (via $\zeta(\cdot)$, $\xi(\cdot)$), and where the constant κ depends only on $\|u_0\|$, Λ_1, Λ_2 and on the same above functions $\alpha_f, \beta_f, \alpha_g, \beta_g$ and the maximal monotone operators $(A_t)_{t\in I}$ (via $\zeta(\cdot)$, $\xi(\cdot)$).*

As Theorem 3.6 the following result with history-dependent operators also holds true.

Theorem 3.18. *Let $I := [0, T]$ and let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators (hence continuous). For each $t \in I$ let $A_t : D(A_t) \subset E \rightrightarrows E$ be a maximal monotone operator such that the multimapping $t \mapsto \operatorname{gph} A_t$ from I into $E \times E$ is Lebesgue measurable and there are $\zeta(\cdot) \in L^1_E(I)$ and $\xi(\cdot) \in L^1_E(I)$ with $\xi(t) \in A_t(\zeta(t))$ for all $t \in I$. Let $g : I \times E \to E$ be a mapping such that $g(\cdot, x)$ is Lebesgue measurable for all $x \in E$ and such that $g(\cdot, 0) \in L^1_E(I)$ and there exist a non-negative function $\gamma(\cdot) \in L^1_{\mathbb{R}_+}(i)$ satisfying for all $t \in I$ and $x, y \in E$*

$$\|g(t, x) - g(t, y)\| \le \gamma(t)\|x - y\|.$$

Let $\Lambda_1, \Lambda_2 : E \to E$ be two positive symmetric bijective linear operators (hence continuous) and let $\mathcal{R} : \mathcal{C}_E(I) \to \mathcal{C}_E(I)$ be an almost history-dependent operator with time-dependent rates. Then given any $u_0 \in E$ the differential inclusion

$$\begin{cases} -\Lambda_2 \frac{du}{dt}(t) \in A_t\left(\Lambda_1 \frac{du}{dt}(t) + g(t, u(t)) + (\mathcal{R}u)(t)\right) & a.e.\ t \in I \\ u(0) = u_0 \end{cases}$$

admits a unique absolutely continuous solution $u(\cdot)$ on I.

The proof of Theorem 3.16 also guarantees the following statement in finite dimensions. Indeed, under the assumptions below of Theorem 3.19 only part (II) in the proof of Theorem 3.16 requires to be modified in the obvious continuity on E of the mapping $F(t, \cdot)$ for any $t \in [0, T]$, so according to the finite dimensional property of E the differential equation

$$\begin{cases} \frac{du}{dt} = F(t, u(t)) \text{ a.e. } t \in [0, T] \\ u(0) = u_0 \end{cases}$$

admits at least one solution, and this solution enjoys all the properties in Theorem 3.16.

Theorem 3.19. *Assume that E is a finite dimensional Euclidean space, $p\in[1,+\infty]$ and let $I := [0,T]$. Let $f, g : I \times E \to E$ be two mappings such that $f(\cdot,x)$ and $g(\cdot,x)$ are Lebesgue measurable for all $x \in E$ and $f(t,\cdot)$ and $g(t,\cdot)$ are continuous for all $t \in I$, and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^p_{\mathbb{R}_+}(I)$ such that for all $(t,x) \in I \times E$*

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t).$$

Let $\Lambda : E \to E$ be a positive symmetric bijective linear operator. For each $t \in I$ let $A_t : D(A_t) \subset E \rightrightarrows E$ be a maximal monotone operator such that the multimapping $t \mapsto \operatorname{gph} A_t$ from I into $E \times E$ is Lebesgue measurable and there are $\zeta(\cdot) \in L^p_E(I)$ and $\xi(\cdot) \in L^p_E(I)$ with $\xi(t) \in A_t(\zeta(t))$ for all $t \in I$.
Then given $u_0 \in E$ the differential inclusion

$$\begin{cases} f(t,u(t)) - \Lambda \frac{du}{dt}(t) \in A_t\left(\frac{du}{dt}(t) + g(t,u(t))\right) \ \ a.e.\ t \in I \\ u(0) = u_0 \end{cases} \tag{3.23}$$

admits at least one absolutely continuous solution $u(\cdot)$ on I.
Further, there are a real constant $\kappa > 0$ and two non-negative functions $a(\cdot)$ and $b(\cdot)$ in $L^p_{\mathbb{R}_+}(I)$ such that for any solution $u(\cdot)$ of (3.23) one has

$$\|u(t)\| \le \kappa \text{ for all } t \in I \quad \text{and} \quad \|\dot u(t)\| \le a(t)\|u(t)\| + b(t) \quad a.e.\ t \in I,$$

where $a(\cdot)$ and $b(\cdot)$ depend only on Λ, on the functions α_f, β_f, α_g, β_g satisfying (3.3), and on the maximal monotone operators $(A_t)_{t\in I}$ (via the mappings $\zeta(\cdot)$, $\xi(\cdot)$), and where the constant κ depends only on $\|u_0\|$ and Λ and on the same above functions α_f, β_f, α_g, β_g and the maximal monotone operators $(A_t)_{t\in I}$ (via $\zeta(\cdot)$, $\xi(\cdot)$).

4. Evolution inclusions and variational inequalities with various perturbations

We pass in this section to the study of diverse evolution inclusions and variational inequalities involving perturbations of various types. We begin with inclusions with operators evaluated only at the state.

4.1. Evolution inclusions under Carathéodory perturbations and with operators evaluated at the state

Two lemmas will be useful. The first is a lemma of compactness.

Lemma 4.1. *Assume that for any $t \in I = [0,T]$, $A_t : D(A_t) \subset E \rightrightarrows E$ is a maximal monotone operator satisfying $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ and such that $\overline{D(A_t)}$ is ball-compact. Let $X : [0,T] \rightrightarrows E$ be a weakly compact convex valued measurable multimapping such that $X(t) \subset m(t)\,\mathbb{B}_E$ for all $t \in I$ with $m(\cdot) \in L^2_{\mathbb{R}_+}(I)$. Let $u_0 \in D(A_0)$ and let S^2_X denote the set of all $L^2_E(I)$-selections of X. For each $h \in S^2_X$ denote u_h the unique absolutely continuous solution of the evolution inclusion*

$$\begin{cases} -\dot u_h(t) \in A_t u_h(t) + h(t) \text{ a.e. } t \in I, \\ u_h(0) = u_0. \end{cases}$$

Then the set $\mathcal{X} := \{u_h : h \in S^2_X\}$ *is compact in* $\mathcal{C}_E(I)$ *for the topology of uniform convergence on* I*, and for each* $h \in S^2_X$

$$\|\dot{u}_h(t)\| \leq K\big(1 + \dot{r}(t) + m(t)\big) + m(t) \quad \text{a.e. } t \in I,$$

where K *is the real constant of Theorem 3.15 (independent of* t*) depending on* $\|u_0\|, c, I, r(\cdot)$.

Proof. Let K be the constant furnished by Theorem 3.15. For each $h \in S^2_X$, one has $\|h(t)\| \leq m(t)$ for a.e. $t \in I$. Then by virtue of Theorem 3.15 the solution set $\mathcal{X} := \{u_h : h \in S^2_X\}$ is bounded and equicontinuous in $C_E(I)$, namely

$$\|\dot{u}_h(t)\| \leq \gamma(t) := K\big(1 + \dot{r}(t) + m(t)\big) + m(t),$$

with $\gamma(\cdot) \in L^2_{\mathbb{R}_+}(I)$, so the set $\{u_h(t) : h \in S^2_X\}$ is bounded in E (uniformly with respect to $t \in I$). Since $u_h(t) \in D(A_t)$ for every $t \in I$ by Theorem 3.15 and since $\overline{D(A_t)}$ is ball-compact for every $t \in I$, we deduce by the Arzela-Ascoli theorem that $\mathcal{X}$ is relatively compact in $\mathcal{C}_E(I)$ for the topology of uniform convergence on I. It remains to see that $\mathcal{X}$ is closed in $\mathcal{C}_E(I)$. Let $(u_{h_n})_n$ be a sequence in $\mathcal{X}$ converging uniformly on I to some u in $\mathcal{C}_E(I)$. We may suppose that $(h_n)_n$ weakly converges in $L^2_E(I)$ to some $h \in S^2_X$. By Lemma 4.2 below we obtain that $u \in W^{1,2}_E(I)$ and that

$$\begin{cases} -\dot{u}(t) \in A_t u(t) + h(t) \ \text{ a.e } t \in I, \\ u(0) = u_0. \end{cases}$$

By uniqueness we have that $u = u_h$, so the set $\mathcal{X}$ is compact in $\mathcal{C}_E(I)$ for the topology of uniform convergence on I. □

The second lemma is concerned with a fundamental convergence property.

Lemma 4.2. *Let* $I = [0, T]$ *and* $A_t : D(A_t) \subset E \rightrightarrows E$ *be a maximal monotone operator for each* $t \in I$*, and let also* $u_0 \in D(A_0)$*. Let* $\Lambda : E \to E$ *be a continuous linear operator. Let* $p \in [1, +\infty[$ *and let* $(h_n)_n$ *be a sequence in* $L^p_E(I)$ *converging weakly in* $L^p_E(I)$ *to a mapping* $h \in L^p_E(I)$*. Assume that for each* $n \in \mathbb{N}$ *there exists an absolutely continuous solution mapping* $u_n : I \to E$ *to the evolution inclusion*

$$\begin{cases} -\Lambda\dot{u}_n(t) \in A_t u_n(t) + h_n(t) \text{ a.e. } t \in I, \\ u_n(0) = u_0. \end{cases}$$

Assume that the sequence $(u_n)_n$ *of these mappings pointwise converges on* I *to a mapping* $u : I \to E$ *and that* $(\dot{u}_n)_n$ *weakly converges in* $L^p_E(I)$*. Assume also that for almost every* $t \in I$ *the sequences* $(\dot{u}_n(t))_n$ *and* $(h_n(t))_n$ *are bounded in* E*. Then* u *is an absolutely continuous solution of the evolution inclusion*

$$\begin{cases} -\Lambda\dot{u}(t) \in A_t u(t) + h(t) \text{ a.e. } t \in I, \\ u(0) = u_0, \end{cases}$$

and one has $u \in W^{1,p}_E(I)$.

Proof. Denote by $v(\cdot)$ the mapping to which $(\dot{u}_n)_n$ weakly converges in $L^p_E(I)$, and notice for $1/p + 1/p' = 1$ that the equality $\langle \Lambda \dot{u}_n(t), y(t)\rangle = \langle \dot{u}_n(t), \Lambda^* y(t)\rangle$ for every $y(\cdot) \in L^{p'}_E(I)$ ensures that $(\Lambda \dot{u}_n(\cdot))_n$ weakly converges in $L^p_E(I)$ to $\Lambda \dot{u}(\cdot)$. For each $t \in I$ writing for every $e \in E$

$$\langle e, u_n(t)\rangle = \langle e, u_0\rangle + \int_0^t \langle e, \dot{u}_n(s)\rangle\, ds = \langle e, u_0\rangle + \int_I \langle \mathbf{1}_{[0,t]}(s)e, \dot{u}_n(s)\rangle\, ds,$$

we see as $n \to \infty$ that

$$\langle e, u(t)\rangle = \langle e, u_0\rangle + \int_I \langle \mathbf{1}_{[0,t]}(s)e, v(s)\rangle\, ds = \langle e, u_0 + \int_I \mathbf{1}_{[0,t]}(s)v(s)\, ds\rangle,$$

and this being true for every $e \in E$ we obtain

$$u(t) = u_0 + \int_I \mathbf{1}_{[0,t]}(s)v(s)\, ds = u_0 + \int_0^t v(s)\, ds.$$

It follows that $u(\cdot)$ is absolutely continuous with $\dot{u} = v$ a.e. and $\dot{u} \in L^p_E(I)$.

Since the sequence $(\Lambda \dot{u}_n + h_n)_n$ weakly converges in $L^p_E(I)$ to $\Lambda \dot{u} + h$, we may suppose that it Komlos converges to $\Lambda \dot{u} + h$. Choose a negligible set N in I such that for each $n \in \mathbb{N}$

$$-\Lambda \dot{u}_n(t) \in A_t u_n(t) + h_n(t), \text{ for every } t \in I \setminus N,$$

$$\lim_n \frac{1}{n}\sum_{j=1}^n [\Lambda \dot{u}_j(t) + h_j(t)] = \Lambda \dot{u}(t) + h(t), \text{ for every } t \in I \setminus N.$$

This gives in particular for each $t \in I \setminus N$ that $u_n(t) \in D(A_t)$ for every $n \in \mathbb{N}$, hence $u(t) \in \overline{D(A_t)}$. According to the assumptions we may also suppose that for each $t \in I \setminus N$ both sequences $(\Lambda \dot{u}_n(t))_n$ and $(h_n(t))_n$ are bounded in E. Fix any $t \in I \setminus N$ and any $y \in D(A_t)$. For any $j \in \mathbb{N}$ the inclusion

$$-\Lambda \dot{u}_j(t) \in A_t u_j(t) + h_j(t)$$

ensures by monotonicity of A_t

$$\langle \Lambda \dot{u}_j(t) + h_j(t), u_j(t) - y\rangle \le \langle A^0_t y, y - u_j(t)\rangle. \tag{4.1}$$

The equality

$$\langle \Lambda \dot{u}_j(t) + h_j(t), u(t) - y\rangle = \langle \Lambda \dot{u}_j(t) + h_j(t), u_j(t) - y\rangle + \langle \Lambda \dot{u}_j(t) + h_j(t), u(t) - u_j(t)\rangle$$

valid for every $j \in \mathbb{N}$, allows us to write for each $n \in \mathbb{N}$

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^n \langle \Lambda \dot{u}_j(t) + h_j(t), u(t) - y\rangle \\ &= \frac{1}{n}\sum_{j=1}^n \langle \Lambda \dot{u}_j(t) + h_j(t), u_j(t) - y\rangle + \frac{1}{n}\sum_{j=1}^n \langle \Lambda \dot{u}_j(t) + h_j(t), u(t) - u_j(t)\rangle,\end{aligned}$$

thus by (4.1) $$\frac{1}{n}\sum_{j=1}^{n}\langle \Lambda\dot{u}_j(t)+h_j(t),u(t)-y\rangle$$
$$\leq \frac{1}{n}\sum_{j=1}^{n}\langle A_t^0 y, y-u_j(t)\rangle + m(t)\frac{1}{n}\sum_{j=1}^{n}\|u(t)-u_j(t)\|,$$

where $m(t) := \sup\{\|\Lambda\|\,\|\dot{u}_n(t)\| + \|h_n(t)\| : n\in\mathbb{N}\} < +\infty$. Passing to the limit as $n\to\infty$ in this inequality yields

$$\langle \Lambda\dot{u}(t)+h(t),u(t)-y\rangle \leq \langle A_t^0 y, y-u(t)\rangle.$$

Acoording to this and the previous inclusion $u(t)\in\overline{D(A_t)}$, Lemma 3.11 entails that in fact $u(t)\in D(A_t)$ and that for every $t\in I\setminus N$

$$-\Lambda\dot{u}(t)\in A_t u(t)+h(t).$$

This inclusion and the evident equality $u(0)=u_0$ finish the proof of the lemma. □

We are now in a position to state and prove the following theorem on evolution inclusions with a single-valued perturbation depending on time and state variable.

Theorem 4.3. *Assume that for every $t\in I=[0,T]$, $A_t : D(A_t)\subset E \rightrightarrows E$ is a maximal monotone operator satisfying $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ and such that $\overline{D(A_t)}$ is ball-compact. Let $f : I\times E\to E$ be a Carathéodory mapping for which there exists a non-negative function $\alpha(\cdot)\in L^2_{\mathbb{R}_+}(I)$ such that, for all $t\in I$ and for all $x\in E$, one has*

$$\|f(t,x)\|\leq\alpha(t). \tag{4.2}$$

Then, given $u_0\in D(A_0)$ there is an absolutely continuous solution to the inclusion

$$\begin{cases} -\dot{u}(t)\in A_t u(t)+f(t,u(t)) & a.e.\ t\in I\\ u(0)=u_0. \end{cases}$$

Moreover, this solution $u\in W_E^{1,2}(I)$ and satisfies $u(t)\in D(A_t)$ for all $t\in I$ and

$$\|\dot{u}(t)\|\leq K\Big(1+\dot{r}(t)+\alpha(t)\Big)+\alpha(t)\ \ a.e.\ t\in I,$$

where K is a positive real constant depending on $\|u_0\|$, c, I, r and α.
If in addition $f(t,\cdot)$ is integrably hypomonotone on bounded sets in the sense that for each real $\eta>0$ there is a non-negative function $\kappa_\eta(\cdot)\in L^1_{\mathbb{R}_+}(I)$ such that

$$\langle f(t,x)-f(t,y),x-y\rangle\geq-\kappa_\eta(t)\|x-y\|^2\quad \text{for all}\, t\in I,\ x,y\in\eta\mathbb{B}_E,$$

then the solution is unique.

Proof. Step 1. We will follow some arguments used in [22].
Let $u_0\in D(A_0)$. Let $w : I\to E$ be the unique absolutely continuous solution to

$$\begin{cases} -\dot{w}(t)\in A_t w(t) & \text{a.e. } t\in I\\ w(0)=u_0. \end{cases}$$

Now, let $\rho : I = [0,T] \to \mathbb{R}_+$ be the unique absolutely continuous solution of the differential equation $\dot\rho(t) = \alpha(t)$ a.e. $t \in I$ with $\rho(0) = \sup_{t\in I} \|w(t)\|$. Since $\dot\rho \in L^2_{\mathbb{R}_+}(I)$, the set

$$L := \{h \in L^2_E(I) : \|h(t)\| \le \dot\rho(t) \text{ a.e. } t \in I\},$$

is clearly convex and $\sigma(L^2_E(I), L^2_E(I))$-compact. For any $h \in L$ denote (see Theorem 3.15) by u_h the unique absolutely continuous solution to the perturbed evolution problem

$$\begin{cases} -\dot u_h(t) \in A_t u_h(t) + h(t) \text{ a.e. } t \in I \\ \ u_h(0) = u_0. \end{cases}$$

This solution satisfies in view of Theorem 3.15

$$\|\dot u_h(t)\| \le K(1 + \dot r(t) + \alpha(t)) + \alpha(t) \text{ a.e. } t \in I,$$

where K is a positive real constant depending only on $\|u_0\|$, c, I, r and $\alpha(\cdot)$. Using the monotonicity of A_t for all $t \in I$, one obtains the estimate

$$\frac{1}{2}\|u_h(t) - w(t)\|^2 \le \int_0^t \|h(s)\|\,\|u_h(s) - w(s)\|\,ds.$$

Thanks to [15, Lemma A.5] we have $\|u_h(t) - w(t)\| \le \displaystyle\int_0^t \|h(s)\|\,ds$, so that

$$\|u_h(t)\| \le \rho(0) + \int_0^t \dot\rho(s)ds = \rho(t). \tag{4.3}$$

Set $M := \{u_h :\ h \in L\}$ and notice that M is compact in $\mathcal{C}_E(I)$ for the topology of uniform convergence on I according to the above estimate and to the compactness property stated in Lemma 4.1.

Step 2. By construction the set L is convex and $\sigma(L^2_E(I), L^2_E(I))$-compact. Further, for every $h \in L$, we note that

$$\|f(t, u_h(t))\| \le \alpha(t) = \dot\rho(t).$$

Let us define the mapping $\Phi : L \to L$ by

$$\Phi(h)(t) = f(t, u_h(t)) \text{ for all } t \in I,$$

where u_h is as above the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} -\dot u_h(t) \in A_t u_h(t) + h(t) \text{ a.e. } t \in I \\ \ u_h(0) = u_0. \end{cases}$$

Theorem 3.15 also says that $u_h(t) \in D(A_t)$ for every $t \in I$. Clearly, if h is a fixed point of Φ (i.e. $h = \Phi(h)$), then $u_h(t) \in D(A_t)$ for all $t \in I$ and u_h is an absolutely continuous solution to the inclusion under consideration, namely

$$\begin{cases} -\dot u_h(t) \in A_t u_h(t) + h(t) \text{ a.e. } t \in I \\ \ u_h(0) = u_0, \end{cases}$$

with $h(t) = f(t, u_h(t))$ for all $t \in I$.

Therefore, let us prove that $\Phi : L \to L$ is continuous when L is endowed with the topology induced by the $\sigma(L^2_E(I), L^2_E(I))$ topology. Recalling that L is convex $\sigma(L^2_E(I), L^2_E(I))$-compact and E is separable, we see that L is a metrizable $\sigma(L^2_E(I), L^2_E(I))$-compact convex set. It is then enough to show that Φ is sequentially $\sigma(L^2_E(I), L^2_E(I))$ continuous on L. Take any $h \in L$ and any sequence $(h_n)_n$ in L converging $\sigma(L^2_E(I), L^2_E(I))$ in $L^2_E(I)$ to h. Keeping in mind that the set $M := \{u_\ell : \ell \in L\}$ is compact in $\mathcal{C}_E(I)$ for the topology of uniform convergence on I, and that u_{h_n} is the solution to

$$\begin{cases} -\dot{u}_{h_n}(t) \in A_t u_{h_n}(t) + h_n(t), \text{ a.e. } t \in I, \\ u_{h_n}(0) = u_0, \end{cases}$$

it is easily seen by Lemma 4.2 that $(u_{h_n})_n$ converges uniformly on I to the solution u_h of the differential inclusion

$$\begin{cases} -\dot{u}_h(t) \in A_t u_h(t) + h(t), \text{ a.e. } t \in I, \\ u_h(0) = u_0. \end{cases}$$

To continue the proof of the required continuity of Φ, consider any $g \in L^2_E(I)$. For any $n \in \mathbb{N}$ we have

$$|\langle g(t), f(t, u_{h_n}(t))\rangle| \le \|g(t)\|\alpha(t).$$

Whence the measurable functions $t \mapsto \langle g(t), f(t, u_{h_n}(t))\rangle$ are dominated by the integrable function $t \to \alpha(t)\|g(t)\|$. By the above uniform convergence of $(u_{h_n})_n$ to u_h and by the continuity of $f(t,\cdot)$ we also have

$$\langle g(t), f(t, u_{h_n}(t))\rangle \to \langle g(t), f(t, u_h(t))\rangle \quad \text{for every } t \in I.$$

As a consequence of Lebesgue's convergence theorem, it follows that

$$\lim_{n\to\infty} \int_0^T \langle g(t), f(t, u_{h_n}(t))\rangle dt = \int_0^T \langle g(t), f(t, u_h(t))\rangle dt.$$

We infer that $(\Phi(h_n))_n$ converges to $\Phi(h)$ on L with respect to the $\sigma(L^2_E(I), L^2_E(I))$ topology. This means that $\Phi : L \to L$ is continuous on L endowed with the topology induced by the $\sigma(L^2_E(I), L^2_E(I))$ topology. Applying the Schauder fixed point theorem to the continuous mapping $\Phi : L \to L$ shows that Φ admits a fixed point, $h = \Phi(h)$, just showing (by the above development) the existence of solution for the differential inclusion in the statement of the theorem.

Now assume that f satisfies the hypomonotonicity condition in the statement. Let $u(\cdot)$ and $v(\cdot)$ be two absolutely continuous solutions. Choose a real number $\eta > 0$ such $\eta\mathbb{B}_E$ contains both $u(I)$ and $v(I)$. For a.e. $t \in I$ we have by monotonicity of the operator A_t

$$\langle \dot{u}(t) + f(t, u(t)) - \dot{v}(t) - f(t, v(t)), u(t) - v(t)\rangle \le 0,$$

which gives
$$\frac{1}{2}\frac{d}{dt}\|u(t) - v(t)\|^2 \le \kappa_\eta(t)\|u(t) - v(t)\|^2,$$

so that the uniqueness property follows from Gronwall's lemma. □

Theorem 4.3, with operators evaluated at the state, is of great interest and it provides the existence and uniqueness of the solution with a precise estimation of the velocity. In Subsection 4.3 (resp. Subsection 4.4) we will deal with differential inclusions associated to operators evaluated at the state (resp. mainly at the velocity) and associated also to perturbations involving rough signals related to the Young integral. Before that, we need to recall various features on the Young integral.

4.2. Young integral

Consider the compact interval $I := [0, T]$ of $\mathbb{R}$. For a mapping $\zeta : J \to \mathbb{R}^d$ on a compact subinterval $J := [s, t]$ of I, its variation (or 1-variation) $|\zeta|_{1-var;J}$ on J is the supremum

$$\sup \sum_{i=1}^{n} \|\zeta(t_i) - \zeta(t_{i-1})\|$$

over all sequences $s = t_0 \leq t_1 \leq \cdots \leq t_n = t$, so the bounded variation property of ζ on J can be translated as $|\zeta|_{1-var;J} < +\infty$.

Let $C^{1-var}(I, \mathbb{R}^d)$ denote the space of *continuous mappings with bounded variation* defined on $I = [0, T]$ with values in $\mathbb{R}^d$. We recall some notations. By $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ we denote the space of linear mappings L from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|L| := \sup_{x \in \mathbb{R}^d, \|x\|_{\mathbb{R}^d} = 1} \|L(x)\|_{\mathbb{R}^e}.$$

Fix $z \in C^{1-var}(I, \mathbb{R}^d)$. For any piecewise continuous mapping $\Lambda : I \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ the Young integral $\int_s^t \Lambda(\tau)\, dz(\tau)$ is well defined in $\mathbb{R}^e$ for any $s \leq t$ in I. It will often be convenient to write $\int_s^t \Lambda_\tau\, dz_\tau$ in place of $\int_s^t \Lambda(\tau)\, dz(\tau)$. If $|\Lambda(\cdot)|_{\infty;[s,t]} \leq \beta$ with $s \leq t$, then (see, e.g., [43, Proposition 2.2])

$$\left\| \int_s^t \Lambda_\tau\, dz_\tau \right\|_{\mathbb{R}^e} \leq \beta |z|_{1-var;[s,t]}. \tag{4.4}$$

Let $\Delta_T := \{(s, t) : 0 \leq s \leq t \leq T\}$. A function $\omega : \Delta_T \to [0, \infty[$ is called a *control function* on Δ_T (see [43, Definition 1.6]) if ω is continuous, superadditive and $\omega(s, s) = 0$ for $0 \leq s \leq T$. By super-additivity of ω we mean

$$\omega(s, \tau) + \omega(\tau, t) \leq \omega(s, t) \quad \text{for all } 0 \leq s \leq \tau \leq t \leq T.$$

Examples of control functions on Δ_T are $(s, t) \mapsto |t - s|^\theta$ for $\theta \geq 1$, and also $(s, t) \mapsto \int_s^t \rho(\tau) d\tau$, where ρ is a positive Lebesgue integrable function on I. Let us consider the class of continuous integrand operators $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying the conditions

($\mathcal{B}_1$) $|b(t, x)| \leq M$ for all $t \in I, x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s, x) - b(t, x)| \leq \omega(s, t)$ for all $s, t \in I$ with $s \leq t$, and all $x \in \mathbb{R}^e$,

($\mathcal{B}_3$) $|b(t, x) - b(t, y)| \leq M\|x - y\|$ for all $t \in I,\ x \in \mathbb{R}^e$,

where ω is a control function on Δ_T and M is a positive constant.

If $\mathcal{X} \subset C(I,\mathbb{R}^e)$ is a set of continuous mappings from I into $\mathbb{R}^e$ controlled by a control function $\alpha(\cdot,\cdot)$ in the sense that for each $x(\cdot) \in \mathcal{X}$

$$\|x(s) - x(t)\| \leq \alpha(s,t) \quad \text{for all } s,t \in I \text{ with } s \leq t,$$

then for $b(\cdot,\cdot)$ as above the set of mappings $\{b(\cdot,x(\cdot)) : x \in \mathcal{X}\}$ from $[0,T]$ into $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ is uniformly bounded and uniformly bounded in variation, in particular $b(\cdot,x(\cdot))$ belongs to $C^{1-var}(I,\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$ for any $x(\cdot) \in \mathcal{X}$. Indeed, we have for all $0 \leq s \leq t \leq T$

$$|b(s,x(s)) - b((t,x(t))| \leq |b(s,x(s)) - b(t,x(s))| + |b(t,x(s)) - b(t,x(t))|$$

as well as

$$|b(s,x(s)) - b(t,x(s))| \leq \omega(s,t) \text{ and } |b(t,x(s)) - b(t,x(t)| \leq M\|x(s) - x(t)\| \leq M\alpha(s,t),$$

so using Proposition 1.11 in [43], we see that

$$|b(\cdot,x(\cdot))|_{1-var;[s,t]} < +\infty, \text{ for any } 0 \leq s \leq t \leq T.$$

Consequently, the Young integral $\int_0^t b(s,x(s))dz_s$ along z is well-defined and the mapping $t \mapsto \int_0^t b(s,x(s))dz_s$ belongs to $C^{1-var}(I,\mathbb{R}^e)$, and according to Friz and Victoir [43, Theorem 6.8] we have the following estimates for all $0 \leq s \leq t \leq T$

$$\begin{aligned}
&\left\| \int_s^t b(\tau,x(\tau))dz_\tau \right\|_{\mathbb{R}^e} \\
&\leq \frac{1}{1-2^{1-\theta}} \mathbb{N}|z|_{1-var;[s,t]}|b(\cdot,x(\cdot))|_{1-var;[s,t]} + |b(s,x(s))|\|z(t) - z(s)\|_{\mathbb{R}^d} \\
&\leq \frac{1}{1-2^{1-\theta}}\ |z|_{1-var;[s,t]}|b(\cdot,x(\cdot))|_{1-var;[s,t]} + M\|z(t) - z(s)\|_{\mathbb{R}^d}
\end{aligned}$$

with $\theta = 2$ and

$$\left| \int_0^{\cdot} b(\tau,x(\tau))dz_\tau \right|_{1-var;[s,t]} \leq C(1,1)|z|_{1-var;[s,t]}\big(|b(\cdot,x(\cdot))|_{1-var;[s,t]} + |b(\cdot,x(\cdot))|_{\infty;[s,t]}\big).$$

In consequence, we see that the set $\mathcal{Y}$ of mappings

$$\mathcal{Y} := \{t \mapsto \int_0^t b(\tau,x(\tau))dz_\tau : x \in \mathcal{X}\} \tag{4.5}$$

in $C^{1-var}(I,\mathbb{R}^e)$ is uniformly bounded, and equicontinuous according to the continuity of $t \mapsto |z|_{1-var;[0,t]}$ since $z \in C^{1-var}(I,\mathbb{R}^d)$; further $\mathcal{Y}$ is additionally uniformly bounded in variation. Altogether, $\mathcal{Y}$ is uniformly bounded, equicontinuous and uniformly bounded in variation. When $\mathcal{X}$ is compact in $C(I,\mathbb{R}^e)$ and equi-absolutely continuous, then

$$\mathcal{Y} \text{ is } \textbf{compact} \text{ with respect to the topology of uniform convergence on } I, \tag{4.6}$$

a property which will be crucial in the next subsection.

4.3. Perturbed evolution inclusions under rough signal and with operators evaluated at the state

Using the properties of Young integral recalled in the previous subsection, Theorem 4.4 below establishes a general existence result for evolution inclusions with perturbations involving the Young integral and with operators evaluated at the state.

Theorem 4.4. *Let $I := [0,T]$ and let $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$, with $t \in I$, be maximal monotone operators satisfying $(\mathcal{A}_1)$-$(\mathcal{A}_2)$. Let $z \in C^{1-var}(I,\mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions*

$(\mathcal{B}_1)$ $\quad |b(t,x)| \le M$ *for all* $t \in I,\ x \in \mathbb{R}^e$,

$(\mathcal{B}_2)$ $\quad |b(s,x) - b(t,x)| \le \omega(s,t)$ *for all* $s,t \in I$ *with* $s \le t$, *and for all* $x \in \mathbb{R}^e$,

$(\mathcal{B}_3)$ $\quad |b(t,x) - b(t,y)| \le M\|x-y\|$ *for all* $t \in I,\ x,y \in \mathbb{R}^e$,

where ω is a control function on Δ_T and M is a positive constant.
Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a Carathéodory mapping for which there are a function $\alpha \in L^2_{\mathbb{R}_+}(I)$ and a function $\kappa_\eta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ for each real $\eta > 0$ such that

(i) $\quad \|f(t,x,y)\| \le \alpha(t)$ *for all* $t \in I,\ x,y \in \mathbb{R}^e$,

(ii) $\quad \langle f(t,x,y_1) - f(t,x,y_2), y_1 - y_2\rangle \ge -\kappa_\eta(t)\|y_1-y_2\|^2$, *for all* $(t,x) \in I \times \mathbb{R}^e$, *and for all* $y_1,y_2 \in \mathbb{R}^e \times \mathbb{R}^e$.

Then, for any $u_0 \in D(A_0)$, the problem

$$\begin{cases} \xi(t) = \int_0^t b(s,u(s))dz_s,\ \forall t \in I \\ -\dot{u}(t) \in A_t u(t) + f(t,\xi(t),u(t))\ a.e.\ t \in I \\ u(0) = u_0 \end{cases}$$

has a solution $(\xi,u) \in C^{1-var}(I,\mathbb{R}^e) \times W^{1,2}(I,\mathbb{R}^e)$. Further, this solution satisfies $u(t) \in D(A_t)$ for every $t \in I$.

Proof. Let $\Gamma : I \rightrightarrows \mathbb{R}^e$ be the measurable multimaping with compact convex values given by $\Gamma(t) := \alpha(t)\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$. Let S_Γ be the set of all measurable selections of Γ, so S_Γ coincides with S^2_Γ. For each $\ell(\cdot) \in S_\Gamma$ denote (according to Theorem 3.15) by u_ℓ the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} -\dot{u}_\ell(t) \in A_t u_\ell(t) + \ell(t) \quad \text{a.e. } t \in I \\ u_\ell(0) = u_0, \end{cases}$$

and set $\mathcal{X} := \{u_\ell : \ell \in S_\Gamma\}$. Lemma 4.1 tells us that $\mathcal{X}$ is compact in $C(I,\mathbb{R}^e)$ and equi-absolutely continuous on I with

$$\|\dot{u}_\ell(t)\| \le \gamma(t) := K(1 + \dot{r}(t) + \alpha(t)) + \alpha(t) \text{ a.e. } t \in I$$

for each $\ell \in S_\Gamma$. As consequence, the set (see (4.6))

$$\mathcal{Y} := \left\{ t \mapsto \int_0^t b(s,x(s))\,dz_s : x \in \mathcal{X} \right\}$$

is **compact** in $C(I,\mathbb{R}^e)$ for the topology of uniform convergence on I, and so is the set $\overline{\text{co}}(\mathcal{Y})$.

By Theorem 4.3, for each $h \in \overline{\mathrm{co}}(\mathcal{Y})$, there is a *unique* $W^{1,2}(I, \mathbb{R}^e)$ solution v_h to the differential inclusion

$$\begin{cases} -\dot{v}_h(t) \in A_t v_h(t) + f(t, h(t), v_h(t)) & \text{a.e. } t \in I \\ v_h(0) = u_0, \end{cases}$$

and this solution satisfies $v_h(t) \in D(A_t)$ for all $t \in I$ along with $\|\dot{v}_h(t)\| \leq \gamma(t)$ for a.e. $t \in I$. Now let us consider the mapping Φ defined on $\overline{\mathrm{co}}(\mathcal{Y})$ with values in $\mathcal{Y}$ by putting for every $h \in \overline{\mathrm{co}}(\mathcal{Y})$

$$\Phi(h)(t) := \int_0^t b(s, v_h(s)) dz_s \quad \text{for all } t \in I.$$

Since $v_h \in \mathcal{X}$, it is clear that $\Phi(h) \in \mathcal{Y}$. Our aim is to show the existence theorem by applying some ideas developed in [28] via a fixed point theorem. For this purpose we first prove that $\Phi : \overline{\mathrm{co}}(\mathcal{Y}) \to \mathcal{Y}$ is continuous with respect to the topology of uniform convergence on I. Take any sequence $(h_n)_n$ in $\overline{\mathrm{co}}(\mathcal{Y})$ uniformly converging on I to $h \in \overline{\mathrm{co}}(\mathcal{Y})$. We claim that the sequence $(v_{h_n})_n$, where v_{h_n} is the absolutely continuous solution associated with h_n of the differential inclusion

$$\begin{cases} v_{h_n}(0) = u_0 \\ -\dot{v}_{h_n}(t) \in A_t v_{h_n}(t) + f(t, h_n(t), v_{h_n}(t)), & \text{a.e. } t \in I, \end{cases}$$

uniformly converges on I to the absolutely continuous solution v_h associated with h of the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ -\dot{v}_h(t) \in A_t v_h(t) + f(t, h(t), v_h(t)) & \text{a.e. } t \in I. \end{cases}$$

Since $(v_{h_n})_n$ is equi-absolutely continuous with for each $n \in \mathbb{N}$ the estimate

$$\|\dot{v}_{h_n}(t)\| \leq \gamma(t) \text{ a.e. } t \in I,$$

we may suppose that $(v_{h_n})_n$ converges uniformly on I to an absolutely continuous mapping $w : I \to \mathbb{R}^e$ with $w(t) = u_0 + \int_0^t \dot{w}(s) ds$. We may also suppose that $(\dot{v}_{h_n})_n$ weakly converges in $L^2(I, \mathbb{R}^e)$ to some $\xi \in L^2(I, \mathbb{R}^e)$ with $\|\xi(t)\| \leq \gamma(t)$ a.e. $t \in I$, so we easily see that $\dot{w} = \xi$. Consequently, for each $t \in I$ we have

$$k_n(t) := f(t, h_n(t), v_{h_n}(t)) \to k(t) := f(t, h(t), w(t)) \quad \text{as } n \to \infty.$$

Remember that $\|k_n(t)\| = \|f(t, h_n(t), v_{h_n}(t))\| \leq \alpha(t)$ for all $t \in I$ with $\alpha \in L^2_{\mathbb{R}_+}(I)$. We deduce that w is absolutely continuous and is solution of the differential inclusion

$$\begin{cases} w(0) = u_0 \\ -\dot{w}(t) \in A_t w(t) + f(t, h(t), w(t)) & \text{a.e. } t \in I \end{cases}$$

according to Lemma 4.2. Then by uniqueness $w = v_h$, which confirms the claim of uniform convergence on I of $(v_{h_n})_n$ to v_h.

Taking into account this uniform convergence of $(v_{h_n})_n$ to v_h on I, we see by applying $(\mathcal{B}_3)$ that $b(\cdot, v_{h_n}(\cdot))_n$ converges uniformly on I to $b(\cdot, v_h(\cdot))$, whence $\big(\Phi(h_n)(\cdot)\big)_n$ converges uniformly on I to $\Phi(h)(\cdot)$ by [43, Proposition 2.7].

This translates the desired continuity of $\Phi : \overline{\mathrm{co}}(\mathcal{Y}) \to \mathcal{Y}$ with respect to the topology of uniform convergence on I. By the Schauder fixed point theorem Φ has a fixed point, say $h = \Phi(h)$, which means

$$h(t) = \Phi(h)(t) = \int_0^t b(s, v_h(s))\, dz_s, \text{ for all } t \in I,$$

$$\begin{cases} v_h(0) = u_0 \\ -\frac{dv_h}{dt}(t) \in A_t v_h(t) + f(t, h(t), v_h(t)) \text{ a.e. } t \in I; \end{cases}$$

further $v_h(t) \in D(A_t)$ for all $t \in I$ as already seen above. Then, taking $\xi(\cdot) = h(\cdot)$ and $u(\cdot) = v_h(\cdot)$, we obtain that (ξ, u) belongs to $C^{1-var}(I, \mathbb{R}^e) \times W^{1,2}(I, \mathbb{R}^e)$ and is a solution of the differential inclusion in the statement of the theorem, and $u(t) \in D(A_t)$ for all $t \in I$. $\square$

4.4. Integro-differential variational inequalities under rough signals and time-dependent subdifferentials at velocity arguments

In addition to the study of convergence in Lemma 4.2 for inclusions in the form $f_n(t) - \Lambda \dot{u}_n(t) \in A_t(u_n(t))$, the following two lemmas consider situations involving the operator along the velocity $\dot{u}_n(t)$ instead of the state $u_n(t)$. More precisely, the lemmas deal with inclusions where $\partial\varphi(t, \dot{u}_n(t))$ or $\partial\varphi(t, \dot{u}_n(t) + g(t, u_n(t))$ replaces $A_t(u_n(t))$.

Lemma 4.5. *Let $I := [0, T]$, let $u_0 \in E$ and let $(u_n, u)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous mappings from I into the separable Hilbert space E in the form*

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s)\, ds, \qquad u(t) = u_0 + \int_0^t \dot{u}(s)\, ds,$$

(so $u_n(0) = u(0) = u_0$), and such that $(u_n)_n$ converges to u uniformly on I and $(\dot{u}_n)_n$ converges to $\dot{u}$ weakly in $L^1_E(I)$. Let $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there exist a non-negative function $\mu(\cdot) \in L^p_{\mathbb{R}_+}(I)$ with $p \in]1, +\infty]$ and a mapping $\pi(\cdot) \in L^{p'}_E(I)$ with $1/p + 1/p' = 1$ such that:

(i) $\|\dot{u}_n(t)\| \le \mu(t)$ *for almost every* $t \in I$,

(ii) $\big(\varphi(\cdot, \dot{u}_n(\cdot)), \varphi(\cdot, \dot{u}(\cdot))\big)_{n\in\mathbb{N}}$ *is uniformly integrable on* I,

(iii) $\varphi(\cdot, \pi(\cdot))$ *is integrable on* I.

Let $\Lambda : E \to E$ be a positive symmetric linear operator (hence continuous). Let $(f_n, f)_{n\in\mathbb{N}}$ be a sequence of Carathéodory mappings from $I \times E \to E$ such that for each real $\eta > 0$ there exists a real constant $\beta_\eta > 0$ for which

$$\|f_n(t, x)\| \le \beta_\eta \quad \textit{for all } n \in \mathbb{N},\ t \in I \textit{ and } x \in \eta\mathbb{B}_E.$$

Let $(v_n, v)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_E(I)$ such that for a.e. $t \in I$ the sequence $(v_n(t))_n$ converges strongly in E to $v(t)$.

Assume that for every $n \in \mathbb{N}$

$$f_n(t, v_n(t)) - \Lambda \dot{u}_n(t) \in \partial\varphi(t, \dot{u}_n(t)) \textit{ a.e. } t \in I. \tag{4.7}$$

Then for a.e. $t \in I$ one has $f(t, v(t)) - \Lambda \dot{u}(t) \in \partial\varphi(t, \dot{u}(t))$.

Proof. Put $\Gamma(t) := \mu(t)\mathbb{B}_E$ for all $t \in I$. Since $\dot{u}_n(t) \in \Gamma(t)$ a.e. $t \in I$ by (i) and since $(\dot{u}_n)_n$ converges weakly to $\dot{u}$ in $L^1_E(I)$, we have $\dot{u}(t) \in \Gamma(t)$ a.e. $t \in I$. Choose a real $\gamma > 0$ and a Lebesgue null set $N \subset I$ such that for each $t \in I \setminus N$ we have $v_n(t) \to v(t)$ as $n \to \infty$ and

$$\|v(t)\| \leq \gamma \text{ and } \|v_n(t)\| \leq \gamma \text{ for all } n \in \mathbb{N}.$$

For each $t \in I \setminus N$ the continuity assumption of $f(t, \cdot)$ implies that $(f(t, v_n(t)))_n$ strongly converges in E to $f(t, v(t))$. Setting $h_n := f_n(\cdot, v_n(\cdot))$ we see that, for each $t \in I \setminus N$,

$$\|h_n(t)\| = \|f_n(t, v_n(t))\| \leq \beta_\gamma \quad \text{for all } n \in \mathbb{N}. \tag{4.8}$$

This uniform boundedness and the a.e. convergence of $(h_n(\cdot))_n$ to $h := f(\cdot, v(\cdot))$ assure us (by the Lebesgue dominated convergence theorem) that $(h_n)_n$ strongly converges to h in $L^{p'}_E(I)$ with $1/p + 1/p' = 1$, and hence also in $L^1_E(I)$. Take any Lebesgue measurable subset Z in I. Keeping in mind that $\dot{u}_n(t) \in \Gamma(t) = \mu(t)\mathbb{B}_E$ a.e. $t \in I$, we observe that

$$\left| \int_Z \langle h_n(t) - h(t), \dot{u}_n(t) \rangle \, dt \right| \leq \|h_n - h\|_{p'} \|\dot{u}_n\|_p \leq \|h_n - h\|_{p'} \left(\int_I \mu(t)^p \, dt \right)^{1/p},$$

thus we deduce that $\int_Z \langle h_n(t) - h(t), \dot{u}_n(t) \rangle \, dt \to 0$ as $n \to \infty$. Noticing $\|h(t)\| \leq \beta_\gamma$ a.e. $t \in I$ by (4.8), it ensues that

$$\begin{aligned} &\lim_{n\to\infty} \int_Z \langle h_n(t), \dot{u}_n(t) \rangle \, dt \\ &= \lim_{n\to\infty} \int_Z \langle h_n(t) - h(t), \dot{u}_n(t) \rangle \, dt + \lim_{n\to\infty} \int_Z \langle h(t), \dot{u}_n(t) \rangle \, dt \\ &= \lim_{n\to\infty} \int_Z \langle h(t), \dot{u}_n(t) \rangle dt = \int_Z \langle h(t), \dot{u}(t) \rangle \, dt. \end{aligned} \tag{4.9}$$

Now define $\psi_\Lambda : I \times E \to \mathbb{R} \cup \{+\infty\}$ by

$$\psi_\Lambda(t, x) = \langle \Lambda x, x \rangle \text{ if } x \in \Gamma(t) \text{ and } \psi_\Lambda(t, x) = +\infty \text{ otherwise.}$$

This function ψ_Λ is a non-negative convex normal integrand, so the lower semicontinuity property in [33, Theorem 8.1.6] yields

$$\liminf_n \int_Z \psi_\Lambda(\dot{u}_n(t)) dt \geq \int_Z \psi_\Lambda(\dot{u}(t)) \, dt,$$

that is,

$$\liminf_n \int_Z \langle \Lambda \dot{u}_n(t), \dot{u}_n(t) \rangle \, dt \geq \int_Z \langle \Lambda \dot{u}(t), \dot{u}(t) \rangle \, dt. \tag{4.10}$$

The same lower semicontinuity property in [33, Theorem 8.1.6], associated with the convex normal integrand φ and with the assumption (ii), also entails that

$$\int_Z \varphi(t, \dot{u}(t)) \, dt \leq \liminf_n \int_Z \varphi(t, \dot{u}_n(t)) \, dt. \tag{4.11}$$

On the other hand, as φ is a convex normal integrand, the conjugate integrand function $\varphi^* : I \times E \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi^*(t, y) = \sup_{x \in E} \big(\langle x, y \rangle - \varphi(t, x) \big)$$

is also a convex normal integrand (see, e.g, [35, 70, 71]). In particular, putting $q_n(t) := h_n(t) - \Lambda\dot{u}_n(t)$ for all $t \in I$ the function $t \mapsto \varphi^*(t, q_n(t))$ is Lebesgue measurable. Note furthermore that $(q_n)_n$ weakly converges in $L^1_E(I)$ to the mapping $q(\cdot) := h(\cdot) - \Lambda\dot{u}(\cdot)$ since $(h_n)_n$ strongly converges in $L^1_E(I)$ as already noticed and $(\Lambda\dot{u}_n)_n$ weakly converges to $\Lambda\dot{u}$ in $L^1_E(I)$ by the assumption of weak convergence in $L^1_E(I)$ of $(\dot{u}_n)_n$ to $\dot{u}$. Further, by assumption we have

$$q_n(t) = h_n(t) - \Lambda\dot{u}_n(t) \in \partial\varphi(t, \dot{u}_n(t)),$$

so that for a.e. $t \in I$

$$\langle h_n(t) - \Lambda\dot{u}_n(t), \pi(t) - \dot{u}_n(t)\rangle \leq \varphi(t, \pi(t)) - \varphi(t, \dot{u}_n(t)),$$

which (taking the positivity of the linear mapping Λ into account) gives

$$0 \leq \langle \Lambda\dot{u}_n(t), \dot{u}_n(t)\rangle \leq \varphi(t, \pi(t)) - \varphi(t, \dot{u}_n(t)) + \beta_\gamma\|\pi(t)\| + \mu(t)\|\Lambda\|\,\|\pi(t)\|,$$

so the sequence $\big(\Lambda\dot{u}_n(\cdot), \dot{u}_n(\cdot)\rangle\big)_n$ is uniformly integrable on I by the assumptions (ii) and (iii). This and the equality

$$\langle q_n(t), \dot{u}_n(t)\rangle = \langle h_n(t), \dot{u}_n(t)\rangle - \langle \Lambda\dot{u}_n(t), \dot{u}_n(t)\rangle$$

tell us that the sequence of measurable mappings $\big(\langle q_n(\cdot), \dot{u}_n(\cdot)\rangle\big)_n$ is uniformly integrable on the interval I. Then, for $\rho_n(t) := -\varphi(t, \dot{u}_n(t)) + \langle \dot{u}_n(t), q_n(t)\rangle$ the sequence $(\rho_n(\cdot))_n$ is uniformly integrable on I by (ii), which combined with the weak convergence in $L^1_E(I)$ of $(q_n)_n$ to q and with the inequality

$$\rho_n(t) \leq \varphi^*(t, q_n(t))$$

(due to the definition of conjugate function) allows us to apply again [33, Theorem 8.1.6] to get

$$\int_Z \varphi^*(t, q(t))\,dt \leq \liminf_n \int_Z \varphi^*(t, q_n(t))\,dt. \tag{4.12}$$

Consequently, integrating on the Lebesgue measurable subset Z of I the equality

$$\varphi(t, \dot{u}_n(t)) + \varphi^*(t, q_n(t)) = \langle \dot{u}_n(t), q_n(t)\rangle$$

(due to the inclusion $q_n(t) \in \partial\varphi(t, \dot{u}_n(t))$) gives

$$\int_Z \varphi(t, \dot{u}_n(t))\,dt + \int_Z \varphi^*(t, q_n(t))\,dt + \int_Z \langle \Lambda\dot{u}_n(t), \dot{u}_n(t)\rangle\,dt = \int_Z \langle \dot{u}_n(t), h_n(t)\rangle\,dt.$$

Passing to the limit as $n \to \infty$ in this equality and using (4.9)–(4.12) furnish the inequality

$$\int_Z \varphi(t, \dot{u}(t))\,dt + \int_Z \varphi^*(t, q(t))\,dt \leq \int_Z \langle \dot{u}(t), q(t)\rangle\,dt.$$

This being true for every Lebesgue measurable set Z in I and the non-negative function $t \mapsto \varphi(t, \dot{u}(t)) + \varphi^*(t, q(t)) - \langle \dot{u}(t), q(t)\rangle$ being Lebesgue measurable, we deduce that for almost every $t \in I$

$$\varphi(t, \dot{u}(t)) + \varphi^*(t, q(t)) - \langle \dot{u}(t), q(t)\rangle \leq 0.$$

It follows for almost every $t \in I$ that $\varphi(t, \dot{u}(t)) + \varphi^*(t, q(t)) = \langle \dot{u}(t), q(t) \rangle$, or equivalently

$$q(t) = f(t, v(t)) - \Lambda \dot{u}(t) \in \partial\varphi(t, \dot{u}(t)).$$

The proof of the lemma is finished. □

Remark 4.6. We point out that an L^1_E-version of the above lemma can also be obtained by combining the arguments therein with [19], if we replace the above assumptions (i), (ii), (iii) by the following ones:

(i) $(\dot{u}_n(\cdot))_n$ is uniformly integrable on I with $\dot{u}_n(t) \in \Gamma(t)$, where $\Gamma(\cdot)$ is a Lebesgue measurable multimapping with weakly compact convex values,

(ii) $(\varphi(\cdot, \dot{u}_n(\cdot)), \varphi(\cdot, \dot{u}(\cdot)))_{n\in\mathbb{N}}$ is uniformly integrable on I,

(iii) there exists $\pi(\cdot) \in L^\infty_E$ such that $\varphi(\cdot, \pi(\cdot))$ is integrable on I. □

Lemma 4.7. *Let $I := [0, T]$, let $u_0 \in E$ and let $(u_n, u)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous mappings from I into the separable Hilbert space E in the form*

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s), \qquad u(t) = u_0 + \int_0^t \dot{u}(s)ds,$$

(so $u_n(0) = u(0) = u_0$) and such that $(u_n)_n$ converges to u uniformly on I and $(\dot{u}_n)_n$ converges to $\dot{u}$ weakly in $L^2_E(I)$, and for which there is a non-negative function $\mu(\cdot) \in L^2_{\mathbb{R}_+}(I)$ satisfying for each $n \in \mathbb{N}$

$$\|\dot{u}_n(t)\| \leq \mu(t) \quad a.e.\ t \in I.$$

Let $\varphi : I \times E \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there exist two non-negative functions $\alpha(\cdot) \in L^2_{\mathbb{R}_+}(I)$ and $\beta(\cdot) \in L^1_{\mathbb{R}_+}(I)$ such that

$$-\alpha(t)\|x\| - \beta(t) \leq \varphi(t, x) \quad \text{for all } (t, x) \in I \times E.$$

Let $\Lambda : E \to E$ be a positive symmetric linear operator (hence continuous). Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of mappings in $L^2_E(I)$ converging strongly to f and g respectively in $L^2_E(I)$, and such that there exists a non-negative function $\gamma(\cdot) \in L^2_{\mathbb{R}_+}(I)$ satisfying for each $n \in \mathbb{N}$ the inequality $\|g_n(t)\| \leq \gamma(t)$ a.e. $t \in I$. Assume that for every $n \in \mathbb{N}$

$$f_n(t) - \Lambda \dot{u}_n(t) \in \partial\varphi\big(t, \dot{u}_n(t) + g_n(t)\big) \ a.e.\ t \in I. \tag{4.13}$$

Then for a.e. $t \in I$ one has $f(t) - \Lambda \dot{u}(t) \in \partial\varphi\big(t, \dot{u}(t) + g(t)\big)$.

Proof. First, note that $(\Lambda \dot{u}_n)_n$ converges weakly in $L^2_E(I)$ to $\Lambda \dot{u}$, and that $(\dot{u}_n)_n$ converges weakly in $L^1_E(I)$ to $\dot{u}$. Modifying g_n, g and γ on a Lebesgue null set, we may and do suppose that the function $\gamma \geq 0$ in $L^2_{\mathbb{R}_+}(I)$ satisfies $\|g(t)\| \leq \gamma(t)$ for all $t \in I$ and also $\|g_n(t)\| \leq \gamma(t)$ for all $n \in \mathbb{N}$ and all $t \in I$.

Put $\mu_0(t) := 1 + \gamma(t) + \mu(t)$ for all $t \in I$. Let $(\pi_k, \sigma_k)_{k\in\mathbb{N}}$ be a Castaing representation of the Lebesgue measurable closed-valued multimapping $M : I \rightrightarrows E \times \mathbb{R}$ defined by

$$M(t) := \operatorname{epi} \varphi(t, \cdot) \cap \big(B_E[0, \mu_0(t)] \times \mathbb{R}\big) \quad \text{for all } t \in I.$$

Take any Lebesgue measurable set Z in I. For each $k \in \mathbb{N}$ we have $\|\pi_k(t)\| \le \mu_0(t)$ for all $t \in I$, then $\pi_k(\cdot) \in L^2_E(I)$ and hence as $n \to \infty$

$$\int_Z \langle f_n(t), \pi_k(t)\rangle \, dt \to \int_Z \langle f(t), \pi_k(t)\rangle \, dt$$

and also
$$\int_Z \langle \Lambda\dot{u}_n(t), \pi_k(t)\rangle \, dt \to \int_Z \langle \Lambda\dot{u}(t), \pi_k(t)\rangle \, dt.$$

Clearly, the assumptions on $(f_n)_n$ and $(g_n)_n$ along with the weak convergence in $L^2_E(I)$ of $(\dot{u}_n)_n$ to $\dot{u}$ ensures as $n \to \infty$

$$\int_Z \langle f_n(t), g_n(t)\rangle \, dt \to \int_Z \langle f(t), g(t)\rangle \, dt,$$

$$\int_Z \langle \dot{u}_n(t), f_n(t)\rangle \, dt \to \int_Z \langle \dot{u}(t), f(t)\rangle \, dt,$$

$$\int_Z \langle \Lambda\dot{u}_n(t), g_n(t)\rangle \, dt \to \int_Z \langle \Lambda\dot{u}(t), g(t)\rangle \, dt.$$

Now define $\psi_\Lambda : I \times E \to \mathbb{R} \cup \{+\infty\}$ by

$$\psi_\Lambda(t, x) = \langle \Lambda x, x\rangle \quad \text{for all } (t, x) \in I \times E.$$

This function ψ_Λ is a non-negative convex normal integrand by the positivity of the linear operator Λ, so the weak convergence of $(\dot{u}_n)_n$ to $\dot{u}$ in $L^1_E(I)$ and the lower semicontinuity property in [33, Theorem 8.1.6] give

$$\liminf_n \int_Z \psi_\Lambda(\dot{u}_n(t))dt \ge \int_Z \psi_\Lambda(\dot{u}(t)) \, dt,$$

that is,
$$\liminf_n \int_Z \langle \Lambda\dot{u}_n(t), \dot{u}_n(t)\rangle \, dt \ge \int_Z \langle \Lambda\dot{u}(t), \dot{u}(t)\rangle \, dt. \tag{4.14}$$

Putting $u_\infty(\cdot) := u(\cdot)$ and $g_\infty(\cdot) := g$, we also observe by the assumption on φ that for each $n \in \mathbb{N} \cup \{\infty\}$

$$\varphi\big(t, \dot{u}_n(t) + g_n(t)\big) \ge -\alpha(t)(\gamma(t) + \mu(t)) - \beta(t) \text{ a.e. } t \in I$$

with the non-negative function $t \mapsto \alpha(t)(\gamma(t) + \mu(t)) + \beta(t)$ in $L^1_{\mathbb{R}_+}(I)$, thus the same lower semicontinuity property in [33, Theorem 8.1.6], associated with the convex normal integrand φ and with the weak convergence of $(\dot{u}_n + g_n)_n$ in $L^1_E(I)$ to $\dot{u} + g$, entails that

$$\int_Z \varphi(t, \dot{u}(t) + g(t)) \, dt \le \liminf_n \int_Z \varphi(t, \dot{u}_n(t) + g_n(t)) \, dt. \tag{4.15}$$

Fix any $k \in \mathbb{N}$ and put

$$\rho_n(t) := \langle f_n(t), \pi_k(t) - \dot{u}_n(t) - g_n(t)\rangle - \langle \Lambda\dot{u}_n(t), \pi_k(t)\rangle + \langle \Lambda\dot{u}_n(t), g_n(t)\rangle$$

$$\rho(t) := \langle f(t), \pi_k(t) - \dot{u}(t) - g(t)\rangle - \langle \Lambda\dot{u}(t), \pi_k(t)\rangle + \langle \Lambda\dot{u}(t), g(t)\rangle.$$

Using for a.e. $t \in I$ the inclusion $f_n(t) - \Lambda \dot{u}_n(t) \in \partial\varphi(t, \dot{u}_n(t) + g_n(t))$, we can write

$$\langle \Lambda \dot{u}_n(t), \dot{u}_n(t)\rangle + \varphi(t, \dot{u}_n(t) + g_n(t)) + \rho_n(t) \le \varphi(t, \pi_k(t)),$$

so integrating on any Lebesgue measurable subset Z of I and invoking (4.14), (4.15) along with the convergences preceding them, we obtain as $n \to \infty$

$$\int_Z \Big(\langle f(t) - \Lambda\dot{u}(t), \pi_k(t) - \dot{u}(t) - g(t)\rangle + \varphi(t, \dot{u}(t) + g(t))\Big)\, dt \le \int_Z \varphi(t, \pi_k(t))\, dt,$$

with $\varphi(t, \pi_k(t)) \ge -\alpha(t)\mu_0(t) - \beta(t)$ and $t \mapsto \alpha(t)\mu_0(t) + \beta(t)$ in $L^1_{\mathbb{R}_+}(I)$. This being true for every Lebesgue measurable set Z in I, it follows that there exists a Lebesgue measurable null set $N \subset I$ such that for all $k \in \mathbb{N}$ and all $t \in I \setminus N$

$$\langle f(t) - \Lambda\dot{u}(t), \pi_k(t) - \dot{u}(t) - g(t)\rangle + \varphi(t, \dot{u}(t) + g(t)) \le \varphi(t, \pi_k(t)). \tag{4.16}$$

According to the weak convergence of $(\dot{u}_n)_n$ to $\dot{u}$ in $L^2_E(I)$ and to the assumption $\|\dot{u}_n(t)\| \le \mu(t)$ a.e. $t \in I$, choose a Lebesgue null set $N_1 \supset N$ with $\|\dot{u}(t)\| \le \mu(t)$ for all $t \in I \setminus N_1$. Fix any $t \in P := I \setminus N_1$. For any $y \in B[0, \mu_0(t)] \cap \operatorname{dom}\varphi(t, \cdot)$ there is a subsequence $(\pi_k(t), \sigma_k(t))_{k \in K}$ with an infinite subset $K \subset \mathbb{N}$ converging in $E \times \mathbb{R}$ to $(y, \varphi(t, y))$. This combined with the inequality $\varphi(t, \pi_k(t)) \le \sigma_k(t)$ and the convergence $\sigma_k(t) \to \varphi(t, y)$ as $K \ni k \to \infty$, implies that

$$\limsup_{K \ni k \to \infty} \varphi(t, \pi_k(t)) \le \varphi(t, y).$$

It follows from this and the lower semicontinuity of $\varphi(t, \cdot)$ that

$$\varphi(t, y) \le \liminf_{K \ni k \to \infty} \varphi(t, \pi_k(t)) \le \limsup_{K \ni k \to \infty} \varphi(t, \pi_k(t)) \le \varphi(t, y),$$

which means that $\lim_{K \ni k \to \infty} \varphi(t, \pi_k(t)) = \varphi(t, y)$. We deduce from this and from (4.16) that for every $y \in B[0, \mu_0(t)]$

$$\langle f(t) - \Lambda\dot{u}(t), y - \dot{u}(t) - g(t)\rangle + \varphi\big(t, \dot{u}(t) + g(t)\big) \le \varphi(t, y). \tag{4.17}$$

The latter inequality is true for any y in the neighborhood $B[0, \mu_0(t)]$ of $\dot{u}(t) + g(t)$ so that for any $t \in P$ we get for the convex function $\varphi(t, \cdot)$

$$f(t) - \Lambda\dot{u}(t) \in \partial\varphi\big(t, \dot{u}(t) + g(t)\big).$$

This finishes the proof of the lemma. □

Theorem 4.8. *Let $I := [0, T]$ and $\varphi : I \times \mathbb{R}^e \to \mathbb{R} \cup \{+\infty\}$ be a convex normal integrand for which there are a mapping $\zeta \in L^2_{\mathbb{R}^e}(I)$ such that $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ is in $L^2_{\mathbb{R}_+}(I)$ and two non-negative functions $\alpha, \beta : I \to \mathbb{R}$ with $\alpha(\cdot) \in L^2_{\mathbb{R}_+}(I)$ and $\beta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that*

$$-\alpha(t)\|x\| - \beta(t) \le \varphi(t, x) \quad \text{for all } (t, x) \in I \times \mathbb{R}^e$$

(note that $\sqrt{\beta} \in L^2_{\mathbb{R}_+}(I)$ since $\beta \in L^2_{\mathbb{R}_+}(I)$). Let $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in \mathbb{R}^e$ and such that

there are non-negative functions α_f, α_g, β_f, β_g in $L^2_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times \mathbb{R}^e$

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x-y\|, \; \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x-y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping such that $F(\cdot,\cdot,x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for each $x \in \mathbb{R}^e$ and such that

(i) *for some non-negative $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable function $\beta_F : I \times I \to \mathbb{R}_+$ with $(t,s) \mapsto (\beta_F(t,s))^2$ Lebesgue integrable on $I \times I$*

$$\|F(t,s,x)\| \le \beta_F(t,s) \quad \textit{for all}\, t,s \in I \textit{ and } x \in \mathbb{R}^e,$$

(ii) *for each $(t,s) \in I \times I$ the mapping $F(t,s,\cdot)$ is continuous on $\mathbb{R}^e$.*

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator.

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

($\mathcal{B}_1$) $|b(t,x)| \le M$ *for all $x \in \mathbb{R}^e$,*

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \le \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s,t \in I$ with $s \le t$,*

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \le M\|x-y\|$ *for all $t \in I$ and $x, y \in \mathbb{R}^e$,*

where ω is a control function on Δ_T and M is a positive constant.

Then for any $u_0 \in \mathbb{R}^e$, there exists an absolutely continuous solution $u : I \to \mathbb{R}^e$ of the evolution variational inequality with rough signal and with Volterra integral

$$\begin{cases} u(0) = u_0 \\ \int_0^t b(s,u(s))dz_s + \int_0^t F(t,s,u(s))ds + f(t,u(t)) - \Lambda \frac{du}{dt}(t) \in \partial\varphi\left(t, \frac{du}{dt}(t) + g(t,u(t))\right), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$.

Proof. The assumptions on β_F directly imply that the function $t \mapsto \int_0^t \beta_F(t,s)\,ds$ is in $L^2_{\mathbb{R}_+}(I)$. Consider the set $\mathcal{E}$ of mappings $e : I \times \mathbb{R}^e \to \mathbb{R}^e$ Lebesgue measurable in t and such that

$$\|e(t,x)\| \le \alpha_f(t)\|x\| + MT|z|_{1-var;I} + \beta_f(t) + \int_0^t \beta_F(t,s)\,ds \quad \text{for all } (t,x) \in I \times \mathbb{R}^e$$

and such that for each real $r > 0$

$$\|e(t,x) - e(t,y)\| \le \gamma_{f,r}(t)\|x-y\| \quad \text{for all } t \in I, x, y \in r\mathbb{B}_{\mathbb{R}^e}.$$

By Theorem 3.1 there is a non-negative function $\mu(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for every mapping $e \in \mathcal{E}$ the solution $\zeta(\cdot)$ of the differential inclusion

$$\begin{cases} e(t,\zeta(t)) - \Lambda \frac{d\zeta}{dt}(t) \in \partial\varphi\left(t, \frac{d\zeta}{dt}(t) + g(t,\zeta(t))\right) \text{ a.e. } t \in I \\ \zeta(0) = u_0 \end{cases}$$

satisfies $\|\dot{\zeta}(t)\| \le \mu(t)$ a.e. $t \in I$. Put $\Gamma(t) := \mu(t)\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$.

Consider the subset $\mathcal{X}$ in the Banach space $C(I,\mathbb{R}^e)$ defined by

$$\mathcal{X} := \{q_\ell : \ell \in S_\Gamma\}$$

where q_ℓ is the mapping from I into $\mathbb{R}^e$ given by

$$q_\ell(t) := u_0 + \int_0^t \ell(s)\, ds$$

and where S_Γ stands here for the set of Lebesgue almost everywhere selections of Γ. We note at first that $\mathcal{X}$ is compact convex and equi-absolutely continuous on I since for any $\ell \in S_\Gamma$

$$\|q_\ell(t) - q_\ell(s)\| \le \int_s^t \mu(\tau)\, d\tau \quad \text{for all } s,t \in I \text{ with } s \le t.$$

Then for any $h \in \mathcal{X}$ and any $s,t \in I$ with $s \le t$, using $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ we can write

$$\begin{aligned}|b(t,h(t)) - b(s,h(s))| &\le |b(t,h(t)) - b(s,h(t))| + |b(s,h(t)) - b(s,h(s))| \\ &\le \omega(s,t) + M\int_s^t \mu(\tau) d\tau\end{aligned}$$

so that $b(\cdot,h(\cdot)) \in C^{1-var}(I,\mathcal{L}(\mathbb{R}^e,\mathbb{R}^e))$ with $|b(t,h(t))| \le M$ for all $t \in I$ by $(\mathcal{B}_1)$. In particular, for any $h \in \mathcal{X}$ and any $t \in I$ the Young integral $\int_0^t b(s,h(s))\, dz_s$ is well defined and $b(\cdot,h(\cdot))$ is bounded in variation uniformly with respect to $h \in \mathcal{X}$. Fix for a moment $h \in \mathcal{X}$ and define $I_h : I \to \mathbb{R}^e$ by $I_h(t) := \int_0^t F(t,s,h(s))\, ds$ and $e_h : I \times \mathbb{R}^e \to \mathbb{R}^e$ by

$$e_h(t,x) = \int_0^t b(s,h(s))\, dz_s + I_h(t) + f(t,x) \quad \text{for all } (t,x) \in I \times \mathbb{R}^e.$$

The inequality $\|I_h(t)\| \le \int_0^t \beta_F(t,s)\, ds$ easily entails that $I_h(\cdot) \in L^2_E(I)$, and clearly $e_h(\cdot,x)$ is Lebesgue measurable for each $x \in \mathbb{R}^e$, and for each real $r > 0$ and each $t \in I$

$$\|e_h(t,x) - e_h(t,y)\| \le \gamma_{f,r}\|x-y\| \quad \text{for all } x,y \in r\mathbb{B}_{\mathbb{R}^e}.$$

Further, since $\left\|\int_0^t b(s,h(s))\, dz_s\right\| \le MT|z|_{1-var;I}$ we have

$$\|e_h(t,x)\| \le \alpha_f(t)\|x\| + MT|z|_{1-var;I} + \beta_f(t) + \int_0^t \beta_F(t,s)\, ds \quad \text{for all } (t,x) \in I \times \mathbb{R}^e.$$

Therefore, $e_h \in \mathcal{E}$ and hence, by what precedes, the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s,h(s))\, dz_s + I_h(t) + f(t,v_h(t)) - \Lambda \frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t,v_h(t))\right), \text{ a.e. } t \in I \end{cases}$$

admits a unique absolutely continuous solution v_h and this solution satisfies the inequality $\|\dot{v}_h(t)\| \le \mu(t)$ a.e. $t \in I$. This ensures that $v_h \in \mathcal{X}$ by definition of $\mathcal{X}$, so we can consider the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ with $\Phi(h) = v_h$ for every $h \in \mathcal{X}$.

Let us show that Φ is continuous with respect to the topology induced on $\mathcal{X}$ by the topology of uniform convergence on I. For this purpose, it is sufficient to show that, if a sequence $(h_n)_n$ in $\mathcal{X}$ uniformly converges on I to $h \in \mathcal{X}$, then for v_{h_n} denoting the absolutely continuous solution of the differential inclusion

$$\begin{cases} v_{h_n}(0) = u_0 \\ \int_0^t b(s, h_n(s))dz_s + I_{h_n}(t) + f(t, v_{h_n}(t)) - \Lambda \frac{dv_{h_n}}{dt}(t) \in \partial\varphi\big(t, \frac{dv_{h_n}}{dt}(t) + g(t, v_{h_n}(t))\big), \end{cases}$$

the sequence $(v_{h_n})_n$ uniformly converges on I to the absolutely continuous solution v_h of the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s, h(s))dz_s + I_h(t) + f(t, v_h(t)) - \Lambda \frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t, v_h(t))\right), \text{ a.e. } t \in I. \end{cases}$$

The inequality $\|\dot{v}_{h_n}(t)\| \leq \mu(t)$ a.e. $t \in I$ yields that $(v_{h_n})_n$ is equi-absolutely continuous, and these features combined with the equality $v_{h_n}(0) = u_0$ allow us by the Ascoli-Arzela theorem to suppose that $(v_{h_n})_n$ uniformly converges on I to an absolutely continuous mapping u on I. Since $v_{h_n}(t) = u_0 + \int_0^t \frac{dv_{h_n}}{ds}(s)\, ds$ for all $t \in I$ and since $\|\dot{v}_{h_n}(t)\| \leq \mu(t)$ for a.e. $t \in I$, we may also suppose that $(\frac{dv_{h_n}}{dt})_n$ weakly converges in $L^1_{\mathbb{R}^e}(I)$ to some $w \in L^1_{\mathbb{R}^e}(I)$ with $\|w(t)\| \leq \mu(t)$ for all $t \in I$, so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s)\, ds \quad \text{for every } t \in I.$$

Identifying the limits yields for every $t \in I$

$$u(t) = u_0 + \int_0^t w(s)\, ds,$$

so $\dot{u} = w$. Furthermore, it is readily seen that the sequence $(t \mapsto \int_0^t b(s, h_n(s))\, dz_s)_n$ uniformly converges on I to $t \mapsto \int_0^t b(s, h(s))\, dz_s$ and the sequences $(f(\cdot, v_{h_n}(\cdot)))_n$ and $(g(\cdot, v_{h_n}(\cdot)))_n$ converge both uniformly on I and strongly in $L^2_{\mathbb{R}^e}(I)$ to $f(\cdot, v(\cdot))$ and $g(\cdot, v(\cdot))$ by the boundedness of $(v_{h_n})_n$ and by the growth conditions of f and g. On the other hand, noticing that

$$F(t, s, h_n(s)) \to F(t, s, h(s)) \text{ and } \|F(t, s, h_n(s))\| \leq \beta_F(t, s) \text{ with } \beta_F \in L^2_{\mathbb{R}_+}(I \times I),$$

the dominated convergence theorem guarantees that

$$\int\int_{I \times I} \|F(t, s, h_n(s)) - F(t, s, h(s))\|^2\, ds dt \to 0,$$

so writing by the Hölder inequality

$$\begin{aligned} \int_0^T \|I_{h_n}(t) - I_h(t)\|^2\, dt &= \int_0^T \left\| \int_0^t (F(t, s, h_n(s)) - F(t, s, h(s)))\, ds \right\|^2 dt \\ &\leq T \int_0^T \left(\int_0^t \|F(t, s, h_n(s)) - F(t, s, h(s))\|^2\, ds \right) dt \\ &\leq T \int\int_{I \times I} \|F(t, s, h_n(s)) - F(t, s, h(s))\|^2\, ds dt, \end{aligned}$$

we deduce that $(I_{h_n}(\cdot))_n$ strongly converges in $L^2_E(I)$ to I_h.

Therefore, by Lemma 4.7 we obtain

$$\int_0^t b(s,h(s))dz_s + I_h(t) + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in \partial\varphi\Big(t, \frac{du}{dt}(t) + g(t,u(t))\Big) \text{ a.e. } t \in I$$

with $u(0) = u_0$, so that by uniqueness we obtain $u = v_h$. This confirms that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous with respect to the topology induced by the topology of uniform convergence on I. This continuity of $\Phi : \mathcal{X} \to \mathcal{X}$ on the compact convex set $\mathcal{X}$ of $C(I, \mathbb{R}^e)$ tells us that Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This implies

$$h(t) = \Phi(h)(t) = v_h(t) \text{ for all } t \in I,$$

hence

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s, v_h(s))dz_s + I_h(t) + f(t, v_h(t)) - \Lambda\frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t, v_h(t))\right), \text{ a.e. } t \in I. \end{cases}$$

So, putting $u(\cdot) = v_h(\cdot)$ we conclude that $u(\cdot)$ solves the evolution variational inequality with rough signal and Volterra integral

$$\begin{cases} u(0) = u_0 \\ \int_0^t b(s, u(s))dz_s + \int_0^t F(t,s,u(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in \partial\varphi\left(t, \frac{du}{dt}(t) + g(t,u(t))\right), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$. The proof of the theorem is complete. $\square$

Taking, in Theorem 4.8 above, $\varphi(t,x) = \delta_{C(t)}(x)$ as the indicator function of $C(t)$ directly yields the following corollary.

Corollary 4.9. *Let $I := [0,T]$ and $C : I \rightrightarrows \mathbb{R}^e$ be a closed convex valued Lebesgue multimapping which has a selection $\zeta \in L^2_{\mathbb{R}^e}(I)$, i.e. $\zeta(t) \in C(t)$ for all $t \in I$. Let $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in \mathbb{R}^e$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^2_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times \mathbb{R}^e$*

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x - y\|, \ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x - y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping such that $F(\cdot,\cdot,x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for each $x \in \mathbb{R}^e$ and such that

(i) *for some non-negative $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable function $\beta_F : I \times I \to \mathbb{R}_+$ with $(t,s) \mapsto (\beta_F(t,s))^2$ Lebesgue integrable on $I \times I$*

$$\|F(t,s,x)\| \le \beta_F(t,s) \quad \textit{for all}\, t, s \in I \textit{ and } x \in \mathbb{R}^e,$$

(ii) *for each $(t,s) \in I \times I$ the mapping $F(t,s,\cdot)$ is continuous on $\mathbb{R}^e$.*

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator.

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ *and let* $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ *be a continuous integrand operator satisfying the conditions*

$(\mathcal{B}_1)$ $\quad |b(t,x)| \le M$ *for all* $x \in \mathbb{R}^e$,

$(\mathcal{B}_2)$ $\quad |b(s,x) - b(t,x)| \le \omega(s,t)$ *for all* $x \in \mathbb{R}^e$ *and* $s, t \in I$ *with* $s \le t$,

$(\mathcal{B}_3)$ $\quad |b(t,x) - b(t,y)| \le M\|x-y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$,

where ω *is a control function on* Δ_T *and* M *is a positive constant.*

Then for any $u_0 \in \mathbb{R}^e$, *there exists an absolutely continuous solution* $u : I \to \mathbb{R}^e$ *of the evolution variational sweeping process with rough signal and Volterra integral*

$$\begin{cases} u(0) = u_0 \\ \int_0^t b(s,u(s))\,dz_s + \int_0^t F(t,s,u(s))\,ds + f(t,u(t)) - \Lambda \frac{du}{dt}(t) \in N_{C(t)}\left(\frac{du}{dt}(t) + g(t,u(t))\right), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$.

4.5. Integro-differential evolution inclusions under rough signals and time-dependent maximal monotone operators evaluated at velocity arguments

We begin with a fundamental convergence lemma.

Lemma 4.10. *Let* $I := [0,T]$ *and let* $(u_n, u)_{n\in\mathbb{N}}$ *be a sequence of absolutely continuous mappings from* I *into the separable Hilbert space* E,

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s), \qquad u(t) = u_0 + \int_0^t \dot{u}(s)ds,$$

(so $u_n(0) = u(0) = u_0$*) such that* $(\dot{u}_n)_n$ *converges to* $\dot{u}$ *weakly in* $L^p_E(I)$ *with* $p \in [1, +\infty[$. *For each* $t \in I$ *let* $A_t : E \rightrightarrows E$ *be a time-dependent maximal monotone operator such that the multimapping* $(t,x) \mapsto A_t(x)$ *is* $\mathcal{L}(I) \otimes \mathcal{B}(E)$*-measurable, and for which there is a Lebesgue measurable selection* (ζ, ξ) *of the multimapping* $t \mapsto \operatorname{gph} A_t$ *with both* $\zeta(\cdot)$ *and* $\xi(\cdot)$ *in* $L^q_E(I)$ *for* $q := \max\{p, p'\}$ *and* $1/p + 1/p' = 1$. *Let* $\Lambda : E \to E$ *be a positive symmetric linear operator (hence continuous).*

Let $(f_n, f)_{n\in\mathbb{N}}$ *be a sequence in* $L^{p'}_E(I)$ *converging strongly in* $L^{p'}_E(I)$ *to* f *and* $(g_n)_n$ *be a sequence in* $L^q_E(I)$ *converging strongly in* $L^q_E(I)$ *to* g, *where as above* $q := \max\{p, p'\}$.

Assume that for every $n \in \mathbb{N}$

$$f_n(t) - \Lambda \frac{du_n}{dt}(t) \in A_t\left(\frac{du_n}{dt}(t) + g_n(t)\right) \text{ a.e. } t \in I. \tag{4.18}$$

Then for a.e. $t \in I$ *one has* $\quad f(t) - \Lambda \dfrac{du}{dt}(t) \in A_t\left(\dfrac{du}{dt}(t) + g(t)\right)$.

Proof. First, note that $(\Lambda\dot{u}_n)_n$ converges weakly in $L^p_E(I)$ to $\Lambda\dot{u}$. Define for each $m \in \mathbb{N}$ the nonempty-valued multimapping $M_m :\rightrightarrows E \times E$ by

$$M_m(t) := \{(\zeta(t), \xi(t))\} \bigcup \left((\operatorname{gph} A^0_t) \cap (m\mathbb{B}_E \times m\mathbb{B}_E)\right) \quad \text{for all } t \in I.$$

Noticing that the function $(t,x) \mapsto d(0, A_t(x))$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable (according to the $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurability of $(t,x) \mapsto A_t(x)$), we see that the set

$$\operatorname{gph} M_m = \big(\operatorname{gph}(\zeta(\cdot), \xi(\cdot))\big) \bigcup \left\{ (t,x,y) : \begin{array}{c} \|x\| \le m,\ \|y\| \le m, \\ y \in A(t,x),\ \|y\| = d(0, A_t(x)) \end{array} \right\}$$

belongs to $\mathcal{L}(I) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E) = \mathcal{L}(I) \otimes \mathcal{B}(E \times E)$. This ensures by [35, Theorem III-22] (already recalled in a partial form in (2.3)) that there exists a sequence of Lebesgue selections $(\pi_{k,m}, \sigma_{k,m})_{k\in\mathbb{N}}$ of M_m such that $(\pi_{k,m}(t), \sigma_{k,m}(t))_k$ is dense in $M_m(t)$ for each $t \in I$. Fix for a moment (k,m) in $\mathbb{N} \times \mathbb{N}$. Take any Lebesgue measurable set Z in I. According to the assumptions on $(f_n)_n$, $(g_n)_n$ and $(\dot{u}_n)_n$, we have as $n \to \infty$

$$\int_Z \langle f_n(t), \pi_{k,m}(t) \rangle\, dt \to \int_Z \langle f(t), \pi_{k,m}(t) \rangle\, dt,$$

$$\int_Z \langle g_n(t), \sigma_{k,m}(t) \rangle\, dt \to \int_Z \langle g(t), \sigma_{k,m}(t) \rangle\, dt,$$

and also
$$\int_Z \langle \Lambda\dot{u}_n(t), \pi_{k,m}(t)) \rangle\, dt \to \int_Z \langle \Lambda\dot{u}(t), \pi_{k,m}(t)) \rangle\, dt,$$

$$\int_Z \langle \dot{u}_n(t), \sigma_{k,m}(t)) \rangle\, dt \to \int_Z \langle \dot{u}(t), \sigma_{k,m}(t)) \rangle\, dt.$$

Clearly, the strong convergence of $(f_n)_n$ and $(g_n)_n$ in $L^{p'}_E(I)$ to f and g respectively along with the weak convergence of $(\dot{u}_n)_n$ and $(\Lambda\dot{u}_n)_n$ in $L^p_E(I)$ to $\dot{u}$ and $\Lambda\dot{u}$ respectively, assures us as $n \to \infty$ that

$$\int_Z \langle \dot{u}_n(t), f_n(t) \rangle\, dt \to \int_Z \langle \dot{u}(t), f(t) \rangle\, dt,$$

$$\int_Z \langle \Lambda\dot{u}_n(t), g_n(t) \rangle\, dt \to \int_Z \langle \Lambda\dot{u}(t), g(t) \rangle\, dt.$$

Further, since $(g_n)_n$ strongly converges in $L^p_E(I)$ to g it also holds that

$$\int_Z \langle f_n(t), g_n(t) \rangle\, dt \to \int_Z \langle f(t), g(t) \rangle\, dt.$$

Now define $\psi_\Lambda : I \times E \to \mathbb{R} \cup \{+\infty\}$ by

$$\psi_\Lambda(t,x) = \langle \Lambda x, x \rangle \quad \text{for all } (t,x) \in I \times E.$$

This function ψ_Λ is a non-negative convex normal integrand by the positivity of the linear operator Λ, so the lower semicontinuity property in [33, Theorem 8.1.6] gives

$$\liminf_n \int_Z \psi_\Lambda(\dot{u}_n(t))\, dt \ge \int_Z \psi_\Lambda(\dot{u}(t))\, dt,$$

that is,
$$\liminf_n \int_Z \langle \Lambda\dot{u}_n(t), \dot{u}_n(t) \rangle\, dt \ge \int_Z \langle \Lambda\dot{u}(t), \dot{u}(t) \rangle\, dt. \tag{4.19}$$

By the differential inclusion assumption and by monotonicity of A_t, for each $n \in \mathbb{N}$ we can write for a.e. $t \in I$

$$\langle \Lambda\dot{u}_n(t) - f_n(t), \dot{u}_n(t) + g_n(t) - \pi_{k,m}(t) \rangle \le \langle \sigma_{k,m}(t), \pi_{k,m}(t) - \dot{u}_n(t) - g_n(t) \rangle\,.$$

By integration of this inequality on Z we obtain for each $n \in \mathbb{N}$

$$\int_Z [\langle \Lambda \dot{u}_n(t) - f_n(t), \dot{u}_n(t) + g_n(t) - \pi_{k,m}(t)\rangle]\, dt \\ \leq \int_Z [\langle \sigma_{k,m}(t)), \pi_{k,m}(t) - \dot{u}_n(t) - g_n(t)\rangle]\, dt,$$

which gives as $n \to \infty$ (by (4.19) and the convergences preceding it)

$$\int_Z [\langle \Lambda \dot{u}(t) - f(t), \dot{u}(t) + g(t) - \pi_{k,m}(t)\rangle]\, dt \leq \int_Z [\langle \sigma_{k,m}(t), \pi_{k,m}(t) - \dot{u}(t) - g(t)\rangle]\, dt.$$

This being true for every Lebesgue measurable subset Z of I, there exists a Lebesgue measurable null subset $N_{k,m}$ in I such that fo every $t \in I \setminus N_{k,m}$

$$\langle \Lambda \dot{u}(t) - f(t), \dot{u}(t) + g(t) - \pi_{k,m}(t)\rangle \leq \langle \sigma_{k,m}(t), \pi_{k,m}(t) - \dot{u}(t) - g(t)\rangle\,. \tag{4.20}$$

On the other hand, we know that a subsequence of $(\dot{u}_n + g_n)_n$ Komlos converges to $\dot{u} + g$ (according to its weak convergence in $L^p_E(I)$), so there exist an increasing function $s : \mathbb{N} \to \mathbb{N}$ and a Lebesgue negligible set $N \subset I$ such that for every $t \in I \setminus N$ (by convexity of $\overline{D(A_t)}$, as recalled in Lemma 3.11(a))

$$\overline{D(A_t)} \ni \frac{1}{n}\sum_{j=1}^{n} \left(\dot{u}_{s(n)}(t) + g_{s(n)}(t)\right) \to \dot{u}(t) + g(t),$$

hence we have $\dot{u}(t) + g(t) \in \overline{D(A_t)}$. Now put $N' := \bigcup_{(k,m)\in\mathbb{N}\times\mathbb{N}} N_{k,m}$ and fix any $t \in P := I \setminus (N \cup N')$. Take any $x \in D(A_t)$ and choose an integer $m \in \mathbb{N}$ satisfying

$$m \geq \max\{\|x\|, \|A^0_t(x)\|, \|\zeta(t)\|, \|\xi(t)\|\},$$

so $(x, A^0_t(x)) \in M_m(t)$. By the density of $(\pi_{k,m}(t), \sigma_{k,m}(t))_{k\in\mathbb{N}}$ in $M_m(t)$ there is a subsequence $(\pi_{k,m}(t), \sigma_{k,m}(t))_{k\in K}$ with an infinite subset $K \subset \mathbb{N}$ such that

$$\lim_{K\ni k\to\infty} (\pi_{k,m}(t), \sigma_{k,m}(t)) = (x, A^0_t(x)).$$

From this and from (4.20) we derive

$$\left\langle \Lambda \frac{du}{dt}(t) - f(t), \frac{du}{dt}(t) + g(t) - x \right\rangle \leq \left\langle A^0_t(x), x - \frac{du}{dt}(t) - g(t) \right\rangle.$$

Since this is true for all $x \in D(A_t)$ and since $\dot{u}(t) + g(t) \in \overline{D(A_t)}$, Lemma 3.11 assures us that

$$f(t) - \Lambda \frac{du}{dt}(t) \in A_t\left(\frac{du}{dt}(t) + g(t)\right),$$

which finishes the proof. □

In complement to Theorem 4.8 and with the help of the above lemma, we pass to the case concerning the situation of differential inclusions involving $A_t\left(\frac{du}{dt}(t) + g(t, u(t))\right)$ with maximal monotone operators A_t in place of $\partial\varphi_t\left(\frac{du}{di}(t) + g(t, u(t))\right)$.

Theorem 4.11. *Let $I := [0,T]$ and let $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in \mathbb{R}^e$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^2_{\mathbb{R}_+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}_+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times \mathbb{R}^e$*

$$\|f(t,x)\| \leq \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \leq \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \leq \gamma_{f,r}(t)\|x - y\|, \ \|g(t,x) - g(t,y)\| \leq \gamma_{g,r}(t)\|x - y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping such that $F(\cdot, \cdot, x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for each $x \in \mathbb{R}^e$ and such that

(i) *for some non-negative $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable function $\beta_F : I \times I \to \mathbb{R}_+$ with $(t,s) \mapsto (\beta_F(t,s))^2$ Lebesgue integrable on $I \times I$*

$$\|F(t,s,x)\| \leq \beta_F(t,s) \quad \textit{for all}\, t,s \in I \textit{ and } x \in \mathbb{R}^e,$$

(ii) *for each $(t,s) \in I \times I$ the mapping $F(t,s,\cdot)$ is continuous on $\mathbb{R}^e$.*

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator. For each $t \in I$ let $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ be a maximal monotone operator such that the multimapping $(t,x) \mapsto A_t(x)$ from $I \times \mathbb{R}^e$ into $\mathbb{R}^e$ is Lebesgue measurable and there are $\pi(\cdot) \in L^2_{\mathbb{R}^e}(I)$ and $\sigma(\cdot) \in L^2_{\mathbb{R}^e}(I)$ with $\sigma(t) \in A_t(\pi(t))$ for all $t \in I$. Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

($\mathcal{B}_1$) $|b(t,x)| \leq M$ *for all $x \in \mathbb{R}^e$,*

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \leq \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s, t \in I$ with $s \leq t$,*

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \leq M\|x - y\|$ *for all $t \in I$ and $x, y \in \mathbb{R}^e$, where ω is a control function on Δ_T and M is a positive constant.*

Then for any $u_0 \in \mathbb{R}^e$, there exists an absolutely continuous solution $u : I \to \mathbb{R}^e$ of the integro-differential evolution inclusion with rough signal and Volterra integral

$$\begin{cases} u(0) = u_0 \\ \int_0^t b(s,u(s))\, dz_s + \int_0^t F(t,s,u(s))\, ds + f(t,u(t)) - \Lambda \frac{du}{dt}(t) \in A_t \left(\frac{du}{dt}(t) + g(t,u(t)) \right), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$.

Proof. First, it can be easily seen that the function $t \mapsto \int_0^t \beta_F(t,s)\, ds$ is in $L^2_{\mathbb{R}_+}(I)$. We continue by employing arguments similar to those for Theorem 4.8. Define $\mathcal{E}$ as the set of mappings $e : I \times \mathbb{R}^e \to \mathbb{R}^e$ Lebesgue measurable in t and such that

$$\|e(t,x)\| \leq \alpha_f(t)\|x\| + MT|z|_{1-var;I} + \beta_f(t) + \int_0^t \beta_F(t,s)\, ds \tag{4.21}$$

for all $(t,x) \in I \times \mathbb{R}^e$ and such that for each real $r > 0$

$$\|e(t,x) - e(t,y)\| \leq \gamma_{f,r}(t)\|x - y\| \quad \text{for all}\, t \in I, x, y \in r\mathbb{B}_{\mathbb{R}^e}. \tag{4.22}$$

By Theorem 3.16 there is a non-negative function $\mu(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for every mapping $e \in \mathcal{E}$ the solution $\zeta(\cdot)$ of the differential inclusion

$$\begin{cases} e(t, \zeta(t)) - \Lambda\dot{\zeta}(t) \in A_t\left(\dot{\zeta}(t) + g(t, \zeta(t))\right) & \text{a.e. } t \in I \\ \zeta(0) = u_0 \end{cases}$$

satisfies $\|\dot{\zeta}(t)\| \leq \mu(t)$ a.e. $t \in I$. Put $\Gamma(t) := \mu(t)\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$. Consider the subset $\mathcal{X}$ in the Banach space $C(I, \mathbb{R}^e)$ defined by

$$\mathcal{X} := \{q_\ell : \ell \in S_\Gamma\}$$

where q_ℓ is the mapping from I into $\mathbb{R}^e$ given by

$$q_\ell(t) := u_0 + \int_0^t \ell(s)\, ds$$

and where we recall that S_Γ stands for the set of Lebesgue almost everywhere selections of the multimapping Γ. We note at first that $\mathcal{X}$ is compact convex and equi-absolutely continuous on I since for any $\ell \in S_\Gamma$

$$\|q_\ell(t) - q_\ell(s)\| \leq \int_s^t \mu(\tau)\, d\tau \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

Then for any $h \in \mathcal{X}$ and any $s, t \in I$ with $s \leq t$, using $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ we can write

$$\begin{aligned} |b(t, h(t)) - b(s, h(s))| &\leq |b(t, h(t)) - b(s, h(t))| + |b(s, h(t)) - b(s, h(s))| \\ &\leq \omega(s, t) + M\int_s^t \mu(\tau)d\tau \end{aligned}$$

so that $b(\cdot, h(\cdot)) \in C^{1-var}(I, \mathcal{L}(\mathbb{R}^e, \mathbb{R}^e))$ with $|b(t, h(t))| \leq M$ for all $t \in I$ by $(\mathcal{B}_1)$. In particular, for any $h \in \mathcal{X}$ and any $t \in I$ the Young integral $\int_0^t b(s, h(s))\, dz_s$ is well defined and $b(\cdot, h(\cdot))$ is bounded in variation uniformly with respect to $h \in \mathcal{X}$. Fix for a moment $h \in \mathcal{X}$ and define $I_h : I \to \mathbb{R}^e$ by $I_h(t) := \int_0^t F(t, s, h(s))\, ds$ and $e_h : I \times \mathbb{R}^e \to \mathbb{R}^e$ by

$$e_h(t, x) = \int_0^t b(s, h(s))\, dz_s + I_h(t) + f(t, x) \quad \text{for all } (t, x) \in I \times \mathbb{R}^e.$$

Clearly, $e_h(\cdot, x)$ is Lebesgue measurable for each $x \in \mathbb{R}^e$, and for each real $r > 0$ and each $t \in I$

$$\|e_h(t, x) - e_h(t, y)\| \leq \gamma_{f,r}\|x - y\| \quad \text{for all } x, y \in r\mathbb{B}_{\mathbb{R}^e}.$$

Further, since $\left\|\int_0^t b(s, h(s))\, dz_s\right\| \leq MT|z|_{1-var;I}$ we have

$$\|e_h(t, x)\| \leq \alpha_f(t)\|x\| + MT|z|_{1-var;I} + \beta_f(t) + \int_0^t \beta_F(s, t)\, ds \quad \text{for all } (t, x) \in I \times \mathbb{R}^e.$$

Therefore, $e_h \in \mathcal{E}$ and hence, by what precedes, the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s,h(s))\,dz_s + I_h(t) + f(t,v_h(t)) - \Lambda\dot{v}_h(t) \in A_t\big(\dot{v}_h(t) + g(t,v_h(t))\big), \text{ a.e. } t \in I \end{cases}$$

admits a unique absolutely continuous solution v_h which satisfies $\|\dot{v}_h(t)\| \le \mu(t)$ a.e. $t \in I$. This ensures that $v_h \in \mathcal{X}$ by definition of $\mathcal{X}$, so we can consider the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ with $\Phi(h) = v_h$ for every $h \in \mathcal{X}$. Let us show that Φ is continuous with respect to the topology induced on $\mathcal{X}$ by the topology of uniform convergence on I. For this purpose, it is sufficient to show that, if a sequence $(h_n)_n$ in $\mathcal{X}$ uniformly converges on I to $h \in \mathcal{X}$, then for v_{h_n} denoting the absolutely continuous solution of the differential inclusion

$$\begin{cases} v_{h_n}(0) = u_0 \\ \int_0^t b(s,h_n(s))dz_s + I_h(t) + f(t,v_{h_n}(t)) - \Lambda\dot{v}_{h_n}(t) \in A_t\big(\dot{v}_{h_n}(t) + g(t,v_{h_n}(t))\big), \text{ a.e. } t \in I, \end{cases}$$

the sequence $(v_{h_n})_n$ uniformly converges on I to the absolutely continuous solution v_h of the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s,h(s))\,dz_s + I_h(t) + f(t,v_h(t)) - \Lambda\dot{v}_h(t) \in A_t\left(t, \dot{v}_h(t) + g(t,v_h(t))\right), \text{ a.e. } t \in I. \end{cases}$$

Since $(v_{h_n})_n$ is equi-absolutely continuous according to the inequality $\|\dot{v}_{h_n}(t)\| \le \mu(t)$ a.e. $t \in I$, we may and do suppose that $(v_{h_n})_n$ uniformly converges on I to an absolutely continuous mapping v on I. Since $v_{h_n}(t) = u_0 + \int_0^t \dot{v}_{h_n}(s)\,ds$ for all $t \in I$ and since $\|\dot{v}_{h_n}(t)\| \le \mu(t)$ for a.e. $t \in I$, we may also suppose that $(\dot{v}_{h_n})_n$ weakly converges in $L^2_{\mathbb{R}^e}(I)$ to some $w \in L^2_{\mathbb{R}^e}(I)$ with $\|w(t)\| \le \mu(t)$ for all $t \in I$, so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s)\,ds \quad \text{for every } t \in I.$$

Identifying the limits yields for every $t \in I$

$$v(t) = u_0 + \int_0^t w(s)\,ds,$$

so $\dot{v} = w$. Furthermore, it is readily seen that the sequence $(t \mapsto \int_0^t b(s,h_n(s))\,dz_s)_n$ uniformly converges on I to $t \mapsto \int_0^t b(s,h(s))\,dz_s$ and the sequences $\big(f(\cdot,v_{h_n}(\cdot))\big)_n$ and $\big(g(\cdot,v_{h_n}(\cdot))\big)_n$ converge both uniformly on I and strongly in $L^2_{\mathbb{R}^e}(I)$ to $f(\cdot,v(\cdot))$ and $g(\cdot,v(\cdot))$ by the boundedness of $(v_{h_n})_n$ and by the growth conditions of f and g. On the other hand, as in Theorem 4.8 we can see that the sequence $(I_{h_n}(\cdot))_n$ strongly converges in $L^2_E(I)$ to $I_h(\cdot)$. Therefore, by Lemma 4.10 we obtain

$$\int_0^t b(s,h(s))dz_s + I_h(t) + f(t,u(t)) - \Lambda\dot{u}(t) \in A_t\big(\dot{u}(t) + g(t,u(t))\big) \text{ a.e. } t \in I$$

with $u(0) = u_0$, so that by uniqueness $u = v_h$. This confirms that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous with respect to the topology induced by the topology of uniform convergence on I.

This continuity of $\Phi : \mathcal{X} \to \mathcal{X}$ on the compact convex set $\mathcal{X}$ of $C(I, \mathbb{R}^e)$ tells us that Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = v_h(t) \quad \text{for all } t \in I \text{ with}$$

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s, v_h(s))dz_s + I_h(t) + f(t, v_h(t)) - \Lambda \dot{v}_h(t) \in A_t\big(\dot{v}_h(t), g(t, v_h(t))\big), \text{ a.e. } t \in I. \end{cases}$$

So, putting $u(\cdot) = v_h(\cdot)$ we conclude that $u(\cdot)$ solves the integro-differential evolution variational inequality with rough signal and Volterra integral

$$\begin{cases} u(0) = u_0 \\ \int_0^t b(s, u(s))dz_s + \int_0^t F(t, s, u(s))\, ds + f(t, u(t)) - \Lambda \dot{u}(t) \in A_t\left(t, \dot{u}(t) + g(t, u(t))\right), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$. The proof of the theorem is finished. $\square$

5. Differential evolution inclusions coupled with rough signals, Volterra-type integrals and fractional differential equations related to Riemann-Liouville fractional derivatives

In this section we are interested in fractional order boundary problems involving evolution differential inclusions associated with rough signals and Volterra integral perturbations.

We will start with such above problems with operators evaluated at velocities and with perturbations involving the Riemann-Liouville fractional derivative. Otherwise stated, we will consider in the setting of the separable Hilbert space E problems of the following type:

$$D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), \ t \in [0,1], \tag{5.1}$$

$$I_{0^+}^{\beta} h(t)\,|_{t=0} := \lim_{t\to 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)\, ds = 0, \tag{5.2}$$

$$h(1) = I_{0^+}^{\gamma} h(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)\, ds, \tag{5.3}$$

$$-\frac{du}{dt}(t) \in A_t\left(\frac{du}{dt}(t) + g(t, u(t))\right) + \int_0^t f(t, s, u(s), h(s))\, ds \ \text{ a.e. } t \in [0,1], \tag{5.4}$$

where $\alpha \in]1, 2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0, \gamma > 0$ are given constants, D^α is the standard *Riemann-Liouville fractional derivative*, Γ is the standard Gamma function and $f : [0,1] \times [0,1] \times E \times E \to E$ is a single-valued mapping, and $A_t : D(A_t) \subset E \rightrightarrows E$ is a maximal monotone operator for each $t \in [0,1]$.

Definition 5.1. Let f be a mapping from $[0,1]$ into the separable Hilbert space E and let a real $\alpha > 0$. The *fractional Bochner-integral of order* α of the mapping f is defined by

$$I_{\theta^+}^{\alpha} f(t) := \int_\theta^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds$$

for any $0 \leq \theta < t \leq 1$ for which the Lebesgue-Bochner integral exists in E. $\square$

When $f \in L^1_E([0,1])$ one knows by [65]:

- if $0 < \alpha < 1$, then $I_{0^+} f(t)$ exists in E for almost every $t \in [0,1]$ and $I_{0^+} f \in L^1_E([0,1])$
- if $\alpha \geq 1$, then $I_{0^+} f(t)$ exists in E for all $t \in [0,1]$ and $I_{0^+} f \in \mathcal{C}_E([0,1])$.

Definition 5.2. Let $\alpha > 0$ and $n := [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. For $f \in L^1_E([0,1])$, its *Riemann-Liouville fractional derivative of order* α at $t \in [0,1]$ is given by

$$D^\alpha f(t) := D^\alpha_{0^+} f(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{0^+} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s)\, ds$$

whenever the latter exists. □

We refer to [50], [54], [66], for the general theory of Fractional Calculus and Fractional Differential Equations.

Following standard notation, $W^{\alpha,1}_{B,E}([0,1])$ denotes the space of all continuous mappings from $[0,1]$ into the separable Hilbert space E such that their Riemann-Liouville fractional derivatives of order $\alpha - 1$ are continuous and their Riemann-Liouville fractional derivatives of order α are Bochner integrable.

We will have to use some elementary features from [24] that we recall in Lemma 5.3 and Theorem 5.4 below.

Green function and its properties. Let us consider

$$\alpha \in]1,2],\ \beta \in [0, 2-\alpha],\ \lambda \geq 0,\ \gamma > 0. \tag{5.5}$$

Define the *Green function* $G : [0,1] \times [0,1] \to \mathbb{R}$ by

$$G(t,s) = \varphi(s) I^{\alpha-1}_{0^+}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I^{\alpha-1}_{s^+}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} \tag{5.6}$$

where $\quad \varphi(s) = \dfrac{\exp(\lambda s)}{\mu_0} \left[\left(I^{\alpha-1+\gamma}_{s^+}(\exp(-\lambda t)) \right)(1) - \left(I^{\alpha-1}_{s^+}(\exp(-\lambda t)) \right)(1) \right]$

with $\quad \mu_0 = \left(I^{\alpha-1}_{0^+}(\exp(-\lambda t)) \right)(1) - \left(I^{\alpha-1+\gamma}_{0^+}(\exp(-\lambda t)) \right)(1).$

Lemma 5.3. *Let E be a separable Hilbert space. Let $\alpha, \beta, \lambda, \gamma$ be as in* (5.5) *and let G be the Green function defined as in* (5.6)*. The following hold.*

(a) *$G(\cdot,\cdot)$ satisfies the estimate*

$$|G(t,s)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1 + \Gamma(\gamma+1)}{|\mu_0| \Gamma(\alpha) \Gamma(\gamma+1)} + 1 \right) =: M_G.$$

(b) *If $u \in W^{\alpha,1}_{B,E}([0,1])$ satisfies boundary conditions* (5.1), (5.2), (5.3) *above, then*

$$u(t) = \int_0^1 G(t,s) \left(D^\alpha u(s) + \lambda D^{\alpha-1} u(s) \right) ds \quad \text{for every } t \in [0,1].$$

(c) *For any* $f \in L^1_E([0,1])$ *and for* $u_f : [0,1] \to H$ *defined by*

$$u_f(t) := \int_0^1 G(t,s) f(s) ds \quad \text{for} \quad t \in [0,1],$$

one has $I^\beta_{0^+} u_f(t)\,|_{t=0} = 0$ *and* $u_f(1) = \left(I^\gamma_{0^+} u_f\right)(1)$.

Moreover, one also has $u_f \in W^{\alpha,1}_{B,E}([0,1])$ *and*

$$\left(D^\alpha u_f\right)(t) + \lambda \left(D^{\alpha-1} u_f\right)(t) = f(t) \quad \text{for all} \quad t \in [0,1].$$

The following theorem characterizes the topological structure of the solution set.

Theorem 5.4. *Let* E *be a separable Hilbert space and Let* $\alpha, \beta, \lambda, \gamma$ *be as in* (5.5). *Let* $X : [0,1] \rightrightarrows E$ *be a measurable multimapping with nonempty compact convex values in* E *such that* $X(t) \subset \ell \mathbb{B}_E$ *for all* $t \in [0,1]$, *where* ℓ *is a positive real constant, and let* S_X *be (as above) the set of all measurable selections of* X. *Then the set of* $W^{\alpha,1}_{B,E}([0,1])$*-solutions of the problems for all* $f \in S_X$

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t), \text{ a.e. } t \in [0,1] \\ I^\beta_{0^+} u(t)\,|_{t=0} = 0, \quad u(1) = I^\gamma_{0^+} u(1) \end{cases} \tag{5.7}$$

is equi-Lipschitz, convex and compact in $\mathcal{C}_E([0,1])$ *relative to the topology of uniform convergence on* $[0,1]$.

We can now state and prove the existence result for the fractional differential inclusion which we are interested in.

Theorem 5.5. *Let* $\alpha, \beta, \lambda, \gamma$ *be as in* (5.5), $I := [0,1]$ *and* $\varphi : I \times \mathbb{R}^e \to \mathbb{R} \cup \{+\infty\}$ *be a normal convex integrand for which there are a mapping* $\zeta \in L^2_{\mathbb{R}^e}(I)$ *such that* $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ *is in* $L^2_{\mathbb{R}_+}(I)$ *and two non-negative functions* $\alpha_\varphi, \beta_\varphi : I \to \mathbb{R}$ *with* $\alpha_\varphi, \sqrt{\beta_\varphi} \in L^2_{\mathbb{R}_+}(I)$ *such that*

$$-\alpha_\varphi(t)\|x\| - \beta_\varphi(t) \le \varphi(t,x) \quad \text{for all } (t,x) \in I \times \mathbb{R}^e$$

(*note that* $\sqrt{\beta_\varphi} \in L^2_{\mathbb{R}_+}(I)$ *since* $\beta_\varphi \in L^2_{\mathbb{R}_+}(I)$). *Let* $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ *be two mappings such that* $f(\cdot, x)$ *and* $g(\cdot, x)$ *are Lebesgue measurable for all* $x \in \mathbb{R}^e$ *and such that there are non-negative functions* α_f, α_g, β_f, β_g *in* $L^2_{\mathbb{R}+}(I)$ *along with* $\gamma_{f,r}$, $\gamma_{g,r}$ *in* $L^1_{\mathbb{R}+}(I)$ *for each real* $r > 0$ *such that for all* $(t,x) \in I \times \mathbb{R}^e$

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \text{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ *and* $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x - y\|, \; \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x - y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ *be a mapping such that* $F(\cdot,\cdot,x)$ *is* $\mathcal{L}(I) \otimes \mathcal{L}(I)$ *measurable for each* $x \in \mathbb{R}^e$ *and such that*

(i) *for some non-negative* $\mathcal{L}(I) \otimes \mathcal{L}(I)$ *measurable function* $\beta_F : I \times I \to \mathbb{R}_+$ *with* $(t,s) \mapsto (\beta_F(t,s))^2$ *Lebesgue integrable on* $I \times I$

$$\|F(t,s,x)\| \le \beta_F(t,s) \quad \text{for all } t, s \in I \text{ and } x \in \mathbb{R}^e,$$

(ii) *for each* $(t,s) \in I \times I$ *the mapping* $F(t,s,\cdot)$ *is continuous on* $\mathbb{R}^e$.

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator.
Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

$(\mathcal{B}_1)$ $\quad |b(t,x)| \le M$ *for all* $x \in \mathbb{R}^e$,

$(\mathcal{B}_2)$ $\quad |b(s,x) - b(t,x)| \le \omega(s,t)$ *for all* $x \in \mathbb{R}^e$ *and* $s, t \in I$ *with* $s \le t$,

$(\mathcal{B}_3)$ $\quad |b(t,x) - b(t,y)| \le M\|x - y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$,

where ω is a control function on I and M is a positive constant.

Then for any $u_0 \in \mathbb{R}^e$, there exist a $W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ mapping $\xi : I \to \mathbb{R}^e$ and an absolutely continuous mapping $u : I \to \mathbb{R}^e$ satisfying the fractional evolutional integro-differential inequality with rough signal

$$\begin{cases} D^\alpha \xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t), \text{ a.e. } t \in I \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

the inclusion being fulfilled for a.e. $t \in I$.

Proof. Consider the set $\mathcal{E}$ of mappings $e : I \times \mathbb{R}^e \to \mathbb{R}^e$ Lebesgue measurable in t and such that

$$\|e(t,x)\| \le \alpha_f(t)\|x\| + MT|z|_{1-var;I} + \beta_f(t) + \int_0^t \beta_F(t,s)\,ds \quad \text{for all } (t,x) \in I \times \mathbb{R}^e$$

and such that for each real $r > 0$

$$\|e(t,x) - e(t,y)\| \le \gamma_{f,r}(t)\|x - y\| \quad \text{for all } t \in I, x, y \in r\mathbb{B}_{\mathbb{R}^e}.$$

By Theorem 3.1 there is a non-negative function $\mu(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for every mapping $e \in \mathcal{E}$ the solution $\zeta(\cdot)$ of the differential inclusion

$$\begin{cases} e(t,\zeta(t)) - \Lambda \frac{d\zeta}{dt}(t) \in \partial\varphi\left(t, \frac{d\zeta}{dt}(t) + g(t,\zeta(t))\right) \text{ a.e. } t \in I \\ \zeta(0) = u_0 \end{cases}$$

satisfies $\|\dot\zeta(t)\| \le \mu(t)$ a.e. $t \in I$. Put $\Delta(t) := u_0 + \left(\int_0^t \mu(s)ds\right)\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$. Recalling that $\zeta(t) = u_0 + \int_0^t \dot\zeta(s)ds$ for all $t \in I$ with $\dot\zeta(t) \in \mu(t)\mathbb{B}_{\mathbb{R}^e}$ we see that $\zeta(t) \in \Delta(t)$ for all $t \in I$. Consider the set $\mathcal{X}$ in the Banach space $C(I, \mathbb{R}^e)$ defined by

$$\mathcal{X} := \{q_\ell : \ell \in S^1_\Delta\},$$

each mapping q_ℓ being given for any $t \in I$ by $q_\ell(t) := \int_0^1 G(t,s)\ell(s)\,ds$, where G is the Green function given in (5.6) and S^1_Δ is the set of all Lebesgue integrable selections of the multimapping $\Delta(\cdot)$ from I into $\mathbb{R}^e$. We note that $\mathcal{X}$ is equi-Lipschitz, convex, compact in $\mathcal{C}_E(I)$ for the topology of uniform convergence (see Theorem 5.4 and Lemma 5.3) and that for every $h \in \mathcal{X}$

$$\|h(t) - h(s)\| \le N|t - s| \quad \text{for all } s, t \in I,$$

where N is a positive constant independent on $h \in \mathcal{X}$.

Then for any $h \in \mathcal{X}$ and any $s,t \in I$ with $s \le t$, using $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ we can write

$$\begin{aligned}|b(t,h(t)) - b(s,h(s))| &\le |b(t,h(t)) - b(s,h(t))| + |b(s,h(t)) - b(s,h(s))| \\ &\le \omega(s,t) + MN|t-s|\end{aligned}$$

so that $b(\cdot,h(\cdot)) \in C^{1-var}(I,\mathcal{L}(\mathbb{R}^e,\mathbb{R}^e))$ with $|b(t,h(t))| \le M$ for all $t \in I$ by $(\mathcal{B}_1)$. In particular, for any $h \in \mathcal{X}$ and any $t \in I$ the Young integral $\int_0^t b(s,h(s))\,dz_s$ is well defined and $b(\cdot,h(\cdot))$ is bounded in variation uniformly with respect to $h \in \mathcal{X}$.

Fix for a moment $h \in \mathcal{X}$ and define $e_h : I \times \mathbb{R}^e \to \mathbb{R}^e$ by

$$e_h(t,x) = \int_0^t b(s,h(s))\,dz_s + \int_0^t F(t,s,h(s))\,ds + f(t,x) \quad \text{for all } (t,x) \in I \times \mathbb{R}^e.$$

Clearly, $e_h(\cdot,x)$ is Lebesgue measurable for each $x \in \mathbb{R}^e$, and for each real $r > 0$ and each $t \in I$

$$\|e_h(t,x) - e(t,y)\| \le \gamma_{f,r}\|x-y\| \quad \text{for all } x,y \in r\mathbb{B}_{\mathbb{R}^e}.$$

Further, since $\| \int_0^t b(s,h(s))\,dz_s\| \le MT|z|_{1-var;I}$ we have

$$\|e_h(t,x)\| \le \alpha_f(t)\|x\| + MT|z|_{1-var;I} + \beta_f(t) + \int_0^t \beta_F(t,s)\,ds \quad \text{for all } (t,x) \in I \times \mathbb{R}^e.$$

Therefore, $e_h \in \mathcal{E}$ and hence, by what precedes, with $I_h : I \to \mathbb{R}^e$ defined by $I_h(t) := \int_0^t F(t,s,h(s))\,ds$ the differential inclusion

$$\begin{cases} v_h(0) = v_0 \\ \int_0^t b(s,h(s))\,dz_s + I_h(t) + f(t,v_h(t)) - \Lambda\frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t,v_h(t))\right), \text{ a.e. } t \in I \end{cases}$$

admits a unique absolutely continuous solution v_h which satisfies the inequality $\|\dot v_h(t)\| \le \mu(t)$ a.e. $t \in I$. This ensures that the mapping $t \mapsto \int_0^1 G(t,s)v_h(s)\,ds$ belongs to $\mathcal{X}$ by definition of $\mathcal{X}$. Then, we can consider the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ with $\Phi(h)(t) = \int_0^1 G(t,s)v_h(s)ds$ for every $h \in \mathcal{X}$ and for every $t \in I$. Let us show that Φ is continuous with respect to the topology induced on $\mathcal{X}$ by the topology of uniform convergence on I. For this purpose, consider any sequence $(h_n)_n$ in $\mathcal{X}$ converging uniformly on I to a mapping $h \in \mathcal{X}$. Denote by v_{h_n} the absolutely continuous solution of the differential inclusion

$$\begin{cases} v_{h_n}(0) = u_0 \\ \int_0^t b(s,h_n(s))dz_s + I_{h_n}(t) + f(t,v_{h_n}(t)) - \Lambda\frac{dv_{h_n}}{dt}(t) \in \partial\varphi\left(t, \frac{dv_{h_n}}{dt}(t) + g(t,v_{h_n}(t))\right), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$. We are going to show that the sequence $(v_{h_n})_n$ uniformly converges on I to the absolutely continuous solution v_h of the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ \int_0^t b(s,h(s))dz_s + I_h(t) + f(t,v_h(t)) - \Lambda\frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t,v_h(t))\right), \text{ a.e. } t \in I. \end{cases}$$

The inequality $\|\dot v_{h_n}(t)\| \le \mu(t)$, valid for a.e. $t \in I$, ensures that $(v_{h_n})_n$ is equi-absolutely continuous, and these two properties allow us to suppose that $(v_{h_n})_n$ uniformly converges on I to an absolutely continuous mapping v on I.

Since $v_{h_n}(t) = v_0 + \int_{]0,t]} \frac{dv_{h_n}}{ds}(s)\, ds$ for all $t \in I$ and since $\|\dot{v}_{h_n}(t)\| \leq \mu(t)$ a.e. $t \in I$, we may also suppose that $(\frac{dv_{h_n}}{dt})_n$ weakly converges in $L^2_{\mathbb{R}^e}(I)$ to some $w \in L^2_{\mathbb{R}^e}(I)$ with $\|w(t)\| \leq \mu(t)$ for all $t \in I$, so that

$$\lim_n v_{h_n}(t) = v_0 + \int_0^t w(s)\, ds \text{ for every } t \in I.$$

It follows that for every $t \in I$

$$v(t) = v_0 + \int_0^t w(s)\, ds,$$

which gives in particular $\dot{v} = w$. Furthermore, it is readily seen that the sequence $(t \mapsto \int_0^t b(s, h_n(s))\, dz_s)_n$ uniformly converges on I to $t \mapsto \int_0^t b(s, h(s))\, dz_s$ and that the sequences $(f(\cdot, v_{h_n}(\cdot)))_n$ and $(g(\cdot, v_{h_n}(\cdot)))_n$ uniformly converge on I to $f(\cdot, v(\cdot))$ and $g(\cdot, v(\cdot))$ respectively. Therefore, by Lemma 4.7 we obtain

$$\int_0^t b(s, h(s))\, dz_s + I_h(t) + f(t, v(t)) - \Lambda \frac{dv}{dt}(t) \in \partial\varphi\Big(t, \frac{dv}{dt}(t) + g(t, v(t))\Big) \text{ a.e. } t \in I$$

with $v(0) = v_0$, so that by uniqueness $v = v_h$. As desired, this confirms the uniform convergence on I of $(v_{h_n}(\cdot))_n$ to $v_h(\cdot)$.

Now, let us write by Lemma 5.3

$$\begin{aligned} \|\Phi(h_n)(t) - \Phi(h)(t)\| &= \left\| \int_0^1 G(t,s) v_{h_n}(s)\, ds - \int_0^1 G(t,s) v_h(s)\, ds \right\| \\ &= \left\| \int_0^1 G(t,s)[v_{h_n}(s) - v_h(s)]\, ds \right\| \\ &\leq \int_0^1 M_G \left\| v_{h_n}(s) - v_h(s) \right\|\, ds. \end{aligned}$$

Since $\|v_{h_n}(\cdot) - v_h(\cdot)\| \to 0$ uniformly on I as $n \to \infty$, we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|v_{h_n}(s) - v_h(s)\|\, ds \to 0,$$

which implies that $(\Phi(h_n))_n$ converges to $\Phi(h)$ uniformly on I. This proves the desired continuity of $\Phi : \mathcal{X} \to \mathcal{X}$ with respect to the topology induced by the topology of uniform convergence on I. This continuity of $\Phi : \mathcal{X} \to \mathcal{X}$ on the compact convex set $\mathcal{X}$ of $C(I, \mathbb{R}^e)$ tells us that Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means, according to the definition of Φ and to Lemma 5.3(c), that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s) v_h(s)) ds \text{ for all } t \in I$$

and also

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = v_h(t), \text{ a.e. } t \in I \\ I^\beta_{0^+} h(t)\,|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+} h(1) \\ v_h(0) = u_0 \\ \int_0^t b(s, h(s)) dz_s + \int_0^t F(t, s, h(s))\, ds + f(t, v_h(t)) - \Lambda \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t) + g(t, v_h(t))), \end{cases}$$

where the inclusion is satisfied for a.e. $t \in I$.

So, by putting $\xi = h$, $u = v_h$ we conclude that the pair (ξ, u) solves the fractional/evolutional inequality with rough signal and Volterra integral

$$\begin{cases} D^\alpha \xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t), \text{ a.e. } t \in [0,1] \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

the inclusion being satisfied for a.e. $t \in I$. The proof of the theorem is finished. $\square$

Sweeping processes under rough signals and coupled with fractional differential equations and Volterra-type integrals easily follow with $\varphi(t,x) = \delta_{C(t)}(x)$ in the following form.

Corollary 5.6. *Let $I := [0,1]$ and let $C : I \rightrightarrows \mathbb{R}^e$ be a closed convex valued Lebesgue multimapping which has a selection $\zeta \in L^2_{\mathbb{R}^e}(I)$, i.e. $\zeta(t) \in C(t)$ for all $t \in I$. Let $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in \mathbb{R}^e$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^e_{\mathbb{R}^+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}^+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times \mathbb{R}^e$*

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x-y\|, \ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x-y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping such that $F(\cdot,\cdot,x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for each $x \in \mathbb{R}^e$ and such that

(i) *for some non-negative $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable function $\beta_F : I \times I \to \mathbb{R}_+$ with $(t,s) \mapsto (\beta_F(t,s))^2$ Lebesgue integrable on $I \times I$*

$$\|F(t,s,x)\| \le \beta_F(t,s) \quad \textit{for all}\, t, s \in I \textit{ and } x \in \mathbb{R}^e,$$

(ii) *for each $(t,s) \in I \times I$ the mapping $F(t,s,\cdot)$ is continuous on $\mathbb{R}^e$.*

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator. Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

($\mathcal{B}_1$) $|b(t,x)| \le M$ *for all $x \in \mathbb{R}^e$,*

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \le \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s, t \in I$ with $s \le t$,*

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \le M\|x-y\|$ *for all $t \in I$ and $x, y \in \mathbb{R}^e$,*

where ω is a control function on I and M is a positive constant.

Then for any $u_0 \in \mathbb{R}^e$, there exist a $W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ mapping $\xi : I \to \mathbb{R}^e$ and an absolutely continuous mapping $u : I \to \mathbb{R}^e$ satisfying the fractional evolutional integro-differential variational inequality with rough signal

$$\begin{cases} D^\alpha\xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t), \text{ a.e. } t \in I \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \in \mathbb{R}^e \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

where the inclusion is required to be fulfilled for a.e. $t \in I$.

Remark 5.7. Theorem 5.5 and its Corollary 5.6 contain several novelties since we consider here the problem driven by a fractional/evolutional dynamic with rough signal by contrast with the problem driven merely by a fractional inclusion in the form (with $I := [0,1]$)

$$D^\alpha u(t) + \lambda D^{\alpha-1}u(t) \in M(t,u(t),D^{\alpha-1}u(t)), \text{ a.e. } t \in I, I^\beta_{0^+}u(t)|_{t=0} = 0, u(1) = I^\gamma_{0^+}u(1),$$

for a multimapping $M : I \times \mathbb{R}^e \times \mathbb{R}^e \rightrightarrows \mathbb{R}^e$. □

It is worth mentioning that when $\alpha = 2, D^\alpha u(t) = \ddot{u}(t), D^{\alpha-1}u(t) = \dot{u}(t)$, so with $I := [0,1]$ the problems in Theorem 5.5 and its Corollary 5.6 respectively are reduced to mechanical problems with dry friction/evolutional dynamic with rough signal

$$\begin{cases} \ddot{\xi}(t) + \lambda\dot{\xi}(t) = u(t), \text{ a.e. } t \in I \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

and respectively

$$\begin{cases} \ddot{\xi}(t) + \lambda\dot{\xi}(t) = u(t), \text{ a.e. } t \in I \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

where both above inclusions are required to hold for a.e. $t \in I$.
Those considerations lead, with a multimapping $M : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ where $I := [0,1]$, to more general problems such as

$$\begin{cases} \ddot{\xi}(t) + \lambda\dot{\xi}(t) \in M(t,\xi(t),u(t)), \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

and

$$\begin{cases} 0 \in \ddot{\xi}(t) + \lambda\dot{\xi}(t) + \partial\Phi(t,\xi(t),u(t)), \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t,u(t))), \end{cases}$$

where all the inclusions are required to hold for a.e. $t \in I$.

Such problems will be the subject of other works. For clarity, let us mention that the problem of fractional inclusion

$$\begin{cases} \ddot{\xi}(t) + \lambda\dot{\xi}(t) \in M(t, \xi(t)), \text{ a.e. } t \in [0,1] \\ I_{0^+}^{\beta}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I_{0^+}^{\gamma}\xi(1) \end{cases}$$

is solved in [24] meanwhile the second order problem of the form

$$\begin{cases} 0 \in \ddot{\xi}(t) + \lambda\dot{\xi}(t) + \partial\Phi(t, \xi(t)), \text{ a.e. } t \in [0,1] \end{cases}$$

is studied in [32]. □

Proceeding as in the proof of Theorem 5.5 above with the statement and arguments of Theorem 3.16 we also have the following theorem concerning differential inclusions governed by time-dependent maximal monotone operators at velocities under rough signals and coupled with fractional differential equations and Volterra-type integrals.

Theorem 5.8. *Let $\alpha, \beta, \lambda, \gamma$ be as in* (5.5)*. Let $I := [0,1]$ and let $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ be two mappings such that $f(\cdot, x)$ and $g(\cdot, x)$ are Lebesgue measurable for all $x \in \mathbb{R}^e$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^2_{\mathbb{R}^+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}^+}(I)$ for each real $r > 0$ such that for all $(t, x) \in I \times \mathbb{R}^e$*

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \text{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x - y\|, \ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x - y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping such that $F(\cdot, \cdot, x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for each $x \in \mathbb{R}^e$ and such that

(i) *for some non-negative $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable function $\beta_F : I \times I \to \mathbb{R}_+$ with $(t,s) \mapsto (\beta_F(t,s))^2$ Lebesgue integrable on $I \times I$*

$$\|F(t,s,x)\| \le \beta_F(t,s) \quad \text{for all } t, s \in I \text{ and } x \in \mathbb{R}^e,$$

(ii) *for each $(t,s) \in I \times I$ the mapping $F(t,s,\cdot)$ is continuous on $\mathbb{R}^e$.*

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator. For each $t \in I$ let $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ be a maximal monotone operator such that the multimapping $(t,x) \mapsto A_t(x)$ from $I \times \mathbb{R}^e$ into $\mathbb{R}^e$ is Lebesgue measurable and there are $\pi(\cdot) \in L^2_{\mathbb{R}^e}(I)$ and $\sigma(\cdot) \in L^2_{\mathbb{R}^e}(I)$ with $\sigma(t) \in A_t(\pi(t))$ for all $t \in I$. Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

$(\mathcal{B}_1)$ $\quad |b(t,x)| \le M$ *for all $x \in \mathbb{R}^e$,*

$(\mathcal{B}_2)$ $\quad |b(s,x) - b(t,x)| \le \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s, t \in I$ with $s \le t$,*

$(\mathcal{B}_3)$ $\quad |b(t,x) - b(t,y)| \le M\|x - y\|$ *for all $t \in I$ and $x, y \in \mathbb{R}^e$,*

where ω is a control function on I and M is a positive constant.

Then for any $u_0 \in \mathbb{R}^e$, there exist a $W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ mapping $\xi : I \to \mathbb{R}^e$ and an absolutely continuous mapping $u : I \to \mathbb{R}^e$ satisfying the fractional evolutional integro-differential system with rough signal

$$\begin{cases} D^{\alpha}\xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t), \text{ a.e. } t \in I \\ I_{0^+}^{\beta}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I_{0^+}^{\gamma}\xi(1) \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in A_t\left(\frac{du}{dt}(t) + g(t,u(t))\right), \end{cases}$$

where the inclusion is required to hold for a.e. $t \in I$.

Remark 5.9. It is worth mentioning that, under the above assumptions on the maximal monotone operator A_t, it can be seen that the mappings $t \mapsto J_{\lambda}^{A_t}(x)$ and $t \mapsto A_{\lambda}(t,x)$ are measurable. □

We pass now to another important variety of problems where the operator A_t is evaluated at the state $u(t)$ instead of the velocity $\frac{du}{dt}(t)$. In the statement we keep $D^{\alpha}\xi$ as the Riemann-Liouville fractional derivative of order α.

Theorem 5.10. *Let $I := [0,1]$ and let $(A_t)_{t\in I}$ be maximal monotone operators on $\mathbb{R}^e$ satisfying the following conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$:*

$(\mathcal{A}_1)$ *There exists a function $r \in W^{1,2}(I)$ which is non-negative on $[0,1[$ and non-decreasing with $r(1) < \infty$ and $r(0) = 0$ such that*

$$dis(A_t, A_s) \le |r(t) - r(s)| \text{ for all } t, s \in I,$$

$(\mathcal{A}_2)$ *there exists a non-negative real number c such that*

$$\|A_t^0 x\| \le c(1 + \|x\|) \text{ for all } t \in I,\ x \in D(A_t).$$

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

$(\mathcal{B}_1)$ $|b(t,x)| \le M$ *for all $x \in \mathbb{R}^e$,*

$(\mathcal{B}_1)$ $|b(s,x) - b(t,x)| \le \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s, t \in I$ with $s \le t$,*

$(\mathcal{B}_1)$ $|b(t,x) - b(t,y)| \le M\|x - y\|$ *for all $t \in I$ and $x, y \in \mathbb{R}^e$,*

where ω is a control function on $[0,T]$ and M is a positive constant.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a Carathéodory mapping for which there is a function $a \in L^2_{\mathbb{R}_+}(I)$ such that

(i) $\|f(t,x,y)\| \le a(t)$ *for all $t \in I$, $x, y \in \mathbb{R}^e$,*

(ii) *for each real $\eta > 0$ there is a non-negative function $\kappa_\eta(\cdot) \in L^1_{\mathbb{R}_+}(I)$ such that for each $(t,x) \in I \times \mathbb{R}^e$ we have*
$\langle f(t,y_1,x) - f(t,y_2,x), y_1 - y_2\rangle \ge -\kappa_\eta(t)\|y_1 - y_2\|^2$ *for all $y_1, y_2 \in \mathbb{R}^e$.*

Assume that $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \ge 0$, $\gamma > 0$. Then for any $u_0 \in D(A_0)$, there exist a $W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ mapping $\xi : I \to \mathbb{R}^e$ and an absolutely continuous mapping $u : I \to \mathbb{R}^e$ satisfying the dynamic FDI/ EVI with rough signal

$$\begin{cases} D^{\alpha}\xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t),\ t \in I \\ I_{0^+}^{\beta}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I_{0^+}^{\gamma}\xi(1) \\ u(t) \in D(A(t)),\ t \in I \\ -\frac{du}{dt}(t) \in A_t u(t) + f\big(t, u(t), \int_0^t b(s,\xi(s))dz_s\big) \quad \text{a.e. } t \in I \\ u(0) = u_0. \end{cases}$$

Proof. For any continuous mapping $g : I \to \mathbb{R}^e$, the mapping $f_g : I \times \mathbb{R}^e \to \mathbb{R}^e$ defined by

$$f_g(t, x) := f(t, x, g(t)) \quad \text{for all } (t, x) \in I \times \mathbb{R}^e$$

is Lebesgue measurable for every $x \in \mathbb{R}^e$, continuous in x on $\mathbb{R}^e$ for every $t \in I$, and satisfies

$$\|f_g(t, x)\| \leq a(t) \quad \text{for all } (t, x) \in I \times \mathbb{R}^e$$

and for each real $\eta > 0$

$$\langle f_g(t, x) - f_g(t, y), x - y\rangle \geq -\kappa_\eta(t)\|x - y\|^2 \quad \text{for all } (t, x, y) \in I \times \mathbb{B}_{\mathbb{R}^e} \times \mathbb{B}_{\mathbb{R}^e}.$$

Then, given $u_0 \in D(A_0)$, by Theorem 4.3 there is a unique absolutely continuous solution u_g to the differential inclusion

$$\begin{cases} u_g(0) = u_0 \\ -\frac{du_g}{dt}(t) \in A_t u_g(t) + f(t, u_g(t), g(t)) \ \text{ a.e. } t \in I, \end{cases}$$

further $u_g(t) \in D(A_t)$ for all $t \in I$ and $u_g(\cdot)$ is uniformly bounded and equi-absolutely continuous with

$$\left\|\frac{du_g}{dt}(t)\right\| \leq \gamma(t) = K(1 + \dot{r}(t) + a(t)) + a(t), \tag{5.8}$$

so for some real constant $L > 0$ one has for every Lebesgue measurable mapping $g : I \to \mathbb{R}^e$

$$\|u_g(t)\| \leq L \quad \text{for all } t \in I.$$

Now let us consider the set $\mathcal{X}$ defined by $\mathcal{X} := \{\zeta_\ell : I \to \mathbb{R}^e : \ \ell \in S^1_{L\mathbb{B}_{\mathbb{R}^e}}\}$, each mapping ζ_ℓ being given for every $t \in I$ by

$$\zeta_\ell(t) = \int_0^1 G(t, s)\ell(s)\, ds,$$

where G is the Green function defined in (5.6). We note that $\mathcal{X}$ is convex, compact and equi-Lipschitz (see Theorem 5.4 and Lemma 5.3) with for every $h \in \mathcal{X}$

$$\|h(t) - h(s)\| \leq N|t - s|,$$

where N is a positive constant independent of h. Then for any $h \in \mathcal{X}$, using $(\mathcal{B}_2), (\mathcal{B}_3)$ we obtain

$$\begin{aligned} |b(t, h(t)) - b(s, h(s))| &\leq |b(t, h(t)) - b(s, h(t))| + |b(s, h(t)) - b(s, h(s))| \\ &\leq \omega(s, t) + N|t - s|, \end{aligned}$$

so that for any $h \in \mathcal{X}$, $b(\cdot, h(\cdot)) \in C^{1-var}(I, \mathcal{L}(\mathbb{R}^e, \mathbb{R}^e))$ with $|b(t, h(t))| \leq M$ for all $t \in I$ by $(\mathcal{B}_1)$. In particular, for each $h \in \mathcal{X}$ and each $t \in I$ the integral $\int_0^t b(s, h(s))\, dz_s$ is well defined for all $h \in \mathcal{X}$ and $b(\cdot, h(\cdot))$ is uniformly bounded in variation with respect to $h \in \mathcal{X}$.

As consequence, it follows that the set

$$\mathcal{Y} := \left\{ t \mapsto \int_0^t b(s, h(s)) dz_s : h \in \mathcal{X} \right\}$$

is compact in $C(I, \mathbb{R}^e)$ for the topology of uniform convergence on I. In particular, if a sequence of mappings $(h_n)_n$ in $\mathcal{X}$ converges uniformly on I to a mapping $h \in \mathcal{X}$, then the sequence $\left(t \mapsto \int_0^t b(s, h_n(s)) dz_s\right)_n$ converges uniformly on I to the mapping $t \mapsto \int_0^t b(s, h(s)) dz_s$. For each $h \in \mathcal{X}$ let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s) u_h(s)\, ds \quad \text{for all } t \in I,$$

where u_h is the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} u_h(0) = a \in D(A(0)) \\ -\frac{du_h}{dt}(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dz_s) \ \text{ a.e. } t \in I. \end{cases}$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly on I to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = u_0 \\ -\dot{u}_{h_n}(t) \in A_t u_{h_n}(t) + f(t, u_{h_n}(s), \int_0^t b(s, h_n(s)) dz_s) \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges on I to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = u_0 \\ -\dot{u}_h(t) \in A_t u_h(t) + f(t, u_h(s), \int_0^t b(s, h(s)) dz_s) \quad \text{a.e. } t \in I. \end{cases}$$

Since $(u_{h_n})_n$ is equi-absolutely continuous for each $n \in \mathbb{N}$ the estimate (see (5.8))

$$\|\dot{u}_{h_n}(t)\| \le \gamma(t) \text{ a.e. } t \in I,$$

holds. We may suppose that $(u_{h_n})_n$ converges uniformly on I to an absolutely continuous mapping $w : I \to \mathbb{R}^e$ with $w(t) = u_0 + \int_0^t \dot{w}(s)\, ds$ for all $t \in I$. We may also suppose that $(\dot{u}_{h_n})_n$ weakly converges to a mapping $q \in L^2(I, \mathbb{R}^e)$ with $\|q(t)\| \le \gamma(t)$, hence $\dot{w} = q$, so for every $t \in I$ we have as $n \to \infty$

$$k_n(t) := f(t, u_{h_n}(t), \int_0^t b(s, h_n(s)) dz_s) \to k(t) := f(t, w(t), \int_0^t b(s, h(s)) dw(s)).$$

Keeping in mind that

$$\|k_n(t)\| = \left\| f\big(t, u_{h_n}(t), \int_0^t b(s, h_n(s))\, dz_s\big) \right\| \le a(t) \quad \text{for all } t \in I$$

with $a \in L^2_{\mathbb{R}_+}(I)$, Lemma 4.2 yields that w is absolutely continuous and solution of the differential inclusion

$$\begin{cases} w(0) = u_0 \\ -\dot{w}(t) \in A_t w(t) + f(t, w(t), \int_0^t b(s, h(s)) dz_s) \quad \text{a.e. } t \in I. \end{cases}$$

Now let us write by Lemma 5.3

$$\begin{aligned}\|\Phi(h_n)(t)-\Phi(h)(t)\| &= \left\|\int_0^1 G(t,s)u_{h_n}(s)\,ds - \int_0^1 G(t,s)u_h(s)\,ds\right\|\\ &= \left\|\int_0^1 G(t,s)[u_{h_n}(s)-u_h(s)]\,ds\right\|\\ &\le \int_0^1 M_G\,\|u_{h_n}(s)-u_h(s)\|\;ds.\end{aligned}$$

Since $\|u_{h_n}(\cdot)-u_h(\cdot)\|\to 0$ uniformly on I as $n\to\infty$, we deduce that

$$\sup_{t\in I}\|\Phi(h_n)(t)-\Phi(h)(t)\|\le \int_0^1 M_G\|u_{h_n}(s)-u_h(s)\|\,ds\to 0,$$

which entails that $\Phi(h_n)\to\Phi(h)$ uniformly on I. This translates the continuity of $\Phi:\mathcal{X}\to\mathcal{X}$ with respect to the topology induced by the topology of uniform convergence on I. Therefore, by the Schauder fixed point theorem Φ has a fixed point, say $h=\Phi(h)\in\mathcal{X}$. This can be translated as

$$h(t)=\Phi(h)(t)=\int_0^1 G(t,s)u_h(s)\,ds \text{ for every } t\in I,$$

with
$$\begin{cases}u_h(0)=u_0\\ -\dot u_h(t)\in A_t u_h(t)+f(t,u_h(t),\int_0^t b(s,h(s))dz_s) \quad \text{a.e. } t\in I.\end{cases}$$

Further, as we saw above by Theorem 4.3, we have $u_h(t)\in D(A_t)$ for all $t\in I$.

According to Lemma 5.3, this means that we have just shown that there exists a mapping $h\in W^{\alpha,1}_{B,\mathbb{R}^e}(I)$ such that

$$\begin{cases}D^\alpha h(t)+\lambda D^{\alpha-1}h(t)=u_h(t),\\ I^\beta_{0^+}h(t)\,|_{t=0}=0, \quad h(1)=I^\gamma_{0^+}h(1)\\ u_h(0)=u_0\\ u_h(t)\in D(A_t),\forall t\in I\\ -\dot u_h(t)\in A_t u_h(t)+f(t,u_h(t),\int_0^t b(s,h(s))dz_s) \quad \text{a.e. } t\in I.\end{cases}$$

This finishes the proof of the theorem with $\xi:=h$ and $u:=u_h$. □

6. Differential evolution inclusions coupled with rough signals and Caputo fractional differential equations

The present section is devoted to problems in the line of those in Section 5 but with the Caputo fractional derivative instead of the Riemann-Liouville one.

We study Caputo fractional differential inclusions coupled with time dependent maximal monotone operators $A_t: D(A_t)\rightrightarrows E$ with $t\in[0,1]$, where E is still a separable Hilbert space. For the sake of completeness, we recall some needed properties for the Caputo fractional calculus and provide a series of lemmas on the corresponding integral.

Definition 6.1. Given a function $h : [0,T] \to E$, its *Caputo fractional derivative of order* $\alpha > 0$ at a point $t \in [0,T]$ is defined by

$$^{c}D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{1-n+\alpha}}\, ds,$$

when the latter integral exists in E. Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real α. $\square$

Throughout we assume $\alpha \in]1,2]$. We denote by $\mathcal{C}^1_E([0,T])$ the vector space of differentiable mappings from $[0,T]$ into E whose first order derivatives are continuous on $[0,T]$, and we also denote

$$^{c}W^{\alpha,\infty}_{B,E}([0,T]) = \{u \in \mathcal{C}^1_E([0,T]) : {}^{c}D^{\alpha-1}u \in \mathcal{C}_E([0,T]),\ {}^{c}D^{\alpha}u \in L^\infty_E([0,T])\},$$

where ${}^{c}D^{\alpha-1}u$ and ${}^{c}D^{\alpha}u$ are the Caputo fractional derivatives of order $\alpha-1$ and α of u, respectively.

We recall and summarize some properties of a Green function given in [23] which is fundamental for many differential equations involving Caputo fractional derivatives.

Lemma 6.2. *Keep* $\alpha \in]1,2]$ *and consider the Green function (relative to the Caputo fractional derivative)* $G : [0,T] \times [0,T] \to \mathbb{R}$ *defined by*

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{1+t}{T+2}\left[\dfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right], & \text{if } 0 \le s < t, \\ -\dfrac{1+t}{T+2}\left[\dfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] & \text{if } 0 \le t \le s < T. \end{cases}$$

The following assertions hold.

(a) *For* $f \in L^\infty_E([0,T])$ *and for the mapping* $u_f : [0,T] \to E$ *defined by*

$$u_f(t) = \int_0^T G(t,s) f(s) ds,\ \forall t \in [0,T],$$

one has
$$u_f(0) - \frac{du_f}{dt}(0) = 0, \qquad u_f(T) + \frac{du_f}{dt}(T) = 0,$$

$$^{c}D^{\alpha-1}u_f(t) = \int_0^t f(s)ds - \frac{I^{\alpha}f(T) + I^{\alpha-1}f(T)}{(T+2)\Gamma(3-\alpha)} t^{2-\alpha},\ \forall\, t \in [0,T],$$

$$^{c}D^{\alpha}u_f(t) = f(t),\ \forall t \in [0,T].$$

(b) *If* $u(\cdot)$ *is a* ${}^{c}W^{\alpha,\infty}_{B,E}([0,T])$*-solution to*

$$\begin{cases} {}^{c}D^{\alpha}u(t) = \sigma(t),\ t \in [0,T] \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0 \end{cases}$$

where $\sigma \in L^\infty_E([0,T])$, *then* $u(t) = \displaystyle\int_0^T G(t,s)\sigma(s)ds$ *for all* $t \in [0,T]$.

We also recall the following lemma ([23, Lemma 3.5]).

Lemma 6.3. *Let $X : [0,T] \rightrightarrows E$ be a compact convex valued measurable multimapping such that there is a real constant $L>0$ for which $\sup\{\|x\| : x \in X(t)\} \leq L < +\infty$ for all $t \in [0,T]$. Then the set $\mathcal{X}$ of ${}^cW^{\alpha,\infty}_{B,E}([0,T])$-solutions to the fractional differential inclusion*

$$\begin{cases} {}^cD^\alpha u(t) \in X(t), \ t \in [0,T] \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0, \end{cases}$$

is bounded, convex, equi-Lipschitz and compact in the Banach space $\mathcal{C}_E([0,T])$ for the topology of uniform convergence on $[0,T]$.

Using in Theorem 5.5 the Green function (relative to the Caputo fractional derivative) in (5.6) in place of that (relative to the Riemann-Liouville fractional derivative) and using also Lemma 6.2 in place of Lemma 5.3 and Lemma 6.3 in place of Theorem 5.4, we obtain our following first result of this section.

Theorem 6.4. *Let $\alpha \in]1,2]$ and $I := [0,1]$. Let $\varphi : I \times \mathbb{R}^e \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand for which there are a mapping $\zeta \in L^2_{\mathbb{R}^e}(I)$ such that $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$ is in $L^2_{\mathbb{R}_+}(I)$ and two non-negative functions $\alpha_\varphi, \beta_\varphi : I \to \mathbb{R}$ with $\alpha_\varphi, \sqrt{\beta_\varphi} \in L^2_{\mathbb{R}_+}(I)$ such that*

$$-\alpha_\varphi(t)\|x\| - \beta_\varphi(t) \leq \varphi(t,x) \quad \text{for all } (t,x) \in I \times \mathbb{R}^e$$

(note that $\sqrt{\beta_\varphi} \in L^2_{\mathbb{R}_+}(I)$ since $\beta_\varphi \in L^2_{\mathbb{R}_+}(I)$). Let $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ be two mappings such that $f(\cdot,x)$ and $g(\cdot,x)$ are Lebesgue measurable for all $x \in \mathbb{R}^e$ and such that there are non-negative functions α_f, α_g, β_f, β_g in $L^2_{\mathbb{R}+}(I)$ along with $\gamma_{f,r}$, $\gamma_{g,r}$ in $L^1_{\mathbb{R}+}(I)$ for each real $r > 0$ such that for all $(t,x) \in I \times \mathbb{R}^e$

$$\|f(t,x)\| \leq \alpha_f(t)\|x\| + \beta_f(t) \quad \text{and} \quad \|g(t,x)\| \leq \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ and $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \leq \gamma_{f,r}(t)\|x-y\|, \ \|g(t,x) - g(t,y)\| \leq \gamma_{g,r}(t)\|x-y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping such that $F(\cdot,\cdot,x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for each $x \in \mathbb{R}^e$ and such that

(i) *for some non-negative $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable function $\beta_F : I \times I \to \mathbb{R}_+$ with $(t,s) \mapsto (\beta_F(t,s))^2$ Lebesgue integrable on $I \times I$*

$$\|F(t,s,x)\| \leq \beta_F(t,s) \quad \text{for all } t,s \in I \text{ and } x \in \mathbb{R}^e,$$

(ii) *for each $(t,s) \in I \times I$ the mapping $F(t,s,\cdot)$ is continuous on $\mathbb{R}^e$.*

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ be a positive symmetric bijective linear operator.

Let $z \in C^{1-var}(I,\mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

($\mathcal{B}_1$) $\quad |b(t,x)| \leq M$ *for all $x \in \mathbb{R}^e$,*

($\mathcal{B}_2$) $\quad |b(s,x) - b(t,x)| \leq \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s,t \in I$ with $s \leq t$,*

($\mathcal{B}_3$) $\quad |b(t,x) - b(t,y)| \leq M\|x-y\|$ *for all $t \in I$ and $x,y \in \mathbb{R}^e$,*

where ω is a control function on I and M is a positive constant.

Then for any $u_0 \in \mathbb{R}^e$*, there exist a* ${}^cW^{\alpha,\infty}_{B,\mathbb{R}^e}(I)$ *mapping* $\xi : I \to \mathbb{R}^e$ *and an absolutely continuous mapping* $u : I \to \mathbb{R}^e$ *satisfying the fractional evolutional integro-differential inequality with rough signal*

$$\begin{cases} {}^cD^\alpha \xi(t) = u(t), \ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(0) = u_0 \\ \int_0^t b(s, \xi(s)) dz_s + \int_0^t F(t, s, \xi(s))\, ds + f(t, u(t)) - \Lambda \frac{du}{dt}(t) \in \partial\varphi\left(t, \frac{du}{dt}(t) + g(t, u(t))\right), \end{cases}$$

where the inclusion is required to hold for a.e. $t \in I$.

The case of sweeping processes follows by taking $\varphi(t, x) = \delta_{C(t)}(x)$.

Corollary 6.5. *Let* $\alpha \in]1, 2]$ *and* $I := [0, 1]$*. Let* $C : I \rightrightarrows \mathbb{R}^e$ *be a closed convex valued Lebesgue multimapping which has a selection* $\zeta \in L^2_{\mathbb{R}^e}(I)$*, i.e.* $\zeta(t) \in C(t)$ *for all* $t \in I$*. Let* $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ *be two mappings such that* $f(\cdot, x)$ *and* $g(\cdot, x)$ *are Lebesgue measurable for all* $x \in \mathbb{R}^e$ *and such that there are non-negative functions* α_f*,* α_g*,* β_f*,* β_g *in* $L^2_{\mathbb{R}+}(I)$ *along with* $\gamma_{f,r}$*,* $\gamma_{g,r}$ *in* $L^1_{\mathbb{R}+}(I)$ *for each real* $r > 0$ *such that for all* $(t, x) \in I \times \mathbb{R}^e$

$$\|f(t, x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t, x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ *and* $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t, x) - f(t, y)\| \le \gamma_{f,r}(t)\|x - y\|, \ \|g(t, x) - g(t, y)\| \le \gamma_{g,r}(t)\|x - y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ *be a mapping such that* $F(\cdot, \cdot, x)$ *is* $\mathcal{L}(I) \otimes \mathcal{L}(I)$ *measurable for each* $x \in \mathbb{R}^e$ *and such that*

(i) *for some non-negative* $\mathcal{L}(I) \otimes \mathcal{L}(I)$ *measurable function* $\beta_F : I \times I \to \mathbb{R}_+$ *with* $(t, s) \mapsto (\beta_F(t, s))^2$ *Lebesgue integrable on* $I \times I$

$$\|F(t, s, x)\| \le \beta_F(t, s) \quad \textit{for all}\ t, s \in I \ \textit{and}\ x \in \mathbb{R}^e,$$

(ii) *for each* $(t, s) \in I \times I$ *the mapping* $F(t, s, \cdot)$ *is continuous on* $\mathbb{R}^e$.

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ *be a positive symmetric bijective linear operator.*
Let $z \in C^{1-var}(I, \mathbb{R}^d)$ *and let* $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ *be a continuous integrand operator satisfying the conditions*

($\mathcal{B}_1$) $|b(t, x)| \le M$ *for all* $x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s, x) - b(t, x)| \le \omega(s, t)$ *for all* $x \in \mathbb{R}^e$ *and* $s, t \in I$ *with* $s \le t$,

($\mathcal{B}_3$) $|b(t, x) - b(t, y)| \le M\|x - y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$,

where ω *is a control function on* I *and* M *is a positive constant.*

Then for any $u_0 \in \mathbb{R}^e$*, there exist a* ${}^cW^{\alpha,\infty}_{B,\mathbb{R}^e}(I)$ *mapping* $\xi : I \to \mathbb{R}^e$ *and an absolutely continuous mapping* $u : I \to \mathbb{R}^e$ *satisfying the fractional/evolutional dynamics with rough signal*

$$\begin{cases} {}^cD^\alpha\xi(t) = u(t),\ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in N_{C(t)}\left(\frac{du}{dt}(t) + g(t,u(t))\right), \end{cases}$$

where the inclusion is required to hold for a.e. $t \in I$.

A result similar to Theorem 5.8 with Caputo fractional derivative instead of Riemann-Liouville fractional derivative also holds true with the same arguments in Theorem 6.6.

Theorem 6.6. *Let* $\alpha \in]1,2]$ *and* $I := [0,1]$. *Let* $f, g : I \times \mathbb{R}^e \to \mathbb{R}^e$ *be two mappings such that* $f(\cdot,x)$ *and* $g(\cdot,x)$ *are Lebesgue measurable for all* $x \in \mathbb{R}^e$ *and such that there are non-negative functions* α_f, α_g, β_f, β_g *in* $L^2_{\mathbb{R}^+}(I)$ *along with* $\gamma_{f,r}$, $\gamma_{g,r}$ *in* $L^1_{\mathbb{R}^+}(I)$ *for each real* $r > 0$ *such that for all* $(t,x) \in I \times \mathbb{R}^e$

$$\|f(t,x)\| \le \alpha_f(t)\|x\| + \beta_f(t) \quad \textit{and} \quad \|g(t,x)\| \le \alpha_g(t)\|x\| + \beta_g(t)$$

and for all $t \in I$ *and* $x, y \in r\mathbb{B}_{\mathbb{R}^e}$

$$\|f(t,x) - f(t,y)\| \le \gamma_{f,r}(t)\|x - y\|,\ \|g(t,x) - g(t,y)\| \le \gamma_{g,r}(t)\|x - y\|.$$

Let $F : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ *be a mapping such that* $F(\cdot,\cdot,x)$ *is* $\mathcal{L}(I) \otimes \mathcal{L}(I)$ *measurable for each* $x \in \mathbb{R}^e$ *and such that*

(i) *for some non-negative* $\mathcal{L}(I) \otimes \mathcal{L}(I)$ *measurable function* $\beta_F : I \times I \to \mathbb{R}_+$ *with* $(t,s) \mapsto (\beta_F(t,s))^2$ *Lebesgue integrable on* $I \times I$

$$\|F(t,s,x)\| \le \beta_F(t,s) \quad \textit{for all}\, t, s \in I \textit{ and } x \in \mathbb{R}^e,$$

(ii) *for each* $(t,s) \in I \times I$ *the mapping* $F(t,s,\cdot)$ *is continuous on* $\mathbb{R}^e$.

Let $\Lambda : \mathbb{R}^e \to \mathbb{R}^e$ *be a positive symmetric bijective linear operator. For each* $t \in I$ *let* $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ *be a maximal monotone operator such that the multimapping* $(t,x) \mapsto A_t(x)$ *from* $I \times \mathbb{R}^e$ *into* $\mathbb{R}^e$ *is Lebesgue measurable and there are* $\pi(\cdot) \in L^2_{\mathbb{R}^e}(I)$ *and* $\sigma(\cdot) \in L^2_{\mathbb{R}^e}(I)$ *with* $\sigma(t) \in A_t(\pi(t))$ *for all* $t \in I$. *Let* $z \in C^{1-var}(I, \mathbb{R}^d)$ *and let* $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ *be a continuous integrand operator satisfying the conditions*

$(\mathcal{B}_1)$ $|b(t,x)| \le M$ *for all* $x \in \mathbb{R}^e$,

$(\mathcal{B}_2)$ $|b(s,x) - b(t,x)| \le \omega(s,t)$ *for all* $x \in \mathbb{R}^e$ *and* $s, t \in I$ *with* $s \le t$,

$(\mathcal{B}_3)$ $|b(t,x) - b(t,y)| \le M\|x - y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$,

where ω *is a control function on* I *and* M *is a positive constant.*
Then for any $u_0 \in \mathbb{R}^e$, *there exist a* ${}^cW^{\alpha,\infty}_{B,\mathbb{R}^e}(I)$ *mapping* $\xi : I \to \mathbb{R}^e$ *and an absolutely continuous mapping* $u : I \to \mathbb{R}^e$ *satisfying the fractional evolutional integro-differential system with rough signal*

$$\begin{cases} {}^cD^\alpha\xi(t) = u(t),\ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(0) = u_0 \\ \int_0^t b(s,\xi(s))dz_s + \int_0^t F(t,s,\xi(s))\,ds + f(t,u(t)) - \Lambda\frac{du}{dt}(t) \in A_t\left(\frac{du}{dt}(t) + g(t,u(t))\right), \end{cases}$$

where the inclusion is required to hold for a.e. $t \in I$.

Regarding the situation where the operator A_t is evaluated at the state $u(t)$ in place of the velocity $\frac{du}{dt}(t)$, the statement and arguments of the next theorem are similar to those of Theorem 5.10. Here we deal with the Caputo fractional derivative instead of the Riemann-Liouville one.

Theorem 6.7. *Let $\alpha \in]1,2]$ and $I := [0,1]$. Let $(A_t)_{t\in I}$ be maximal operators on $\mathbb{R}^e$ satisfying the following conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$:*

$(\mathcal{A}_1)$ *There exists a function $r \in W^{1,2}(I)$ which is non-negative on $[0,1[$ and non-decreasing with $r(1) < \infty$ and $r(0) = 0$ such that*

$$dis(A_t, A_s) \le |r(t) - r(s)|, \text{ for all } t, s \in I;$$

$(\mathcal{A}_2)$ *there exists a non-negative real number c such that*

$$\|A_t^0 x\| \le c(1 + \|x\|) \text{ for all } t \in I,\ x \in D(A_t).$$

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions

$(\mathcal{B}_1)$ $|b(t,x)| \le M$ *for all $x \in \mathbb{R}^e$,*

$(\mathcal{B}_2)$ $|b(s,x) - b(t,x)| \le \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s,t \in I$,*

$(\mathcal{B}_3)$ $|b(t,x) - b(t,y)| \le M\|x - y\|$ *for all $t \in I$ and $x, y \in \mathbb{R}^e$,*

where ω is a control function on $[0,T]$ and M is a positive constant.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a Carathéodory mapping for which there is a function $a \in L^2_{\mathbb{R}_+}(I)$ such that

(i) $\|f(t,x,y)\| \le a(t)$ *for all $t \in I$, $x, y \in \mathbb{R}^e$,*

(ii) *for each real $\eta > 0$ there is a non-negative function $\kappa_\eta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for each $(t,x) \in I \times \mathbb{R}^e$*

$$\langle f(t,y_1,x) - f(t,y_2,x), y_1 - y_2\rangle \ge \kappa_\eta(t)||y_1 - y_2||^2 \text{ for all } y_1, y_2 \in \mathbb{R}^e.$$

Then for any $u_0 \in D(A(0))$, there exist a ${}^cW^{\alpha,\infty}_{B,\mathbb{R}^e}(I)$ mapping $\xi : I \to \mathbb{R}^e$ and an absolutely continuous mapping $u : I \to \mathbb{R}^e$ satisfying the Caputo fractional differential inclusion with rough signal

$$\begin{cases} {}^cD^\alpha\xi(t) = u(t),\ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(t) \in D(A(t)),\ t \in I \\ -\frac{du}{dt}(t) \in A_t u(t) + f\left(t, u(t), \int_0^t b(s,\xi(s))dz_s\right) \quad \text{a.e. } t \in I \\ u(0) = u_0. \end{cases}$$

7. BVRC evolution inclusions coupled with rough signals and fractional differential equations

First, we summarize and state an important existence result in [21] in the form of the following theorem. When a vector measure m on a measurable space $(\mathcal{I}, \mathfrak{T})$ is absolutely continuous with respect to a positive measure ν on the same measurable space, it will be convenient (as usual) to say that m is ν-absolutely continuous.

Theorem 7.1. *Let E be a separable Hilbert space and for each $t \in I := [0, T]$ let $A(t) : D(A(t)) \subset E \rightrightarrows E$ be a maximal monotone operator satisfying:*

$(\mathcal{A}_1^*)$ *There exits a real constant $c > 0$ such that*

$$\|A_t^0 x\| \le c(1 + \|x\|) \quad \text{for every } (t, x) \in I \times D(A(t)),$$

$(\mathcal{A}_2^*)$ $\mathrm{dis}(A_t, A_s) \le dr(]s, t])$, *for all $0 \le s \le t \le T$, where $r : I \to \mathbb{R}_+$ is a on-decreasing right continuous function on I with $r(0) = 0$ and $r(T) < +\infty$,*

$(\mathcal{A}_4^*)$ *for each $t \in I$ the set $D(A(t))$ is ball-compact.*

Let $f : I \times E \to E$ be a map with $f(\cdot, x)$ Borel measurable *for all $x \in E$ and for which there is a real constant $M > 0$ such that $\|f(t, x)\| \le M$ for all $(t, x) \in I \times E$ and*

$$\|f(t, x) - f(t, y)\| \le M\|x - y\| \quad \text{for all } t \in I, x, y \in E.$$

Consider the measure $\nu := dr + dt$ and assume further that there is $\delta \ge 0$ such that $0 \le 2M \frac{dt}{d\nu}(t)\nu(\{t\}) \le \delta < 1$ for all $t \in I$.
Then for each $u_0 \in D(A(0))$ there is a unique bounded in variation and right continuous (BVRC) solution to the problem

$$\begin{cases} u(0) = u_0 \\ u(t) \in D(A(t)) \quad \forall t \in I \\ \text{the differential measure } du \text{ is } \nu\text{-absolutely continuous} \\ -\dfrac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t))\dfrac{dt}{d\nu}(t) \quad \nu - a.e.\, t \in I, \end{cases}$$

further, $\left\|\frac{du}{d\nu}\right\|_{L_E^\infty(I,\nu)} \le K$, where K is a real constant depending only on u_0, on the real constants c, M and on the non-decreasing function $r(\cdot)$.

The next theorem establishes the existence of BVRC solutions to differential inclusions as in Theorem 7.1 with the additional presence of rough signals and with the coupling with fractional differential equations involving fractional Riemann-Liouville derivatives. It is worth recalling that BVRC mappings from $I = [0, T]$ into $\mathbb{R}^e$ are Borel measurable, a property which will be implicitly used inside the proof of the theorem. In the statement and beyond we keep, as in Section 5, the notation $D^\alpha \xi$ for the Riemann-Liouville fractional derivative of order α of a mapping $\xi : [0, 1] \to \mathbb{R}^e$.

Theorem 7.2. *Let $I := [0, 1]$ and $\alpha \in]1, 2]$, $\beta \in [0, 2 - \alpha]$, $\lambda \ge 0$, $\gamma > 0$. For each $t \in I$ let $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ be a maximal monotone operator satisfying:*

$(\mathcal{A}_1^*)$ *There is a real constant $c > 0$ such that*

$$\|A_t^0 x\| \le c(1 + \|x\|) \quad \text{for every } (t, x) \in I \times D(A_t),$$

($\mathcal{A}_2^*$) $\mathrm{dis}(A_t, A_s) \le dr(]s,t])$, *for all* $0 \le s \le t \le 1$, *where* $r : I \to \mathbb{R}^+$ *is a non-decreasing right continuous function on* I *with* $r(0) = 0$ *and* $r(1) < +\infty$,

($\mathcal{A}_3^*$) $D(A(t))$ *is closed for each* $t \in I$.

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ *and let* $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ *be a continuous integrand operator satisfying for some real constant* $M_b > 0$ *and some control function* ω *on* I *the conditions*

($\mathcal{B}_1$) $|b(t,x)| \le M_b$ *for all* $x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \le \omega(s,t)$ *for all* $x \in \mathbb{R}^e$ *and* $s,t \in I$ *with* $s \le t$,

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \le M_b\|x - y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ *be a mapping with* $f(\cdot, x, y)$ *Borel measurable for all* $(x,y) \in \mathbb{R}^e \times \mathbb{R}^e$ *and* $f(t,\cdot,\cdot)$ *continuous for all* $t \in I$, *and which satisfies for some real constant* $M_f > 0$:

(i) $\|f(t,x,y) - f(t,x',y)\| \le M_f\|x - x'\|$ *for all* $(t,x,x',y) \in I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e$,

(ii) $\|f(t,x,y)\| \le M_f$ *for all* $(t,x,y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.

Consider the measure $\nu := dr + dt$ *and assume that there is a real constant* $\delta \ge 0$ *such that for* $M := \max\{M_b, M_f\}$

$$0 \le 2M \frac{dt}{d\nu}(t)\nu(\{t\}) \le \delta < 1 \quad \textit{for all}\, t \in I.$$

Then for any $u_0 \in D(A(0))$, *there exist a* $W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ *mapping* $\xi : I \to \mathbb{R}^e$ *and a BVRC mapping* $u : I \to \mathbb{R}^e$ *whose differential measure* du *is* ν*-absolutely continuous and which satisfies the evolution system with rough signal*

$$\begin{cases} D^\alpha \xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t), \ t \in I \\ I^\beta_{0^+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+}\xi(1) \\ u(t) \in D(A(t)), \, t \in I \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f\left(t, u(t), \int_0^t b(s,\xi(s))\,dz_s\right)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I \\ u(0) = u_0. \end{cases}$$

Proof. The first part (I) of the proof will consist in adaptations of arguments in the proof of Theorem 5.10.

(I) Put $M := \max\{M_b, M_f\}$ and fix any $u_0 \in D(A(0))$. Notice that for any continuous mapping $g : I \to \mathbb{R}^e$ by Theorem 7.1 there is a **unique** BVRC solution u_g to the differential inclusion

$$\begin{cases} u_g(0) = u_0 \\ u_g(t) \in D(A(t)), \ \forall t \in I \\ du_g \ \nu\text{-absolutely continuous} \\ -\frac{du_g}{d\nu}(t) \in A_t u_g(t) + f(t, u_g(t), g(t))\frac{dt}{d\nu}(t) \quad \nu - \text{a.e. } t \in I, \end{cases}$$

and notice also that by the same Theorem 7.1 there is a real constant $\eta > 0$ independent on $g \in \mathcal{C}_{\mathbb{R}^e}(I)$ such that u_g is uniformly bounded and equi-BVRC with

$$\left\|\frac{du_g}{d\nu}(t)\right\| \le \eta \quad \nu - \text{a.e. } t \in I.$$

So, for some constant $K > 0$ (independent on $g \in \mathcal{C}_{\mathbb{R}^e}(I)$) one has $\|u_g(t)\| \leq K$ for all $t \in I$ and for all continuous mappings $g \in \mathcal{C}_{\mathbb{R}^e}(I)$. Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_q : I \to \mathbb{R}^e : \ q \in S^1_{K\mathbb{B}_{\mathbb{R}^e}}\},$$

each mapping ξ_q being given for every $t \in I$ by

$$\xi_q(t) = \int_0^1 G(t,s)q(s)\,ds,$$

where G is the Green function defined in (5.6). We note that $\mathcal{X}$ is convex, compact (relative to the topology of uniform convergence on I) and equi-Lipschitz, i.e., for some real constant $N > 0$

$$\|h(t) - h(s)\| \leq N|t-s| \quad \text{for all } h \in \mathcal{X} \text{ and } s,t \in I.$$

Then for any $h \in \mathcal{X}$, using $(\mathcal{B}_2), (\mathcal{B}_3)$

$$\begin{aligned}|b(t,h(t)) - b(s,h(s))| &\leq |b(t,h(t)) - b(s,h(t))| + |b(s,h(t)) - b(s,h(s))| \\ &\leq \omega(s,t) + N|t-s|\end{aligned}$$

so $h \in \mathcal{X}$, $b(\cdot,h(\cdot)) \in C^{1-var}(I, \mathcal{L}(\mathbb{R}^e,\mathbb{R}^e))$ with $|b(\cdot,h(\cdot))| \leq M$ by $(\mathcal{B}_1)$.
In particular, for each $h \in \mathcal{X}$ the integral $\int_0^t b(s,h(s))dz_s$ has a meaning for all $t \in I$ with $b(\cdot,h(\cdot))$ uniformly bounded in variation with respect to $h \in \mathcal{X}$. As a consequence, the set

$$\mathcal{Y} := \{t \mapsto \int_0^t b(s,h(s))dz_s : h \in \mathcal{X}\}$$

is **compact** in $C(I,\mathbb{R}^e)$ for the topology of uniform convergence on I. In particular, if a sequence $(h_n)_n$ in $\mathcal{X}$ converges uniformly on I to a mapping $h \in \mathcal{X}$, then the sequence of mappings $(t \mapsto \int_0^t b(s,h_n(s))\,dz_s)_n$ converges uniformly on I to the mapping $t \mapsto \int_0^t b(s,h(s))\,dz_s$.
For each $h \in \mathcal{X}$ let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds \quad \text{for all } t \in I,$$

where u_h is the **unique** BVRC solution to the differential inclusion

$$\begin{cases} u_h(0) = u_0 \\ u_h(t) \in D(A(t)), \ \forall t \in I \\ du_h \ \nu\text{-absolutely continuous} \\ -\frac{du_h}{d\nu}(t) \in A_t u_h(t) + f(t,u_h(t),\int_0^t b(s,h(s))\,dz_s)\frac{dt}{d\nu}(t) \quad \nu-\text{a.e. } t \in I. \end{cases}$$

Clearly, we have $\Phi(h) \in \mathcal{X}$ for every $h \in \mathcal{X}$.

(II) Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly on I to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique BVRC solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = u_0 \\ u_{h_n}(t) \in D(A(t)),\ \forall t \in I \\ du_{h_n}\ \ \nu\text{-absolutely continuous} \\ -\frac{du_{h_n}}{d\nu}(t) \in A_t u_{h_n}(t) + f(t, u_{h_n}(s), \int_0^t b(s, h_n(s))dz_s)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I, \end{cases}$$

converges uniformly on I to the unique BVRC solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = u_0 \\ u_h(t) \in D(A(t)),\ \forall t \in I \\ du_h\ \ \nu\text{-absolutely continuous} \\ -\frac{du_h}{d\nu}(t) \in A_t u_h(t) + f(t, u_h(s), \int_0^t b(s, h(s))dz_s)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

This needs a careful look. Since $(u_{h_n})_n$ is equi-BVRC with for each $n \in \mathbb{N}$ the estimate

$$\left\|\frac{du_{h_n}}{d\nu}(t)\right\| \le \eta \quad \nu - \text{a.e. } t \in I,$$

we may suppose that $(u_{h_n})_n$ pointwise converges to a BVRC mapping $w : I \to \mathbb{R}^e$ with $w(t) = u_0 + \int_0^t \frac{dw}{d\nu}(s)\, d\nu(s)$, and hence the differential measure dw is ν-absolutely continuous. We note by closedness of $D(A(t))$ that

$$w(t) \in D(A(t)) \quad \text{for all } t \in I. \tag{7.1}$$

We also suppose that $(\frac{du_{h_n}}{d\nu})_n$ weakly converges to $\frac{dw}{d\nu}$ in $L^1_{\mathbb{R}^e}(I, \nu)$ with $\left\|\frac{dw}{d\nu}(t)\right\| \le \eta$, so for every $t \in I$ we have as $n \to \infty$

$$k_n(t) := f\Big(t, u_{h_n}(t), \int_0^t b(s, h_n(s))dz_s\Big)\frac{dt}{d\nu}(t) \to k(t) := f\Big(t, w(t), \int_0^t b(s, h(s))dz_s\Big)\frac{dt}{d\nu}(t).$$

Keeping in mind that

$$\left\| f\Big(t, u_{h_n}(t), \int_0^t b(s, h_n(s))\, dz_s\Big)\right\| \le M \quad \text{for all } t \in I,$$

we show that $w(\cdot)$ is solution of the differential inclusion

$$\begin{cases} w(0) = u_0 \\ w(t) \in D(A(t)),\ \forall t \in I \\ dw\ \ \nu\text{-absolutely continuous} \\ -\frac{dw}{d\nu}(t) \in A_t w(t) + f\left(t, w(t), \int_0^t b(s, h(s))\, dz_s\right)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

By construction $w(0) = u_0$, and we have already seen that $w(\cdot)$ is BVRC and its differential measure dw is ν-absolutely continuous. The ν-a.e. inclusion will be shown by applying an argument of Komlos convergence. Indeed, since $(\frac{du_{h_n}}{d\nu} + k_n)_n$ weakly converges to $\frac{dw}{d\nu} + k$ in $L^1_{\mathbb{R}^e}(I, \nu)$ we may suppose that $(\frac{du_{h_n}}{d\nu} + k_n)_n$ Komlos converges to $\frac{dw}{d\nu} + k$. There is a ν-negligible set Q of I such that

$$-\frac{du_{h_n}}{d\nu}(t) \in A_t u_{h_n}(t) + k_n(t), \text{ for all } n \in \mathbb{N} \text{ and } t \in I \setminus Q,$$

$$\lim_n \frac{1}{n}\sum_{j=1}^n \left[\frac{du_{h_j}}{d\nu}(t) + k_j(t)\right] = \frac{dw}{d\nu}(t) + k(t), \quad \text{for all } t \in I \setminus Q.$$

Let $t \in I \setminus Q$ and let $\zeta \in D(A_t)$. According to the inclusion

$$-\frac{du_{h_n}}{d\nu}(t) \in A_t u_{h_n}(t) + k_n(t)$$

and the monotonicity of A_t we have

$$\left\langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), u_{h_n}(t) - \zeta \right\rangle \leq \langle A_t^0 \zeta, \zeta - u_{h_n}(t) \rangle.$$

Writing

$$\begin{aligned} &\left\langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), w(t) - \zeta \right\rangle \\ &= \left\langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), u_{h_n}(t) - \zeta \right\rangle + \left\langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), w(t) - u_{h_n}(t) \right\rangle \end{aligned}$$

yields

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), w(t) - \zeta \right\rangle \\ &= \frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), u_{h_j}(t) - \zeta \right\rangle + \frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), w(t) - u_{h_j}(t) \right\rangle, \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), w(t) - \zeta \right\rangle \\ &\leq \frac{1}{n}\sum_{j=1}^{n} \langle A_t^0 \zeta, \zeta - u_{h_j}(t) \rangle + (\eta + M)\frac{1}{n}\sum_{j=1}^{n} \|w(t) - u_{h_j}(t)\|. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the latter inequality gives

$$\left\langle \frac{dw}{d\nu}(t) + k(t), w(t) - \zeta \right\rangle \leq \langle A_t^0 \zeta, \zeta - w(t) \rangle.$$

It follows by Lemma 1.1 that

$$-\frac{dw}{d\nu}(t) \in A_t w(t) + k(t) \quad \nu - \text{a.e. } t \in I.$$

Further, we have already seen in (7.1) that $w(t) \in D(A_t)$ for all $t \in I$. Consequently, the mapping $w(\cdot)$ is a BVRC solution of the aforementioned differential inclusion, so that by uniqueness $w = u_h$.

Now let us write by Lemma 5.3

$$\begin{aligned} \Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t,s) u_{h_n}(s)\, ds - \int_0^1 G(t,s) u_h(s)\, ds \\ &= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]\, ds \leq \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\|\, ds. \end{aligned}$$

Using for each $s \in I$ the convergence $\|u_{h_n}(s) - u_h(s)\| \to 0$ as $n \to \infty$ and using also the inequality $\|u_{h_n}(s) - u_h(s)\| \leq 2K$ for all $s \in I$, we deduce by the Lebesgue convergence theorem that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\| \, ds \to 0,$$

which entails that $(\Phi(h_n))_n$ converges uniformly on I to $\Phi(h)$, as desired. Then $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous for the topology induced by the topology of uniform convergence on I, hence by the Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$ we have

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s) u_h(s) \, ds,$$

with du_h ν-absolutely continuous and

$$\begin{cases} u_h(0) = u_0, \quad u_h(t) \in D(A(t)), \, \forall t \in I \\ -\frac{du_h}{d\nu}(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dz_s) \frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

According to Lemma 5.3(c), this translates that we have just shown that there exists a mapping $h \in W^{\alpha,1}_{B,\mathbb{R}^e}(I)$ such that

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u_h(t), \; t \in I \\ I^\beta_{0^+} h(t) \,|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+} h(1) \end{cases}$$

and such that du_h is ν-absoluetly continuous with

$$\begin{cases} u_h(0) = u_0, \quad u_h(t) \in D(A(t)), \forall t \in I \\ -\frac{du_h}{d\nu}(t) \in A_t u_h(t) + f\big(t, u_h(t), \int_0^t b(s, h(s)) dz_s\big) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

This finishes the proof of the theorem. □

Corollary 7.3. *Let us keep notation and assumptions as in Theorem* 7.2 *and let* $L : [0,1] \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0, \infty[$ *be a lower semicontinuous integrand such that* $L(t, x, y, .)$ *is convex on* $\mathbb{R}^e$ *for every* $(t, x, y) \in [0,1] \times \mathbb{R}^e \times \mathbb{R}^e$*, and let* $u_0 \in D(A(0))$*. Then the problem of minimizing the cost function*

$$\int_0^1 L\big(t, \int_0^t b(s, \xi(s)) \, dz_s, u(t), \frac{du}{d\nu}(t)\big) \, d\nu(t)$$

over the pairs (ξ, u) *with* $\xi \in W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ *and* $u : I := [0,1] \to \mathbb{R}^e$ *BVRC, and subject to*

$$\begin{cases} D^\alpha \xi(t) + \lambda D^{\alpha-1} \xi(t) = u(t), \; t \in I \\ I^\beta_{0^+} \xi(t) \,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0^+} \xi(1) \\ u(0) = u_0 \\ u(t) \in D(A(t)), \forall t \in I \\ du \;\; \nu\text{-}\textit{absolutely continuous} \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f\left(t, u(t), \int_0^t b(s, x(s)) \, dz_s\right) \frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I \end{cases}$$

has an optimal solution.

Proof. Let $(h_n, u_n)_n$ be a minimizing sequence, namely

$$\lim_n \int_0^T L(t, \int_0^t b(s, h_n(s))dz_s, u_n(t), \dot{u}_n(t))d\nu = \inf_{(k,v)} \Big[\int_0^T L(t, \int_0^t b(s, k(s))dz_s, v(t), \dot{v}(t))d\nu \Big]$$

$$\begin{cases} D^\alpha h_n(t) + \lambda D^{\alpha-1} h_n(t) = u_n(t), \\ I^\beta_{0+} h_n(t)\mid_{t=0} = 0, \quad h_n(1) = I^\gamma_{0+} h_n(1) \\ u_n(0) = u_0, \quad u_n(t) \in D(A(t)), \forall t \in I \\ du_n \ \ \nu\text{-absolutely continuous} \\ -\frac{du_n}{d\nu}(t) \in A_t u_n(t) + f(t, u_n(t), \int_0^t b(s, h_n(s))dz_s)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

First, by compactness of the set $\mathcal{X}$ (as seen in the proof of Theorem 7.2), there is a (not relabeled) subsequence of $(h_n)_n$ in $\mathcal{X}$ converging uniformly on I to $h \in \mathcal{X}$. Second, by the proof of Theorem 7.2 the sequence $(u_n)_n$ of solutions to

$$\begin{cases} u_n(0) = u_0, \quad u_n(t) \in D(A(t)), \forall t \in I \\ du_n \ \ \nu\text{-absolutely continuous} \\ -\frac{du_n}{d\nu}(t) \in A_t u_n(t) + f(t, u_n(t), \int_0^t b(s, h_n(s))dz_s)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I \end{cases}$$

admits a subsequence (not relabeled) converging pointwise to a BVRC mapping $u(\cdot)$ with $\frac{du_n}{d\nu} \to \frac{du}{d\nu}$ weakly in $L^1_{\mathbb{R}^e}(I, \nu)$, and the differential measure du is ν-absolutely continuous. By compactness of the set (again seen in the proof of Theorem 7.2)

$$\mathcal{Y} := \{t \mapsto \int_0^\cdot b(s, x(s))dz_s : x \in \mathcal{X}\}$$

for the topology of uniform convergence on I, we may suppose that the sequence of mappings $\big(t \mapsto \int_0^t b(s, h_n(s))\, dz_s\big)_n$ converges uniformly on the interval I to the mapping $t \mapsto \int_0^t b(s, h(s))dz_s$. So, by repeating the arguments in the proof of Theorem 7.2, it ensues that $u(\cdot)$ satisfies the inclusion

$$\begin{cases} u(0) = u_0, \quad u(t) \in D(A(t)), \forall t \in I \\ du \ \ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f\left(t, u(t), \int_0^t b(s, h(s))\, dz_s\right)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

Further, by Lemma 5.3 the equalities

$$\begin{cases} D^\alpha h_n(t) + \lambda D^{\alpha-1} h_n(t) = u_n(t), \ \ t \in I \\ I^\beta_{0+} h_n(t)\mid_{t=0} = 0, \quad h_n(1) = I^\gamma_{0+} h_n(1) \end{cases}$$

are equivalent to
$$h_n(t) = \int_0^1 G(t, s)u_n(s)\, ds,$$

again with the Green function G considered in (5.6). Therefore, by passing to the limit in this equality as $n \to \infty$, we obtain

$$h(t) = \int_0^1 G(t, s)u(s)\, ds \text{ for all } t \in I.$$

Altogether, we see that the pair (h,u), with $\xi \in W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ and $u : I \to \mathbb{R}^e$, satisfies the system

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t),\\ I^\beta_{0^+} h(t)\mid_{t=0} = 0, \quad h(1) = I^\gamma_{0^+} h(1)\\ u(0 = u_0, \quad u(t) \in D(A(t)), \forall t \in I\\ du \ \nu\text{-absolutely continuous}\\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s,h(s))dz_s)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

According to the lower semicontinuity of the above integral functional (see [33, Theorem 8.16]) it ensues that

$$\liminf_n \int_0^1 L(t, \int_0^t b(s,h_n(s))dz_s, u_n(t), \frac{du}{d\nu}(t))d\nu \geq \int_0^1 L(t, \int_0^t b(s,h(s))dz_s, u(t), \frac{du}{d\nu}(t))d\nu,$$

so we see that the pair (h,u) is an optimal solution. □

Several variants of Theorem 7.2 and its Corollary 7.3 can be obtained with either other classes of time dependent maximal monotone operators or with other fractional-type derivatives. For example, regarding the case of dynamic systems with Caputo fractional differential equations and evolution inclusions with rough signals, we state the following variant. As in the previous Section 6, ${}^cD^\alpha\xi$ denotes the Caputo fractional derivative of order α of a mapping $\xi : [0,1] \to \mathbb{R}^e$.

Theorem 7.4. *Let $I := [0,1]$ and $\alpha \in]1,2]$. For each $t \in I$ let $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ be a maximal monotone operator satisfying:*

($\mathcal{A}_1^*$) *There is a real constant $c > 0$ such that*

$$\|A^0_t x\| \leq c(1 + \|x\|) \quad \textit{for every } (t,x) \in I \times D(A_t),$$

($\mathcal{A}_2^*$) $\mathrm{dis}(A_t, A_s) \leq dr(]s,t])$, *for all $0 \leq s \leq t \leq 1$, where $r : I \to \mathbb{R}^+$ is a non-decreasing right continuous function on I with $r(0) = 0$ and $r(1) < +\infty$,*

($\mathcal{A}_3^*$) ($\mathcal{A}_3^*$) *$D(A(t))$ is closed for each $t \in I$.*

Let $u_0 \in D(A(0))$ and let $z \in C^{1-var}(I, \mathbb{R}^d)$. Let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying for some real constant $M_b > 0$ and some control function ω on I the conditions

($\mathcal{B}_1$) $|b(t,x)| \leq M_b$ *for all $x \in \mathbb{R}^e$,*

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \leq \omega(s,t)$ *for all $x \in \mathbb{R}^e$ and $s,t \in I$ with $s \leq t$,*

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \leq M_b\|x - y\|$ *for all $t \in I$ and $x,y \in \mathbb{R}^e$.*

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping with $f(\cdot,x,y)$ Borel measurable for all $(x,y) \in \mathbb{R}^e \times \mathbb{R}^e$ and $f(t,\cdot,\cdot)$ continuous for all $t \in I$, and which satisfies for some real constant $M_f > 0$:

(i) $\|f(t,x,z) - f(t,y,z)\| \leq M_f\|x - y\|$ *for all $(t,x,y,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e$,*

(ii) $\|f(t,x,z)\| \leq M_f$ *for all $(t,x,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.*

Consider the measure $\nu := dr + dt$ and assume that there is $\delta \geq 0$ such that for $M := \{M_b, M_f\}$

$$0 \leq 2M\frac{dt}{d\nu}(t)\nu(\{t\}) \leq \delta < 1 \quad \textit{for all } t \in I.$$

Then there exist a ${}^cW^{\alpha,\infty}_{B,\mathbb{R}^e}([0,1])$ *mapping* $\xi : I \to \mathbb{R}^e$ *and a BVRC mapping* $u : I \to \mathbb{R}^e$ *satisfying the dynamic system with rough signal*

$$\begin{cases} {}^cD^\alpha\xi(t) = u(t),\ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(t) \in D(A(t)),\, t \in I \\ du\ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f\left(t, u(t), \int_0^t b(s,\xi(s))\, dz_s\right)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I \\ u(0) = u_0. \end{cases}$$

Let $L : I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0,\infty[$ *be a lower semicontinuous integrand such that* $L(t,x,y,.)$ *is convex on* $\mathbb{R}^e$ *for every* $(t,x,y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$. *Then the problem of minimizing the cost function* $\int_0^1 L\big(t, \int_0^t b(s,\xi(s))dz_s, u(t), \frac{du}{d\nu}(t)\big)\, d\nu(t)$ *over the pairs* (ξ,u) *with* $\xi \in {}^cW^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ *and* $u : I \to \mathbb{R}^e$ *BVRC, and subject to*

$$\begin{cases} {}^cD^\alpha\xi(t) = u(t),\ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(t) \in D(A(t)),\, t \in I \\ du\ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f\left(t, u(t), \int_0^t b(s,\xi(s))\, dz_s\right)\frac{dt}{d\nu}(t) \quad \nu\text{-a.e. } t \in I \\ u(0) = u_0 \end{cases}$$

has an optimal solution.

Proof. The proof is omitted. It is sufficient to repeat the arguments in the proofs of the previous Theorem 7.2 and Corollary 7.3 with suitable modifications using the properties of Caputo fractional differential equations given in Theorem 3.1. $\square$

Direct applications to convex sweeping processes hold true. It is enough in Theorem 7.2 and Theorem 7.4 to take the normal cone $N_{C(t)}(\cdot)$ in place of the operator $A_t(\cdot)$ since it is known (see [82]) that for closed convex sets S, S' of E one has $\operatorname{dis}(N_S, N_{S'}) = \operatorname{haus}(S,S')$ for the pseudo-distance between the maximal monotone operators N_S and $N_{S'}$, representing the normal cones to the closed convex sets S and S' respectively. First, we state the result related to the coupling with fractional differential equations with Riemann-Liouville fractional derivatives.

Theorem 7.5. *Let* $I := [0,1]$ *and* $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$. *For each* $t \in I$ *let* $C(t)$ *be a closed convex set in* $\mathbb{R}^e$ *such that*

$$\operatorname{haus}(C(t), C(s)) \leq dr(]s,t]) \quad \textit{for all } s,t \in I \textit{ with } s \leq t,$$

where $r : I \to \mathbb{R}_+$ *is a non-decreasing right-continuous function on* I *with* $r(0) = 0$ *and* $r(1) < +\infty$, *and let* $u_0 \in C(0)$. *Let* $z \in C^{1-var}(I, \mathbb{R}^d)$ *and let* $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ *be a continuous integrand operator satisfying for some real constant* $M_b > 0$ *and some control function* ω *on* I *the conditions*

$(\mathcal{B}_1)$ $\quad |b(t,x)| \leq M_b$ *for all* $x \in \mathbb{R}^e$,

$(\mathcal{B}_2)$ $\quad |b(s,x) - b(t,x)| \leq \omega(s,t)$ *for all* $x \in \mathbb{R}^e$ *and* $s, t \in I$ *with* $s \leq t$,

$(\mathcal{B}_3)$ $\quad |b(t,x) - b(t,y)| \leq M_b\|x - y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ *be a mapping with* $f(\cdot, x, y)$ *Borel measurable for all* $(x, y) \in \mathbb{R}^e \times \mathbb{R}^e$ *and* $f(t, \cdot, \cdot)$ *continuous for all* $t \in I$*, and which satisfies for some real constant* $M_f > 0$*:*

(i) $\|f(t, x, z) - f(t, y, z)\| \leq M_f\|x - y\|$ *for all* $(t, x, y, z) \in I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e$,

(ii) $\|f(t, x, z)\| \leq M_f$ *for all* $(t, x, z) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.

Consider the measure $\nu := dr + dt$ *and assume that for there is* $\delta \geq 0$ *such that for* $M := \{M_b, M_f\}$

$$0 \leq 2M\frac{dt}{d\nu}(t)\nu(\{t\}) \leq \delta < 1 \quad \text{for all } t \in I.$$

Then there exist a $W^{\alpha,1}_{B,\mathbb{R}^e}([0, 1])$ *mapping* $\xi : I \to \mathbb{R}^e$ *and a BVRC mapping* $u : I \to \mathbb{R}^e$ *satisfying to the dynamic system with rough signal*

$$\begin{cases} D^\alpha\xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t),\ t \in I \\ I^\beta_{0+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0+}\xi(1) \\ u(t) \in C(t),\ t \in I, \quad u(0) = u_0, \\ du\ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in N_{C(t)}u(t) + f\left(t, u(t), \int_0^t b(s, \xi(s))\,dz_s\right)\frac{dt}{d\nu}(t) \qquad \nu\text{−a.e. } t \in I. \end{cases}$$

Let $L : I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0, \infty[$ *be a lower semicontinuous integrand such that* $L(t, x, y, .)$ *is convex on* $\mathbb{R}^e$ *for every* $(t, x, y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$*. Then the problem of minimizing the cost function*

$$\int_0^1 L\Big(t, \int_0^t b(s, \xi(s))\,dz_s, u(t), \frac{du}{d\nu}(t)\Big)\,d\nu(t)$$

over pairs (ξ, u) *with* $\xi \in W^{\alpha,1}_{B,\mathbb{R}^e}([0, 1])$ *and* $u : I \to \mathbb{R}^e$ *BVRC, and subject to*

$$\begin{cases} D^\alpha\xi(t) + \lambda D^{\alpha-1}\xi(t) = u(t),\ t \in [0, 1] \\ I^\beta_{0+}\xi(t)\,|_{t=0} = 0, \quad \xi(1) = I^\gamma_{0+}\xi(1) \\ u(t) \in C(t),\ t \in I, \quad u(0) = u_0 \\ du\ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in N_{C(t)}u(t) + f(t, u(t), \int_0^t b(s, \xi(s))dz_s)\frac{dt}{d\nu}(t) \quad \nu\text{−a.e. } t \in I. \end{cases}$$

has an optimal solution.

The following theorem states the similar result for the coupling of BV sweeping processes with differential equations with Caputo fractional derivatives.

Theorem 7.6. *Let* $I := [0, 1]$ *and* $\alpha \in]1, 2]$*. For each* $t \in I$ *let* $C(t)$ *be a closed convex set in* $\mathbb{R}^e$ *such that*

$$\text{haus}(C(t), C(s)) \leq dr(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \leq t,$$

where $r : I \to \mathbb{R}_+$ *is a non-decreasing right-continuous function on* I *with* $r(0) = 0$ *and* $r(1) < +\infty$*, and let* $u_0 \in C(0)$*.*

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying for some real constant $M_b > 0$ and some control function ω on I the conditions

($\mathcal{B}_1$) $|b(t,x)| \leq M_b$ *for all* $x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \leq \omega(s,t)$ *for all* $x \in \mathbb{R}^e$ *and* $s, t \in I$ *with* $s \leq t$,

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \leq M_b \|x - y\|$ *for all* $t \in I$ *and* $x, y \in \mathbb{R}^e$.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a mapping with $f(\cdot, x, y)$ Borel measurable for all $(x,y) \in \mathbb{R}^e \times \mathbb{R}^e$ and $f(t, \cdot, \cdot)$ continuous for all $t \in I$, and which satisfies for some real constant $M_f > 0$:

(i) $\|f(t,x,z) - f(t,y,z)\| \leq M_f \|x - y\|$ *for all* $(t,x,y,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e$,

(ii) $\|f(t,x,z)\| \leq M_f$ *for all* $(t,x,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.

Consider the measure $\nu := dr + dt$ and assume that there is $\delta \geq 0$ such that for $M := \{M_b, M_f\}$

$$0 \leq 2M \frac{dt}{d\nu}(t)\nu(\{t\}) \leq \delta < 1 \quad \text{for all } t \in I.$$

Then there exist a ${}^cW^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ mapping $\xi : I \to \mathbb{R}^e$ and a BVRC mapping $u : I \to \mathbb{R}^e$ satisfying to the dynamic system with rough signal

$$\begin{cases} {}^cD^\alpha \xi(t) = u(t), \ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(t) \in C(t), \, t \in I, \quad u(0) = u_0 \\ du \ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in N_{C(t)} u(t) + f\left(t, u(t), \int_0^t b(s, \xi(s))\, dz_s\right) \frac{dt}{d\nu}(t) \quad \nu\text{–a.e. } t \in I. \end{cases}$$

Let $L : I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0, \infty[$ be a lower semicontinuous integrand such that $L(t, x, y, .)$ is convex on $\mathbb{R}^e$ for every $(t, x, y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$. Then the problem of minimizing the cost function $\int_0^1 L\big(t, \int_0^t b(s, \xi(s))\, dz_s, u(t), \frac{du}{d\nu}(t)\big)\, d\nu(t)$ over pairs (ξ, u) subject to

$$\begin{cases} {}^cD^\alpha \xi(t) = u(t), \ t \in I \\ \xi(0) - \frac{d\xi}{dt}(0) = 0 \\ \xi(1) + \frac{d\xi}{dt}(1) = 0 \\ u(t) \in C(t) \, t \in I, \quad u(0) = u_0 \\ du \ \nu\text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in N_{C(t)} u(t) + f\left(t, u(t), \int_0^t b(s, \xi(s))\, dz_s\right) \frac{dt}{d\nu}(t) \quad \nu\text{–a.e. } t \in I. \end{cases}$$

has an optimal solution.

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