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Continuity Phenomenon of Kenderov and Porosity: the Case of Countable Systems

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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Kenderov [*Continuity-like properties of set-valued mappings*, Serdica Bulg. Math. Publ. 9 (1983) 149–160] proved a general result stating that an arbitrary multivalued mapping from a topological space X to a set Y has some properties resembling continuity at every point of a residual subset $X_0 \subset X$ (i.e. its complement $X \setminus X_0$ is of first Baire category). This statement has far-reaching consequences and can be called a "continuity phenomenon", since it proves and unifies in a general approach several different results in topology and functional analysis, mainly concerning single-valuedness almost everywhere (in topological sense) of multivalued mappings. In this paper we show that, in the case when X is a metric space and Y is a compact separable topological space, the set X_0 is even σ-full cone porous (a notion introduced here). It implies that the above (and other) "generic" results have "σ-full cone porous" versions, with unified proofs.

Keywords: Porous sets, fully cone porous set, generalized monotone mapping, submonotone mapping, lower almost continuity.

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1. Introduction and notation

Baire categories play an important role in several problems of analysis. The definition of a set of the first Baire category is based on the notion of a nowhere dense set. The latter was strengthened by Dolzenko [2], who introduced the notion of a porous set and started the theory of σ-porous sets. After Dolzenko's paper, both porosity and related σ-porosity have been used to strengthen theorems involving Baire categories, see expository papers on the subject [9], [12], [13], [14].

Let $F : X \to 2^Y$ be a multivalued mapping from a topological space X into a set Y and A be a subset of Y. Define

$$H(A) = \left\{ x \in X : \begin{array}{ll} (a) & Fx \cap A \neq \emptyset \ \text{ and} \\ (b) & \forall \text{ open } U \ni x,\ \exists \text{ open } U' \subset U : F(U') \cap A = \emptyset \end{array} \right\}$$

We say, following [5], that F is *A-lower almost continuous* (A-lac) at some point x_0 of X if x_0 does not belong to $H(A)$.

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Equivalently, F is A-lac at x_0 if either $F(x_0) \cap A = \emptyset$ or there exists an open neighborhood U of x_0 such that $Fx \cap A \neq \emptyset$ for all x from some dense subset of U.

The same can be expressed also in the following way: F is A-lac at x_0 if from $F(x_0) \cap A \neq \emptyset$ it follows that the closure in X of the set $F^{-1}(A)$ is dense in some open neighborhood of x_0.

Lemma 1.1. (Kenderov [5]) *Let $\mathcal{A} := \{A_i\}_{i=1}^{\infty}$ be a countable family of subsets of the set Y. Then every mapping $F : X \to 2^Y$ from the topological space X into Y is A-lac (for every A from $\mathcal{A}$) at the points of some residual subset of X.*

The aim of this paper is to generalize this lemma and other results from [5] to porous set and present some far reaching consequences.

Let X be a metric space. An open (resp. closed) ball in X with center x and radius $r > 0$ is denoted by $B(x, r)$ (resp. $\overline{B}(x, r)$). If X is a normed space, B_X (resp. S_X) will denote its closed unit ball (resp. unit sphere), i.e. $B_X := \overline{B}(0, 1)$ (resp. $S_X := \{x \in X : \|x\| = 1\}$).

Definition 1.2. A subset A of X is said to be *porous* in X, if there exist $\alpha > 0$ and $r_0 > 0$ such that for every $r \in (0, r_0)$ and $x \in A$ there is $z \in X$ such that $B(z, \alpha r) \subset B(x, r) \setminus A$. A set A is called *σ-porous* in X, if it is a countable union of sets, which are porous in X. □

Clearly, a set which is σ-porous in X is of first category, the converse being false in general. Given a normed space E, for $v \in E$ and $c > 0$ define the cone

$$C(v, c) = \{x \in E : x = \lambda v + w, \lambda > 0, \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(v, c).$$

Definition 1.3. A subset M of a normed space E is called *cone porous (resp. fully cone porous) at* $x \in M$, if there exist $r > 0, v \in E, \|v\| = 1$ and $c \in (0, 1)$ such that

$$M \cap B(x, r) \cap \big(x + C(v, c)\big) = \emptyset,$$

(resp. $M \cap \big(x + C(v, c)\big) = \emptyset$).

The set M is called *cone porous* (resp. fully cone porous), if it is cone porous (resp. fully cone porous) at every of its point. M is called *$\sigma-$cone porous* (resp. σ-fully cone porous), if it is an union of countably many cone porous (resp. σ-fully cone porous) sets. □

It is clear that every fully cone porous set is cone porous and every cone porous set is porous.

The cone porous sets were defined firstly in [13] under the name cone supported sets, however the definition of fully cone supported sets introduced here seams to be new.

Proposition 1.4. *Let X be a metric space, $P \subset X$ be a non-empty subset and $\alpha \in (0, 1)$. Then the following set is porous:*

$$X_\alpha(P) := \left\{ x \in X \setminus P : \begin{array}{l} \exists r_0 > 0 : \forall r \in (0, r_0) \\ \exists z \in B(x, r(1-\alpha)) : B(z, \alpha r) \subset P \end{array} \right\}$$

Proof. Take $x \in X_\alpha(P)$. Then $x \notin P$ and

$$\exists r_0 > 0 : \forall r \in (0, r_0) \ \ \exists z \in B\Big(x, r(1-\alpha)\Big) : B(z, \alpha r) \subset P,$$

which means that $B(z, \alpha r) \subset B(x, r) \setminus X_\alpha(P)$. □

Similarly we have the following.

Proposition 1.5. *Let P be a non-empty subset of a normed space E and assume that $r > 0$ and $c > 0$. Then:*

(a) *the set $C_{r,c}(P) := \Big\{x \in E \setminus P : \exists\, v \in E, \|v\| = 1, B(x;r) \cap \big(x + C(v,c)\big) \subset P\Big\}$ is cone porous.*

(b) *the set $C_c(P) := \Big\{x \in E \setminus P : \exists\, v \in E, \|v\| = 1, \big(x + C(v,c)\big) \subset P\Big\}$ is fully cone porous.*

Definition 1.6. The multivalued mapping $F : X \to 2^Y$ from a metric space X into a set Y will be called *A-lac* for a non-empty subset $A \subset Y$ (A-lower almost continuous) at $x_0 \in X$ if either $Fx_0 \cap A = \emptyset$, or

$$\forall \alpha > 0,\ \forall r_0 > 0,\ \exists r \in (0, r_0) : \quad F\big(B(x, \alpha r)\big) \cap A \neq \emptyset, \quad \forall x \in B\big(x_0, r - \alpha r\big).$$

Definition 1.7. The multivalued mapping $F : E \to 2^Y$ from a normed space E into a set Y will be called

(a) *A-cone lac* for a non-empty subset $A \subset Y$ (A-cone lower almost continuous) at $x \in E$ if either $F(x) \cap A = \emptyset$, or

$$\forall r > 0,\ \forall c > 0,\ \forall v \in S_E, \quad F\big(B(x;r) \cap \big(x + C(v,c)\big)\big) \cap A \neq \emptyset,$$

(b) *A-fully cone lac* for a non-empty subset $A \subset Y$ (A-fully cone lower almost continuous) at $x \in E$ if either $F(x) \cap A = \emptyset$, or

$$\forall c > 0, \forall v \in S_E, \quad F\big(x + C(v,c)\big) \cap A \neq \emptyset.$$

Theorem 1.8. *Let $\{A_i\}_{i=1}^{\infty}$ be a countable family of subsets of the set Y. Then every multivalued mapping $F : X \to 2^Y$ from the metric space X into Y is A_i-lac (for every i) on the complement of some σ-porous subset of X.*

Proof. For $P_i = \big\{x \in X : Fx \cap A_i = \emptyset\big\}$ the set $X_{1/n}(P_i)$ is porous for every $n \geq 2$ by Proposition 1.4. Therefore the set

$$X_i = \bigcup_{n=2}^{\infty} X_{1/n}(P_i)$$

is σ-porous for every $i \geq 1$. Therefore the set $X_0 = \bigcup_{i=1}^{\infty} X_i$ is σ-porous as well.

We have

$$X \setminus X_0 = X \setminus \Big(\bigcup_{i=1}^{\infty} \bigcup_{n=2}^{\infty} X_{1/n}(P_i)\Big) \tag{1}$$

$$= \bigcap_{i=1}^{\infty} \bigcap_{n=2}^{\infty} \Big(X \setminus X_{1/n}(P_i)\Big). \tag{2}$$

Let $x_0 \in X \setminus X_0, \alpha > 0$ and i be fixed. If $x_0 \in P_i$ the proof is completed. Assume that $x_0 \notin P_i$, so $F(x_0) \cap A_i \neq \emptyset$. Choose $n > 1/\alpha$. Since $x_0 \notin X_{1/n}(P_i)$, by definition of $X_{1/n}(P_i)$, we have

$$\forall r_0 > 0, \exists r \in (0, r_0) \text{ such that } B(z, 1/n) \not\subset P_i \quad \forall z \in B\Big(x_0, r(1-1/n)\Big)$$

which means

$$\forall z \in B\Big(x_0, r(1-1/n)\Big) \quad \exists x \in B(z, 1/n) \subset B(z, \alpha) \text{ such that } x \notin P_i.$$

Hence $F(x) \cap A_i \neq \emptyset$ and the proof is complete. □

Similarly we have the following.

Theorem 1.9. *Let $\{A_i\}_{i=1}^{\infty}$ be a countable family of subsets of the set Y. Then every multivalued mapping $F : E \to 2^Y$ from a Banach space E into Y is*

(a) *A_i-cone lac (for every i) on the complement of some σ-cone porous subset of E.*

(b) *A_i-full cone-lac (for every i) on the complement of some σ-full cone porous subset of E.*

2. Applications to submonotone mappings

Let E be a Banach space and E^* be its topological dual. We recall that S_E denotes the unit sphere of E.

A multivalued operator $T : Dom(T) \to 2^{E^*}$ is called *submonotone* if

$$\liminf_{\substack{x_0 \neq x \to_e x_0 \\ y \in T(x),\ y_0 \in T(x_0)}} \frac{\langle x - x_0, y - y_0 \rangle}{\|x - x_0\|} \geq 0, \quad \forall e \in E,\ \|e\| = 1,\ \forall x_0 \in Dom(T),$$

which means $\quad \forall \varepsilon > 0,\ \forall e \in S,\ \exists\, \delta > 0 : \dfrac{\langle x - x_0, y - y_0 \rangle}{\|x - x_0\|} \geq -\varepsilon$

$$\forall x \in U(x_0, e, \delta) \cap Dom(T),\ \forall y \in Tx,\ \forall y_0 \in Tx_0,$$

where $Dom(T) = \{x \in E : T(x) \neq \emptyset\}$ and

$$U(x_0, e, \delta) = \Big\{x \in E : x \neq x_0,\ \|x - x_0\| < \delta,\ \Big\|\frac{x - x_o}{\|x - x_0\|} - e\Big\| < \delta\Big\}.$$

A multivalued operator $T : Dom(T) \to 2^{E^*}$ is called *strictly submonotone* if

$$\liminf_{\substack{x_i \to x_0 \\ y_i \in T(x_i),\ i=1,2 \\ x_1 \neq x_2,\ x_1 - x_2 \to_e 0}} \frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\|x_1 - x_2\|} \geq 0, \quad \forall e \in S,\ \forall x_0 \in E,$$

which means $\quad \forall \varepsilon > 0,\ \forall e \in S,\ \exists\, \delta > 0 : \dfrac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\|x_1 - x_2\|} \geq -\varepsilon$

whenever $\ y_i \in T(x_i),\ x_1 \neq x_2,\ i = 1, 2,\ \|x_i - x_0\| < \delta,\ \Big\|\dfrac{x_1 - x_2}{\|x_1 - x_2\|} - e\Big\| < \delta.$

The operator T is called *uniformly strictly submonotone* if δ does not depend on $e \in S$. T is called *locally bounded*, if for every $x_0 \in E$ there exists $\gamma > 0$ and $M > 0$ such that

$$\|x^*\| < M \quad \forall x^* \in T(x),\ \forall x \in B(x_0, \gamma).$$

Theorem 2.1. *Suppose that E is a Banach space such that E^* is separable, and $T : E \to 2^{E^*}$ is a locally bounded uniformly strictly submonotone operator with domain $D(T) \neq \emptyset$. Then there exists a σ-cone porous set $X_0 \subset D(T)$ such that T is single-valued and upper semicontinous on $D(T) \setminus X_0$.*

Proof. Let $\{x_n^*\}_{n=1}^{\infty}$ be a dense subset of E^* and define

$$A_{n,k} = x_n^* + \frac{1}{k} B_{E^*},$$

where B_{E^*} is the closed unit ball in E^*.

By Theorem 1.9(a), T is $A_{n,k}$-cone-lac for every $n, k \geq 1$ on a complement of a σ-cone-porous set X_0.

Let $\bar{x}_0 \in D(T) \setminus X_0$ and $\varepsilon > 0$. Since T is locally bounded at $\bar{x}_0$, there exists $\gamma > 0$ and $M > 0$ such that

$$\|x^*\| < M \quad \forall x^* \in T(x),\ \forall x \in B(\bar{x}_0, \gamma).$$

Assume that $T(\bar{x}_0) \cap A_{n_0,k_0} \neq \emptyset$ for some $n_0, k_0 \geq 1$. Put

$$\delta = \frac{\varepsilon}{2(M + \max\{\|a\| : a \in A_{n_0,k_0}\})}. \tag{3}$$

and
$$r = \min\{\delta, \gamma\}.$$

Assume that $T\big(B(\bar{x}_0, r)\big) \not\subset A_{n_0,k_0} + \varepsilon B_{E^*}$.

Then there exist $x_0 \in D(T) \cap B(\bar{x}_0, r)$ and $x_0^* \in T(x_0) \setminus \big(A_{n_0,k_0} + \varepsilon B_{E^*}\big)$. By the separation theorem, there exists $e_0 \in E$, $\|e_0\| = 1$, such that

$$\big\langle e_0, x_0^* \big\rangle > \max \big\langle e_0, A_{n_0,k_0} \big\rangle + \varepsilon. \tag{4}$$

We will prove that

$$\big\langle e, x_0^* \big\rangle > \max \big\langle e, A_{n_0,k_0} \big\rangle\ + \varepsilon/2, \qquad \forall e \in B(e_0, \delta). \tag{5}$$

Take $e \in B(e_0, \delta)$ and $x_1^* \in A_{n_0,k_0}$ such that

$$\langle e, x_1^* \rangle > \max \langle e, A_{n_0,k_0} \rangle - \delta.$$

Then:

$$\begin{aligned}
\langle e, x_0^* \rangle &= \langle e - e_0, x_0^* \rangle + \langle e_0, x_0^* \rangle \\
&\geq -\delta \|x_0^*\| + \max \langle e_0, A_{n_0,k_0} \rangle + \varepsilon \\
&\geq -\delta \|x_0^*\| + \langle e_0 - e + e, x_1^* \rangle + \varepsilon,
\end{aligned}$$

hence

$$\begin{aligned}\langle e, x_0^*\rangle &> -\delta\|x_0^*\| - \delta\|x_1^*\| + \max\langle e, A_{n_0,k_0}\rangle - \delta + \varepsilon\\ &\geq -\delta(M + \|x_{n_0}^*\| + \frac{1}{k_0} + 1) + \max\langle e, A_{n_0,k_0}\rangle + \varepsilon\\ &> \max\langle e, A_{n_0,k_0}\rangle + \varepsilon/2 \quad \text{by} \ \ (3)\end{aligned}$$

and (5) is shown. We will prove that

$$B(x_0, \delta) \cap C(e_0, \delta/3) \subset U(x_0, e_0, \delta). \tag{6}$$

Let $z \in B(x_0, \delta) \cap C(e_0, \delta/3)$. Then

$$\|z - x_0\| < \delta$$

and

$$z = x_0 + a, \quad a \in C(e_0, \delta/3) = \bigcup_{\lambda > 0} \lambda B(e_0, \delta/3).$$

So $a = \lambda y$ for some $\lambda > 0$ and some $y \in B(e_0, \delta/3)$, and

$$\|a\| < \delta, \quad \|y - e_0\| < \delta/3, \|y\| \geq 1 - \delta/3.$$

Then

$$\begin{aligned}\left\|\frac{z - x_0}{\|z - x_0\|} - e_0\right\| &= \left\|\frac{a}{\|a\|} - e_0\right\|\\ &= \left\|\frac{y}{\|y\|} - e_0\right\|\\ &= \left\|\frac{y - e_0}{\|y\|} + \frac{e_0}{\|y\|} - e_0\right\|\end{aligned}$$

hence

$$\begin{aligned}\left\|\frac{z - x_0}{\|z - x_0\|} - e_0\right\| &\leq \frac{\delta/3}{1 - \delta/3} + \frac{1}{1 - \delta/3}\Big|1 - \|y\|\Big|\\ &\leq \frac{2\delta/3}{1 - \delta/3} < \delta.\end{aligned}$$

So we proved (6).

Since T is A_{n_0,k_0}-cone lac at $\bar{x}_0$, by Definition 1.7

$$\forall r > 0, \ \forall v \in S, \forall c > 0, \ \ T\Big(B(\bar{x}_0, r) \cap \big(x_0 + C(v, c)\big)\Big) \cap A_{n_0,k_0} \neq \emptyset. \tag{7}$$

In particular, for $r = \delta$, $v = e_0$, $c = \delta/3$ there exists

$$z_0 \in B(\bar{x}_0, \delta) \cap \big(\bar{x}_0 + C(e_0, \delta/3)\big)$$

such that $T(z_0) \cap A_{n_0,k_0} \neq \emptyset$.

Take $z_0^* \in Tz_0 \cap A_{n_0,k_0}$ and put $e_1 = \frac{z_0 - x_0}{\|z_0 - x_0\|}$. Then

$$\begin{aligned}\left\langle e_1, z_0^* \right\rangle &\geq \inf \left\langle e_1, Tz_0 \right\rangle \\ &\geq \sup \left\langle e_1, Tx_0 \right\rangle - \varepsilon \ \big(\text{using submonotonicity, since } z_0 \in U(x_0, e_0, \delta) \text{ by (6)}\big),\end{aligned}$$

which gives

$$\left\langle e_1, z_0^* \right\rangle \geq \left\langle e_1, x_0^* \right\rangle > \max \left\langle e_1, A_{k_0} \right\rangle + \varepsilon/2 \quad \big(\text{by (5)}\big) \geq \left\langle e_1, z_0^* \right\rangle + \varepsilon/2,$$

a contradiction.

So we proved that $diam Tx_0 < 2/k_0 + 2\varepsilon$. Since this is true for every $\varepsilon > 0$ and for every $k_0 \geq 1$ (for possibly different n_0), it follows that T is single-valued and upper semi-continuous on $D(T) \setminus X_0$, and the theorem is proved. □

Similarly we have the following.

Theorem 2.2. *Suppose that E is a Banach space such that E^* is separable, and $T : E \to 2^{E^*}$ is a locally bounded monotone operator with domain $D(T) \neq \emptyset$. Then there exists a σ-full cone porous set $X_0 \subset D(T)$ such that T is single-valued and upper semicontinous on $D(T) \setminus X_0$.*

The proof is the same as those of Theorem 2.1 as the only difference is that we use Theorem 1.9 (b) instead of Theorem 1.9 (a).

Remark 2.3. Theorem 2.1 is a generalization to submonotone operators of a Preiss-Zajicek's result [7], [8] with respect to the notion of σ-cone porous set. The result in [8] for monotone operators is with respect to an angle small set, a notion defined there and, in general, not comparable with the notion of σ-cone porous set, see [13], Note 4. The difference of Theorem 2.2 and the result in [8] is only with respect of the (incompatible) notions of porosity. It is worth to mention that a Zajicek's result [13], Theorem 1 contains a partial generalization to Asplund spaces of Theorem 2.2 with respect to a σ-cone porous set and without upper semi-continuity.

3. Generalized monotone mappings

Definition 3.1. A multivalued mapping $F : X \to 2^Y$, where X and Y are sets is called *monotone with respect to (w.r.t., for short) a function* $h : X \times Y \to \mathbb{R}$ if

$$h(x_1, y_1) - h(h_2, y_1) \geq h(x_1, y_2) - h(x_2, y_2) \quad \forall x_i \in X, \forall y_i \in Fx_i. \tag{8}$$

Example 3.2. The multivalued mapping $Fx = \text{argmax}_{y \in Y}\{h(x, y) + f(y)\}$, where $f : Y \to \mathbb{R}$ is any function, is monotone w.r.t. h. □

Further we assume that X is a Banach space, Y is a separable compact topological space, $h : X \times Y \to \mathbb{R}$ is continuous and $h(\cdot, y)$ is linear for every $y \in Y$. Then (8) is equivalent to

$$\inf_{y_1 \in Fx_1} h(x_1 - x_2, y_1) \geq \sup_{y_1 \in Fx_2} h(x_1 - x_2, y_2). \tag{9}$$

Definition 3.3. We say that h has the separation property (SP) if

$$\forall A \subset Y, \text{ closed}, \ \forall y \in Y \setminus A, \ \ \exists\, x \in X : h(x,y) > \sup_{a \in A} h(x,a).$$

Theorem 3.4. *Assume that $F : X \to 2^Y$ is a monotone multivalued mapping w.r.t. $h : X \times Y \to \mathbb{R}$, X is a Banach space, Y is a separable compact topological space, h is continuous and $h(.,y)$ is linear for every $y \in Y$. Assume that h has the (SP) property. Then there exists a σ-full cone porous set $X_0 \subset X$ such that for every $x \in X \setminus X_0$, F is single-valued and upper semi-continuous at x.*

Proof. This is a routine matter to prove.

Claim: *F has a maximal monotone w.r.t. h extension $\overline{F}$, which is upper semi-continuous.*

Let $\{A_i\}_{i=1}^{\infty}$ be a countable base of Y. By the porous version of the continuous phenomenon (see Theorem 1.9) there exists a σ-cone porous subset $X_0 \subset X$ such that $\overline{F}$ is $\overline{A_i}$-cone lac at every point of $X \setminus X_0$ for every $i \in \mathbb{N}$. This means by Definition 1.6 and Definition 1.7

$$X_0 = \bigcup_{n=1}^{\infty} X_n, \quad X_n \ \text{ is } \ \text{full cone porous}, \tag{10}$$

$$\forall x \in X \setminus X_0, \ \forall r > 0, \ \forall i = 1,2,...,\forall c > 0, \ \forall v \in S : \tag{11}$$

$$\overline{A_i} \cap \overline{F}(x) \neq \emptyset \ \ \Rightarrow \ \overline{A_i} \cap \overline{F}\Big(x + C(v,c)\Big) \neq \emptyset. \tag{12}$$

Assume that $\overline{A_{i_0}} \cap \overline{F}(x_0) \neq \emptyset$ but $\overline{F}(x_0) \not\subset \overline{A_{i_0}}$ for some $x_0 \in X \setminus X_0$ and some $i_0 \in \mathbb{N}$. Take $y_0 \in \overline{F}(x_0) \setminus \overline{A_{i_0}}$.

Then by property (SP)

$$\exists\, e_0 \in X, \ \|e_0\| = 1 : \ 2\varepsilon := h(e_0, y_0) - \sup h(e_0, \overline{A_{i_0}}) > 0. \tag{13}$$

By the continuity of $h(.,y)$,

$$\exists \delta > 0 : h(e_0, y_0) - h(e, y_0) < \varepsilon \quad \forall e \in B(e_0, \delta) \tag{14}$$

and

$$\sup h(e, \overline{A_{i_0}}) - \sup h(e_0, \overline{A_{i_0}}) < \varepsilon, \quad \forall e \in B(e_0, \delta). \tag{15}$$

Take $c = \delta$, $v = e_0$. By (11) and (12) there exists $x_1 \in x_0 + C(v,c)$ such that $\overline{F}(x_1) \cap \overline{A_{i_0}} \neq \emptyset$. Let $y_1 \in \overline{F}(x_1) \cap \overline{A_{i_0}}$ and $e_1 = \frac{x_1 - x_0}{\|x_1 - x_0\|}$. Then $\|e_1 - e_0\| < \delta$ and

$$\begin{aligned} h(e_1, y_1) &\geq \inf h(e_1, \overline{F}(x_1)) \\ &\geq \sup h(e_1, \overline{F}x_0) \quad \text{(by generalized monotonicity)} \\ &> h(e_1, y_0), \end{aligned}$$

so we have

$$\begin{aligned} h(e_1, y_1) &> h(e_0, y_0) - \varepsilon \quad (\ \text{by } (14)) \\ &= \sup h(e_0, \overline{A}_{i_0}) + \varepsilon \quad (\ \text{by } \ (13)) \\ &> \sup h(e_1, \overline{A}_{i_o}) \ \ (\ \text{by } (15)). \end{aligned}$$

This is a contradiction, because $y_1 \in \overline{A_{i_0}}$. So we proved that $\overline{F}$ is single-valued and upper semi-continuous on $X \backslash X_0$, therefore F possesses also the latter properties. $\square$

The following proposition is a variant of Proposition 4 in [6] for the Gâteaux differentiable case. In its statement, given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ on a Banach space X, as usual f^* denotes its Legendre-Fenchel conjugate defined on X^* by $f^*(x^*) = \sup_{y \in X} \big(\langle x^*, y\rangle - f(y)\big)$ for all $x^* \in X^*$.

Proposition 3.5. *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous function, $x^* \in intDom f^*$. Then the following statements are equivalent:*

(1) *∂f^* is single-valued and w-upper semi-continuous at x^*, and $\partial f^*(x^*) = \{x\}$, $x \in X$,*

(2) *f^* is Gâteaux differentiable at x^*, and $\nabla f^*(x^*) = x$,*

(3) *$f - x^*$ attains its w-minimum on X at x, which means $f - x^*$ attains its minimum at x and*

$$(f - x^*)(x_n) \to \min_{z \in X} (f - x^*)(z) \quad \textit{implies} \quad x_n \xrightarrow{w} x.$$

Proof. The equivalence of (1) and (2) is widely known.

(2) $\Longrightarrow$ (3)

Let $x = \nabla f^*(x^*)$ and $\{x_n\}$ be such that

$$f(x_n) - \langle x^*, x_n\rangle \to \inf_{z \in X} (f - x^*)(z) = -f^*(x^*). \tag{16}$$

Let $\varepsilon > 0, h^* \in S_{X^*}$, where we recall that S_{X^*} denotes the closed unit sphere of X^*. Since $x = \nabla f^*(x^*)$, there exists $t > 0$ such that

$$f^*(x^* + th^*) - f^*(x^*) - \langle th^*, x\rangle \leq \varepsilon t. \tag{17}$$

Further, we have

$$\begin{gathered}\langle x^* + th^*, x_n\rangle - f(x_n) \leq f^*(x^* + th^*) \ \text{ by definition of } f^*,\\ f(x_n) - \langle x^*, x_n\rangle \leq -f^*(x^*) + \varepsilon t \quad \text{for large } n \ \text{ by (16).}\end{gathered}$$

Adding these two inequalities (noting that $f(x_n)$ is finite for large n by (16)) we obtain

$$\begin{aligned}\langle th^*, x_n\rangle &\leq f^*(x^* + th^*) - f^*(x^*) + t\varepsilon\\ &\leq \langle th^*, x\rangle + 2\varepsilon t \quad \text{for large } n \quad \text{by (17),}\end{aligned}$$

which implies

$$\langle h^*, x_n - x\rangle \leq 2\varepsilon \ \text{ for large } n.$$

Similarly,

$$\langle -h^*, x_n - x\rangle \leq 2\varepsilon \ \text{ for large } n,$$

hence $x_n \xrightarrow{w} x$.

(3) $\Longrightarrow$ (2). The proof is similar to the proof of the implication (iii) $\Longrightarrow$ (ii) in Proposition 4, [6], and we addopt its notation, namely we put $y := x^*$, so $f - y$ attains its w-minimum at x by assumption.

Let $h \in E^*, \|h\| = 1, \varepsilon > 0$ and sequence $\{t_n\}_{n=1}^{\infty}, t_n > 0$ be given.

We put $\lambda_n = \varepsilon t_n/2$ and $y_n = y + t_n h$. We need to proof that

$$\|f^*(y_n) - f^*(y) - t_n\langle h, x\rangle\| \le \varepsilon t_n, \quad \text{for large } n.$$

Further the proof is the same as in [6], Proposition 4, (iii) $\Longrightarrow$ (ii). □

Similarly we have the following.

Proposition 3.6. *Let X be a Banach space, $g : X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous function, $x \in intDom(g^*|_X)$. Then the following statements are equivalent:*

(1) *$\partial g^*|_X$ is single-valued and w^*-upper semi-continuous at $x \in X$, and we have $\partial g^*|_X(x) = \{x^*\}$,*

(2) *$g^*|_X$ is Gâteaux differentiable at x, and $\nabla g^*|_X(x) = x^*$,*

(3) *$g - x$ attains its w^*-minimum on X^* at x^*, which means that $g - x$ attains its minimum at x^* and*

$$(g - x)(x_n^*) \to \min_{z^* \in X^*} (g - x)(z^*) \quad \textit{implies} \quad x_n^* \xrightarrow{w^*} x^*.$$

Particular cases

The following corollary could be considered as a weak version of Stegall's type variational principle [10] in the dual of a separable Banach space.

Corollary 3.7. *Let X be a separable Banach space and $g : X^* \to \mathbb{R}$ be a w^*-upper semi-continuous function. Then: For every w^*-compact subset $K^* \subset E^*$ there exists a σ-full cone porous set $X_0 \subset X$ such that for every $x \in X \setminus X_0$, $g + x$ attains its w^*-maximum on K^*,*

Proof: Define $F(x) = \operatorname{argmax}_{y \in K^*}\{g(y) + \langle x, y\rangle\}$.

Then F is monotone according to the definition 3.1 with $Y = X^*$, $h : X \times Y \to \mathbb{R}$, $h(x, y) = \langle x, y\rangle$. Since B_{X^*} is w^*-compact and separable with respect to the w^*-topology, by Theorem 3.4 there exists a σ-full cone porous subset $X_0 \subset X$ such that F is single-valued and w^*-upper-semicontinuous on $X \setminus X_0$.

It is a routine matter to prove that $g + \langle \cdot, x\rangle$ attains its w^*-maximum for every $x \in X \setminus X_0$. □

Corollary 3.8. *Suppose that a Banach space E admits an equivalent norm whose dual is strictly convex. If $T : E \to 2^{E^*}$ is a monotone operator, then there is a σ-full cone porous set $E_0 \subset E$ such that for every $x \in E, T(x)$ contains at most one point.*

Proof: Using Theorem 1.9 (b), the proof is the same as those in [5], where the set E_0 is proved to be of first Baire category. □

This corollary is a generalization to a σ-full cone porous set of a result of Zajicek [13], Theorem 2 (where the result is proved for a σ-cone porous set). In particular, the assertion in this corollary is valid in separable Banach spaces.

Corollary 3.9. *Let Y be a separable compact topological space, $C(Y)$ be the space of all continuous functions on Y with the sup norm. Then there exists a σ-full cone porous subset X_0 of $C(Y)$ such that for every $x \in C(Y) \setminus X_0$ the minimization problem*

$$\text{minimize}_{\,y\in Y} \quad f(y) + x(y)$$

is well posed (it means it has unique minimum point and every minimizing sequesnce converges to it).

Proof. The result follows from Theorem 3.4 if we define $h(x, y) = x(y)$ where $x \in X, y \in Y$ and $X = C(Y)$. □

Remark 3.10. Corollary 3.9 stated for the usual notion of porosity is a particular case of a result of Deville-Revalski [1].

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