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On Second Order Sufficient Conditions for a Minimum

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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The paper offers second order necessary and sufficient conditions for a sufficiently general class of optimization problems in Banach spaces. The reader can easily see that this is a "no gap" pair of conditions, although sufficient conditions (as usual) hold under some additional assumptions, mainly aimed to guarantee that tangent sets to the non-functional constraint set approximate the latter sufficiently well. In the last section we apply the results to a mathematical programming problem that contains, along with functional equality and inequality constraints, also a non-functional set constraint.

Keywords: Optimization problem, mathematical programming, local solution, tangent cone, critical cone.

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1. Introduction

The paper basically deals with second order sufficient conditions for a local minimum in optimization problems in Banach spaces. The key result relates to minimization of a reasonably good functional on an arbitrary set. Specifically, we shall consider the problem of minimizing

$$J(x) = g(F(x)), \quad \text{s.t. } x \in U \subset X, \tag{1}$$

where g is a sublinear function on a Banach space Y and F is a mapping from another Banach space X into Y.

This is a particular case of a problem considered in [7], where a second order necessary optimality conditions for the problem were obtained and applied to a general class of optimal control problems. Necessary conditions for (1) is a direct consequence of the corresponding result of [7]. But an independent proof of the necessary condition for (1) is fairly simple, so we give it here – see Theorem 3.4 below. Sufficient conditions however require more efforts.

We note that a problem with constraints, e.g.

$$\text{minimize} \quad f(x), \quad \text{s.t. } x \in U,\ G(x) \in K$$

can be reduced to the form (1), provided f is Lipschitz in a neighborhood of the point of interest (see e.g. [6], Lemma 4.38).

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Note also that the statement of the problem contains no assumptions on U which therefore can be an arbitrary set. Needless to say that there is a huge literature on higher order conditions in optimization problems. But to the best of my knowledge sufficient conditions for an infinite dimensional problem with an arbitrary constraint set were for the first time studied in a recent paper by Frankowska and Osmolovskii [5]. Their paper offers second order necessary and sufficient conditions for the problem of minimizing a smooth function $f(x)$ on an arbitrary subset of a Banach space X (see Example 4.2 below in the forth section for more details). Among other earlier publications relevant to the present study I would mention the books by Aubin and Frankowska [1], Bonnans and Shapiro [2] and Thibault [10] and papers by Cominetti [3], Gfrerer, Ye and Zhow [4] and Penot [9] where many further references can be found.

The main result of the present paper is stated and proved in the fourth section. The third section, as we have mentioned, is devoted to necessary conditions for a minimum in the problem and the next, second, section contains several simple or known results needed for the proof of the main theorems. In the last fifth section the results obtained in the two previous sections are applied to a problem of mathematical programming including, along with equality and inequality constraints, also a nonfunctional set constraint involving an arbitrary set. I also plan to apply elsewhere the main results of this paper to optimal control problems with state constraints.

2. Preliminaries

In this section we shall state and/or prove several results to be used in the proof of the main theorem in the next sections. In what follows X is a Banach space, X^* is the dual space and f is a function on X.

Let f be a convex function continuous at x. Recall that the *subdifferential* of f at x is the collection of x^* such that $\langle x^*, h\rangle \le f(x+h) - f(x)$ for all $h \in X$ and the *ε-subdifferential* of f at x is

$$\partial_\varepsilon f(x) = \{x^* : \ f(x+h) - f(x) \ge \langle x^*, h\rangle - \varepsilon, \ \forall h\}.$$

The inclusion $\partial f(u) \subset \partial_\varepsilon f(x)$ if $|f(x) - f(u)| \le \varepsilon$ is immediate.

Recall also that a function f is *sublinear* if $f(x+y) \le f(x)+f(y)$ and $f(\lambda x) = \lambda f(x)$ for all x, y and $\lambda \ge 0$.

Proposition 2.1. *Let f be a continuous sublinear function. Then for any $\varepsilon > 0$*

$$\sup_{x^* \in \partial_\varepsilon f(\overline{x})} \langle x^*, x - \overline{x}\rangle \ge f(x) - f(\overline{x})$$

for all x of a neighborhood of $\overline{x}$ (depending on ε).

To prove the proposition we should only take into account that, for a sublinear function f and $x^* \in \partial f(0)$, the equality $\langle x^*, x\rangle = f(x)$ holds if and only if $x^* \in \partial f(x)$ and $\langle x^*, x\rangle \le f(x)$ otherwise.

Proposition 2.2. *Let X and Y be Banach spaces, and let $\varphi(x)$ be a continuous sublinear function on X. Set $P = \{x : \ \varphi(x) \le 0\}$. If $\varphi(\overline{x}) < 0$ for some $\overline{x} \in X$, then there is an $K > 0$ such that $K\varphi(x)^+ \ge d(x, P)$ for all $x \in X$.*

Proof. Indeed, assuming the contrary, we have to admit that there is a sequence (x_n) such that $\|x_n\| = 1$ and $0 < n\varphi(x_n) \le d(x_n, P)$ for all n. Set $\lambda = -\varphi(\overline{x}) > 0$. We may assume without loss of generality that $\|\overline{x}\| = 1$.

Then taking $\xi_n = \lambda^{-1}\varphi(x_n)$, we get (as φ is a sublinear function)

$$\varphi(x_n + \xi_n\overline{x}) \le \varphi(x_n) - \lambda\xi_n = 0.$$

This means that $x_n + \xi_n\overline{x} \in P$ and therefore $d(x_n, P) \le \xi_n = \lambda^{-1}\varphi(x_n)$ in contradiction with the assumption. □

Let $U \subset X$ (and X as above a Banach space) and $\overline{x} \in U$. The *tangent cone* $T(U, \overline{x})$ to U at $\overline{x}$ is the collection of $h \in X$ such that $d(\overline{x} + t_m h, U) = o(t_m)$ for some positive $t_m \to 0$. Here and below $d(z, U)$ is the distance from z to U. The *second order tangent set to U at $\overline{x}$ along $h \in T(U, \overline{x})$* is the collection of $v \in X$ such that $d(\overline{x} + t_m h + t_m^2 v, U) = o(t_m^2)$ for some $t_m > 0$ converging to zero.

Proposition 2.3. *If $h \in T(U, \overline{x})$ and $v \in T^2(U, \overline{x}; h)$ then*

$$v + \lambda h \in T^2(U, \overline{x}, h), \qquad \forall \lambda \in \mathbb{R}.$$

Proof. We have $\overline{x} + t_m h + t_m^2 v + r_0(t_m) \in U$ for a sequence of positive $t_m \to 0$, where $\|r_0(t)\| = o(t^2)$. Hence

$$\overline{x} + t_m(h - \lambda t_m h) + t_m^2(v + \lambda h) + r_0(t_m) \in U.$$

Set $\tau(t) = t - \lambda t^2$. Then $\tau(t)$ is a strictly increasing in a neighborhood of zero in $\mathbb{R}_+$ and $\frac{d\tau}{dt}(0) = 1$. Hence $t(\tau)$ is also strictly increasing near zero in $\mathbb{R}_+$ and and $\frac{dt}{d\tau}(0) = 1$. In other words, $t(\tau) = \tau + \xi(\tau)$, where $|\xi(\tau)| = o(\tau)$. Therefore we can rewrite the above inclusion as follows

$$\overline{x} + \tau_m h + \tau_m^2(v + \lambda h) + r(\tau_m) \in U,$$

where $\tau_m = \tau(t_m)$ and $\|r(\tau)\| = o(\tau^2)$, whence the claim. □

The set $$N(U, \overline{x}) = \{x^* \in X^* : \langle x^*, h\rangle \le 0,\ \forall h \in T(U, \overline{x})\}$$

(the polar cone to $T(U, \overline{x})$) is called the *(Dini-Hadamard) normal cone to U at $\overline{x}$*.

Proposition 2.4. (Lyusternik-Graves theorem) *Let X and Y be Banach spaces and $F : X \to Y$ a continuous and continuously differentiable mapping defined in a neighbourhood of $\overline{x} \in X$. Assume that $F'(\overline{x})$ maps X onto Y.*

Set $Q = \{x : F(x) = F(\overline{x})\}$. Then $T(Q, \overline{x}) = \ker F'(\overline{x})$.

Moreover, there are positive K and ε such that for any $y \in Y$ with $\|y - F(\overline{x})\| < \varepsilon$ there is an $x \in X$ such that

$$F(x) = y \quad \text{and} \quad \|x - \overline{x}\| \le K\|y - F(\overline{x})\|.$$

For the proof see [8] (Theorem of Lyusternik) and [6] (Theorem 1.20)

Closed sets $Q_1, \dots, Q_k$ in X are *subtransversal* at $\overline{x} \in Q = Q_1 \cap \dots \cap Q_k$ if there is an $N > 0$ such that

$$d(x, Q) \leq N(d(x, Q_1) + \cdots + d(x, Q_k))$$

for all x in a neighborhood of $\overline{x}$. We refer to [6] for more details concerning subtransversality and transversality properties.

Proposition 2.5. *Consider the problem of minimizing $f(x)$ on a set $Q \subset X$, where X is a metric space and f is Lipschitz in a neighborhood of $\overline{x} \in Q$. If $\overline{x}$ is a local solution of the problem, then it is a point of unconditional local minimum of the function $f(x) + Kd(x, Q)$ whenever K is greater than the Lipschitz constant of f.*

3. Basic model – necessary condition

Assume that

(H_1) g is continuous and sublinear;

(H_2) F is continuous and continuously differentiable near $\overline{x}$ and has a second Fréchet derivatives at $\overline{x}$.

The necessary condition for a minimum in the problem stated below is closely connected with (and can be basically viewed as an immediate consequence of) Theorem 3.4 in [7].

Proposition 3.1. *Set $\overline{y} = F(\overline{x})$ and assume that J attains a local minimum on U at $\overline{x}$. Let finally the assumptions (H_1),(H_2) hold. Then*

$$g'(\overline{y}, F'(\overline{x})h) \geq 0, \quad \forall h \in T(U, \overline{x}) \tag{2}$$

Moreover, if

(H_3) the set $F'(\overline{x})(T(U, \overline{x}))$ is convex

then the set

$$\Lambda = \{\langle y^* \in \partial g(\overline{y}) : \ \langle y^*, F'(\overline{x})h\rangle \geq 0, \quad \forall h \in T(U, \overline{x})\} \tag{3}$$

is nonempty.

The first statement is obvious. To prove the second, it is sufficient to note that the convex sets $\mathrm{int}(\mathrm{epi}\, g)$ and $\{0\} \times F'(\overline{x})(T(U, \overline{x}))$ do not meet and the first is open. So separating the sets, we immediately get the result.

Remark 3.2. Note in connection with the last statement of the proposition that for any $h \in T(U, \overline{x})$ there is some $y^* \in \partial g(\overline{y})$ such that $\langle y^*, F'(\overline{x})h\rangle \geq 0$. This is immediate from (2). Moreover, if g is itself a linear functional, that is, $g(y) = \langle y^*, y\rangle$, then ($H_3$) is not needed for $\Lambda \neq \emptyset$. □

To state second order conditions we need some additional concepts. The first is the concept of the *critical cone* of the problem at $\overline{x}$:

$$K_c = \{h \in T(U, \overline{x}) : \ g'(\overline{y}, F'(\overline{x})h) \leq 0\}. \tag{4}$$

If $\overline{x}$ is a point of a local minimum of J, then, as immediately follows from Proposition 3.1, $g'(\overline{y}, F'(\overline{x})h) = 0$ for any $h \in K_c$.

Theorem 3.3. *If J attains a local minimum on U at $\overline{x}$, then*

$$\max_{y^*\in\partial g(\overline{y})} \big\langle y^*, F''(\overline{x})(h,h) + 2F'(\overline{x})v\big\rangle \geq 0 \tag{5}$$

if $h \in K_c$ and $v \in T^2(U,\overline{x},h)$. In particular, $g(F''(\overline{x})(h,h) + 2F'(\overline{x})v) \geq 0$ for such h and v if $\overline{y} = F(\overline{x}) = 0$.

Proof. The proof is elementary. Fix some $\varepsilon > 0$, $h \in K_c$ and $v \in T^2(U,\overline{x};h)$. In view of Proposition 2.1, we have for sufficiently small $t > 0$

$$\begin{aligned} 0 &\leq g(F(\overline{x}+th+t^2v)) - g(F(\overline{x})) \leq \sup_{y^*\in\partial_\varepsilon g(\overline{y})} \langle y^*, F(\overline{x}+th+t^2v) - F(\overline{x}\rangle) \\ &= \sup_{y^*\in\partial_\varepsilon g(\overline{y})} \langle y^*, tF'(\overline{x})h + t^2F'(\overline{x})v + \frac{t^2}{2}\, F''(\overline{x})(h,h)\rangle + o(t^2) \\ &\leq \frac{t^2}{2} \sup_{y^*\in\partial_\varepsilon g(\overline{y})} \langle y^*, 2F'(\overline{x})v + F''(\overline{x})(h,h)\rangle + \sup_{y^*\in\partial_\varepsilon g(\overline{y})} t\langle y^*, F'(\overline{x})h\rangle + o(t^2) \end{aligned}$$

By (4) $\langle y^*, F'(\overline{x})h\rangle \leq 0$ if $y^* \in \partial g(\overline{y})$ and $h \in K_c$, and $\partial_\varepsilon g(\overline{y})$ are weak* compact sets, decreasing when ε goes to zero to $\partial g(\overline{y})$. Thus

$$\lim_{\varepsilon\to 0} \sup_{y^*\in\partial_\varepsilon g(\overline{y})} \langle y^*, F'(\overline{x})h\rangle \leq 0$$

for $h \in K_c$. It follows that

$$0 \leq t^2 \sup_{y^*\in\partial g(\overline{y})} \langle y^*, 2F'(\overline{x})v + F''(\overline{x})(h,h)\rangle + o(t^2).$$

It remains to note that the upper bound is attained as $\partial g(\overline{y})$ is a weak* compact set. $\square$

Note that the theorem contains Proposition 3.1 as a particular case. To see this, we only need to observe that $T^2(U,\overline{x};0) = T(U,\overline{x})$.

4. Basic model – sufficient condition

As we have mentioned, the theorem just proved is closely connected with Theorem 3.4 in [7]. (The definition of critical cone we use here is a bit different from the definition of critical set in [7] but the difference practically does not affect the proofs.) But we can hardly hope to get a sufficient condition by simply replacing zero at the right side of (5) by something definitely positive valued. We need to be sure that the tangent sets well approximate the original sets near the point. The following condition guarantees the needed quality of approximation.

(H_4) for any $\lambda > 0$ there is an $\varepsilon > 0$ such that for any $u\in U - \overline{x}$ satisfying $\|u\| \leq \varepsilon$ and $g'(\overline{y}, F'(\overline{x})u) \leq \varepsilon\|u\|$ there are $h \in K_c$ and $v \in T^2(U,\overline{x};h)$ such that

$$\|h-u\| \leq \lambda\|u\|, \quad \|h+v-u\| \leq \lambda\|u\|^2. \tag{6}$$

Note that a union of sets satisfying (H_4) also has this property.

Here is the statement of the main result.

Theorem 4.1. *Assume* (H_1), (H_2), (H_4). *Assume further that there is a* $c_0 > 0$ *such that*

$$\sup_{y^* \in \partial g(\overline{y})} \langle y^*, F'(\overline{x})v + \frac{1}{2}\, F''(\overline{x})(h,h)\rangle \geq c_0\|h\|^2. \tag{7}$$

if $h \in K_c$ *and* $v \in T^2(U,\overline{x};h)$. *Then* $\overline{x}$ *is a local minimum of* J *subject to* $x \in U$.

Proof. As follows from (H_4), there is a positive valued function $r(t)$ on $\mathbb{R}_+$ such that $r(t) \to 0$ as $t \to 0$ and (6) holds with λ replaced by $r(\varepsilon)$, provided $\|u\| \leq \varepsilon$ and $g'(\overline{y}, F'(\overline{x})u) \leq \varepsilon\|u\|$. By ($H_2$)

$$F(\overline{x}+u) = F(\overline{x}) + F'(\overline{x})u + \frac{1}{2}\, F''(\overline{x})(u,u) + a(u),$$

where $\|a(u)\| = o(\|u\|^2)$.

We may assume that $\|a(u)\| \leq r(\varepsilon)\|u\|^2$ if $\|u\|$ is sufficiently small without loss of generality. Thus for such u

$$\begin{aligned} J(\overline{x}+u) - J(\overline{x}) &\geq g\big(F(\overline{x}) + F'(\overline{x})u + \frac{1}{2}\, F''(\overline{x})(u,u)\big) - Nr(\varepsilon)\|u\|^2 - g(F(\overline{x})) \\ &\geq g'(\overline{y}, F'(\overline{x})u) - O(\|u\|^2) \end{aligned} \tag{8}$$

where N is the Lipschtz constant of g.

Assume now that $\overline{x}$ is not an isolated local minimum of J. This means that there is a sequence of nonzero $u_m \in U - \overline{x}$ converging to zero such that $J(\overline{x}+u_m)) \leq J(\overline{x})$. As immediately follows from (8), for any $\varepsilon > 0$ there is an $\delta > 0$ such that

$$g'(\overline{y}, F'(\overline{x})u) \leq J(\overline{x}+u) - J(\overline{x}) + \varepsilon\|u\|$$

if $\|u\| \leq \delta$. Thus, we can be sure that $\|u_m\| \leq \varepsilon_m$, $g'(\overline{y}, F'(\overline{x})u_m) \leq \varepsilon_m\|u_m\|$ with some $\varepsilon_m \to 0$ and there are $h_m \in K_c$ and $v_m \in T^2(U,\overline{x};h_m)$ such that

$$\|h_m - u_m\| \leq r(\varepsilon_m)\|u_m\| \quad \text{and} \quad \|h_m + v_m - u_m\| \leq r(\varepsilon_m)\|u_m\|^2. \tag{9}$$

It follows from (9) that $\|h_m - u_m\|$ is $o(\|u_m\|)$ and hence $o(\|h_m\|)$, so that $\|v_m\| = o(\|h_m\|)$ as well. Taking this into account along with the fact that g is a sublinear function (which by Proposition 2.1 means in particular that $\langle y^*, y\rangle = g(y)$ whenever $y^* \in \partial g(y)$), we get from (8)

$$\begin{aligned} 0 &\geq g(F(\overline{x}) + F'(\overline{x})(h_m+v_m) + \frac{1}{2}\, F''(\overline{x})(h_m,h_m)) - g(F(\overline{x})) - o(\|h_m\|^2) \\ &\geq g'(\overline{y}, F'(\overline{x})(h_m+v_m) + \frac{1}{2}\, F''(\overline{x})(h_m,h_m)) - o(\|h_m\|^2). \\ &= \sup_{y^* \in \partial g(\overline{y})} \langle y^*, F'(\overline{x})(h_m+v_m) + \frac{1}{2}\, F''(\overline{x})(h_m,h_m)\rangle - o(\|h_m\|^2). \end{aligned} \tag{10}$$

But $\langle y^*, F'(\overline{x})h_m\rangle \leq g'(\overline{y}, F'(\overline{x})h_m) \leq 0$ as $h_m \in K_c$. Hence

$$o(\|h_m\|^2) \geq \sup_{y^* \in \partial g(\overline{y})} \langle y^*, F'(\overline{x})v_m + \frac{1}{2}\, F''(\overline{x})(h_m,h_m)\rangle,$$

in contradiction to (7). □

As follows from the proof, the key assumption is (H_4). We shall see that verification of this assumption may require some efforts even in reasonably simple situations.

Example 4.2. Consider the problem

$$\text{minimize } f(x), \quad \text{s.t. } x \in U,$$

obviously a simple version of (1): enough to set $g(y) = y$ and $F(x) = f(x)$ (with $Y = \mathbb{R}$). The problem was thoroughly studied by Frankowska and Osmolovskiy in [5]. They showed that

$$f''(\overline{x})(h,h) + f'(\overline{x})v \geq 0, \quad \forall\, h \in K_c \text{ and } \forall\, v \in T^2(U,\overline{x},h)$$

if $\overline{x}$ is a local minimum of f on U, and, conversely, under certain additional assumptions, $\overline{x}$ is a local minimum in the problem, provided

$$f''(\overline{x})(h,h) + f'(\overline{x})v \geq c_0\|h\|^2$$

for all $h \in K_c$ and $v \in T^2(U,\overline{x};h)$.
Note that in this case $K_c = \{h \in T(U,\overline{x}) :\ f'(\overline{x})h \leq 0\}$.

It is an easy matter to see that the stated necessary condition follows from Theorem 3.3. But verification that the sufficient condition is a consequence of Theorem 4.1 requires some work.

The three basic assumptions applied in [5] to the components of the problem (to get the sufficient condition) can be stated in our terms as follows:

(A_1) for any $\lambda > 0$ there is a $\varepsilon > 0$ such that $d(u, T(U,\overline{x})) \leq \lambda\|u\|$ for any $u \in U - \overline{x}$ with $\|u\| \leq \varepsilon$;

(A_2) there is a $k > 0$ such that the inequality $d(w, K_c) \leq k\langle f'(\overline{x}), w\rangle$ holds for any $w \in T(U,\overline{x})$;

(A_3) for any $\lambda > 0$ there is an $\varepsilon > 0$ such that for any $u \in U - \overline{x}$ with $\|u\| \leq \varepsilon$ and $d(u, K_c) \leq \varepsilon\|u\|$ there are $h \in K_c$ and $v \in T^2(U,\overline{x};h)$ such that (6) holds.

It is to be mentioned that the third condition is formulated in [5] a bit differently. Namely, instead of the second relation in (6), it is required (in our terms) that the distance from u to $T^2(U,\overline{x};h)$ was not greater than $\lambda\|u\|^2$. However, as easily follows from Proposition 2.3, the assumptions are equivalent.

There is also no loss of generality in assuming that the ε in (A_1) and (A_3)) coincide. We claim that (A_1)–(A_3) imply (H_4), so that Theorem 3.4 in [5] follows from the just proved Theorem 4.1. Indeed, (H_4) states in this case that

(H_{4a}) for any $\lambda > 0$ there is an $\varepsilon > 0$ such that for any $u \in U - \overline{x}$ satisfying $\|u\| \leq \varepsilon$ and $\langle f'(\overline{x}), u\rangle \leq \varepsilon\|u\|$ there is a pair (h,v) with $h \in K_c$ and $v \in T^2(U,\overline{x};h)$ such that (6) holds.

Now given a $\lambda > 0$, we take a $\mu \leq \lambda$ (to be specified a bit later) and choose $\varepsilon > 0$ according to (A_1) with λ replaced by μ. Without any loss of generality, we may assume that $\varepsilon \leq \mu$.

Take a $u \in U - \overline{x}$ with $\|u\| \le \varepsilon$ satisfying also $\langle f'(\overline{x}), u\rangle \le \varepsilon\|u\|$, and let $w \in T(U, \overline{x})$ be such that $\|w - u\| \le \mu\|u\|$. Such a w does exist by (A_1). Then applying (A_2) we get

$$\begin{aligned} d(u, K_c) &\le d(w, K_c) + \|u - w\| \le k\langle f'(\overline{x}), w\rangle + \|u - w\| \\ &\le k\langle f'(\overline{x}), u\rangle + (1 + k\|f'(\overline{x})\|)\|u - w\| \\ &\le (k\varepsilon + \mu(1 + k\|f'(\overline{x})\|))\|u\| \\ &\le \mu(1 + k(1 + \|f'(\overline{x})\|))\|u\|. \end{aligned}$$

Hence, if the chosen μ satisfies $(1 + k(1 + \|f'(\overline{x})\|))\mu \le \lambda$, we get $d(u, K_c) \le \lambda\|u\|$, so by ($A_3$) there is an $h \in K_c$ and a $v \in T^2(U, \overline{x}; h)$ such that (6) holds as claimed by (H_{4a}).

On the other hand, not all of the three conditions (A_1)–(A_3) (more specifically, (A_2)) seem to follow from (H_{4a}). In other words, (H_{4a}) may be a weaker assumption. It would be interesting to find an example, if so. □

Thus, even in a simple problem considered in the example, verification of (H_4) requires some efforts. The following proposition offers a sufficient condition for (H_4) to hold.

Proposition 4.3. *Assume that the following two properties are satisfied at $\overline{x}$:*

(i) *for any $\lambda > 0$ there is an $\varepsilon > 0$ such that for any $u \in U - \overline{u}$, $\|u\| < \varepsilon$ there is an $h \in T(U, \overline{x})$ such that $\|u - h\| \le \lambda\|u\|$ and for any such h there is a $v \in T^2(U, \overline{x}; h)$ such that $\|h + v - u\| \le \lambda\|u\|^2$.*

(ii) *there is an $N > 0$ such that*

$$d(u, K_c) \le N(g'(\overline{y}, F'(\overline{x})u)^+ + d(u, T(U, \overline{x}))) \tag{11}$$

for all u. (Here $\alpha^+ = \max\{\alpha, 0\}$.)

Then (H_4) *holds true.*

Proof. Fix a positive λ and take a $\mu > 0$ to be specified a bit later. Let now $\varepsilon > 0$ be chosen according to (i) with λ replaced by μ. We can take ε satisfying $\varepsilon \le \mu$. Take next a $u \in U - \overline{x}$ such that $\|u\| < \varepsilon$ and $g'(\overline{y}, F'(\overline{x})u) \le \varepsilon\|u\|$. Then $d(u, T(U, \overline{x})) \le \mu\|u\|$ by (i) and $d(u, K_c) \le 2N\mu\|u\|$ by (ii), so that there is an $h \in K_c$ such that $\|u - h\| \le 2N\mu\|u\|$ and by (i) there is a $v \in T^2(U, \overline{x}; h)$ such that $\|h + v - u\| \le 2N\mu\|u - h\|^2$. It is clear now that ($H_4$) holds with the given λ if $2N\mu \le \lambda$. □

Note that although (i) looks very similar to the assumption (A_3) in the example, the conditions are substantially different: (i) is an assumption only on the constraint set U while (A_3) applies both to the constraint set and the cost function (through the condition $h \in K_c$). Here are two important situations in which the property (i), that characterizes the needed quality of approximation of U by the tangent cones to U at $\overline{x}$, is satisfied:

(a) U is a union of finitely many closed convex sets, or more generally, there is an $\varepsilon > 0$ such that $(U - \overline{x}) \cap B(\overline{x}, \varepsilon) \subset T(U, \overline{x})$: we can take $h = u$ and $v = 0$ in this case;

(b) $U = \{x : G(x) = 0, \ g_i(x) \le 0\}$, where $G : X \to Y$ (with Y also being a Banach space) and the functions g_i are twice continuously differentiable, $G'(\overline{x})$ maps X onto Y and $g_i'(\overline{x})$ are positively independent on the kernel of $G'(\overline{x})$. It should be also noted that union of finitely many sets satisfying (i) also has this property

To see that (i) holds in (a) it is enough to set $h = u$, and in case of (b) the property, although not explicitely mentioned, was basically proved in [5].

The second property (ii) rather relates to the mutual geometry of the function g, the mapping F and the set U. Indeed, set $Q = \{h : g'(\overline{y}, F'(\overline{x})h) \le 0\}$. Then $K_c = Q \cap T(U, \overline{x})$ and $g'(\overline{y}, F'(\overline{x})h) \le Nd(h, Q)$ if N is sufficiently large. (Indeed, if $g'(\overline{y}, F'(\overline{x})h) > 0$ and $h' \in Q$ with $\|h' - h\| \le (1+\varepsilon)d(h, Q)$, then

$$g'(\overline{y}, F'(\overline{x})h) \le g'(\overline{y}, F'(\overline{x})h) - g'(\overline{y}, F'(\overline{x})h') \le L\|F'(\overline{x})\|\|h - h'\| \le L'd(h, Q),$$

where L is the Lipschitz constant of $g'(\overline{y}, \cdot)$ and $L' = (1+\varepsilon)L$.)

The inequality in (ii) now implies that

$$d(h, K_c) \le N(d(h, Q) + d(h, T(U, \overline{x}))) \tag{12}$$

with some $N > 0$, that is that the sets $T(U, \overline{x})$ and Q are subtransversal at $\overline{x}$. An important observation is that, vice versa, *(12) implies (11) if* $g'(\overline{y}, F'(\overline{x})u) < 0$ *for some* u. This is immediate from Proposition 2.2. Thus, (12) can replace (11) in (ii) in this case.

In case when $\Lambda \ne \emptyset$ (see (3)) we can state a weaker sufficient condition for $\overline{x}$ to be a local minimum which, however, can be easier to verify in concrete situations. Namely, set

$$K_c(\Lambda) = \{h \in T(U, \overline{x}) : \ \langle y^*, F'(\overline{x})h\rangle \le 0, \quad \forall\, y^* \in \Lambda\},$$

and consider the following version of (H_4):

$(\mathrm{H}_4(\Lambda))$ (H_4) holds with K_c replaced by $K_c(\Lambda)$.

Theorem 4.4. *Assume* (H_1), (H_2). *Assume further that* $\Lambda \ne \emptyset$, $(\mathrm{H}_4(\Lambda))$ *holds and there is a* $c_0 > 0$ *such that*

$$\sup_{y^* \in \Lambda} \langle y^*, F''(\overline{x})(h, h) + 2F'(\overline{x})v\rangle \ge c_0\|h\|^2. \tag{13}$$

if $h \in K_c(\Lambda)$ *and* $v \in T^2(U, \overline{x}; h)$. *Then* $\overline{x}$ *is a local minimum of* J *on* U.

This result is an immediate consequence of Theorem 4.1. Indeed, $\langle y^*, y\rangle \le g'(\overline{y}, y)$ for any y as $\Lambda \subset \partial g(\overline{y})$. It follows that $K_c \subset K_c(\Lambda)$ and (13) implies (7) for any h and v. To get a direct proof of this theorem we only need to slightly modify the proof of Theorem 4.1. Indeed, applying $(\mathrm{H}_4(\Lambda))$, we get $h_m \in K_c(\Lambda)$ and $v_m \in T^2(U, \overline{x}; h_m)$ for which (10) holds. Furthermore, as $\Lambda \subset \partial g(\overline{y})$, we get from (10)

$$0 \ge \sup_{y^* \in \Lambda} \langle y^*, F'(\overline{x})(h_m + v_m) + \frac{1}{2}\, F''(\overline{x})(h_m, h_m)\rangle - o(\|h_m\|^2).$$

But (2) obviously holds for $h \in T(U, \overline{x})$ as $y^* \in \Lambda$, so that $\langle y^*, F'(\overline{x})h_m\rangle = 0$ since $h_m \in K_c(\Lambda)$ and we again arrive at a contradiction.

5. A mathematical programming problem

Here we shall apply the results of the previous section to the problem

$$\text{minimize } f_0(x), \quad \text{s.t. } f_i(x) \le 0,\ i = 1, \dots, k;\ G(x) = 0,\ x \in U. \tag{14}$$

We assume that the functions f_i and mapping $G : X \to Y$ are twice continuously differentiable near $\overline{x} \in U$ and U is a closed convex set. Let $\overline{x} \in X$ satisfy all constraints in (14). We can also assume without loss of generality that $f_i(\overline{x}) = 0$ for all $i = 0, 1, \dots, k$.

Theorem 5.1. *Let $\overline{x}$ be a local minimum in* (14). *If $G'(\overline{x})$ maps X onto Y and the sets $G^{-1}(0)$ and U are subtransversal at $\overline{x}$, then there is an $\gamma > 0$ such that*

$$\max_{0 \le i \le k} f_i'(\overline{x})h + \gamma \|G'(\overline{x})h\| \ge 0, \quad \forall\, h \in T(U, \overline{x}). \tag{15}$$

Moreover, there are $\lambda_i \ge 0$ and y^* with $\|y^*\| \le \gamma$, $\lambda_0 + \dots + \lambda_k = 1$ and

$$0 \in \sum_{i=0}^{k} \lambda_i f_i'(\overline{x}) + G'^*(\overline{x})y^* + N(U, \overline{x}). \tag{16}$$

Proof. Set $V = G^{-1}(0) \cap U$ and $f(x) = \max_{0 \le i \le 1} f_i(x)$. Then $\overline{x}$ is a local minimum in the problem of minimizing $f(x)$ on V. Taking into account that f is Lipshitz near $\overline{x}$, we conclude that $\overline{x}$ is an unconditional local minimum of the function $f(x) + Ld(x, V)$, where L is any number greater than the Lipschtz constant of f at $\overline{x}$ (see e.g. [6], Lemma 4.38). Furthermore, as the sets $G^{-1}(0)$ and U are subtransversal at $\overline{x}$, there is a $\rho > 0$ such that $d(x, V) \le \rho(d(x, G^{-1}(0)) + d(x, U))$ for all x close to $\overline{x}$.

Finally, as $G'(\overline{x})$ maps X onto Y, by the Lyusternik-Graves theorem there is a $\xi > 0$ such that $d(x, G^{-1}(0)) \le \xi \|G(x)\|$ for x of a neighborhood of $\overline{x}$.

Combining these three facts, we conclude that $\overline{x}$ is a local minimum of the function $g(x) = f(x) + \gamma(\|G(x)\| + d(x, U))$ with some $\gamma > 0$. It follows that $g'(\overline{x}; h) \ge 0$ for all h. The first statement is now immediate (as $T(U, \overline{x})$ is the collection of h such that the directional derivative of $d(\cdot, U)$ at $\overline{x}$ along h equals zero).

In view of convexity of U, the second part of the theorem follows from the fact that zero is an absolute minimum minimum of the function $g'(\overline{x}, \cdot)$ along with standard calculus rules of convex analysis. □

Remark 5.2. The theorem extends to the case of a non-convex U with ∂ being the G-subdifferential (see [6]) and $N(U, \cdot)$ the cone generated by the G-subdifferential of the distance function $d(U, \cdot)$. □

To state the second order necessary condition, we set as before

$$K_c = \{h \in T(U, \overline{x}) : \max_{0 \le i \le k} f'(\overline{x})h + \gamma \|G'(\overline{x})h\| = 0\} \tag{17}$$

and denote by Λ the collection of $(\lambda_0, \dots, \lambda_k, y^*)$ such that $\lambda_i \ge 0$, $\lambda_0 + \dots + \lambda_k = 1$ and $\|y^*\| \le \gamma$.

Theorem 5.3. *Under the assumptions of Theorem* 5.1*,*

$$\max_{0\le i\le k}(f_i''(\overline{x})(h,h) + f'(\overline{x})v) + \gamma\|G''(\overline{x})(h,h) + G'(\overline{x})v\| \ge 0$$

for all $h \in K_c$ *and* $v \in T^2(U,\overline{x};h)$*. In particular, for all such* h *and* v

$$\sup_{(\lambda_0,\dots,\lambda_k,y^*)\in\Lambda}\Big(\sum_{i=0}^{k}\big(\lambda_i f_i''(\overline{x})(h,h) + f_i'(\overline{x})v\big) + \langle y^*, G''(\overline{x})(h,h) + G'(\overline{x})v\rangle\Big) \ge 0$$

Proof. As we have seen in the proof of the previous theorem, under the assumptions $\overline{x}$ is a local minimum in the problem

$$\text{minimize } \max_{0\le i\le k} f_i(x) + \gamma\|G(x)\| \quad \text{s.t. } x \in U \tag{18}$$

This problem already has the same structure as (1) with $y = (\alpha_0,\dots,\alpha_k,z)$ and

$$g(y) = \max_{0\le i\le k}\alpha_i + \gamma\|z\|, \quad F(x) = (f_0(x),\dots,f_k(x),G(x)). \tag{19}$$

It remains to apply Theorem 3.3 taking into account that $f_i(\overline{x}) = 0$ for all i and $G(\overline{x}) = 0$ so that Λ is precisely the subdifferential of g at zero. □

Let us pass to sufficient conditions. Consider again the problem (18) with some positive γ. An obvious observation is that $\overline{x}$ is a local solution of (14) if it is a local minimum in the problem (18) (even without any additional assumptions as e.g. in Theorem 5.1), so that any sufficient condition for a minimum in (18) is automatically a sufficient condition for a minimum in (14).

We shall start with reformulations of Theorems 4.1 and 4.4 for (18). Note that $g'(0,y) = g(y)$ as g is a sublinear function. Then the K_c for the problem is

$$K_c = \{h \in T(U,\overline{x}) : \max_{0\le i\le k}\langle f_i'(\overline{x}),h\rangle + \gamma\|G'(\overline{x})h\| \le 0\}, \tag{20}$$

and the condition (H$_4$) is stated as follows:

(H$_5$) for any $\lambda > 0$ there is an $\varepsilon > 0$ such that for any $u \in U - \overline{x}$ such that

$$\|u\| \le \varepsilon \quad \text{and} \quad \max_{0\le i\le k}\langle f_i'(\overline{x}),u\rangle + \gamma\|G'(\overline{x})u\| \le \varepsilon\|u\|$$

there are $h \in K_c$ and $v \in T^2(U,\overline{x};h)$ such that (6) holds (with g and F defined by (19)).

So applying Theorem 4.1, we get the following result

Theorem 5.4. *Let* $\overline{x}$ *satisfy the constraints of* (14) *and* (H$_5$) *hold with* K_c *defined by* (17) *and some* $\gamma > 0$*. Let further*

$$\max_{0\le i\le 1}(f_i''(\overline{x})(h,h) + 2\langle f_i'(\overline{x}),v\rangle) + \gamma\|G''(\overline{x})(h,h) + 2G'(\overline{x})v)\| \ge c\|h\|^2$$

for all $h \in K_c$*,* $v \in T^2(U,\overline{x};h)$ *and some* $c > 0$*. Then* $\overline{x}$ *is a local minimum in the problem.*

As follows from Proposition 4.3 and convexity of U, (H_5) is satisfied if there are $K > 0$ and $\gamma > 0$ such that

$$d(h, K_c) \le K\Big(\sum_{i=0}^{k}(f_i(\overline{x})h)^+ + \gamma(\|G'(\overline{x})h\| + d(h, T(U, \overline{x})))\Big) \tag{21}$$

for all h of a neighborhood of zero. We refer to [10] for a thorough study of such inequalities. Note that *the inequality (21) holds if $\overline{x} \in$ int U, in particular when $U = X$, and $G'(\overline{x})$ maps X onto Y*. This fact is implicit in the proof of Theorem 4.4 in [5]. Here are the main arguments leading to it.

The first and the most obvious is that $T(U, \overline{x}) = X$ in this case and $d(h, T(U, \overline{x}))$ is identical zero. The second argument, following from the fact that $G'(\overline{x})$ is onto, is that there is a $\gamma > 0$ such that for any $y \in Y$ there is an x such that $G'(\overline{x})x = y$ and $\|x\| \le \gamma\|y\|$. It follows that that for any $h \in X$ there is an $h' \in \ker G'(\overline{x})$ such that $\|h - h'\| \le \gamma\|G'(\overline{x})h\|$. And the final argument is the following lemma (see [5], Lemma 2.2):

Lemma 5.5. *Given a finite collection $\{x_1^*, \dots, x_m^*\}$ of elements of X^*. Then there is a constant $C > 0$ such that for any $x \in X$ there is an h such that*

$$\|h\| \le C\sum_{i=1}^{m}\langle x_i^*, x\rangle^+; \quad \text{and} \quad \langle x_i^*, x + h\rangle \le 0, \quad \forall\ i = 1, \dots, m.$$

Now, given an h, we find an $h' \in \ker G'(\overline{x})$ as above (that is such that we have $\|h - h'\| \le \gamma\|G'(\overline{x})h\|$) and then, applying the lemma with $X = \ker G'(\overline{x})$ and $x_i^* = f_i'(\overline{x})$ find an $h'' \in \ker G'(\overline{x})$ such that $f_i'(\overline{x})h'' \le 0$ and $\|h' - h''\| \le C\sum(f_i'(\overline{x})h')^+$. Then $h'' \in K_c$ and the required estimate (21) is immediate.

Reformulation of Theorem 4.4 for the problem causes no difficulties as well. Denote by Λ_γ the collection of $(\lambda_1, \dots, \lambda_k, y^*)$ such that $\lambda_i \ge 0$, $\lambda_0 + \dots + \lambda_k = 1$, $\|y^*\| \le \gamma$ and

$$\sum_{i=0}^{k}\lambda_i f_i'(\overline{x})h + \langle y^*, G'(\overline{x})h\rangle \ge 0, \quad \forall\ \ h \in T(U, \overline{x}). \tag{22}$$

Set further

$$K_c(\gamma) = \{h \in T(U, \overline{x}) : \ \sum_{i=0}^{k}\lambda_i f_i'(\overline{x})h + \langle y^*, G'(\overline{x})h\rangle = 0, \ \forall(\lambda_0, \dots, \lambda_k, y^*) \in \Lambda_\gamma\}$$

and consider the property

(H_6) (H_5) holds with K_c replaced by $(K_c(\Lambda_\gamma))$.

Theorem 4.4 now implies

Theorem 5.6. *Let $\overline{x}$ satisfy the constraints of (14), and let (H_6)) hold true. Assume also that*

$$\sup_{(\lambda_1,\dots,\lambda_k,y^*)\in\Lambda_\gamma}\Big(\sum_{i=0}^{k}\lambda_i(f_i''(\overline{x})(h,h) + 2\langle f_i'(\overline{x})v\rangle) + \langle y^*, G''(\overline{x})(h,h) + 2G'(\overline{x})v\rangle\Big) \ge c\|h\|^2$$

for all $h \in K_c(\Lambda_\gamma)$, $v \in T^2(U, \overline{x}; h)$ and some $c > 0$. Then $\overline{x}$ is a local minimum in the problem.

In case when $T(U,\overline{x}) = X$ the inequality in (22) reduces to the equality

$$\sum \lambda_i f_i'(\overline{x}) + G'^*(\overline{x})y^* = 0$$

and the inequality in the theorem reduces to the "pure" second order sufficient condition

$$\sum_{i=0}^{k} \lambda_i f_i''(\overline{x})(h,h) + \langle y^*, G''(\overline{x})(h,h)\rangle \geq c\|h\|^2 \quad \forall\ h.$$

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