Journal of Convex Analysis
Volume 31 (2024), No. 4, 1165–1188

Regularization of Nonconvex Integro-Differential Inclusions

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

Received: February 9, 2024
Accepted: April 21, 2024

This paper is concerned with the regularization of the dynamical differential inclusion governed by the subdifferential of a function Φ perturbed both by a Carathéodory mapping and by an integral forcing term. The integrand of the forcing term depends on two time-variables. It is shown that when the function Φ is primal lower regular, one can regularize the differential inclusion by a family of new integro-differential equations which are well posed. The family of solutions of the regularized integro-differential equations is shown to converge uniformly to a (local and global) solution of the differential inclusion.

Keywords: Subdifferential, regularization, primal lower regular function, Moreau envelope, proximal mapping, Volterra integro-differential equation.

2020 Mathematics Subject Classification: 49A52, 49J53, 34A60, 34G25, 34G20.

1. Introduction

Let H be a real Hilbert space and $I = [T_0, T]$, $(T > T_0 > 0)$. Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and $f_1 : I \times H \to H$, $f_2 : I^2 \times H \to H$ be two mappings. The paper is concerned with the differential inclusion

$$(\mathcal{P}_{f_1,f_2}) \begin{cases} -\dot{x}(t) \in \partial\,\Phi(x(t)) + f_1(t, x(t)) + \displaystyle\int_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s & \text{a.e. } t \in [T_0, +\infty[, \\ x(T_0) = x_0 \in \operatorname{dom}\Phi, \end{cases}$$

where $\dot{x}(\cdot) := \frac{dx}{dt}(\cdot)$ and $\partial\,\Phi$ denotes the proximal subdifferential of the function Φ (see the definition in the preliminary section). Recently, with a discretization procedure which was used in the integro-differential sweeping process case by Bouach et al [2, 3], the same authors [4] proved that under the primal lower regularity (plr, for short) of Φ one can obtain the existence and uniqueness of the solution for the differential inclusion $(\mathcal{P}_{f_1,f_2})$. They used in [4] a new Gronwall-like inequality and a new scheme related to the existence of absolutely continuous solutions for the quasi-autonomous differential inclusions

ISSN 0944-6532 / \$ 2.50

$$\begin{cases} -\dot{x}_n(t) \in \partial\,\Phi(x(t)) + f_1(t, x_n(t_k)) + \sum\limits_{j=0}^{k-1} \displaystyle\int_{t_j}^{t_{j+1}} f_2(t, s, x_n(t_j))\,\mathrm{d}s \\ + \displaystyle\int_{t_j}^{t} f_2(t, s, x_n(t_k))\,\mathrm{d}s \quad \text{a.e. } t \in [t_k, t_{k+1}]\,, \;\; x_n(T_0) = x_0 \in \operatorname{dom}\Phi, \end{cases}$$

where $T_0 = t_0 < t_1 < ... < t_n = T$ is a discretization of the interval $[T_0, T]$. Their results have been applied to the study of non-regular electrical circuits with transmission line and a semilinear heat equation with memory [4].

Kecis and Thibault [12] showed the existence and the uniqueness of the solution of the differential inclusion $(\mathcal{P}_{f_1,0})$, i.e., the problem $(\mathcal{P}_{f_1,f_2})$ with $f_2 \equiv 0$ when the function Φ is plr. In [12] the method is based merely on the reduction to ordinary differential equations (ODE, for short) through the Moreau envelope of the function Φ. Our aim in this work is to show how a reduction approach with the ordinary integro-differential equations (OIDE, for short)

$$\dot{u}_\lambda(t) + \nabla e_\lambda \Phi(u_\lambda(t)) + f_1(t, u_\lambda(t)) + \int_{T_0}^{t} f_2(t, s, u_\lambda(s))\mathrm{d}s = 0,$$

also allows us to establish the existence and uniqueness of solution for $(\mathcal{P}_{f_1,f_2})$ and give some important properties of the solution and the perturbations f_1 and f_2. We must also say that our approach only assumes for $(\mathcal{P}_{f_1,f_2})$ the growth condition $\|f_2(t,s,x)\| \le \beta(t,s)(1+\|x\|)$ (see $(\mathcal{Q}_{2,1})$ below), while for the same inclusion $(\mathcal{P}_{f_1,f_2})$ the authors of [4] require the following strengthened form of our assumption $(\mathcal{Q}_{2,1})$:

$$\|f_2(t,s,x)\| \le g(t,s) + \gamma(t)\|x\| \text{ (see } (\mathcal{A}'_{2,1}) \text{ in [4])}.$$

Our regularization approach will serve some purposes in Control Theory, precisely looking for necessary optimality conditions. We associate with the original optimal control problem governed by $(\mathcal{P}_{f_1,f_2})$ a sequence of approximating optimal control problems governed by (OIDE). We also state the necessary conditions for such sequence of problems and we show that their solutions converge to the solution of the original optimal control.

The outline of the paper is as follows. In Section 2, we recall some preliminary results that we use throughout. Section 3 fixes the assumptions. Section 4 provides our main existence and uniqueness result of local solution, while the similar result for global solution is established in Section 5.

2. Preliminaries

In what follows, we suppose that H is a *Hilbert space* endowed with the inner product $\langle\cdot,\cdot\rangle$. In this setting, $B(x,r)$ and $B[x,r]$ denote the open and the closed balls with center x and radius r, respectively. The *indicator function* of a subset C of H will be denoted by $\delta(\cdot, C)$ that is, $\delta(x,C) = 0$ if $x \in C$ and $\delta(x,C) = +\infty$ if $x \in H \setminus C$.

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Then we denote by $\operatorname{epi} f := \{(x,r) \in H \times \mathbb{R} : f(x) \le r\}$ its *epigraph*, and its *(effective) domain* by $\operatorname{dom} f := \{x \in H : f(x) < +\infty\}$. The function f is said to be *proper* if $\operatorname{dom} f \neq \emptyset$.

A vector $\xi \in H$ is called a *proximal subgradient* of f at $\overline{x} \in \operatorname{dom} f$ (see, e.g., [21, 22]) provided there exist positive numbers σ and δ such that

$$\langle \xi, x - \overline{x} \rangle \leq f(x) - f(\overline{x}) + \sigma \|x - \overline{x}\|^2 \text{ for all } x \in B(\overline{x}, \delta).$$

The set of all such vectors ξ is called the *proximal subdifferential* of f at $\overline{x}$ and denoted by $\partial_P f(\overline{x})$. In a similar way, one says that a vector $\xi \in H$ is a *Fréchet subgradient* of f at $\overline{x}$, if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\langle \xi, x - \overline{x} \rangle \leq f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\| \text{ for all } x \in B(\overline{x}, \delta).$$

The set of all Fréchet subgradients of f at $\overline{x}$ is called the *Fréchet subdifferential* of f at $\overline{x}$ and is denoted by $\partial_F f(\overline{x})$.

Further, the *Mordukhovich limiting subdifferential* of f at $\overline{x}$ is given by

$$\partial_L f(\overline{x}) = \{\xi \in H : \exists\, x_n \to_f \overline{x},\, \xi_n \stackrel{w}{\to} \xi : \xi_n \in \partial_P f(x_n), \forall n = 1, 2..\},$$

where the symbol $x_n \to_f \overline{x}$ means that $\|x_n - \overline{x}\| \to 0$, $f(x_n) \to f(\overline{x})$, and $\stackrel{w}{\to}$ signifies the *sequential convergence* in the weak topology of H. Denoting by ∂f any subdifferential defined above, the set $\operatorname{Dom} \partial f := \{x \in H : \partial f(x) \neq \emptyset\}$ is the *domain* of the set-valued mapping ∂f, and its *graph* is the set

$$\operatorname{gph} \partial f := \{(x, \xi) \in H \times H : \xi \in \partial f(x)\}.$$

When $f(\overline{x}) = +\infty$, by convention $\partial f(\overline{x}) = \emptyset$. Obviously, we have

$$\partial_P f(\overline{x}) \subset \partial_F f(\overline{x}) \subset \partial_L f(\overline{x}).$$

For all the above concepts we refer to [6, 16, 18, 21, 22].

Theorem 2.1. (see, e.g., [16, 22]) *If $f : \mathcal{O} \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semicontinuous (lsc, for short) function on a nonempty open set $\mathcal{O} \subset H$, then for anyone of the above subdifferentials,* $\operatorname{Dom} \partial f$ *is graphically dense in* $\operatorname{dom} f$*, that is, for any $x \in \operatorname{dom} f$ there exists a sequence $(x_n)_n$ in* $\operatorname{Dom} \partial f$ *such that $(x_n, f(x_n)) \to (x, f(x))$.*

We recall now the concept of primal lower regular functions.

Definition 2.2. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a function which is lsc on an open convex set $\mathcal{O}$ of H with $\mathcal{O} \cap \operatorname{dom} f \neq \emptyset$. We say that the function f is

(a) *primal lower regular* (plr, for short) on $\mathcal{O}$, if there exists some real number $c \geq 0$ such that for all $x \in \mathcal{O} \cap \operatorname{Dom} \partial_P f$ and for all $\zeta \in \partial_P f(x)$, one has

$$f(y) \geq f(x) + \langle \xi, y - x \rangle - c\,(1 + \|\xi\|)\,\|x - y\|^2, \ \forall y \in \mathcal{O}. \tag{1}$$

(b) *plr uniformly* with respect to subgradients with a constant $c \geq 0$ on $\mathcal{O}$, if for every $x \in \mathcal{O} \cap \operatorname{Dom} \partial_P f$ and for every $\zeta \in \partial_P f(x)$, one has

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - c\|x - y\|^2 \text{ for all } y \in \mathcal{O}.$$

The real $c \geq 0$ will be called a *plr constant* for f over $\mathcal{O}$ and we will say that f is c-plr on $\mathcal{O}$.

One also says that f is *c-plr* (respectively, uniformly with respect to subgradients) on $\mathcal{O}$ when the constant c needs to be stated. For $x_0 \in \operatorname{dom} f$, we say that f is *plr at* x_0 whenever it is plr on some open convex set containing the point x_0. It is clear that every function which is plr uniformly with respect to subgradients is plr (see [12] for more details). □

Obviously, lsc convex functions, $\mathcal{C}^2$ functions, and lower-$\mathcal{C}^2$ functions (that is, functions in the form $g + h$ where $g : H \to \mathbb{R} \cup \{+\infty\}$ is lsc and convex, and $h : H \to \mathbb{R}$ is $\mathcal{C}^2$) are primal lower regular (see [20, 19]). Moreover, Poliquin [19] proved that convexly composite functions defined on finite dimensional spaces are primal lower regular whenever a qualification condition is fulfilled. The same plr property holds true for convexly composite functions in the Hilbert and Banach settings whenever an appropriate qualification condition is assumed. We refer to [24, 13, 9, 22] for more details in the Hilbert and Banach space settings. Given two Banach spaces X and Y, a convexly composite function f on an open convex set U of X is defined as $f = h \circ F$, where $h : Y \to \mathbb{R} \cup \{+\infty\}$ is a lsc convex function and $F : U \to Y$ is a differentiable mapping whose derivative DF is locally Lipschitzian on the open set U.

Remark 2.3. By [15] we obtain an equivalent definition for each above plr property if we replace the proximal subdifferential by the Fréchet subdifferential or the limiting subdifferential. So, for such functions, all three above subdifferentials coincide, hence it will be convenient for a plr function f on an open set U to *only write* $\partial f(x)$ *in the rest of the paper*, for all $x \in U$. □

The following theorem states an important property that characterizes the plr functions by some local hypomonotocity of its subdifferential ∂f. This property has been established for the first time in [19] in the finite dimensional setting with the original definition of [19]. As stated in the Hilbert setting, it has been established in [15]. The property and others can also be found in [22, 23].

Theorem 2.4. *Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a function on the Hilbert space H which is finite at $\overline{x}$ and lsc near $\overline{x}$. The following are equivalent:*

(a) *Φ is plr near $\overline{x}$;*

(b) *there exist real numbers $\varepsilon > 0$ and $c \geq 0$ such that for all $\zeta_i \in \partial\Phi(x_i)$ with $\|x_i - \overline{x}\| < \varepsilon$, $i = 1, 2$, one has*

$$\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle \geq -c(1 + \|\zeta_1\| + \|\zeta_2\|)\|x_1 - x_2\|^2.$$

In the next proposition, we recall some closure property of the graph of the subdifferential of a plr function.

Proposition 2.5. (see [15]) *Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lsc function on some open convex set $\mathcal{O}$ which satisfies the inequality (2.2) for any $y \in \mathcal{O} \cap \operatorname{Dom} \partial\Phi$ and any $\xi \in \partial\Phi(y)$, where $\partial\Phi$ is anyone of the three subdifferentials above. Let $x \in \mathcal{O} \cap \operatorname{Dom} \partial\Phi$ and $(x_n)_n$ be a sequence converging strongly to x in H and let $(\xi_n)_n$ be a sequence which converges weakly to some ξ in H with $\xi_n \in \partial\Phi(x_n)$. Then*

$$\xi \in \partial\Phi(x) \quad \text{and} \quad \lim_{n\to+\infty} \Phi(x_n) = \Phi(x).$$

We will also need the following lemmas in the proof of existence of solution for the evolution differential inclusion.

Lemma 2.6. (see [14]) *Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lsc function which is plr on some open convex set $\mathcal{O}$ containing $B[u_0, s_0]$. Let T_0 and T be real numbers with $T_0 < T$ and let $\eta_0 \in]0, s_0[$. Consider $v(\cdot) \in L^2([T_0, T], H)$ and let $u(\cdot)$ be a mapping from $[T_0, T]$ into H. Let $(u_n(\cdot))_n$ be a sequence of mappings from $[T_0, T]$ into H and $(v_n(\cdot))_n$ be a sequence in $L^2([T_0, T], H)$. Assume that*

(j_1) $\{u_n(t) : n \in \mathbb{N}\} \subset B[u_0, \eta_0] \cap \operatorname{dom} \Phi$ *for almost every* $t \in [T_0, T]$*;*

(j_2) $(u_n(\cdot))_n$ *converges almost everywhere to some mapping* $u(\cdot)$ *with* $u(t) \in \operatorname{dom} \Phi$ *for almost every* $t \in [T_0, T]$*;*

(j_3) $v_n \to v$ *weakly in* $L^2([T_0, T], H)$*;*

(j_4) *for each* $n \geq 1$, $v_n(t) \in \partial \Phi(u_n(t))$ *for almost every* $t \in [T_0, T]$.

Then, $v(t) \in \partial \Phi(u(t))$ *for almost every* $t \in [T_0, T]$.

Lemma 2.7. *Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ and let $x(\cdot)$ be a mapping defined on the interval $[a, b] \subset \mathbb{R}$. Assume that there exists some $t_0 \in]a, b[$ such that $x(\cdot)$ and $(\varphi \circ x)(\cdot)$ are derivable at t_0 and $\varphi(\cdot)$ is c-plr on an open convex set containing $x(t_0)$ along with $\partial \varphi(x(t_0)) \neq \emptyset$. Then*

$$(\varphi \circ x)'(t_0) = \{\langle \xi, \dot{x}(t_0) \rangle : \xi \in \partial \varphi(x(t_0))\},$$

that is, for any $\xi \in \partial \varphi(x(t_0))$ one has $\langle \xi, \dot{x}(t_0) \rangle = (\varphi \circ x)'(t_0)$.

Another important concept is required for the proof of our main existence result. It concerns some regular mappings introduced by Moreau which are given in the following definition.

Definition 2.8. (see [17]) Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and $\lambda > 0$ be a real positive number. The *Moreau envelope function* $e_\lambda f$ and *the proximal mapping* $P_\lambda f$ with index λ of the function f are defined by

$$e_\lambda f(x) = \inf_{y \in H} \Big(f(y) + \frac{1}{2\lambda} \|x - y\|^2 \Big), \quad \text{and}$$

$$P_\lambda f(x) = \operatorname{argmin}_{y \in H} \Big(f(y) + \frac{1}{2\lambda} \|x - y\|^2 \Big) = \Big\{ z \in H : e_\lambda f(x) = f(z) + \frac{1}{2\lambda} \|x - z\|^2 \Big\}.$$

The next theorem summarizes the main properties of the Moreau envelope of plr functions, which will be needed in the sequel. We refer the reader to [15] and [23] for the proof of this result.

Theorem 2.9. *Let H be a Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Assume that f is c-plr on an open convex set containing $B[u_0, s_0]$ for a positive constant c satisfying $c < \frac{1}{2s_0}$. Let $(x_0, y_0) \in \operatorname{gph} \partial f$ with $\|x_0 - u_0\| < \frac{s_0}{18}$ and let*

$$\overline{f}(\cdot) := f(\cdot) + \delta(\cdot, B[x_0, \frac{s_0}{2}]).$$

Then there exists some threshold $\lambda_0 > 0$ *such that for every* $\lambda \in]0, \lambda_0[$ *the following properties hold:*

(a) $e_\lambda \overline{f}$ *is* $\mathcal{C}^{1,1}$ *on* $B(u_0, \frac{s_0}{18})$ *with* $\nabla e_\lambda \overline{f}$ d_λ*-Lipschitz continuous on* $B(u_0, \frac{s_0}{18})$ *for*

$$d_\lambda := \frac{1}{\lambda\big(1 - 2c(\lambda(1 + \|y_0\|) + s_0/2)\big)} + \|y_0\|;$$

(b) $P_\lambda \overline{f}$ *is nonempty, single valued and* κ_0*-Lipschitz continuous on* $B(u_0, \frac{s_0}{18})$ *for*

$$\kappa_0 := 1 + \lambda_0 + \frac{1}{1 - 2c(\lambda_0 + s_0/2)};$$

(c) $P_\lambda \overline{f}(x_0 + \lambda y_0) = x_0$;

(d) $\nabla e_\lambda \overline{f} = \lambda^{-1}(I - P_\lambda \overline{f})$ *on* $B(u_0, \frac{s_0}{18})$;

(e) $\|\nabla e_\lambda \overline{f}(x_0)\| \leq \kappa_0 \|y_0\|$;

(f) $P_\lambda \overline{f}(B(u_0, \frac{s_0}{18})) \subset B(u_0, \frac{7}{16} s_0)$.

(g) *For any* $x \in B(u_0, \frac{s_0}{18}) : \nabla e_\lambda \overline{f}(x) \in \partial_P f(P_\lambda \overline{f}(x))$.

(h) *For any* $x \in B(u_0, \frac{s_0}{18})$ *one has*

$$\|x - x_0\| \geq \big(1 - c\lambda(2 + \|\nabla e_\lambda \overline{f}(x)\| + \kappa_0 \|y_0\|)\big) \|P_\lambda \overline{f}(x) - P_\lambda \overline{f}(x_0)\|.$$

It is well-known from [5] that, for any lsc convex function f, the mapping $\nabla e_\lambda f(\cdot)$ is bounded on H independently of λ by $\|\partial^\circ f(\cdot)\|$, where $\partial^\circ f(x)$ denotes the element of minimal norm of $\partial^\circ f(x)$. The latter property plays a crucial role in stating several nice properties of the solution of some evolution problems involving the subdifferential of an extended real-valued function (e.g., the inclusion $(\mathcal{P}_{f_1,f_2})$ with $f_2 \equiv 0$), especially the asymptotic behavior of the solution as well as the absolute continuous property of f along the solution. Unfortunately, such a boundedness has not yet been shown for plr functions. Nevertheless, if the primal lower regularity property of the function f is uniform with respect to subgradients, we can establish a local boundedness of $\nabla e_\lambda \overline{f}(\cdot)$ for some threshold $\lambda_1 > 0$, which has been shown in [12].

Lemma 2.10. [See [12]] *Let* $f : H \to \mathbb{R} \cup \{+\infty\}$ *be a function which is plr uniformly with respect to subgradients on an open convex set containing* $B[u_0, s_0]$ *with a constant* $c \geq 0$. *Assume that the other assumptions of Theorem* 2.9 *are satisfied . Then there exists a real* $\lambda_1 \in]0, \lambda_0[$ *(independent of* λ *and* x*) such that for any* $\lambda \in]0, \lambda_1[$ *one has*

$$\|\nabla e_\lambda \overline{f}(x)\| \leq \frac{1}{1 - 2c\lambda_1} \|\partial^\circ f(x)\|, \text{ for all } x \in B(u_0, \frac{s_0}{18}), \tag{2}$$

in particular, the family $(\nabla e_\lambda \overline{f}(\cdot))_{\lambda \in]0,\lambda_1[}$ *is bounded on* $B(u_0, \frac{s_0}{18})$ *independent of* λ.

We finish this section with a version of Gronwall's lemma for absolutely continuous solutions of differential inequalities which directly follows from [5]

Lemma 2.11. [Gronwall's lemma] *Let $b, c, \zeta : [T_0, T] \to [0, +\infty[$ be three real valued Lebesgue integrable functions. If the function $\zeta(\cdot)$ is absolutely continuous on the interval $[T_0, T]$ and if for almost all $t \in [T_0, T]$*

$$\dot{\zeta}(t) \leq b(t) + c(t)\zeta(t),$$

then for all $t \in [T_0, T]$

$$\zeta(t) \leq \zeta(T_0) \exp\left(\int_{T_0}^{t} c(\tau)\mathrm{d}\tau \right) + \int_{T_0}^{t} b(s) \exp\left(\int_{s}^{t} c(\tau)\mathrm{d}\tau \right) \mathrm{d}s.$$

3. Technical assumptions and notations

For the sake of readability, in this section we collect the hypotheses used along the paper. For some real numbers $T_0, \tau_0 \in [0, +\infty[$ and $s_0, \eta \in]0, +\infty[$ such that

$$T_0 < \tau_0 \quad \text{and} \quad B[u_0, s_0] \subset B[0, \eta],$$

we will consider the following assumptions. Before going on, let Δ_{τ_0} and Δ_∞ be the triangles defined by

$$\Delta_{\tau_0} := \{(t, s) \in [T_0, \tau_0] \times [T_0, \tau_0] \ : \ s \leq t\},$$

and
$$\Delta_\infty := \{(t, s) \in [T_0, +\infty[\times [T_0, +\infty[\ : \ s \leq t\}. \tag{3}$$

($\mathcal{H}_\Phi$) $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function with $u_0 \in \operatorname{dom}\Phi$ such that Φ is c-plr on some open convex set containing $B[u_0, s_0]$.

($\mathcal{H}_1$) $f_1 : [T_0, \tau_0] \times H \to H$ is Bochner measurable in time and such that

- ($\mathcal{H}_{1,1}$) there exists a non-negative function $\beta_1(\cdot) \in L^2([T_0, \tau_0], \mathbb{R}_+)$ such that $\|f_1(t, x)\| \leq \beta_1(t)$ for any $t \in [T_0, \tau_0]$ and for any $x \in B[0, \eta] \cap \operatorname{dom}\Phi$.
- ($\mathcal{H}_{1,2}$) there exists a non-negative function $L_{1,\eta}(\cdot) \in L^2([T_0, \tau_0], \mathbb{R}_+)$ such that for any $t \in [T_0, \tau_0]$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$\|f_1(t, x) - f_1(t, y)\| \leq L_{1,\eta}(t)\|x - y\|.$$

($\mathcal{H}_2$) $f_2 : [T_0, \tau_0]^2 \times H$ is (Bochner) measurable in $(t, s) \in \Delta_{\tau_0}$ (i.e., $f_2(\cdot, \cdot, x)$ is measurable for each $x \in H$), and such that

- ($\mathcal{H}_{2,1}$) there exists a non-negative function $\beta_2(\cdot, \cdot) \in L^2(\Delta_{\tau_0}, \mathbb{R}_+)$ such that $\|f_2(t, s, x)\| \leq \beta_2(t, s)$ for any $(t, s) \in \Delta_{\tau_0}$ and $x \in B[0, \eta] \cap \operatorname{dom}\Phi$,
- ($\mathcal{H}_{2,2}$) there exists a non-negative function $L_{2,\eta}(\cdot) \in L^2([T_0, \tau_0], \mathbb{R}_+)$ such that for any $(t, s) \in \Delta_{\tau_0}$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$\|f_2(t, s, x) - f_2(t, s, y)\| \leq L_{2,\eta}(t)\|x - y\|.$$

It is well known that the regularization approach using Moreau envelope reduces the constrained inclusion differential to an ordinary differential equation which admits a unique solution thanks to the classical Cauchy-Lipschitz's theorem under some regular conditions. In our case, the obtained regularized problem is perturbed by an integral forcing term depending on the state function, which leads to an integro-differential equation in the infinite dimensional Hilbert H. In order to overcome the existence problem we are going to establish an auxiliary result that will be needed later.

Lemma 3.1. *Assume that the functions f_1 and f_2 satisfy the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ stated above. Then, there exists some real $\theta \in]T_0, \tau_0[$ such that the following integro-differential equation*

$$\begin{cases} -\dot{x}(t) = f_1(t, x(t)) + \displaystyle\int_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s \qquad a.e.\ t \in [T_0, \tau], \\ x(T_0) = u_0 \end{cases} \tag{4}$$

admits a unique absolutely continuous solution $x : [T_0, \theta[\to B(u_0, r)$ for some positive number $r \in]0, s_0[$.

Proof. It is well known that the normal cone $N_{C(t)}(x)$ equals the single-valued set $\{0\}$ if we suppose that $C(t)$ is the whole space H. In such a case, the Voltera sweeping process, studied in [2], reduces to the integro-differential equation (4). Employing the existence and uniqueness result established in Theorem 4.2 in the same reference and using the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, we deduce that (4) admits a unique absolutely continuous solution $x : [T_0, \tau_0] \to H$. Since the solution $x(\cdot)$ is continuous at u_0 then, we can choose some positive number r such that the solution x is defined over $[T_0, \theta[$ with values in the ball $B(u_0, r)$. □

Remark 3.2. It is also worth noticing that the existence and uniqueness result in Lemma 3.1 can also be proved by a classical method of fixed point without using [2]. □

4. Local existence

This section is devoted to the study of local solution of $(\mathcal{P}_{f_1,f_2})$ which is defined on some bounded interval $[T_0, T]$. To establish the desired result, our method consists in taking the Moreau approximation of the function Φ and developing some suitable estimates on the approximate system; by passing to the limit we obtain the existence and certain desired properties of the solution. This process uses various properties of Moreau envelope and its proximal mapping which are recalled in Theorem 2.9 above.

Theorem 4.1. *Suppose that the assumptions $(\mathcal{H}_\Phi), (\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are fulfilled. Then, there exists a real number $T \in]T_0, \tau_0[$ such that the problem $(\mathcal{P}_{f_1,f_2})$ admits a unique absolutely continuous solution $u : [T_0, T] \to B(u_0, \eta_0)$. Further*

(a_1) $u([T_0, T]) \subset \operatorname{dom}\Phi$;

(a_2) $\dot{u}(\cdot) \in L^2([T_0, T], H)$;

(a_3) *For any $s, t \in [T_0, T]$ with $s \le t$*

$$\int_s^t \|\dot{u}(r)\|^2\,\mathrm{d}r \le 2(\Phi(u(s)) - \Phi(u(t))) + \int_s^t \left\| f_1(r, u(r)) + \int_{T_0}^{r} f_2(r, \sigma, u(\sigma))\,\mathrm{d}\sigma \right\|^2 \mathrm{d}r.$$

(a_4) *If we strengthen the plr property of the function Φ to the following condition:*

$(\mathcal{H}'_\Phi)$: *$\Phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper lsc function with $u_0 \in \operatorname{dom}\Phi$ such that Φ is* c-plr uniformly with respect to subgradients *on some open convex set containing $B[u_0, s_0]$,* *then, the mapping $\Phi \circ u$ is absolutely continuous on $[T_0, T]$, and for almost every $t \in [T_0, T]$*

$$(\Phi \circ u)'(t) = \langle \zeta, \dot{u}(t) \rangle \text{ for any } \zeta \in \partial\Phi(u(t)), \tag{5}$$

and for any $s, t \in [T_0, T[$ with $s \leq t$

$$\begin{aligned} &\Phi(u(s)) - \Phi(u(t)) \\ &= \int_s^t \|\dot{u}(r)\|^2 \,\mathrm{d}r + \int_s^t \Big\langle f_1(r, u(r)) + \int_{T_0}^r f_2(r, \sigma, u(\sigma)) \,\mathrm{d}\sigma, \dot{u}(r) \Big\rangle \,\mathrm{d}r. \end{aligned} \tag{6}$$

Before starting with the proof of the above theorem and in order to keep things simple, let us fix some notations that will be used in the proof.

- Let $s_0' \in]0, s_0[$ be such that $\inf_{B[u_0, s_0']} \Phi \in \mathbb{R}$ with $s_0' < \frac{1}{2c}$. This is clearly possible invoking the lower semicontinuity of $\Phi(\cdot)$ at u_0.
- Let $(x_0, y_0) \in \operatorname{gph} \partial\Phi$ be such that

$$\|x_0 - u_0\| < \eta_0 \text{ where } \eta_0 := \frac{s_0'}{18}, \tag{7}$$

  by virtue of the density (graphically) of $\operatorname{Dom} \partial\Phi$ in $\operatorname{dom} \Phi$ (see [7]).
- Let $\overline{\Phi}(\cdot)$ be the function defined by

$$\overline{\Phi}(\cdot) := \Phi(\cdot) + \delta(\cdot, B[x_0, \frac{s_0'}{2}]), \tag{8}$$

$$\text{it follows that} \quad e_\lambda \overline{\Phi}(x) \geq \inf_{B(u_0, s_0')} \Phi, \text{ for all } x \in B(u_0, s_0'). \tag{9}$$

Proof. We refer to [4] for the local uniqueness. Our aim here is to provide another proof of existence based on the use of the Moreau envelope.

Denote by λ_0 the threshold provided by Theorem 2.9 such that $(a) - (h)$ in that theorem hold whenever $\lambda \in]0, \lambda_0[$.

Step 1. A family of approximate solutions.

Fix any $\lambda \in]0, \lambda_0[$ and consider the approximate problem:

$$(\mathcal{P}_\lambda) \quad \begin{cases} \text{Find a real } T_\lambda \in [T_0, \tau_0[\text{ and a locally absolutely continuous mapping} \\ u_\lambda : [T_0, T_\lambda[\to B(u_0, \eta_0) \text{ such that} \\ \dot{u}_\lambda(t) + \nabla e_\lambda \overline{\Phi}(u_\lambda(t)) + f_1(t, u_\lambda(t)) + \displaystyle\int_{T_0}^t f_2(t, s, u_\lambda(s)) \,\mathrm{d}s = 0 \text{ for all } t \in [T_0, T_\lambda[\\ u_\lambda(T_0) = u_0. \end{cases}$$

Let h be the function defined on $[T_0, \tau_0] \times B(u_0, \eta_0)$ by

$$h(t, x) := \nabla e_\lambda \overline{\Phi}(x) + f_1(t, x).$$

The mapping h enjoys the following properties:

- the mapping $t \mapsto h(t, x)$ is Bochner measurable thanks to that of f_1 on $[T_0, \tau_0]$,

- the mapping h satisfies the hypothesis $(\mathcal{H}_1)$ on $[T_0, \tau_0] \times B(u_0, \eta_0)$. Indeed, firstly, let $x \in B(u_0, \eta_0)$ then according to inclusion $x_0 \in B(u_0, \eta_0)$ (due to (7)) and properties (a) and (e) of Theorem 2.9

$$\begin{aligned}\|\nabla e_\lambda \overline{\Phi}(x)\| &\leq \|\nabla e_\lambda \overline{\Phi}(x) - \nabla e_\lambda \overline{\Phi}(x_0)\| + \|\nabla e_\lambda \overline{\Phi}(x_0)\| \\ &\leq d_\lambda \|x - x_0\| + \kappa_0 \|y_0\| \\ &\leq d_\lambda \left(\eta_0 + \|u_0\| + \|x_0\|\right) + \kappa_0 \|y_0\| =: K_\lambda.\end{aligned}$$

  This entails that, through the inclusions $B(u_0, \eta_0) \subset B(u_0, s_0) \subset B[0, \eta]$ and the hypothesis $(\mathcal{H}_1)$ assumed for f_1

$$\|h(t,x)\| \leq \|\nabla e_\lambda \overline{\Phi}(x)\| + \|f_1(t,x)\| \leq K_\lambda + \beta_1(t). \tag{10}$$

  Secondly, for any $t \in [T_0, \tau_0]$ and for any $x, y \in B(u_0, \eta_0)$

$$\begin{aligned}\|h(t,x) - h(t,y)\| &\leq \|\nabla e_\lambda \overline{\Phi}(x) - \nabla e_\lambda \overline{\Phi}(y)\| + \|f_1(t,x) - f_1(t,y)\| \\ &\leq (d_\lambda + L_{1,\eta}(t)) \|x - y\|. \end{aligned}\tag{11}$$

Putting (10) and (11) and $(\mathcal{H}_2)$ together and applying Lemma 3.1, we conclude that there exists some $T_\lambda \in]T_0, \tau_0[$ and a unique mapping $u_\lambda : [T_0, T_\lambda[\to B(u_0, \eta_0)$ as solution of the problem $(\mathcal{P}_\lambda)$ and *defined on its maximal interval of existence*, that is

$$T_\lambda = \sup\{\theta \in]T_0, \tau_0[\, : u_\lambda(\cdot) \text{ is the solution of } (\mathcal{P}_\lambda) \text{ and } u_\lambda([T_0, \theta]) \subset B(u_0, \eta_0)\}.$$

For $s \in [T_0, T_\lambda[$, taking the inner product in $(\mathcal{P}_\lambda)$ with $-\dot{u}_\lambda(s)$ we obtain

$$\begin{aligned}-\|\dot{u}_\lambda(s)\|^2 &= \left\langle \nabla e_\lambda \overline{\Phi}(u_\lambda(s)) + f_1(s, u_\lambda(s)) + \int_{T_0}^s f_2(s, r, u_\lambda(r))\, \mathrm{d}r, \dot{u}_\lambda(s) \right\rangle \\ &= \frac{d}{\mathrm{d}s}\left(e_\lambda \overline{\Phi} \circ u_\lambda\right)(s) + \left\langle f_1(s, u_\lambda(s)) + \int_{T_0}^s f_2(s, r, u_\lambda(r))\, \mathrm{d}r, \dot{u}_\lambda(s) \right\rangle.\end{aligned}$$

By integrating over s from T_0 to $t \in [T_0, T_\lambda[$ we get

$$\begin{aligned}\int_{T_0}^t \|\dot{u}_\lambda(s)\|^2 \,\mathrm{d}s &= e_\lambda \overline{\Phi}(u_0) - e_\lambda \overline{\Phi}(u_\lambda(t)) \\ &\quad + \int_{T_0}^t \left\langle f_1(s, u_\lambda(s)) + \int_{T_0}^s f_2(s, r, u_\lambda(r))\, \mathrm{d}r, -\dot{u}_\lambda(s) \right\rangle \mathrm{d}s.\end{aligned}\tag{12}$$

On the other hand, one has from (9) and the inclusion $u_0 \in B[x_0, \frac{s_0'}{2}]$

$$e_\lambda \overline{\Phi}(u_\lambda(t)) \geq \inf_{B(u_0, s_0')} \Phi \quad \text{and} \quad e_\lambda \overline{\Phi}(u_0) \leq \overline{\Phi}(u_0) = \Phi(u_0),$$

further
$$\begin{aligned}&\int_{T_0}^{t}\Big\langle f_1(s,u_\lambda(s))+\int_{T_0}^{s} f_2(s,r,u_\lambda(r))\,\mathrm{d}r,-\dot u_\lambda(s)\Big\rangle \mathrm{d}s\\ &\le \int_{T_0}^{t}\Big(\|f_1(s,u_\lambda(s))\|+\int_{T_0}^{s}\|f_2(s,r,u_\lambda(r))\|\,\mathrm{d}r\Big)\|\dot u_\lambda(s)\|\mathrm{d}s\\ &\le \int_{T_0}^{t}\Big(\beta_1(s)+\int_{T_0}^{s}\beta_2(s,r)\,\mathrm{d}r\Big)\|\dot u_\lambda(s)\|\mathrm{d}s,\end{aligned}$$

then
$$\begin{aligned}&\int_{T_0}^{t}\Big\langle f_1(s,u_\lambda(s))+\int_{T_0}^{s} f_2(s,r,u_\lambda(r))\,\mathrm{d}r,-\dot u_\lambda(s)\Big\rangle \mathrm{d}s\\ &\le \Big(\int_{T_0}^{t}\Big(\beta_1(s)+\int_{T_0}^{s}\beta_2(s,r)\,\mathrm{d}r\Big)^2\mathrm{d}s\Big)^{\frac12}\Big(\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\mathrm{d}s\Big)^{\frac12}\\ &\le \sqrt2\Big(\int_{T_0}^{t}\beta_1^2(s)\mathrm{d}s+\int_{T_0}^{t}\Big(\int_{T_0}^{s}\beta_2(s,r)\,\mathrm{d}r\Big)^2\mathrm{d}s\Big)^{\frac12}\Big(\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\mathrm{d}s\Big)^{\frac12}\\ &\le \sqrt2\Big(\int_{T_0}^{t}\beta_1^2(s)\mathrm{d}s+\int_{T_0}^{t}(s-T_0)\int_{T_0}^{s}\beta_2^2(s,r)\,\mathrm{d}r\,\mathrm{d}s\Big)^{\frac12}\Big(\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\mathrm{d}s\Big)^{\frac12},\end{aligned}\tag{13}$$

which implies that

$$\int_{T_0}^{t}\Big\langle f_1(s,u_\lambda(s))+\int_{T_0}^{s} f_2(s,r,u_\lambda(r))\,\mathrm{d}r,-\dot u_\lambda(s)\Big\rangle \mathrm{d}s\le \rho_0\Big(\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\mathrm{d}s\Big)^{1/2},$$

where $\rho_0 := \sqrt2\Big(\|\beta_1\|^2_{L^2([T_0,\tau_0];\mathbb{R}_+)}+(\tau_0-T_0)\int_{T_0}^{\tau_0}\int_{T_0}^{s}\beta_2^2(s,r)\,\mathrm{d}r\,\mathrm{d}s\Big)^{1/2}.$

Consequently, it follows from (12) that

$$\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\,\mathrm{d}s-\rho_0\Big(\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\mathrm{d}s\Big)^{1/2}-(\Phi(u_0)-\inf_{B(u_0,s_0')}\Phi)\le 0,$$

which represents a polynomial equation of the second degree whose discriminant $\Delta=\rho_0^2+4(\Phi(u_0)-\inf_{B(u_0,s_0')}\Phi)$ is nonnegative.

Hence, there exists some nonnegative number ρ_1 such that

$$\Big(\int_{T_0}^{t}\|\dot u_\lambda(s)\|^2\mathrm{d}s\Big)^{1/2}\le\rho_1,\qquad \text{for any } t\in[T_0,T_\lambda[.\tag{14}$$

This ensures that the mapping u_λ is locally Hölderian on $[T_0,T_\lambda[$. More precisely

$$\|u_\lambda(t)-u_\lambda(s)\|\le\rho_1\sqrt{t-s},\qquad\text{for all}\quad T_0\le s\le t<T_\lambda.\tag{15}$$

So, we choose some real number $T\in]T_0,\tau_0]$ such that

$$\rho_1\sqrt{T-T_0}<\frac{\eta_0}{2}.\tag{16}$$

The maximality of T_λ implies that for any $\lambda \in]0, \lambda_0[$, we have $T < T_\lambda$. Indeed, assume that there exists a real $\lambda_1 \in]0, \lambda_0[$ such that $T_{\lambda_1} \leq T$. It results from (15) that the family $(u_{\lambda_1}(t))_{t \in [T_0, T_{\lambda_1}[}$ is a Cauchy net in the Hilbert space H, hence $\lim_{t \uparrow T_{\lambda_1}} u_{\lambda_1}(t)$ exists in H. From (15) and (16) and since $T \geq T_{\lambda_1}$ we obtain that

$$u_{\lambda_1}([T_0, T_{\lambda_1}[) \subset B(u_0, \frac{\eta_0}{2}) \quad \text{and} \quad \overline{u}_{\lambda_1} := \lim_{t \uparrow T_{\lambda_1}} u_{\lambda_1}(t) \in B[u_0, \frac{\eta_0}{2}],$$

this allows us to extend the mapping $u_{\lambda_1}(\cdot)$ on a right neighborhood of T_{λ_1}, and by consequence find a real $T'_{\lambda_1} > T_{\lambda_1}$ such that $u_{\lambda_1}(\cdot)$ is a solution of (P_{λ_1}) on $[T_0, T'_{\lambda_1}[\to B(u_0, \eta_0)$ with initial condition $u_{\lambda_1}(T_0) = u_0$. This contradicts the maximality of T_{λ_1}. Therefore, from (15) and (16) we obtain

$$u_\lambda([T_0, T]) \subset B(u_0, \frac{\eta_0}{2}) \text{ for all } \lambda \in]0, \lambda_0[. \tag{17}$$

Then, we can restrict the study on the interval $[T_0, T]$.

Step 2. We show that

(Π_1) $(u_\lambda(\cdot))_{\lambda \in]0, \lambda_0[}$ converges uniformly on $[T_0, T]$ to some absolutely continuous mapping $u(\cdot) : [T_0, T] \to B[u_0, \frac{\eta_0}{2}] \subset B(u_0, s_0)$ with

$$u(T_0) = u_0 \quad \text{and} \quad u([T_0, T]) \subset \operatorname{dom} \Phi.$$

(Π_2) $\lim_{\lambda \downarrow 0} P_\lambda \overline{\Phi}(u_\lambda(t)) = u(t)$ and $P_\lambda \overline{\Phi}(u_\lambda([T_0, T])) \subset \operatorname{dom} \Phi$ for any $\lambda \in]0, \lambda_0[$.

It results from (17) and the properties (b), (f) and (d) in Theorem 2.9 that

$$\begin{aligned} e_\lambda \overline{\Phi}(u_\lambda(t)) &= \overline{\Phi}(P_\lambda \overline{\Phi}(u_\lambda(t))) + \frac{1}{2\lambda} \|u_\lambda(t) - P_\lambda \overline{\Phi}(u_\lambda(t))\|^2 \\ &= \Phi(P_\lambda \overline{\Phi}(u_\lambda(t))) + \frac{\lambda}{2} \|\nabla e_\lambda \overline{\Phi}(u_\lambda(t))\|^2, \end{aligned} \tag{18}$$

by the property (f) of the same theorem, one has $\Phi(P_\lambda \overline{\Phi}(u_\lambda(t))) \geq \inf_{B[u_0, s'_0]} \Phi$.

Then
$$\begin{aligned} \|u_\lambda(t) - P_\lambda \overline{\Phi}(u_\lambda(t))\|^2 &= 2\lambda \left(e_\lambda \overline{\Phi}(u_\lambda(t)) - \Phi(P_\lambda \overline{f}(u_\lambda(t))) \right) \\ &\leq 2\lambda \big(e_\lambda \overline{\Phi}(u_\lambda(t)) - \inf_{B[u_0, s'_0]} \Phi \big). \end{aligned}$$

On the other hand, through (12) and (13), we have the estimation

$$e_\lambda \overline{\Phi}(u_\lambda(t)) \leq \Phi(u_0) + \rho_0 \rho_1. \tag{19}$$

Thus
$$\|u_\lambda(t) - P_\lambda \overline{\Phi}(u_\lambda(t))\| \leq \sqrt{2\lambda \rho_3}, \tag{20}$$

where $\rho_3 := \Phi(u_0) + \rho_0 \rho_1 - \inf_{B[u_0, s'_0]} \Phi$.

Now fix any $\lambda, \mu \in]0, \lambda_0[$. For almost every $t \in [T_0, T]$ we have

$$\frac{1}{2} \frac{d}{dt} \|u_\lambda(t) - u_\mu(t)\|^2 = \langle \dot{u}_\lambda(t) - \dot{u}_\mu(t), u_\lambda(t) - u_\mu(t) \rangle = \varrho_1^{\lambda,\mu}(t) + \varrho_2^{\lambda,\mu}(t) + \varrho_3^{\lambda,\mu}(t),$$

where

$$\varrho_1^{\lambda,\mu}(t) = \langle -\nabla e_\lambda \overline{\Phi}(u_\lambda(t)) + \nabla e_\mu \overline{\Phi}(u_\mu(t)), P_\lambda \overline{\Phi}(u_\lambda(t)) - P_\mu \overline{\Phi}(u_\mu(t))\rangle$$
$$\varrho_2^{\lambda,\mu}(t) = \langle -\nabla e_\lambda \overline{\Phi}(u_\lambda(t)) + \nabla e_\mu \overline{\Phi}(u_\mu(t)), u_\lambda(t) - P_\lambda \overline{\Phi}(u_\lambda(t)) + P_\mu \overline{\Phi}(u_\mu(t)) - u_\mu(t)\rangle$$
$$\begin{aligned}\varrho_3^{\lambda,\mu}(t) = \Big\langle & f_1(t, u_\lambda(t)) - f_1(t, u_\mu(t)) \\ & + \int_{T_0}^t \big(f_2(t,s,u_\lambda(s)) - f_2(t,s,u_\mu(s))\big)\,\mathrm{d}s, u_\mu(t) - u_\lambda(t)\Big\rangle.\end{aligned}$$

Now, we are going to make some estimations on the three latter functions. To this end, let us put

$$\xi_{\lambda,\mu}(t) := \|\nabla e_\lambda \overline{\Phi}(u_\lambda(t))\| + \|\nabla e_\mu \overline{\Phi}(u_\mu(t))\|.$$

It results from the inclusions (f) and (g) of Theorem 2.9 and the subdifferential characterization of the c-plr property of Φ that

$$\begin{aligned}\varrho_1^{\lambda,\mu}(t) &\le 2c(1+\xi_{\lambda,\mu}(t))\|P_\lambda \overline{\Phi}(u_\lambda(t)) - P_\mu \overline{\Phi}(u_\mu(t))\|^2 \\ &\le 4c(1+\xi_{\lambda,\mu}(t))\Big(\big(\|P_\lambda \overline{\Phi}(u_\lambda(t)) - u_\lambda(t)\| + \|P_\mu \overline{\Phi}(u_\mu(t)) - u_\mu(t)\|\big)^2 \\ &\qquad + \|u_\lambda(t) - u_\mu(t)\|^2\Big) \\ &\le 8c\rho_3(\sqrt{\lambda}+\sqrt{\mu})^2(1+\xi_{\lambda,\mu}(t)) + 4c(1+\xi_{\lambda,\mu}(t))\|u_\lambda(t) - u_\mu(t)\|^2,\end{aligned}$$

where the latter inequality is due to (20). Also

$$\varrho_2^{\lambda,\mu}(t) \le \sqrt{2\rho_3}(\sqrt{\lambda}+\sqrt{\mu})\xi_{\lambda,\mu}(t).$$

On the other hand, according to the hypotheses $(\mathcal{H}_{1,2})$ and $(\mathcal{H}_{2,2})$

$$\varrho_3^{\lambda,\mu}(t) \le L_{1,\eta}(t)\|u_\lambda(t) - u_\mu(t)\|^2 + L_{2,\eta}(t)\|u_\lambda(t) - u_\mu(t)\| \int_{T_0}^t \|u_\lambda(s) - u_\mu(s)\|\,\mathrm{d}s.$$

Putting these estimations together we get

$$\begin{aligned}\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 \;\le\; & (8c(1+\xi_{\lambda,\mu}(t)) + 2L_{1,\eta}(t))\,\|u_\lambda(t) - u_\mu(t)\|^2 \\ & + \big(16c\rho_3(\sqrt{\lambda}+\sqrt{\mu}) + 2\sqrt{2\rho_3}\big)(1+\xi_{\lambda,\mu}(t))(\sqrt{\lambda}+\sqrt{\mu}) \\ & + 2L_{2,\eta}(t)\|u_\lambda(t) - u_\mu(t)\| \int_{T_0}^t \|u_\lambda(s) - u_\mu(s)\|\,\mathrm{d}s.\end{aligned}$$

By integrating over s from T_0 to $t \in [T_0, T]$ and using the following inequality

$$\begin{aligned}&\int_{T_0}^t 2L_{2,\eta}(s)\|u_\lambda(s) - u_\mu(s)\|\Big(\int_{T_0}^s \|u_\lambda(r) - u_\mu(r)\|\,\mathrm{d}r\Big)\,\mathrm{d}s \\ &\qquad \le 2\sqrt{T - T_0}\|L_{2,\eta}\|_{L^2} \int_{T_0}^t \|u_\lambda(s) - u_\mu(s)\|^2 \mathrm{d}s,\end{aligned}$$

we get $$\|u_\lambda(t) - u_\mu(t)\|^2 \le C_{\lambda,\mu} + \int_{T_0}^{t} \Theta_{\lambda,\mu}(s)\|u_\lambda(s) - u_\mu(s)\|^2 \mathrm{d}s,$$

where $$C_{\lambda,\mu} := \|u_\lambda(T_0) - u_\mu(T_0)\|^2 + \big(16c\rho_3(\sqrt{\lambda} + \sqrt{\mu}) + 2\sqrt{2\rho_3}\big)(T - T_0 + \|\xi_{\lambda,\mu}\|_{L^1([T_0,T],\mathbb{R}_+)})(\sqrt{\lambda} + \sqrt{\mu}),$$

and $$\Theta_{\lambda,\mu}(s) := 8c(1 + \xi_{\lambda,\mu}(s)) + 2L_{1,\eta}(s) + 2\sqrt{T - T_0}\|L_{2,\eta}\|_{L^2([T_0,\tau_0],\mathbb{R}_+)}.$$

Noting that $\|\xi_{\lambda,\mu}\|_{L^1([T_0,T],\mathbb{R}_+)}$ is bounded independently of λ and μ. It follows from the Gronwall's inequality that

$$\|u_\lambda(t) - u_\mu(t)\|^2 \le C_{\lambda,\mu} + C_{\lambda,\mu}\int_{T_0}^{t} \Theta_{\lambda,\mu}(s)\exp\Big(\int_{T_0}^{t}\Theta_{\lambda,\mu}(r)\mathrm{d}r\Big)\mathrm{d}s.$$

Recalling that $u_\lambda(T_0) = u_\mu(T_0)$ and $\Theta_{\lambda,\mu}(\cdot)$ is bounded in L^1 independently of λ and μ, we conclude that the uniform Cauchy criterion is verified. Then $(u_\lambda(\cdot))_{\lambda\in]0,\lambda_0[}$ converges uniformly on $[T_0, T]$ as $\lambda \downarrow 0$ to some continuous mapping $u : [T_0, T] \to H$ such that $u(T_0) = u_0$. In addition

$$u([T_0, T]) \subset B[u_0, \tfrac{\eta_0}{2}] \subset B(u_0, s_0), \tag{21}$$

since $u_\lambda([T_0, T]) \subset B(u_0, \frac{\eta_0}{2})$ according to (17). It follows from (20) that

$$\sup_{t\in[T_0,T]} \|u_\lambda(t) - P_\lambda\overline{\Phi}(u_\lambda(t))\| \underset{\lambda\downarrow 0}{\longrightarrow} 0, \tag{22}$$

which implies that $P_\lambda\overline{\Phi}(u_\lambda(t)) \to u(t)$. On the other hand, from (19) and (18) we get

$$\Phi(P_\lambda\overline{\Phi}(u_\lambda(t))) \le e_\lambda\overline{\Phi}(u_\lambda(t)) \le \Phi(u_0) + \rho_0\rho_1, \tag{23}$$

hence $P_\lambda\overline{\Phi}(u_\lambda([T_0, T])) \subset \operatorname{dom}\Phi$ and by the lower semicontinuity of Φ

$$\Phi(u(t)) \le \liminf_{\lambda\downarrow 0} \Phi(P_\lambda\overline{\Phi}(u_\lambda(t))) \le \Phi(u_0) + \rho_0\rho_1, \tag{24}$$

which translates the property (a_1). It remains to show the absolute continuity of $u(\cdot)$ as well as properties $(a_2) - (a_4)$. The inequality (14) guarantees that $(\dot{u}_\lambda)_{\lambda\in]0,\lambda_0[}$ is bounded in $L^2([T_0, T], H)$ independently of λ. Therefore we can extract a subsequence $(\lambda_n)_n$ in $]0, \lambda_0[$ converging to 0 such that $(u_{\lambda_n})_{n\ge 1}$ converges weakly to some $\chi(\cdot) \in L^2([T_0, T], H)$, that is

$$\lim_{n\to+\infty} \int_{T_0}^{T} \langle \dot{u}_{\lambda_n}(r), w(r)\rangle\,\mathrm{d}r = \int_{T_0}^{T} \langle \chi(r), w(r)\rangle\,\mathrm{d}r, \forall w \in L^2([T_0, T]; H),$$

in particular, for any $z \in H$, taking $w(\cdot) := z1_{[T_0,t]}, t \in [T_0, T]$ then

$$\Big\langle \int_{T_0}^{t} \dot{u}_{\lambda_n}(r)\,\mathrm{d}r, z\Big\rangle = \int_{T_0}^{T} \langle \dot{u}_{\lambda_n}(r), z1_{[0,t]}(r)\rangle\,\mathrm{d}r$$
$$\underset{n\to+\infty}{\longrightarrow} \int_{T_0}^{T} \langle \chi(r), z1_{[0,t]}(r)\rangle\,\mathrm{d}r = \Big\langle \int_{0}^{t} \chi(r)\,\mathrm{d}r, z\Big\rangle,$$

thus $\quad u_{\lambda_n}(t) = u_0 + \displaystyle\int_{T_0}^{t} \dot{u}_{\lambda_n}(r)\,\mathrm{d}r$ converges weakly in H to $u_0 + \displaystyle\int_0^t \chi(r)\,\mathrm{d}r$.

The uniform congergence of $(u_{\lambda_n})_n$ to u implies $u(t) = u_0 + \int_0^t \chi(r)\,\mathrm{d}r$. This gives the absolute continuity property of $u(\cdot)$ on $[T_0, T]$ as well as the equality $\dot{u} = \chi$ a.e. on $[T_0, T]$ and the inclusion $\dot{u}(\cdot) \in L^2([T_0, T], H)$.

Step 3. Let us show property (a_3). First, put

$$\varphi_n(t) := f_1(t, u_{\lambda_n}(t)) + \int_{T_0}^{t} f_2(t, s, u_{\lambda_n}(s))\,\mathrm{d}s \quad \text{and}$$

$$\varphi(t) := f_1(t, u(t)) + \int_{T_0}^{t} f_2(t, s, u(s))\,\mathrm{d}s, \quad t \in [T_0, T].$$

Then

$$\begin{aligned}
&\int_{T_0}^{T} \|\varphi_n(t) - \varphi_n(t)\|^2\,dt \\
&\le \int_{T_0}^{T} \Big(\|f_1(t, u_{\lambda_n}(t)) - f_1(t, u(t))\| + \int_{T_0}^{t} \|f_2(t, s, u_{\lambda_n}(s)) - f_2(t, s, u(s))\|\,\mathrm{d}s \Big)^2 dt \\
&\le 2 \int_{T_0}^{T} \Big[L_{1,\eta}^2(t) \|u_{\lambda_n}(t) - u(t)\|^2 + L_{2,\eta}^2(t) \Big(\int_{T_0}^{t} \|u_{\lambda_n}(s) - u(s)\|\,\mathrm{d}s \Big)^2 \Big] dt \\
&\le 2 \|L_{1,\eta}(\cdot)(u_{\lambda_n} - u)\|^2_{L^2([T_0,T];H)} + 2(T - T_0) \int_{T_0}^{T} L_{2,\eta}^2(t) \Big(\int_{T_0}^{t} \|u_{\lambda_n}(s) - u(s)\|^2\,\mathrm{d}s \Big) dt.
\end{aligned}$$

In consequence we obtain

$$\begin{aligned}
&\|\varphi_n - \varphi\|^2_{L^2([T_0,T];H)} \\
&\quad \le 2 \|L_{1,\eta}(\cdot)(u_{\lambda_n} - u)\|^2_{L^2([T_0,T];H)} + 2(T - T_0) \|L_{2,\eta}\|^2_{L^2([T_0,T];\mathbb{R}_+)} \|u_{\lambda_n} - u\|^2_{L^2([T_0,T];H)},
\end{aligned}$$

implying that $(\varphi_n)_n$ converges strongly to φ in $L^2([T_0, T]; H)$. So, for any $t \in [T_0, T]$

$$\int_{T_0}^{t} \langle \varphi_n(s), \dot{u}_{\lambda_n}(s) \rangle\,\mathrm{d}s \underset{n \to +\infty}{\longrightarrow} \int_{T_0}^{t} \langle \varphi(s), \dot{u}(s) \rangle\,\mathrm{d}s.$$

On the other hand, It results from (12) and (18) that

$$\int_{T_0}^{t} \|\dot{u}_{\lambda_n}(s)\|^2\,\mathrm{d}s \le \Phi(u_0) - \Phi(P_{\lambda_n} \overline{\Phi}(u_{\lambda_n}(t))) - \int_{T_0}^{t} \Big\langle \varphi_n(r), \dot{u}_{\lambda_n}(s) \Big\rangle \mathrm{d}s. \tag{25}$$

Taking the inferior limit in (25) as $n \to +\infty$ and using the weak lower semicontinuity property of $\|\cdot\|_{L^2([T_0,T],H)}$, the lower semicontinuity property of Φ and (22) entails

$$\int_{T_0}^{t} \|\dot{u}(r)\|^2\,\mathrm{d}r \le \Phi(u_0) - \Phi(u(t)) - \int_{T_0}^{t} \langle \varphi(r), \dot{u}(r) \rangle\,\mathrm{d}r,$$

which leads to $\displaystyle\int_{T_0}^{t} \|\dot{u}(r)\|^2 \,\mathrm{d}r \leq 2(\Phi(u(T_0)) - \Phi(u(t))) + \int_{T_0}^{t} \|\varphi(r)\|^2 \mathrm{d}r,$

and the latter inequality still holds for s instead of T_0 for any for any $T_0 \leq s \leq t \leq T$.

Step 4. Let us prove that $u(\cdot) : [T_0, T] \to B(u_0, \eta_0)$ is the solution of $(\mathcal{P}_{f_1,f_2})$. Based on the above results, one can write

(1) $\{P_{\lambda_n}\overline{\Phi}(u_{\lambda_n}(t)) : t \in [T_0, T], n \in \mathbb{N}\} \subset B(u_0, \frac{7}{16}s_0) \cap \operatorname{dom}\Phi$ via (f) of Theorem 2.9 and (23).

(2) $P_{\lambda_n}\overline{\Phi}(u_{\lambda_n}(\cdot)) \to u(\cdot)$ according to (20) and $u([T_0, T]) \subset B[u_0, \frac{\eta_0}{2}] \cap \operatorname{dom}\Phi$ via (21) and (24).

(3) $\dot{u}_{\lambda_n}(\cdot) + \varphi_n(\cdot) \to \dot{u}(\cdot) + \varphi(\cdot)$ weakly in $L^2([T_0, T], H)$,

(4) $-\dot{u}_{\lambda_n}(t) - \varphi_n(t) = \nabla e_{\lambda_n}\overline{\Phi}(u_{\lambda_n}(t)) \in \partial\Phi(P_{\lambda_n}\overline{\Phi}(u_{\lambda_n}(t)))$ via (g) of Theorem 2.9 and the inclusion $u_\lambda([T_0, T]) \subset B(u_0, \eta_0)$.

Those four properties and Lemma 2.6 lead to the desired inclusion in $(\mathcal{P}_{f_1,f_2})$.

Step 5. Let us prove property (a_4).

First, under the strengthened condition $(\mathcal{H}'_\Phi)$, the function Φ satisfies the assumption $(\mathcal{H}'_\Phi)$, which ensures the well posedness of $(\mathcal{P}_{f_1,f_2})$. Select arbitrary $t \in [T_0, T]$ where

$$-\dot{u}(t) - f_1(t, u(t)) - \int_{T_0}^{t} f_2(t, s, u(s)) \,\mathrm{d}s \in \partial\Phi(u(t)).$$

Fix any $\lambda \in]0, \lambda_1[$ where $\lambda_1 \in]0, \lambda_0[$ given by Lemma 2.10. From previous analysis, one has $u(t) \in B(u_0, \frac{s_0}{18})$ then Lemma 2.10 gives the following estimates

$$\begin{aligned}
\|\nabla e_\lambda\overline{\Phi}(u(t))\| &\leq \frac{1}{1 - 2c\lambda_1}\Big(\|\dot{u}(t)\| + \|f_1(t, u(t))\| + \int_{T_0}^{t} \|f_2(t, s, u(s))\| \,\mathrm{d}s\Big) \\
&\leq \frac{1}{1 - 2c\lambda_1}\Big(\|\dot{u}(t)\| + \beta_1(t) + \int_{T_0}^{t} \beta_2(t, s) \,\mathrm{d}s\Big)
\end{aligned}$$

Therefore, recalling that β_1 and $\beta_2(\cdot, \cdot)$ are in L^2 yields the weak compactness of $(\nabla e_\lambda\overline{\Phi}(u(\cdot)))_\lambda$. Then, we can extract a subsequence, denoted $(\nabla e_{\lambda_m}\overline{f}(u(\cdot)))_m$, which converges weakly to some mapping $v(\cdot)$ in $L^2([T_0, T], H)$. We have also that $e_{\lambda_m}\overline{\Phi}(u(\cdot))$ is absolutely continuous on $[T_0, T]$ because $e_{\lambda_m}\overline{\Phi}(\cdot)$ is $\mathcal{C}^1$ on $B(u_0, \frac{s_0}{18})$ according to (a) of Theorem 2.9. It results that for any $T_0 \leq s \leq t \leq T$

$$e_{\lambda_m}\overline{\Phi}(u(t)) - e_{\lambda_m}\overline{\Phi}(u(s)) = \int_{s}^{t} \langle\nabla e_{\lambda_m}\overline{\Phi}(u(r)), \dot{u}(r)\rangle \,\mathrm{d}r$$

and passing to the limit on m, we get

$$\Phi(u(t)) - \Phi(u(s)) = \int_{s}^{t} \langle v(r), \dot{u}(r)\rangle \,\mathrm{d}r$$

which means that $\Phi \circ u$ is absolutely continuous on $[T_0, T]$, hence it is derivable a.e. on $[T_0, T]$. The equality (5) is a direct consequence of Lemma 2.7.

Fix now any $t \in]T_0, T[$ such that $\Phi \circ u$ and $u(\cdot)$ are derivable and

$$-\dot{u}(t) - f_1(t, u(t)) - \int_{T_0}^{t} f_2(t, s, u(s))\,\mathrm{d}s \in \partial\Phi(u(t)).$$

Using Lemma 2.7 again and taking into account the latter inclusion, it ensures that

$$(\Phi \circ u)'(t) = -\|\dot{u}(t)\|^2 - \Big\langle f_1(t, u(t)) + \int_{T_0}^{t} f_2(t, s, u(s))\,\mathrm{d}s, \dot{u}(t)\Big\rangle \text{ a.e } t \in [T_0, T],$$

we integrate this latter equality to obtain

$$\Phi(u(s)) - \Phi(u(t)) = \int_s^t \|\dot{u}(r)\|^2\,\mathrm{d}r + \int_s^t \Big\langle f_1(r, u(r)) + \int_{T_0}^{r} f_2(r, \sigma, u(\sigma))\,\mathrm{d}\sigma, \dot{u}(r)\Big\rangle\,\mathrm{d}r$$

for any $s, t \in [T_0, T]$ with $s \leq t$ and we are done with the proof of the theorem. □

5. Existence of global solution

Our next goal in this work is to establish the existence of a global locally absolutely continuous solution over all the interval $[T_0, +\infty[$. This existence issue can be obtained through the existence of the truncated interval above under some additional conditions. Especially, the boundness from below of the function Φ by a function of affine type. Before the formulation of this theorem, recall that a function $W(\cdot)$ is said to be locally absolutely continuous on $[T_0, +\infty[$, when it is absolutely continuous on each compact subinterval $[a, b] \subset [T_0, +\infty[$.

Proposition 5.1. *Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. Consider $u_0 \in \operatorname{dom}\Phi$, real numbers $s_0 > 0, T_0 \in [0, +\infty[$ such that Φ is plr on $B[u_0, s_0]$ and consider also two mappings $f_1 : [T_0, +\infty[\times H \to H$ and $f_2 : \Delta_\infty \times H \to H$ where Δ_∞ is defined in* (3). *Assume that the following hold*

($\mathcal{Q}_1$) *f_1 is (Bochner) measurable in time (i.e., $f_1(\cdot, x)$ is measurable for each $x \in H$), such that*

- ($\mathcal{Q}_{1,1}$) *there exist non-negative functions $\beta_1(\cdot) \in L^2_{loc}([T_0, +\infty[, \mathbb{R}_+)$ such that $\|f_1(t, x)\| \leq \beta_1(t)(1 + \|x\|)$, for any $t \in [T_0, +\infty[$ and for any $x \in H$;*
- ($\mathcal{Q}_{1,2}$) *for each real number $\eta > 0$ there exists a non-negative function $L_{1,\eta}(\cdot) \in L^2_{loc}([T_0, +\infty[, \mathbb{R}_+)$ such that for any $t \in [T_0, +\infty[$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$ we have*

$$\|f_1(t, x) - f_1(t, y)\| \leq L_{1,\eta}(t)\|x - y\|.$$

($\mathcal{Q}_2$) *f_2 is (Bochner) measurable in Δ_∞ (i.e., $f_2(\cdot, \cdot, x)$ is measurable for each $x \in H$) such that*

- ($\mathcal{Q}_{2,1}$) *there exists a non-negative function $\beta_2(\cdot, \cdot) \in L^2_{loc}(\Delta_\infty, \mathbb{R}_+)$ such that $\|f_2(t, s, x)\| \leq \beta_2(t, s)(1 + \|x\|)$ for any $(t, s) \in \Delta_\infty$ and for any $x \in H$;*
- ($\mathcal{Q}_{2,2}$) *for each real number $\eta > 0$ there exists a non-negative function $L_{2,\eta}(\cdot) \in L^2_{loc}([T_0, +\infty[, \mathbb{R}_+)$ such that for any $(t, s) \in \Delta_\infty$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,*

$$\|f_2(t, s, x) - f_2(t, s, y)\| \leq L_{2,\eta}(t)\|x - y\|.$$

Denote by $\mathcal{E}$ any subset of H containing $B[u_0, s_0]$. Then there exist a real number $T \in]T_0, +\infty[$ and a locally absolutely continuous mapping $u : [T_0, T[\to \mathcal{E}$ satisfying

(e_1) $-\dot{u}(t) \in \partial\Phi(u(t)) + f_1(t, u(t)) + \int_{T_0}^{t} f_2(t, s, u(s))\,\mathrm{d}s$ *a.e.* $t \in [T_0, T[$;

(e_2) $u(T_0) = u_0$ *and* $x([T_0, T[) \subset \operatorname{dom}\Phi$;

(e_3) $\dot{u} \in L^2_{\mathrm{loc}}([T_0, T[, H)$ *and for any* $s, t \in [T_0, T[$ *with* $s \le t$

$$\int_s^t \|\dot{u}(r)\|^2\,\mathrm{d}r \le 2(\Phi(u(s)) - \Phi(u(t))) + \int_s^t \left\| f_1(r, u(r)) + \int_{T_0}^{r} f_2(r, \sigma, u(\sigma))\,\mathrm{d}\sigma \right\|^2 \mathrm{d}r,$$

(e_4) *If we strengthen the plr property of the function Φ to the following condition:*

$(\mathcal{Q}'_\Phi)$ $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ *is a proper lsc function with* $u_0 \in \operatorname{dom}\Phi$ *such that* Φ *is* c-plr uniformly with respect to subgradients *on some open convex containing* $B[u_0, s_0]$,

then, the mapping $\Phi \circ u$ is absolutely continuous on $[T_0, T[$, and for almost every $t \in [T_0, T[$ we have

$$(\Phi \circ u)'(t) = \langle \zeta, \dot{u}(t) \rangle \quad \textit{for any} \quad \zeta \in \partial\Phi(u(t)),$$

and for any $s, t \in [T_0, T[$ with $s \le t$

$$\Phi(u(s)) - \Phi(u(t)) = \int_s^t \|\dot{u}(r)\|^2\,\mathrm{d}r + \int_s^t \left\langle f_1(r, u(r)) + \int_{T_0}^{r} f_2(r, \sigma, u(\sigma))\,\mathrm{d}\sigma, \dot{u}(r) \right\rangle \mathrm{d}r.$$

(e_5) *As a solution, $x(\cdot)$ cannot be extended in a right neighborhood of T.*

Proof. Denote by Λ the collection of pairs $(\tau, x(\cdot))$ such that $\tau \in]T_0, +\infty]$ and $x(\cdot) : [T_0, \tau[\to \mathcal{E}$ is locally absolutely continuous, and such that the properties (e_1), (e_2), (e_3) and (e_4) are satisfied with $T = \tau$. This is clearly possible thanks to Theorem 4.1 and then, the nonemptiness of the set Λ is guaranteed. Endow Λ with the partial ordering " $\preceq$ " defined by

$$(T_1, x_1) \preceq (T_2, x_2) \;\Leftrightarrow\; T_1 \le T_2 \quad \text{and} \quad x_1(t) = x_2(t) \;\; \text{for all } t \in [T_0, T_1[,$$

for $(T_i, x_i) \in \Lambda, i = 1, 2$. Let $\{(T_\alpha, x_\alpha) : \alpha \in A\}$ be a totally ordered family in Λ. Putting $\widehat{T} := \sup_{\alpha \in A} T_\alpha$, then the mapping $\widehat{x}(\cdot) : [T_0, \widehat{T}[\to \mathcal{E}$ given by $\widehat{x}(t) := x_\alpha(t)$ if $t \in [T_0, T_\alpha[$ is well defined, locally absolutely continuous on this interval and fulfills the properties (e_1), (e_2), (e_3) and (e_4) on $[T_0, \widehat{T}[$. In order to prove the latter result, let us consider a sequence $(\theta_n)_{n \ge 1} \subset [T_0, \widehat{T}[$ which converges increasingly to $\widehat{T}$, then

$$[T_0, \widehat{T}[= \bigcup_{n \ge 1} [T_0, \theta_n].$$

Given any integer $n \ge 1$, by definition of the supremum $\widehat{T}$, there is some $\alpha_n \in A$ such that $\theta_n < T_{\alpha_n} < \widehat{T}$, and by the definition of $\widehat{x}(\cdot)$ we get that for any $t \in [T_0, T_{\alpha_n}[$, $\widehat{x}(t) = x_{\alpha_n}(t)$.

In contrast we have $\{(T_\alpha, x_\alpha) : \alpha \in A\} \subset \Lambda$, which ensures that for any $n \geq 1$

$$\begin{array}{l}\text{the mapping } \widehat{x}(\cdot) \text{ fulfills the properties } (e_2), (e_3) \text{ and } (e_4)\\ \text{on } [T_0, T_{\alpha_n}[\text{ and the property } (e_1) \text{ a.e. } t \in [T_0, T_{\alpha_n}[\end{array} \tag{26}$$

and this entails that $\widehat{x}([T_0, \theta_n]) \subset \operatorname{dom}\Phi$ hence $\widehat{x}([T_0, \widehat{T}[) \subset \operatorname{dom}\Phi$. Let $t \in [T_0, \widehat{T}[$. By definition of $\widehat{T}$, there exists $\overline{\alpha} \in A$ such that $t < T_{\overline{\alpha}} \leq \widehat{T}$ with $x_{\overline{\alpha}}(\cdot)$ satisfying (e_3) and locally absolutely continuous on $[T_0, T_{\overline{\alpha}}[$. Since $t < T_{\overline{\alpha}}$, the mapping $x_{\overline{\alpha}}(\cdot)$ is absolutely continuous on $[T_0, t]$ with $\dot{x}_{\overline{\alpha}}(\cdot) \in L^2([T_0, t], H)$ and

$$\int_s^t \|\dot{x}_{\overline{\alpha}}(r)\|^2 \,\mathrm{d}r \leq 2(\Phi(x_{\overline{\alpha}}(s)) - \Phi(x_{\overline{\alpha}}(t))) + \int_s^t \left\| f_1(r, x_{\overline{\alpha}}(r)) + \int_{T_0}^r f_2(r, \sigma, x_{\overline{\alpha}}(\sigma)) \,\mathrm{d}\sigma \right\|^2 \mathrm{d}r,$$

By definition, for any $t \in [T_0, T_{\overline{\alpha}}[$ we have $\widehat{x}(t) = x_{\overline{\alpha}}(t)$ which ensures that $\widehat{x}(\cdot)$ is locally absolutely continuous on $[T_0, \widehat{T}[$ and satisfies (e_3) on $[T_0, \widehat{T}[$. Repeating the reasoning just given, we deduce that $\widehat{x}(\cdot)$ fulfills (e_4) on $[T_0, \widehat{T}[$.

On the other hand, according to (26), $\widehat{x}(\cdot)$ satisfies the property (e_1) almost everywhere on $[T_0, T_{\alpha_n}[$ then, it results from the inequality $\theta_n < T_{\alpha_n}, n \geq 1$ that for any integer $n \geq 1$ there exists some Lebesgue-null set $\mathcal{N}_n \subset [T_0, \theta_n]$ such that

$$-\dot{\widehat{x}}(t) \in \partial\Phi(\widehat{x}(t)) + f_1(t, \widehat{x}(t)) + \int_{T_0}^t f_2(t, s, \widehat{x}(s)) \,\mathrm{d}s \quad \text{for any} \quad t \in [T_0, \theta_n] \setminus \mathcal{N}_n;$$

which yields

$$-\dot{\widehat{x}}(t) \in \partial\Phi(\widehat{x}(t)) + f_1(t, \widehat{x}(t)) + \int_{T_0}^t f_2(t, s, \widehat{x}(s)) \,\mathrm{d}s \quad \text{for any} \quad t \in [T_0, \widehat{T}[\setminus \Big(\bigcup_n \mathcal{N}_n\Big)$$

which translates the property (e_1) almost everywhere on $[T_0, \widehat{T}[$. Consequently, the pair $(\widehat{T}, \widehat{x}) \in \Lambda$ with $(T_\alpha, x_\alpha) \preceq (\widehat{T}, \widehat{x})$ for all $\alpha \in A$, which means that $(\widehat{T}, \widehat{x})$ is an upper bound for the family $\{(T_\alpha, x_\alpha) : \alpha \in A\}$. According to Zorn's lemma, there exists a maximal element in Λ that we call $(T, u(\cdot))$, so $u(\cdot) : [T_0, T[\to \mathcal{E}$ is locally absolutely continuous on $[T_0, T[$ and has no other extension satisfying (e_1), (e_2), (e_3) and (e_4) than itself. □

Theorem 5.2. *Let $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function which is finite at $u_0 \in H$. Let any real $T_0 \in [0, +\infty[$ and consider the mappings $f_1 : [T_0, +\infty[\times H \to H$ and $f_2 : \Delta_\infty \times H \to H$ which satisfy the assumptions $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$ of Proposition 5.1. Assume also that we have the following conditions*

$(\mathcal{Q}_{\Phi,1})$ *The function Φ is plr around each point of its domain;*

$(\mathcal{Q}_{\Phi,2})$ *There exists a positive number α such that*

$$\Phi(x) \geq -\alpha(1 + \|x\|) \quad \textit{for any } x \in H.$$

Then, there exists a unique locally absolutely continuous mapping $u : [T_0, +\infty[\to H$ which satisfies

$$(\mathcal{P}^{\infty}_{f_1,f_2}) \quad \begin{cases} -\dot{u}(t) \in \partial\Phi(u(t)) + f_1(t,u(t)) + \displaystyle\int_{T_0}^{t} f_2(t,s,u(s))\,\mathrm{d}s \quad a.e.\ \ t \in [T_0,+\infty[, \\ u(T_0) = u_0 \in \operatorname{dom}\Phi, \end{cases}$$

along with the inclusion $u([T_0,+\infty[) \subset \operatorname{dom}\Phi$. *Further, this solution satisfies also:*

$(\mathcal{E}_1)$ $\dot{u} \in L^2_{loc}([T_0,+\infty[, H)$;

$(\mathcal{E}_2)$ *For any* $s,t \in [T_0,+\infty[$ *with* $s \le t$

$$\int_s^t \|\dot{u}(r)\|^2 \mathrm{d}r \le 2(\Phi(u(s)) - \Phi(u(t))) + \int_s^t \Big\| f_1(r,u(r)) + \int_{T_0}^{r} f_2(r,\sigma,u(\sigma))\,\mathrm{d}\sigma \Big\|^2 \mathrm{d}r. \quad (27)$$

In addition, if Φ *is plr uniformly with respect to subgradients around each point of its domain, then the function* $\Phi \circ u$ *is locally absolutely continuous on* $[T_0,+\infty[$, *and for almost every* $t \in [T_0,+\infty[$

$$(\Phi \circ u)'(t) = \langle \zeta, \dot{u}(t) \rangle \ \textit{for any}\ \zeta \in \partial\Phi(u(t)),$$

and for any $s,t \in [T_0,+\infty[$ *with* $s \le t$

$$\Phi(u(s)) - \Phi(u(t)) = \int_s^t \|\dot{u}(r)\|^2\,\mathrm{d}r + \int_s^t \Big\langle f_1(r,u(r)) + \int_{T_0}^{r} f_2(r,\sigma,u(\sigma))\,\mathrm{d}\sigma, \dot{u}(r) \Big\rangle\,\mathrm{d}r. \quad (28)$$

Proof. Again we refer to [4] for the uniqueness and hence continue with another approach for the existence.

Denote by $u(\cdot) : [T_0,T[\to H$, with $T \le +\infty$, the maximal solution in the sense of Proposition 5.1 (with $\mathcal{E} = H$) of the differential inclusion

$$\begin{cases} -\dot{u}(t) \in \partial\Phi(u(t)) + f_1(t,u(t)) + \displaystyle\int_{T_0}^{t} f_2(t,s,u(s))\,\mathrm{d}s \qquad \text{a.e.}\ \ t \in [T_0,T[, \\ u(T_0) = u_0 \in \operatorname{dom}\Phi, \end{cases}$$

then, the solution $u(\cdot)$ enjoys the properties $(e_1), (e_2), (e_3)$ and (e_5) stated in Proposition 5.1 on the interval $[T_0,T[$. Proceeding by contradiction, we are going to prove that $T = +\infty$. Supposing therefore that $T < +\infty$ so, to get the absurdity, it suffices to prove that (e_5) is not valid. Given any $s \in [T_0,T[$, the local absolute continuity of $u(\cdot)$ on $[T_0,T[$ together with the property (e_3) tell us that

$$\begin{aligned} &\|u(s) - u_0\|^2 \le \Big(\int_{T_0}^{s} \|\dot{u}(r)\|\,\mathrm{d}r \Big)^2 \le (s - T_0) \int_{T_0}^{s} \|\dot{u}(r)\|^2\,\mathrm{d}r \\ &\le (s - T_0)\Big(2(\Phi(u_0) - \Phi(u(s))) + \int_{T_0}^{s} \Big\| f_1(r,u(r)) + \int_{T_0}^{r} f_2(r,\sigma,u(\sigma))\,\mathrm{d}\sigma \Big\|^2 \mathrm{d}r \Big) \\ &\le 2(s - T_0)\,\alpha\,\|u(s) - u_0\| + 2(s - T_0)\Big\{ \Phi(u_0) + \alpha(1 + \|u_0\|) + \int_{T_0}^{s} \beta_1^2(r)(1 + \|u(r)\|)^2 \mathrm{d}r \\ &\qquad + \int_{T_0}^{s} \Big(\int_{T_0}^{r} \beta_2(r,\sigma)(1 + \|u(\sigma)\|)\mathrm{d}\sigma \Big)^2 \mathrm{d}r \Big\}, \end{aligned}$$

where the latter inequality is due to conditions $(\mathcal{Q}_{\Phi,2}), (\mathcal{Q}_{1,1}), (\mathcal{Q}_{2,1})$ and the inequality $(p+q)^2 \le 2(p^2+q^2)$. This entails that

$$(\|u(s)-u_0\| - \alpha(s-T_0))^2 \le \alpha^2(s-T_0)^2 + 2(s-T_0)\Big\{\Phi(u_0) + \alpha(1+\|u_0\|) + \int_{T_0}^{s} \beta_1^2(r)(1+\|u(r)\|)^2 \mathrm{d}r + \int_{T_0}^{s}\Big(\int_{T_0}^{r}\beta_2(r,\sigma)(1+\|u(\sigma)\|)\mathrm{d}\sigma\Big)^2 \mathrm{d}r\Big\},$$

which gives

$$\|u(s)-u_0\|^2 \le 4\alpha^2(s-T_0)^2 + 4(s-T_0)(\Phi(u_0)+\alpha(1+\|u_0\|)) + 4(s-T_0)\int_{T_0}^{s}\beta_1^2(r)(1+\|u(r)\|)^2\mathrm{d}r + 4(s-T_0)\int_{T_0}^{s}\Big(\int_{T_0}^{r}\beta_2(r,\sigma)(1+\|u(\sigma)\|)\mathrm{d}\sigma\Big)^2\mathrm{d}r. \tag{29}$$

However, using Hölder's inequality, one obtains by elementary calculations

$$\int_{T_0}^{s}\beta_1^2(r)(1+\|u(r)\|)^2\mathrm{d}r \le 2(1+\|u_0\|)^2\int_{T_0}^{s}\beta_1^2(r)\mathrm{d}r + 2\int_{T_0}^{s}\beta_1^2(r)\|u(r)-u_0\|^2\mathrm{d}r,$$

and

$$\int_{T_0}^{s}\Big(\int_{T_0}^{r}\beta_2(r,\sigma)(1+\|u(\sigma)\|)\mathrm{d}\sigma\Big)^2\mathrm{d}r \le 2(1+\|u_0\|)^2\int_{T_0}^{s}\int_{T_0}^{r}(r-T_0)\beta_2^2(r,\sigma)\mathrm{d}\sigma\mathrm{d}r + 2\int_{T_0}^{s}\int_{T_0}^{r}(r-T_0)\beta_2^2(r,\sigma)\|u(\sigma)-u_0\|^2\mathrm{d}\sigma\mathrm{d}r.$$

Combining these inequalities with (29) gives

$$\begin{aligned}\|u(s)-u_0\|^2 \le A(s) &+ 8(s-T_0)(1+\|u_0\|)^2\int_{T_0}^{s}\Big(\beta_1^2(r)+\int_{T_0}^{r}(r-T_0)\beta_2^2(r,\sigma)\mathrm{d}\sigma\Big)\mathrm{d}r\\ &+ 8(s-T_0)\int_{T_0}^{s}\beta_1^2(r)\|u(r)-u_0\|^2\mathrm{d}r\\ &+ 8(s-T_0)\int_{T_0}^{s}\int_{T_0}^{r}(r-T_0)\beta_2^2(r,\sigma)\|u(\sigma)-u_0\|^2\mathrm{d}\sigma\mathrm{d}r,\end{aligned}$$

where $A(s) := 4\alpha^2(s-T_0)^2 + 4(s-T_0)(\Phi(u_0)+\alpha(1+\|u_0\|))$.

Since s is chosen arbitrary in the interval $[T_0, T[$, then the latter inequality can be written as follows

$$\vartheta(s) \le \int_{T_0}^{s} b(r)\mathrm{d}r + \int_{T_0}^{s} c(r)\vartheta(r)\mathrm{d}r + \int_{T_0}^{s}\int_{T_0}^{r}\omega(r,\sigma)\vartheta(\sigma)\mathrm{d}\sigma\mathrm{d}r$$

where

$$\begin{aligned}&\vartheta(r) := \|u(r)-u_0\|^2\\ &b(r) := \dot{A}(r) + 8(T-T_0)(1+\|u_0\|)^2\Big(\beta_1^2(r)+\int_{T_0}^{r}(T-T_0)\beta_2^2(r,\sigma)\mathrm{d}\sigma\Big)\mathrm{d}r\\ &c(r) := 8(T-T_0)\beta_1^2(r), \quad \omega(r,\sigma) := 8(T-T_0)^2\beta_2^2(r,\sigma).\end{aligned}$$

It is clear that $b(\cdot)$, $c(\cdot)$ and $\omega(\cdot,\cdot)$ are nonnegative integrable functions, then the following function

$$E(s) := \int_{T_0}^{s} b(r)\mathrm{d}r + \int_{T_0}^{s} c(r)\vartheta(r)\mathrm{d}r + \int_{T_0}^{s}\int_{T_0}^{r} \omega(r,\sigma)\vartheta(\sigma)\mathrm{d}\sigma\mathrm{d}r,$$

is a non-decreasing absolutely continuous function with $\vartheta(s) \le E(s)$ and for almost every $s \in [0,T[$ we have

$$\begin{aligned}\dot{E}(s) &= b(s) + c(s)\vartheta(s) + \int_{T_0}^{s} \omega(s,\sigma)\vartheta(\sigma)\mathrm{d}\sigma \\ &\le b(s) + c(s)E(s) + \int_{T_0}^{s} \omega(s,\sigma)E(\sigma)\mathrm{d}\sigma \\ &\le b(s) + \Big(c(s) + \int_{T_0}^{s} \omega(s,\sigma)\mathrm{d}\sigma\Big)E(s).\end{aligned}$$

Applying Gronwall's lemma (see Lemma 2.11) brings us to

$$\|u(s)-u_0\|^2 = \vartheta(s) \le E(s) \le \int_{T_0}^{s} b(r)\exp\Big\{\int_{r}^{s}\Big(c(\tau)+\int_{T_0}^{\tau}\omega(\tau,\sigma)\mathrm{d}\sigma\Big)\mathrm{d}\tau\Big\}\mathrm{d}r. \quad (30)$$

The finiteness of T, inequality (30) and the inclusions $\beta_1(\cdot) \in L^2_{\mathrm{loc}}([T_0,+\infty[$ and $\beta_2(\cdot,\cdot) \in L^2_{\mathrm{loc}}(\Delta_\infty,\mathbb{R}_+)$ tell us that

$$\varrho := \sup_{s\in[T_0,T[} \|u(s)\| < +\infty. \quad (31)$$

It follows from (31), inequality (27) and equality (31) that for any $t,s \in [T_0,T[$ with $s \le t$ we obtain

$$\begin{aligned}&\|u(t)-u(s)\| \le \sqrt{t-s}\Big(\int_{T_0}^{t} \|\dot{u}(r)\|^2\,\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\le \sqrt{t-s}\Big(2(\Phi(u_0)+\alpha(1+\varrho)) + \int_{T_0}^{t}\Big\|f_1(r,u(r)) + \int_{T_0}^{r} f_2(r,\sigma,u(\sigma))\,\mathrm{d}\sigma\Big\|^2\mathrm{d}r\Big)^{\frac{1}{2}} \\ &\le \sqrt{t-s}\Big(2(\Phi(u_0)+\alpha(1+\varrho)) + (1+\varrho)^2\int_{T_0}^{T}\Big(\beta_1(r)+\int_{T_0}^{r}\beta_2(r,\sigma)\mathrm{d}\sigma\Big)^2\mathrm{d}r\Big)^{\frac{1}{2}},\end{aligned}$$

which ensures that the family $(u(t))_{t\in[T_0,T[}$ fulfills the Cauchy's criterion, and then $\overline{u} := \lim_{t\uparrow T} u(t)$ exists. It remains to prove that $\overline{u} \in \mathrm{dom}\, f$. To proceed, fix arbitrary element $s \in [T_0,T[$ then, by (27)

$$\Phi(u(s)) \le \Phi(u_0) + \frac{1}{2}(1+\varrho)^2\int_{T_0}^{T}\Big(\beta_1(r)+\int_{T_0}^{r}\beta_2(r,\sigma)\mathrm{d}\sigma\Big)^2\mathrm{d}r,$$

passing to the limit $s \uparrow T$ and taking into account the lower semicontinuity of Φ, we deduce that

$$\Phi(\overline{u}) \le \Phi(u_0) + \frac{1}{2}(1+\varrho)^2\int_{T_0}^{T}\Big(\beta_1(r)+\int_{T_0}^{r}\beta_2(r,\sigma)\mathrm{d}\sigma\Big)^2\mathrm{d}r,$$

which translates the inclusion $\overline{u} \in \operatorname{dom} f$. Using the plr property of Φ around $\overline{u} \in \operatorname{dom} f$ and the local existence result established in Theorem 4.1 with initial time T and initial state $\overline{u}$, we can extend $u(\cdot)$ as a solution on a right neighborhood of T such that the properties of Proposition 5.1 remain satisfied, which is a contradiction. Finally, $T = +\infty$ and $u(\cdot)$ is a solution of $(\mathcal{P}^{\infty}_{f_1,f_2})$ with $u([T_0, +\infty[) \subset \operatorname{dom} f$ and (e_2) with $T = +\infty$ furnishes that $\dot{u} \in L^2_{loc}([T_0, +\infty[, H)$ as well as the inequality in $(\mathcal{E}_2)$.

Further, if at each point of its domain the function f is plr uniformly with respect to subgradients, by the property (e_4) of Proposition 5.1, it follows that the solution $u(\cdot)$ is additionally such that the function $\Phi \circ u$ is locally absolutely continuous on $[T_0, +\infty[$ and satisfies the equality (28). $\square$

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