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Second-Order Necessary and Sufficient Efficiency Conditions for Nonsmooth Semi-Infinite Multiobjective Optimization Problems

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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We present second-order necessary and sufficient conditions for weak efficiency of nonsmooth semi-infinite multiobjective programming involving inequality, equality and set constraints in terms of the Páles-Zeidan second-order directional derivatives. Sufficient optimality conditions are established in two cases. They include the case without assumptions on generalized convexity and the case with assumptions on the 2-generalized convexity together with second-order duality theorems of Mond-Weir type.

Keywords: Second-order necessary efficiency conditions, Pales-Zeidan second-order directional derivatives, first and second-order tangent vectors, second-order constraint qualifications, nonsmooth semi-infinite multiobjective optimization problem.

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1. Introduction

Necessary and sufficient conditions for efficiency are important field of multiobjecive optimization problems. In recent years, many works on nonsmooth analysis have studied a lot of subdifferentials, especially, convexificators to derive optimality conditions. The theory of nonsmooth vector optimality conditions has been achieved many beautiful and plentiful results in terms of subdifferentials such as the Clarke subdifferential [2], the Michel-Penot subdifferential [16], the Mordukhovich subdifferential [18], the Treiman subdifferential [23], convexificators [10], approximate Jacobians [9]. Convexificators are efficient tools which have attracted extensive attention by many authors. On multiobjecive semi-infinite optimization problems, O. Stein and G. Still [21] derived optimality conditions via directional derivatives. N. Kanzi [11, 12] gave optimality conditions for multiobjective semi-infinite optimization problems with Lipschitz functions via the Clarke subdifferentials. S. K. Mishra and M. Jaiswal [17] derived optimality conditions for differ-

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entiable scalar semi-infinite programming problems with equilibrium constraints. E. Constantin [3] established second-order necessary optimality conditions for scalar optimization problems with inequality constraints via Páles–Zeidan and second-order directional derivatives. D. V. Luu [15] established Fritz John conditions of primal and dual forms for vector equilibrium problems involving equality, inequality and set constraints in terms of second-order Páles–Zeidan derivatives. Recently, E. Constantin [4] established optimality conditions for strict local Pareto minima of second-order of inequality-constrained multiobjective optimization problems with local Lipschitz functions in terms of Páles–Zeidan second-order derivatives.

This paper presents second-order Fritz John necessary conditions for weak efficiency of nonsmooth semi-infinite multiobjective programmig involving inequality, equality and set constraints in terms of the Päle s-Zeidan second-order directional derivatives. Second-order Karush-Kuhn-Tucker necessary conditions for weak efficiency are established under suitable second-order constraint qualifications.

The paper is organized as follows. After some preliminaries, Section 3 is devoted to developing second-order necessary conditions for local weak efficient solutions of nonsmooth multiobjective semi-infinite programming problem (MSP1) involving equality, inequality and set constraints. In Section 4, we derive sufficient optimality conditions for (MSP1) without assumptions on generalized convexity. Section 5 presents sufficient optimality conditions with the 2-genealized convexity. Finally, Section 6 deals with second-order duality theorems of Mond-Weir type.

2. Preliminaries

Let f be a real-valued functio defineded on a Banach space X, which is Lipschitz near $\overline{x} \in X$. We recall [2] that the *Clarke generalized derivative* of f at $\overline{x} \in X$ in a direction $v \in X$ is defined as

$$f^0(\overline{x}; v) := \limsup_{x \to \overline{x}, t \downarrow 0} \frac{1}{t}[f(x + tv) - f(x)].$$

Following [19], the *Páles-Zeidan second-order upper (lower) generalized directional derivative* of f at $\overline{x}$ in the direction v is defined as

$$f^{00}_+(\overline{x}; v) := \limsup_{t \downarrow 0} \frac{2}{t^2}\, [f(\overline{x} + tv) - f(\overline{x}) - tf^0(\overline{x}; v)].$$

$$\Big(\text{resp. } f^{00}_-(\overline{x}; v) := \liminf_{t \downarrow 0} \frac{2}{t^2}\, [f(\overline{x} + tv) - f(\overline{x}) - tf^0(\overline{x}; v)]\Big).$$

Let $f : X \to \mathbb{R}$ be Fréchet differentiable at $\overline{x} \in X$, and $\nabla f(\overline{x})$ be its gradient at $\overline{x}$. The *upper (lower) second-order directional derivative* at $\overline{x}$ in the direction $v \in X$ are defined as

$$f"_+(\overline{x}; v) := \limsup_{t \downarrow 0, v' \to v} \frac{2}{t^2}\; [f(\overline{x} + tv') - f(\overline{x}) - t\nabla f(\overline{x})v].$$

$$\Big(\text{resp. } f"_-(\overline{x}; v) := \liminf_{t \downarrow 0, v' \to v} \frac{2}{t^2}\, [f(\overline{x} + tv') - f(\overline{x}) - t\nabla f(\overline{x})v]\Big).$$

If $f"_+(\overline{x}; v) = f"_-(\overline{x}; v)$, the comon value is denoted by $f"(\overline{x}; v)$ and is called *second-order directional derivative* of f at $\overline{x}$ in the direction v.

Remark 2.1. (a) If f is continuously Fréchet differentiable near $\overline{x}$ with the Fréchet derivative $\nabla f(\overline{x})$, then f is Lipchitz near $\overline{x}$, and $f^0(\overline{x}; v) = \nabla f(\overline{x})v$ $(\forall v \in X)$. This is false, when $\nabla f(x)$ is not continuous at $\overline{x}$.

(b) If $f : X \to \mathbb{R}$ is continuously Fréchet differentiable near $\overline{x}$ and second-order directionally differentiable at $\overline{x}$ in a direction $v \in X$, then (see [4]) $f_+^{00}(\overline{x}; v) = f"_+(\overline{x}; v)$. □

Let f be a mapping from $X \to \mathbb{R}^m$. Recall that f is *Gâteaux differentiable* at $\overline{x}$ if there exists a continuous linear mapping Λ from $X \to \mathbb{R}^m$ such that

$$f(\overline{x} + tv) = f(\overline{x}) + t\Lambda(v) + o(t) \ (\forall \ v \in \mathbb{R}^n),$$

where $\|o(t)\|/t \ \to 0$ as $t \to 0$. The mapping Λ is said to be the *Gâteaux derivative* of f at $\overline{x}$ and is denoted by $f'(\overline{x})$. Note that a mapping which is Gâteaux differentiable at $\overline{x}$ may be not continuous at $\overline{x}$.

Let C be a nonempty subset of X. Following [3], a vector $u \in X$ is said to be a *tangent vector* to C at $\overline{x} \in \mathrm{cl}C$ iff

$$\lim_{t\downarrow 0} \frac{1}{t} d(\overline{x} + tu; C) = 0\,, \tag{1}$$

where $d(x; C)$ stands for the distance from x to C, cl$C$ is the closure of C. The set of all tangent vectors to C at $\overline{x}$ is denoted by $T_{\overline{x}}C$, and is called the tangent cone to C at $\overline{x}$. Note that $T_{\overline{x}}C$ is a nonempty closed cone containing $0 \in X$. Moreover, (1) is equivalent to the existence of a function $\gamma : (0, +\infty) \to X$ with $\gamma(t) \to 0$ as $t \downarrow 0$, and

$$\overline{x} + t(u + \gamma(t)) \in C \quad (\forall t > 0).$$

Adapting the definition in [22], a vector $v \in X$ is said to be a *second-order tangent vector* to C at $\overline{x}$ iff there exist $u \in X$ such that

$$\lim_{t\downarrow 0} \frac{1}{t^2}\, d(\overline{x} + tu + \frac{t^2}{2} v; C) = 0\,. \tag{2}$$

The vector u satisfying (2) are said to be *associated* with v. Denote by $T^2_{\overline{x}}C$ the set of all second-order tangent vectors to C at $\overline{x}$. Observe that $v \in T^2_{\overline{x}}C$ with the associated vector u is equivalent to the existence of a function $\gamma_1 : (0, +\infty) \to X$ with $\gamma_1(t) \to 0$ as $t \downarrow 0$, and

$$\overline{x} + tu + \frac{t^2}{2}(v + \gamma_1(t)) \in C \ (\forall t > 0).$$

Moreover, $v \in T^2_{\overline{x}}C$ implies that $u \in T_{\overline{x}}C$, and $T^2_{\overline{x}}C$ is a nonempty cone containing $0 \in X$ (see [22]).

Remark 2.2. Assume that C is convex. Then, for $\overline{x} \in C$, $C - \overline{x} \subset T^2_{\overline{x}}C$; for $x \in C$, $x - \overline{x} \in T^2_{\overline{x}}C$ with the associated vector $x - \overline{x} \in T_{\overline{x}}C$.

Proof. Since C in convex, for $x \in C, t \in [0, 1], \overline{x} + t(x - \overline{x}) \in C$. Hence, for $\gamma(t) \equiv 0$, one gets that $\overline{x} + t(x - \overline{x} + \gamma(t)) \in C$, which means that $x - \overline{x} \in T_{\overline{x}}C$.

Since C is convex, for $x \in C$, for t small enough, $t_1 := t + \frac{t^2}{2} \in [0,1]$, and so, for $\gamma_1(t) \equiv 0$, $\overline{x} + t(x - \overline{x}) + \frac{t^2}{2}(x - \overline{x} + \gamma_1(t)) \in C$, which means that $x - \overline{x} \in T^2_{\overline{x}}C$ with the associated vector $x - \overline{x} \in T_{\overline{x}}C$. $\square$

Consider the following *multiobjective semi-infinite programming* problem:

(MSP1) $\quad \min f(x)$, s.t. $g_t(x) \leq 0\ (\forall t \in T)$, $h_j(x) = 0\ (\forall j \in L)$, $x \in C$,

where $f = (f_1 \ldots, f_r)$, $f_i\ (i = 1, \ldots, r)$, $g_i\ (i \in T)$ are locally Lipschitz functions defined on X. The index set T is arbitrary, not necessary finite (but nonempty).

Let denote $J := \{1, \ldots, r\}$, $L = \{1, \ldots, \ell\}$. Let M_1 denote the set of feasible points of (MSP1):

$$M_1 := \Big\{x \in C : \ g_i(x) \leqslant 0,\ i \in T,\ h_j(x) = 0,\ j \in L\Big\}.$$

For $\overline{x} \in M_1, u \in X$, the index set of active constraints at $\overline{x}$ as

$$T(\overline{x}) := \{t \in T : g_t(\overline{x}) = 0\}.$$

For $\overline{x} \in M_1, u \in X$, we define the *second-order contingent set* to M_1 at $(\overline{x}, u)$:

$$T^2(M, \overline{x}, u) := \{v \in C : \exists\, t_n \downarrow 0, \exists\, v_n \to v \text{ such that } \overline{x} + t_n u + \frac{t_n^2}{2} v \in M_1,\ \forall n \in \mathbb{N}\},$$

and
$$F(\overline{x}, u) := \{v \in T^2_{\overline{x}}C : f'_k(\overline{x})v + \frac{1}{2} f_k^{00}(\overline{x}; u) < 0\ \forall k \in J\},$$

$$G(\overline{x}, u) := \left\{ v \in T^2_{\overline{x}}C : \begin{array}{l} g'_i(\overline{x})(v) + \frac{1}{2} g_i^{00}(\overline{x}; u) \leq 0\ \forall i \in T(\overline{x}), \\ \nabla h_j(\overline{x})(v) + \nabla^2 h_j(\overline{x})(u, u) = 0\ \forall j \in L \end{array} \right\}.$$

We introduce the second-order constraint qualification (SCQ):

(SACQ) $\qquad G(\overline{x}, u) \subset T^2(M_1, \overline{x}, u).$

To derive second-order necessary conditions for (MSP1), we introduce the following assumptions.

Assumption 2.3. (a) For $\overline{x} \in M$, the functions $f_k\ (k \in J)$ and $g_i\ (i \in T(\overline{x}))$ are locally Lipschitz, Gâteaux differentiable, with Gâteaux derivative $f'_k(\overline{x})$ and $g'_i(\overline{x})$, respectively; regular in the Clarke sense; C is convex.
(b) $h_j\ (j \in L)$ are second-order Fréchet differentiable with Hessian at $\overline{x}$ is $\nabla^2 h_j(\overline{x})$.

A vector $\overline{x}$ is called *local weak efficient solution* iff there is a number $\delta > 0$ such that there is no $x \in M_1 \cap B(\overline{x}; \delta)$ such that

$$f_k(x) < f_k(\overline{x}) \quad (\forall\, k \in J)$$

where $B(\overline{x}; \delta)$ stands for the open ball of radius δ around $\overline{x}$. Finally, the vector $\overline{x}$ is called *local efficient solution* iff there is a number $\delta > 0$ such that there is no $x \in M_1 \cap B(\overline{x}; \delta)$ such that

$$f_k(x) \leq f_k(\overline{x})\ (\forall k \in J), \text{ and } f_s(x) < f_s(\overline{x}) \text{ for at least one } s \in J.$$

Adapting a definition of [4], a vector $u \in X$ is called *critical* at $\overline{x} \in M$ iff

$$f'_k(\overline{x}; u) \leqslant 0\ (\forall k \in J),\ g'_i(\overline{x}; u) \leqslant 0\ (\forall i \in T(\overline{x})),\ \nabla h_j(x) = 0\ (\forall j \in L),\ u \in T_{\overline{x}}C.$$

3. Kuhn-Tucker necessary conditions for weak efficiency

Denote by $\mathcal{S}$ the set of all collections S of finitely many elements of $T(\overline{x})$ and set

$$S(\overline{x}, u) := \left\{ v \in T^2_{\overline{x}}C : \begin{array}{l} g_i'(\overline{x})(v) + \frac{1}{2}\, g_i^{00}(\overline{x}; u) \leq 0 \ (\forall i \in S), \\ \nabla h_j(\overline{x})(v) + \nabla^2 h_j(\overline{x})(u, u) = 0 \ (\forall j \in L) \end{array} \right\}.$$

Theorem 3.1. *Let $\overline{x} \in M_1$ be a weakly efficient solution of* (MSP1) *for which Assumption* 2.3 *and* (SCQ) *hold. Then for every critical direction $u \in X$, there exist $S \in \mathcal{S}$ such that there is no $v \in T^2_{\overline{x}}C$ with the associated vector u satisfying the following system:*

$$f_k'(\overline{x})v + \frac{1}{2}\, f_k^{00}(\overline{x}; u) < 0 \ (\forall k \in J(\overline{x}, u) := \{k \in J : f_k'(\overline{x}, u) \geq 0\}),$$

$$g_t'(\overline{x})(v) + \frac{1}{2}\, g_t^{00}(\overline{x}; u) \leq 0 \ (t \in S),$$

$$\nabla h_j(\overline{x})(v) + \frac{1}{2} \nabla^2 h_j(\overline{x})(u, u) = 0 \ (\forall j \in L)$$

Remark 3.2. The point $\overline{x} \in M_1$ is a weakly efficient solution of (MSP1), but $\overline{x}$ is not a solution of the following problem with finitely many inequality constraints:

$$\min f(x), \quad \text{s.t.} \ \ g_t(x) \leq 0 \ (t \in S), \ h_j(x) = 0 \ (j \in L), \ x \in C.$$

Proof of Theorem 3.1. We show that there exists $S \in \mathcal{S}$ such that

$$S(\overline{x}, u) \cap F(\overline{x}, u) = \emptyset. \tag{3}$$

Assume the contrary, that for every $S \in \mathcal{S}$

$$S(\overline{x}, u) \cap F(\overline{x}, u) \neq \emptyset. \tag{4}$$

Hence, $\cup_{S \in \mathcal{S}} S \cap F(\overline{x}, u) \neq \emptyset$. Since $G(\overline{x}, u) = \cup_{S \in \mathcal{S}} S(\overline{x}, u)$ we have

$$G(\overline{x}, u) \cap F(\overline{x}, u) \neq \emptyset. \tag{5}$$

Consequently, there exists $v_0 \in F(\overline{x}, u) \cap G(\overline{x}, u)$. It follows from (SACQ) that for every $n \in \mathbb{N}$, there exists $t_n \downarrow 0, v_n \to v$ such that $\forall t \in T(\overline{x})$,

$$g_t(\overline{x} + t_n u + \frac{{t_n}^2}{2} v_0) \leq 0 \ (\forall n \in \mathbb{N}, \ \forall t \in T(\overline{x})),$$

$$h_j(\overline{x} + t_n u + \frac{{t_n}^2}{2} v_0) = 0 \ (\forall j \in L).$$

Thus for every $n \in \mathbb{N}$, $\overline{x} + t_n u + \frac{t_n^2}{2} v_0$ is a feasible point of the following problem:

$$\min f(x), \quad \text{s.t.} \ \ g_t(x) \leq 0 \ (\forall t \in T(\overline{x})), \ h_j(x) = 0 \ (j \in L), \ x \in C.$$

Therefore, there exists $k_0 \in J$ such that

$$f_{k_0}(\overline{x} + t_n u + \frac{t_n}{2} v_0) - f_{k_0}(\overline{x}) \geq 0.$$

Hence, for every $n \in \mathbb{N}$, we have

$$f_{k_0}'(\overline{x}, u + \frac{t_n}{2}\, v_0) \geq 0 \tag{6}$$

which implies that $k_0 \in J(x, u)$.

It follows from (6) that for every $n \in \mathbb{N}$,

$$\begin{aligned} & f'_{k_0}(\overline{x}, u) + \frac{1}{2} f^{00}_{k_0}(\overline{x}, u) \\ & \geq \frac{1}{2t_n^2} \left[f_{k_0}(\overline{x} + t_n u + \frac{t_n^2}{2} v_0) - f_{k_0}(\overline{x}) - t_n f'_{k_0}(\overline{x}, u + \frac{t_n}{2} v_0) \right] \geq 0. \end{aligned} \tag{7}$$

Since $u \in T^2_{\overline{x}} C$ and we obtain by (4): $f'_{k_0}(\overline{x}, u) + \frac{1}{2} f^{00}_{k_0}(\overline{x}, u)) < 0$.

This stands in conflict with (7). Hence, (4) is false, and (3) holds:

$$\text{there exists } S \in \mathcal{S} \text{ such that } \ F(\overline{x}, u) \cap S = \emptyset,$$

Consequently, there is no $v \in T^2_{\overline{x}} C$ with the asociated vector u satisfying the following system:

$$\begin{aligned} & f'_k(\overline{x}) v + \frac{1}{2} f^{00}_k(\overline{x}; u) < 0 \ (\forall k \in J(\overline{x}, u)\}, \\ & g'_t(\overline{x})(v) + \frac{1}{2} g^{00}_t(\overline{x}; u) \leq 0 \ (\forall t \in S), \\ & \nabla h_j(\overline{x})(v) + \frac{1}{2} \nabla^2 h_j(\overline{x})(u, u) = 0 \ (\forall j \in L). \end{aligned}$$

□

Theorem 3.3. (i) *Let $\overline{x} \in M_1$ be a weakly efficient solution of* (MSP1) *for which Assumption* 2.3 *and* (SCQ) *hold. Then for every critical direction $u \in X$, there exist $S \in \mathcal{S}$, $\lambda_k \geqslant 0$ $(k \in J(\overline{x}, u))$, $\mu_t \geqslant 0$ $(\forall t \in S$ with $\mu_t = 0$ $\forall t \notin S$, $\nu_j \in \mathbb{R}$ $(j \in L)$, the Lagrange multipliers λ_k $(k \in J)$, μ_t $(t \in S)$, ν_j $(j \in L)$ are not all zero, such that*

$$\sum_{k \in J(\overline{x}, u)} \lambda_k f'_k(\overline{x}, v) + \sum_{t \in S} \mu_t g'_t(\overline{x}, v + \sum_{j \in L} \nu_j \nabla h_j(\overline{x})(v) \geq 0 \quad (\forall v \in T^2_{\overline{x}} C) \qquad (8),$$

and

$$\sum_{k \in J(\overline{x}, u)} \lambda_k f^{00}_k(\overline{x}, u) + \sum_{t \in S)} \mu_t g^{00}_t(\overline{x}, u) + \sum_{j \in L} \nu_j \nabla^2 h_j(\overline{x})(u, u) > 0. \tag{9}$$

(ii) *If the following regularity condition* (CQ1) *holds: There exists $(u_0, v_0) \in X \times T^2_{\overline{x}} C$ such that*

$$\begin{aligned} & g'_{t_i}(\overline{x}) v_0 + g^{00}_{t_i}(\overline{x}, u_0) < 0 \ (i = 1, ..., s), \\ & \nabla h_j(\overline{x}) v_0 + \nabla^2 h_j(\overline{x})(u_0, u_0) = 0 \ (\forall j \in L), \end{aligned}$$

then $\lambda \neq 0$.

Proof. We set

$$\begin{aligned} & \mathcal{F}(\overline{x}, u) := \prod_{k \in J(\overline{x}, u)} (f'_k(\overline{x})(T^2_{\overline{x}} C), \qquad \mathcal{G}(\overline{x}, u) := \prod_{t \in S} g'_t(\overline{x})(T^2_{\overline{x}} C), \\ & \mathcal{H}(\overline{x}, v) := \prod_{j \in L} \nabla h_j(\overline{x}) T^2_{\overline{x}} C), \qquad \mathcal{B} := \mathcal{F}(\overline{x}, u) \times \mathcal{G}(\overline{x}, u) \times \mathcal{H}(\overline{x}, v), \\ & b := \Big(-\frac{1}{2} f^{00}_1(\overline{x}; u), \ldots, -\frac{1}{2} f^{00}_r(\overline{x}; u), -\frac{1}{2} g^{00}_t(\overline{x}; u), \ t \in S, \\ & \qquad -\frac{1}{2} \nabla^2 h_1(\overline{x})(u, u), \ldots, -\frac{1}{2} \nabla^2 h_\ell(\overline{x}, u)(u, u) \Big). \end{aligned}$$

Note that $\mathcal{B}$ is convex. Taking Theorem 3.1 into account, we get $b \notin \mathcal{B}$. We invoke Corollary 1 of Theorem 3.4 and a Corollary of Lemma 5.1 [6] to deduce that for every critical direction $u \in X$, there exist $\lambda_k \geqslant 0$ $(k \in J(\overline{x}, u))$, $\mu_t \geqslant 0$ $(\forall t \in T(\overline{x}))$ with $\mu_t = 0$ $\forall t \notin S$, $\nu_j \in \mathbb{R}$, $j \in L$. The Lagrange multipliers λ_k $(k \in J)$, μ_t $(t \in S)$, ν_j $(j \in L)$ are not all zero such that $\forall v \in T^2_{\overline{x}}C$,

$$\begin{aligned}&\sum_{k\in J(\overline{x},u)} \lambda_k f'_k(\overline{x}, v) + \sum_{t\in S} \mu_{t_i} g'_{t_i}(\overline{x}, v) + \sum_{j\in L} \nu_j \nabla h_j(\overline{x})(v) \geq 0\\ &> -\frac{1}{2} \sum_{k\in J(\overline{x},u)} \lambda_k f^{00}_k(\overline{x}, u) - \frac{1}{2} \sum_{t\in S} \mu_{t_i} g^{00}_{t_i}(\overline{x}, u) - \frac{1}{2} \sum_{j\in L} \nu_j \nabla^2 h_j(\overline{x})(u, u)\ (\forall v \in T^2_{\overline{x}}C),\end{aligned}$$

We take $\mu_t = 0$ if $t \in T(\overline{x}), t \notin S$. Therefore (8) and (9) are fulfilled. Hence,

$$\begin{aligned}&\sum_{k\in J(\overline{x},u)} \lambda_k f'_k(\overline{x}, v) + \sum_{t\in S)} \mu_t g'_t(\overline{x}, v) + \sum_{j\in L} \nu_j \nabla h_j(\overline{x})(v) + \sum_{k\in J(\overline{x},u)} \lambda_k f^{00}_k(\overline{x}, u)\\ &+ \sum_{t\in S} \mu_t g^{00}_t(\overline{x}, u) + \sum_{j\in L} \nu_j \nabla^2 h_j(\overline{x})(u, u) > 0\ (\forall v \in T^2_{\overline{x}}C).\end{aligned} \tag{10}$$

(ii) If (CQ1) holds, but $\lambda = 0$. This conficts with (11). Hence $\lambda \neq 0$. The proof is complete. □

The following example illustrates Theorem 3.1.

Example 3.4. Given $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$, where $f_1(x, y) = |x| + y^2$, $f_2(x, y) = x^2 y$, and $f = (f_1, f_2)$, $g_i(x, y) = x + y - i$ $(i \in \mathbb{N})$, $C = [0, 1] \times [0, 1]$, where $\mathbb{N}$ is the set of natural numbers. Consider the following semi-infinite programming problem:

$$(\mathrm{P}_1) \qquad \min f(x, y), \quad \text{s.t.} \quad g_i(x, y) \leq 0\ (i \in \mathbb{N}),\ (x, y) \in C.$$

Then the feasible set of (P_1) is $M = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\}$. It can be seen that $(\overline{x}, \overline{y}) = (0, 0)$ is a weak efficient solution of Problem (P_1). We have

$$\begin{aligned}&f'_1((0, 0), (u, v)) = |u|, && f'_2((0, 0), (u, v)) = 0,\\ &f_1\text{''}((0, 0), (u, v)) = 2v^2, && f_2\text{''}((0, 0), (u, v)) = 0,\\ &g'((0, 0), (u, v)) = u + v, && g\text{''}((0, 0), (u, v)) = 0.\end{aligned}$$

It can be seen that $T^2_{\overline{x}}C = \mathbb{R}^2_+$. Condition (SACQ) is satisfied:

$$G((\overline{x}, \overline{y}), (u, v)) = \{0\} \subset T^2(M, \overline{x}, \overline{y}), (u, v)).$$

Condition (5) of Theorem 3.1 is satisfied:

$$\begin{aligned}&|u| + u + v \geq 0 \quad (\forall (u, v) \in \mathbb{R}^2_+),\\ &2v^2 + 0 > 0 \quad (\forall (u, v) \in \mathbb{R}^2_+, v \neq 0).\end{aligned}$$

In case $C = X$, we obtain the following

Corollary 3.5. *Let $\overline{x}$ be a weak efficient solution of* (MSP1) *for which all hypotheses of Theorem* 4.1 *hold; f_k, $k \in J$, g_j, $j \in T(\overline{x})$, are Gâteaux differentiable at $\overline{x}$; the functions f_k $(k \in J)$, g_j $(j \in T(\overline{x}))$ are regular in Clarke's sense at $\overline{x}$ and $C = X$.*

Then for every critical direction $u \in X$, there exist $\lambda_k \geqslant 0$ $(k \in J(\overline{x}, u))$, $\mu_t \geqslant 0$ $(\forall t \in T(\overline{x}))$ with $\mu_t = 0$ $\forall t \notin S$, $\nu_j \in \mathbb{R}$, $j \in L$, the Lagrange multipliers λ_k $(k \in J)$, μ_t $(t \in S)$, ν_j $(j \in L)$ are not all zero such that

$$\sum_{k \in J(\overline{x},u)} \lambda_k f_k'(\overline{x}) + \sum_{i \in T(\overline{x})} \mu_i g_i'(\overline{x}) + \sum_{j \in L} \nu_j \nabla h_j(\overline{x}) = 0,$$
$$\sum_{k \in J(\overline{x},u)} \lambda_k f^{00}(\overline{x}, u) + \sum_{i \in T(\overline{x})} \mu_{g_i}^{00}(\overline{x}, u) + \sum_{j \in L} \nu_j h_j{"}(\overline{x}, u) > 0.$$

Proof. Since the functions $f_k (k \in J), g_(, i \in T(\overline{x}))$ are regular in Clarke's sense at $\overline{x}$, we get that $f_k^0 = f_k'(k \in J), g_i^0 = g_i'(i \in T(\overline{x}))$, where f_i' stands for directional derivative of f_i at $\overline{x}$. Hence, we invoke Theorem 3.1 to deduce that there exist $\lambda_k \geq 0\,(k \in J(\overline{x}, u))$, $\mu_i \geq 0\,(i \in T(\overline{x}))$, $\mu_t = 0$ for $t \notin S$, and $\nu_j \in \mathbb{R}\,(j \in L)$ such that

$$\begin{aligned} &\sum_{k \in J(\overline{x},u)} \lambda_k f_k'(\overline{x})v + \sum_{t \in T(\overline{x})} \mu_t g_t'(\overline{x})v + \sum_{j \in L} \nu_j h_j^{'}(\overline{x})v \geq 0 \quad (\forall v \in T_{\overline{x}}^2 C), \\ &\sum_{k \in J(\overline{x},u)} \lambda_k f^{00}(\overline{x}, u) + \sum_{t \in T(\overline{x})} \mu_t g_t^{00}(\overline{x}, u) + \sum_{j \in L} \nu_j h_j{"}(\overline{x}, u) > 0. \end{aligned} \tag{11}$$

Since for $C = X$, we get $T_{\overline{x}}^2 C = X$, and

$$\sum_{k \in J(\overline{x},u)} \lambda_k f_k'(\overline{x}) + \sum_{t \in T(\overline{x})} \mu_t g_t'(\overline{x}) + \sum_{j \in L} \nu_j \nabla h_j(\overline{x}) = 0.$$

The conclusion follows. □

Another consequence of Theorem 4.1 can be stated as follows.

Corollary 3.6. *Let $\overline{x}$ be a weak efficient solution of* (MSP1) *for which all hypotheses of Theorem* 3.1 *hold and C is convex. Then for every critical direction $u \in X$, there exist $\lambda_k \geqslant 0$ $(k \in J(\overline{x}, u))$, $\mu_t \geqslant 0$ $(\forall t \in T(\overline{x}))$ with $\mu_t = 0$ $\forall t \notin S$, $\nu_j \in \mathbb{R}$, $j \in L$, the Lagrange multipliers λ_k $(k \in J)$, μ_t $(t \in S)$, ν_j $(j \in L)$ are not all zero, such that*

$$\sum_{k \in J(\overline{x},u)} \lambda_k f_k'(\overline{x}, x - \overline{x}) + \sum_{t \in S} \mu_t g_t'(\overline{x}, x - \overline{x}) + \sum_{j \in L} \nu_j \nabla h_j(\overline{x}), (x - \overline{x}) \geq 0 \quad (\forall x \in C),$$
$$\sum_{k \in J(\overline{x},u)} \lambda_k f_k^{00}(\overline{x}, u) + \sum_{t \in S} \mu_t g_t^{00}(\overline{x}, u) + \sum_{j \in L} \nu_j h_j{"}(\overline{x}, u) > 0. \tag{12}$$

Proof. Applying Theorem 3.1 yields that for every critical direction $u \in X$, there exist $\lambda_k \geq 0$ $(k \in J)$, $\mu_i \geq 0$ $(i \in T(\overline{x}))$, $\mu_t = 0$ for $t \notin S$, and $\nu_j \in \mathbb{R}\,(j \in L)$ such that

$$\sum_{k \in J(\overline{x},u)} \lambda_k f_k'(\overline{x}, v) + \sum_{t \in S} \mu_t g_t'(\overline{x}, v) + \sum_{j \in L} \nu_j \nabla h_j(\overline{x})v \geq 0 \quad (\forall v \in T_{\overline{x}}^2 C),$$
$$\sum_{k \in J(\overline{x},u)} \lambda_k f_k^{00}(\overline{x}, u) + \sum_{t \in S} \mu_t g_t^{00}(\overline{x}, u) + \sum_{j \in L} \nu_j h_j{"}(\overline{x}, u) > 0.$$

Taking into account Remark 2.2, $(C - \overline{x}) \subset T_{\overline{x}}^2 C$. Hence, combining this result and Theorem 3.1 we obtain the conclusion. □

4. Sufficient optimality conditions for (MP1) without assumptions on generalized convexity

Theorem 4.1. *Let $\overline{x} \in M_1$. Assume that Assumption 2.3 holds and C is convex. Suppose that for any critical direction $u \in M_1$, there exist $\lambda_k \geqslant 0$ $(k \in J)$, $\mu_t \geq 0$ $(t \in T(\overline{x}))$ where $\mu_t = 0$ $(\forall t \notin S)$ and $\nu_j \in \mathbb{R}$ $(j \in J)$ such that*

$$\sum_{k \in J} \lambda_k f'_k(\overline{x}, v) + \sum_{t \in S} \mu_t g'_t(\overline{x}, v) + \sum_{j \in L} \nu_j \nabla h_j(\overline{x}) v \geq 0 \quad (\forall v \in T^2_{\overline{x}} C), \tag{13}$$

$$\sum_{k \in J} \lambda_{k,} f^{00}_{k,-}(\overline{x}, u) + \sum_{t \in S} \mu_t g^{00}_{t,-}(\overline{x}, u) + \sum_{j \in L} \nu_j \nabla^2 h_{j,-}(\overline{x}, u) > 0. \tag{14}$$

Then $\overline{x}$ is a local weak efficient solution of (MSP1).

Proof. Assume the contrary, that $\overline{x}$ is not a local weak efficient solution of (MSP1). Then $\forall n \in \mathbb{N}, \exists x_n \in M_1, x_n \to \overline{x}$ such that

$$f_k(x_n) < f_k(\overline{x}) \quad (\forall k \in J). \tag{15}$$

Taking into account Remark 2.2 , since C is convex, one has $C - \overline{x} \subset T^2_{\overline{x}} C$. Putting $t_n = ||x_n - \overline{x}||$, $v_n = \frac{x_n - \overline{x}}{||x_n - \overline{x}||}$, one gets $x_n = \overline{x} + t_n v_n$, Hence,

$$f_k(\overline{x} + t_n v_n) \leq f_k(\overline{x}). \tag{16}$$

Since $x_n \in M_1$, for $t \in T(\overline{x})$, since $x_n \in M_1$ one has

$$g'_t(\overline{x}, x_n - \overline{x})) \leq 0. \tag{17}$$

Due to $x_n \in M_1$, we also have

$$\nabla h_j(\overline{x}, x_n - \overline{x}) = 0. \tag{18}$$

It follows from (15)–(18) that $x_n - \overline{x}$ is a critical direction. Putting $t_n = ||x_n - \overline{x}||$, $v_n = \frac{x_n - \overline{x}}{||x_n - \overline{x}||}$, one gets $x_n = \overline{x} + t_n v_n$. By assumption and (16)–(18) we get that

$$\begin{aligned}
0 &< \sum_{k \in J} f^{00}_{k,-}(\overline{x}, x_n - \overline{x}) + \sum_{t \in S} \mu_t g^{00}_{t,-}(\overline{x}, x_n - \overline{x}) + \sum_{j \in L} \nu_j \nabla^2 h_{j,-}(\overline{x}, x_n - \overline{x}) \\
&\leq \sum_{k \in J} \lambda_k \liminf_{n \to \infty} \{ \frac{2}{t_n^2} [f_k(\overline{x} + t_n v_n) - f_k(\overline{x})] - \liminf_{n \to \infty} \sum_{k \in J} \lambda_k f'_k(\overline{x}, v_n) \} \\
&+ \sum_{t \in T(\overline{x})} \mu_i \liminf_{n \to \infty} \{ \frac{2}{t_n^2} [g_t(\overline{x} + t_n v_n) - g_t(\overline{x})] - \liminf_{n \to \infty} \sum_{t \in S} \mu_t t_n g'_i(\overline{x}, v_n) \} \\
&+ \sum_{j \in L} \nu_j \liminf_{n \to \infty} \{ \frac{2}{t_n^2} [h_j(\overline{x} + t_n v_n) - h_j(\overline{x})] - \sum_{j \in L} \nu_j t_n \nabla h_j(\overline{x}) v_n \} \\
&\leq - \liminf_{n \to \infty} \frac{2}{t_n^2} [\sum_{k \in J} \lambda_k t_n f'_k(\overline{x}) v_n - \liminf_{n \to \infty} \sum_{t \in S} \mu_t t_n g'_t(\overline{x}) v_n - \sum_{j \in L} \nu_j t_n \nabla h_j(\overline{x}) v_n] \\
&\leq 0.
\end{aligned}$$

Absurd. Hence, $\overline{x}$ is a local weak efficient solution of (MSP1). □

5. Second-order sufficient optimality conditions with the 2-generalized convexity

To derive second-order sufficient optimality conditions for (MSP1), we introduce some notions of 2-generalized convexity (which means second-order generalized convex types). Given a subset $C \subset X$ and $\overline{x} \in C$.

Definition 5.1. (i) A function $f : X \to \mathbb{R}$, which is locally Lipschitz at $\overline{x}$, is called *2-convex* (second-order convex type) at $\overline{x}$ on $C \subset X$, iff for all $x \in C$,

$$f(x) - f(\overline{x}) \geq f'(\overline{x}, x - \overline{x}) + \frac{1}{2} f^{00}(\overline{x}; x - \overline{x});$$

(ii) The function f is called *2-pseudoconvex* (second-order pseudoconvex type) at $\overline{x}$ on $C \subset X$, iff for all $x \in C$, the following implications hold:

$$f'(\overline{x}, x - \overline{x}) + \frac{1}{2} f^{00}(\overline{x}; x - \overline{x}) \geq 0 \Longrightarrow f(x) \geq f(\overline{x}).$$

(iii) The function f is called *2-quasiconvex* (second-order quasiconvex type) at $\overline{x}$ on C, iff for all $x \in C$, the following implication hold:

$$f(x) \leq f(\overline{x}) \Longrightarrow f'(\overline{x}, x - \overline{x}) + \frac{1}{2} f^{00}(\overline{x}; x - \overline{x}) \leq 0.$$

(iv) The function f is called *strictly 2-pseudoconvex* at $\overline{x}$ on $C \subset X$ iff for all $x \in C, x \neq \overline{x}$, the following implications hold:

$$f'(\overline{x}, x - \overline{x}) + \frac{1}{2} f^{00}(\overline{x}; x - \overline{x}) \geq 0 \Longrightarrow f(x) > f(\overline{x}). \quad \square$$

Remark 5.2. It should be noted here that a 2-convex function is also 2-quasiconvex and 2-pseudoconvex. For f is continuous Fréchet differentiable, if for $x \in C$, $f^{00}(\overline{x}; x - \overline{x}) \geq 0 (\forall x \in C)$, then f is 2-convex at $\overline{x}$ on C implies that it is convex at $\overline{x}$ on C. $\square$

Theorem 5.3. *Let $\overline{x} \in M_1$. Suppose that g_i $(i \in T(\overline{x}))$, $\pm h_j$ $(j \in L)$ are 2-quasiconvex at $\overline{x}$ on M_1; C is convex. Assume that for every $v \in C$, there exist $\lambda_k \geq 0$ $(\forall\, k \in J)$, $\mu_t \geq 0$ $(t \in T(\overline{x}))$ where $\mu_t = 0$ $\forall t \notin S$ and ν_j $(j \in L)$ such that for every $x \in M_1$,*

$$\sum_{k \in J} \lambda_k f'_k(\overline{x}, v) + \sum_{t \in S} \mu_t g'_t(\overline{x}, v) + \sum_{j \in L} \nu_j \nabla h_j(\overline{x})(v) \geq 0, \tag{19}$$

$$\sum_{k \in J} \lambda_k f^{00}_k(\overline{x}; x - \overline{x}) + \sum_{t \in S} \mu_t g^{00}_t(\overline{x}; x - \overline{x}) + \sum_{j \in L} \nu_j \nabla^2 h_j(\overline{x})(x - \overline{x}, x - \overline{x}) > 0. \tag{20}$$

Moreover, suppose that λf is 2-pseudoconvex at $\overline{x}$ on M_1. Then $\overline{x}$ is a weakly efficient solution of (MSP1).

Proof. It follows from (19) and (20) that

$$\begin{aligned} &\sum_{k \in J} \lambda_k f'_, k(\overline{x}) v + \sum_{t \in S} \mu_t g'_t(\overline{x}) v + \sum_{j \in L} \nu_j \nabla h_j(\overline{x}) v \\ &+ \sum_{k \in J} \lambda_k f^{00}_k(\overline{x}; x - \overline{x})) + \sum_{t \in S} \mu_t g^{00}_t(\overline{x}; x - \overline{x})) + \sum_{j \in L} \nu_j \nabla^2 h_j(\overline{x}, x - \overline{x}) \geqslant 0. \end{aligned} \tag{21}$$

For $x \in M_1$, $t \in T(\overline{x})$, one has $g_t(x) - g_t(\overline{x}) \leq 0$.

In view of the 2-generalized convexity of g_t $(t \in T(\overline{x}))$, we get that for $t \in T(\overline{x})$, $x \in M_1$,

$$g_t'(\overline{x}, x-\overline{x}) + \frac{1}{2} g_t^{00}(\overline{x}, u) \leq 0. \tag{22}$$

For $x \in M_1$, $j \in L$, we have $h_j(x) - h_j(\overline{x}) = 0$. By virtue of the 2-quasigeneralized convexity of $\pm h_j$ $(j \in L)$, we obtain

$$\nabla h_j(\overline{x})(x-\overline{x}) + \frac{1}{2} f_k^{00}(\overline{x}, x-\overline{x}) = 0. \tag{23}$$

Combining the formulas (21)–(23) yields that $\lambda f(x) \geq \lambda f(\overline{x})$ $(\forall k \in J, \forall x \in D)$, where $\lambda := (\lambda_1, \ldots, \lambda_r)$, $f := (f_1, \ldots, f_r)$. Therefore, $\overline{x}$ is a weak efficient solution of the problem:

(P2) $\quad \min \lambda f(x), \quad \text{s.t.} \;\; g_t(x) \leq 0 \; (\forall t \in T), \quad h_j(x) = 0 \; (j \in L), \quad x \in C.$

Note that the feasible set of (P2) is also M_1. Hence, $\overline{x}$ is a weak efficient solution of (MSP1). $\square$

6. Second-order duality

We consider the following second-order dual problem of Mond-Weir type:

$$\text{(MWD)} \quad \begin{aligned} &\max f(u) := (f_1(u), \ldots, f_r(u)), \\ &\text{s.t.} \quad \sum_{k \in J} f_k'(u, x-u) + \sum_{t \in T(u)} \mu_t g_t'(u, x-u) + \sum_{j \in L} \nu_j \nabla h_j(u)(x-u) \\ &+ \frac{1}{2} \sum_{k \in J} \lambda_k f_k^{00}(u; x-u) + \frac{1}{2} \sum_{t \in T(u)} \mu_t g_t^{00}(u; x-u) \\ &+ \frac{1}{2} \sum_{j \in L} \nu_j \nabla^2 h_j(u, x-u) \geqslant 0, \\ &\mu_t \geq 0 \; (t \in T(u)) \text{ with } \mu_t \in T(u), \; \mu_t = 0 \; \forall t \notin S. \end{aligned} \tag{24}$$

where S is available in Theorem 3.1 ($\overline{x}$ instead of u). Denote by M_D the feasible set of (MWD).

To derive duality theorems we introduce the following assumption:

Assumption 6.1. (a) The functions $f_k \, (k \in J := \{1, \ldots, r\}$; $g_t \, (t \in T(\overline{x})$ are locally Lipschitz at $u \in M_1$, Gâteaux differentiable, with Gâteaux derivative $f_k'(u)$ and $g_i'(u)$, respectively; regular in the Clarke sense; C is convex.
(b) $h_j (j \in L)$ are second-order Fréchet differentiable with Hessian $\nabla^2 h_j(u)$.

We now can state a second-order weak duality theorem for (MSP2) and (MWD).

Theorem 6.2. (Weak Duality) *Let x and (u, λ, μ, ν) be the feasible points for Problems* (MSP1) *and* (MWD)*, respectively. Assume that Assumption* 6.1 *is fulfilled, and λf is 2-pseudoconvex at u on C; g_t $(t \in T(u))$, $\pm h_j$ $(j \in L)$ are 2-quasiconvex at u on M_1; C is convex. Suppose that h is twice directionally differentiable at u.*

Then
$$f(x) \nleq f(u). \tag{25}$$

Proof. For every $t \in T(u)$, $g_t(x) \leq 0 = g_t(u)$ $(\forall x \in M_1)$. By the 2-quasiconvexity of g_i at u on M_1, one gets

$$g_t'(u, x-u) + \frac{1}{2} g_t^{00}(u; x-u) \leq 0 \, (\forall x \in M_1). \tag{26}.$$

Due to the 2-quasiconvexity of $\pm h_j$ at u on M_1, we deduce that

$$\nabla h_j(u)(x-u) + \frac{1}{2} \nabla^2 h_j(u, x-u) = 0 \, (\forall x \in M_1). \tag{27}$$

Combining (25)–(27) yields that

$$\lambda f'(u, x-u) + \frac{1}{2} \lambda f^{00}(u; x-u) \geq 0 \ (\forall x \in M_1).$$

Due to the 2-pseudoconvexity of λf at u on M, it follows that

$$\lambda f(x) \geq \lambda f(u) \ (\forall x \in M_1).$$

Hence, u is a solution of the following problem:

$$\min \lambda f(x), \quad \text{s.t.} \ \ g_t(x) \leq 0 \ (t \in T), \quad h_j(x) = 0 \ (j \in L), \quad x \in C.$$

Hence, $f(x) \not< f(u)$ $(\forall x \in M_1)$. The proof is complete. $\square$

We give a strong duality theorem for (MSP1) and (MWD).

Theorem 6.3. (Strong duality) *Let $\overline{x}$ be a local weak efficient solution of* (MSP1) *for which all hypotheses of Corollary* 3.6 *are fulfilled. Then there exist $\overline{\lambda} := (\overline{\lambda}_1, ..., \overline{\lambda}_r)$, $\overline{\mu} := (\overline{\mu}_1, \ldots, \overline{\mu}_m)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in T(\overline{x}))$, $\mu_t = 0$ for $t \notin S$, $\overline{\nu} := (\overline{\nu}_1, \ldots, \overline{\nu}_\ell)$, $\overline{\nu}_j \in \mathbb{R}$ $(\forall j \in L)$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a feasible point of* (MWD)*, and the value of the objective functions of* (MSP1) *and* (MWD) *at $\overline{x}$ and $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$, respectively, are equal. Moreover, if the assumptions of Theorem* 6.1 *hold, then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a weakly efficient solution of* (MWD).

Proof. Since $\overline{x}$ is a local weakly efficient solution of (MSP1), we can invoke Theorem 3.1 to deduce that there exist $\overline{\lambda} := (\overline{\lambda}_1, \ldots, \overline{\lambda}_r)$, $\overline{\mu} := (\overline{\mu}_1, \ldots, \overline{\mu}_r)$, $\overline{\mu}_t \geqslant 0$ $(\forall t \in T(\overline{x}))$ with $\mu_t = 0$ $\forall t \notin S$, $\overline{\nu} := (\overline{\nu}_1, \ldots, \overline{\nu}_\ell)$, $\overline{\nu}_j \in \mathbb{R}$ such that

$$\begin{aligned} &\sum_{k \in J} \overline{\lambda}_k f_k'(\overline{x}, x - \overline{x}) + \sum_{t \in T(\overline{x})} \overline{\mu}_t g_t'(\overline{x}, x - \overline{x}) + \sum_{j \in L} \overline{\nu}_j \nabla h_j(\overline{x})(x - \overline{x}) \\ &+ \frac{1}{2} \sum_{k \in J} \overline{\lambda}_k f_k^{00}(\overline{x}; x - \overline{x}) + \frac{1}{2} \sum_{t \in T(\overline{x})} \overline{\mu}_t g_t^{00}(\overline{x}; x - \overline{x}) + \frac{1}{2} \sum_{j \in L} \overline{\nu}_j \nabla^2 h_j(\overline{x}; x - \overline{x}) \geqslant 0. \end{aligned}$$

Then, $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a feasible point of (MWD), and the value of the objective functions of (MSP1) and (MWD) at $\overline{x}$ and $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$, respectively, are equal.

If the assumptions of Theorem 6.1 hold, we can deduce that for every feasible points (u, λ, μ, ν) of (MWD),

$$f(\overline{x}) \not< f(u).$$

Hence, there is no $(u, \lambda, \mu, \nu) \in M_1$ such that $f(\overline{x}) < f(u)$. Consequently, $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a weakly efficient solution of (MWD). $\square$

Theorem 6.4. (Converse duality) *Let* $\overline{x}$ *and* $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ *be feasible points of* (MSP1) *and* (MWD), *respectively for which Assumption* 6.1 *is fulfilled at* $\overline{u}$; $\overline{\lambda}f$ *is strict second-order 2-pseudoconvex at* $\overline{u}$ *on* M_D; g_i $(i \in I(u))$, $\pm h_j$ $(j \in L)$ *are second-order 2-quasiconvex at* $\overline{u}$ *on* M_D. *Suppose also that*

$$\sum_{k=1}^{r} \overline{\lambda}_k f_k(\overline{x}) \leq \sum_{k=1}^{r} \overline{\lambda}_k f_k(\overline{u}). \tag{27}$$

Then $\overline{u} = \overline{x}$.

Proof. Assume the contrary, that $\overline{u} \neq \overline{x}$. Since $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{\nu})$ is a feasible of (MWD), it follows that for every $x \in M_1$,

$$\sum_{k \in J} \overline{\lambda}_k f_k'(\overline{u}, x - \overline{u}) + \sum_{t \in S} \overline{\mu}_t \nabla g_t'(\overline{u}, x - \overline{u}) + \sum_{j \in L} \overline{\nu}_j \nabla h_j(\overline{u})(x - \overline{u})$$
$$+ \frac{1}{2} \sum_{k \in J} \overline{\lambda}_k f_k^{00}(\overline{u}; x - \overline{u}) + \frac{1}{2} \sum_{t \in S} \mu_t g_t^{00}(\overline{u}; x - \overline{u}) + \frac{1}{2} \sum_{j \in L} \nu_j \nabla^2 h_j(\overline{u}; x - \overline{u}) > 0. \tag{28}$$

We have $g_t(x) \leq 0 = g_t(\overline{u})$ for every $t \in T(\overline{u})$, $x \in C$. In view of the 2-quasiconvexity of g_i at $\overline{u}$ on M_D,

$$g_t'(\overline{u}, x - \overline{u}) + \frac{1}{2} g_t^{00}(\overline{u}; x - \overline{u}) \leq 0. \tag{29}$$

We also have $h_j(x) = 0 = h_j(\overline{u})$, for all $j \in L, x \in C$. By virtue of the 2-quasiconvexity of $\pm h_j$ at $\overline{u}$ on M_D,

$$\nabla h_j(\overline{u}) + \frac{1}{2} \nabla^2 h_j(\overline{u})(x - \overline{u}) = 0. \tag{30}$$

Combining (28)–(30) yields that

$$\sum_{k \in J} \overline{\lambda}_k f_k'(\overline{u})(x - \overline{u}) + \frac{1}{2} \sum_{i \in J)} \overline{\lambda_k} f_k^{00}(\overline{u}; x - \overline{u}) \geq 0.$$

By the strict second-order 2-pseudoconvex of $\overline{\lambda}f$ at $\overline{u}$ on M_D, we get for every $x \in C$,

$$\sum_{k=1}^{r} \overline{\lambda}_k f_k(x) > \sum_{k=1}^{r} \overline{\lambda}_k f_k(\overline{u}).$$

But this contradicts (27). Hence, $\overline{u} = \overline{x}$. The proof is complete. □

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