Journal of Convex Analysis
Volume 31 (2024), No. 4, 1203–1212

Dual Characterizations of Three Distance Functions

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

Received: January 5, 2024
Accepted: February 24, 2024

Ordinary distance functions, oriented distance functions and farthest distance functions are characterized in terms of convex conjugates and Fenchel subdifferentials.

Keywords: Distance function, oriented distance function, farthest distance function, Fenchel conjugate, Fenchel subdifferential.

2020 Mathematics Subject Classification: 46B20, 26B25.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space. The distance function to a set $C \subseteq X$ is the function $d_C : X \to \mathbb{R}$ defined by

$$d_C(x) := \inf_{c \in C} \|x - c\|.$$

This is a very fundamental notion in Functional Analysis, Approximation Theory and other fields. Other related, though less popular, functions are the oriented distance function

$$\Delta_C := d_C - d_{X \setminus C}$$

and the farthest distance function $F_C^{\|\cdot\|} : X \to \mathbb{R}$ to a compact set $C \subset X$, defined by

$$F_C^{\|\cdot\|}(x) := \max_{c \in C} \|x - c\|.$$

As is well known and easy to prove, $F_C^{\|\cdot\|}$ is convex, and so are d_C and Δ_C when the closure of C is a convex set; therefore, it is quite natural to analyze them from a convex analytic viewpoint. The two most important tools in the study and applications of convex functions are the notions of Fenchel conjugate and Fenchel subdifferential. The aim of this paper is to use such tools to characterize the three functions defined above. Each of these functions will be considered in the most general context that makes its analysis possible without much functional analytic

* I gratefully acknowledge financial support by Grant PID2022-136399NB-C22 from MICINN, Spain, and by ERDF, "A way to make Europe", European Union.

ISSN 0944-6532 / \$ 2.50

technicalities. Thus the ordinary distance function will be analyzed in the context of real normed spaces, whereas the oriented distance function and the farthest distance function will be considered only in a Hilbertian and a finite dimensional context, respectively, since their proofs rely upon a finite dimensional result [3, Theorem 4.1] (which can be extended to a Hilbert space setting), in the case of the oriented distance function, and on upper semicontinuity of the norm, in the case of the farthest distance function (notice that, to ensure existence of a farthest point in a compact set one needs to assume upper semicontinuity of the norm with respect to the topology under which the given set is compact). The analysis will consist in characterizing each function, in its own context, in terms of Fenchel conjugates and Fenchel subdifferentials.

The three distance functions considered in this paper are standard concepts, but it seems that their dual characterizations have not been considered so far, in spite of the fact that they have already been analyzed from a convex analytic perspective. In particular, this is the case of the ordinary distance function d_C, a classical tool not only in Functional Analysis, but also in Variational Analysis [8, 11, 13, 15]. The conjugate of d_C is computed in [14] using the fact that it is the infimal convolution of the norm with the indicator function of C, an important fact that we will also use. A formula for the subdifferential of d_C is given in [2], a paper that also considers the Clarke subdifferential of d_C when C is not convex. The oriented distance function Δ_C has been considered in the more general context of metric spaces since long ago; its introduction in Convex Analysis is due to Hiriart-Urruty [5, 6], who observed that Δ_C is convex if and only if the set C is convex. The convex conjugate and the subdifferential of Δ_C were computed in [3], and a detailed variational analysis on the oriented distance function is provided by [10], where many references on its applications are listed. The farthest distance function $F_C^{\|\cdot\|}$ appears not to have been much studied in the functional analytic literature, even though it can be regarded as a particular case of the Hausdorff distance, since $F_C^{\|\cdot\|}(x)$ is precisely the Hausdorff distance between $\{x\}$ and C. A few references on the farthest distance function are [1, 4, 9, 12]; in [12], the authors give a characterization of the R-convexity of a set in a real Hilbert space in terms of its associated farthest distance function (recall that a set is said to be R-convex if it is the intersection of a collection of closed balls with radius R).

We will use the following, rather standard, terminology and notation. The *dual space* of the normed space $(X, \|\cdot\|)$ is $(X^*, \|\cdot\|_*)$; its *unit ball* and its *unit sphere* will be denoted $B_{\|\cdot\|_*}$ and $S_{\|\cdot\|_*}$, respectively, and $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ will denote the *duality product*, defined by $\langle x, x^*\rangle := x^*(x)$. As usual, in the case when $X = \mathbb{R}^n$, the dual space X^* will be identified with $\mathbb{R}^n$; under this identification, $\langle\cdot,\cdot\rangle$ will reduce to the usual Euclidean scalar product. $B(x,\delta)$ will denote the *closed ball with center* $x \in \mathbb{R}^n$ and *radius* $\delta > 0$. The *domain* of a function $g : X^* \to \mathbb{R}\cup\{+\infty\}$ is *dom* $g := g^{-1}(\mathbb{R})$. The *Fenchel conjugate* of a lower semicontinuous (lsc, in brief) proper convex function $f : X \to \mathbb{R}\cup\{+\infty\}$ or $g : X^* \to \mathbb{R}\cup\{+\infty\}$ is

$$f^* : X^* \to \mathbb{R}\cup\{+\infty\} \quad \text{or} \quad g^* : X \to \mathbb{R}\cup\{+\infty\},$$

respectively, given by

$$f^*(x^*) := \sup_{x\in X}\{\langle x, x^*\rangle - f(x)\} \quad \text{and} \quad g^*(x^*) := \sup_{x^*\in X^*}\{\langle x, x^*\rangle - g(x^*)\},$$

and the *Fenchel subdifferential operators* $\partial f : X \rightrightarrows X^*$ and $\partial g : X^* \rightrightarrows X$ are defined by

$$\partial f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x^*\rangle \ \ \forall y \in X\}$$

and $$\partial g(x^*) := \{x \in X : g(y^*) \geq g(x^*) + \langle x, y^* - x^*\rangle \ \ \forall y^* \in X^*\}.$$

The *one-sided directional derivative* of g at $x^* \in X^*$ in the direction $d^* \in X^*$ is

$$g'(x^*, d^*) := \lim_{t\to 0^+} \frac{g(x^* + td^*) - g(x^*)}{t}.$$

The *infimal convolution* of two lsc proper convex functions $f_1, f_2 : X \to \mathbb{R}\cup\{+\infty\}$ is

$$f_1 \Box f_2 : X \to \mathbb{R}\cup\{+\infty\}, \quad (f_1 \Box f_2)(x) := \inf_{y\in X}\{f_1(y) + f_2(x - y)\}.$$

The *indicator function* of $C \subseteq X$ or $C \subseteq X^*$ is the function $\delta_C : X \to \mathbb{R}\cup\{+\infty\}$ or $\delta_C : X^* \to \mathbb{R}\cup\{+\infty\}$, respectively, that takes the value 0 on C and the value $+\infty$ on its complement. We will use the *support function* $\tau_C : \mathbb{R}^n \to \mathbb{R}$ of a compact set $C \subseteq \mathbb{R}^n$ defined by

$$\tau_C(x^*) := \min_{c\in C}\langle c, x^*\rangle.$$

Notice that τ_C is related to the classical support function $\sigma_C : \mathbb{R}^n \to \mathbb{R}$, defined by $\sigma_C(x^*) := \max_{c\in C}\langle c, x^*\rangle$, by the obvious equality $\tau_C(x^*) = -\sigma_C(-x^*)$. To denote the *interior* of a set we will use *int*.

Given a lsc proper convex function $f : X \to \mathbb{R}\cup\{+\infty\}$, we can define the function $\eta_f : X^* \to \mathbb{R}\cup\{+\infty\}$ by

$$\eta_f(x^*) := \begin{cases} \|x^*\|_* f^*\left(\frac{x^*}{\|x^*\|_*}\right) & \text{if } x^* \neq 0, \\ 0 & \text{if } x^* = 0. \end{cases}$$

Clearly, the function η_f is the positively homogeneous extension of the restriction of f^* to the unit sphere. Interestingly, our three characterizations will involve this function, even though Δ_C and $F_C^{\|\cdot\|}$, for instance, are of quite a different nature. Notice that, for $x^* \neq 0$, one has $\eta_f(x^*) = \widetilde{f^*}(\|x^*\|_*, x^*)$, with $\widetilde{f^*} : \mathbb{R}\times X^* \to \mathbb{R}\cup\{+\infty\}$ being the perspective function corresponding to f^*. Recall that, following the definition introduced in [7, p. 160] for functions defined on $\mathbb{R}^n$, the perspective function corresponding to a lsc proper convex function $h : \mathbb{R}\times X^* \to \mathbb{R}\cup\{+\infty\}$ is given by

$$\widetilde{h}(u, x^*) := \begin{cases} uh\left(\frac{x^*}{u}\right) & \text{if } u > 0, \\ +\infty & \text{if not.} \end{cases}$$

The rest of the paper will consist of three sections: Sections 2, 3 and 4 will characterize the ordinary distance function, the oriented distance function and the farthest distance function, respectively.

2. The distance function to a closed subset of a real normed space

In this section, the distance function to a nonempty closed convex subset of a normed space is characterized. Our first characterization is in terms of conjugate functions.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a real normed space. For $f : X \to \mathbb{R}$, the following statements are equivalent:*

(1) *There exists a nonempty closed convex set $C \subseteq X$ such that $f = d_C$.*

(2) *The following conditions hold:*

  (a) *f is proper, convex and lsc;*

  (b) *$dom\ f^* \subseteq B_{\|\cdot\|_*}$;*

  (c) *$\eta_f(x^*) = f^*(x^*)$ for every $x^* \in B_{\|\cdot\|_*}$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Property (a) is obvious (in fact, f is continuous, even 1-Lipschitz). Property (b) follows from the equality

$$d_C = \|\cdot\| \,\square\, \delta_C,$$

which implies that $f^* = \|\cdot\|^* + \delta_C^* = \delta_{B_{\|\cdot\|_*}} + \delta_C^*$; hence

$$dom\ f^* \subseteq dom\ \delta_{B_{\|\cdot\|_*}} = B_{\|\cdot\|_*}.$$

Moreover, for $x^* \in B_{\|\cdot\|_*} \setminus \{0\}$ we have

$$\eta_f(x^*) = \|x^*\|_* f^*\left(\frac{x^*}{\|x^*\|_*}\right) = \|x^*\|_* \delta_C^*\left(\frac{x^*}{\|x^*\|_*}\right) = \delta_C^*(x^*) = f^*(x^*),$$

which, together with the equalities $\eta_f(0) = 0 = \delta_C^*(0) = f^*(0)$, proves (c).

(2) $\Rightarrow$ (1) Assume that $f : X \to \mathbb{R}$ satisfies (a), (b) and (c). Then, for every $x \in X$, one has

$$\begin{aligned} f(x) = f^{**}(x) &= \sup_{x^* \in dom\ f^*} \{\langle x, x^* \rangle - f^*(x^*)\} = \sup_{x^* \in B_{\|\cdot\|_*}} \{\langle x, x^* \rangle - f^*(x^*)\} \\ &= \sup_{x^* \in B_{\|\cdot\|_*}} \{\langle x, x^* \rangle - \eta_f(x^*)\}. \end{aligned}$$

By (c), the positively homogeneous function η_f is lsc and convex on $B_{\|\cdot\|_*}$, which clearly implies that it is so on the whole of X^*. Hence, there exists a nonempty closed convex set $C \subseteq X$ such that $\eta_f = \delta_C^*$. Therefore

$$\begin{aligned} f(x) &= \sup_{x^* \in B_{\|\cdot\|_*}} \{\langle x, x^* \rangle - \delta_C^*(x^*)\} = \left(\delta_C^* + \delta_{B_{\|\cdot\|_*}}\right)^*(x) \\ &= \left(\delta_C^{**} \square \delta_{B_{\|\cdot\|_*}}^*\right)^{**}(x) = (\delta_C \square \|\cdot\|)^{**}(x) = (d_C)^{**}(x) = d_C(x), \end{aligned}$$

which proves (1). $\square$

Remark 2.2. The properties (b) and (c) are independent for lsc proper convex functions f. Indeed, the function $f := \|\cdot\| \,\square \frac{1}{2}\|\cdot\|^2$ satisfies (b) (with equality, because $f^* = \delta_{B_*} + \frac{1}{2}\|\cdot\|_*^2$, but not (c), since $\eta_f = \frac{1}{2}\|\cdot\|_*$; on the other hand, the function $f := 2\|\cdot\|$, satisfies (c), because $f^* = \delta_{2B_*}$ and $\eta_f \equiv 0$, but not (b), because $dom f^* = 2B_*$.

Corollary 2.3. *Let* $(X, \|\cdot\|)$ *be a real normed space. The mapping* $C \mapsto d_C$ *is a bijection from the set of nonempty closed convex subsets of* X *onto the set of functions* $f : X \to \mathbb{R}$ *satisfying* (a), (b) *and* (c) *of Theorem* 2.1.
The inverse mapping is $f \mapsto f^{-1}(0)$.

The following characterization of the distance function differs from the one in Theorem 2.1 in that it involves subdifferentials.

Corollary 2.4. *Let* $(X, \|\cdot\|)$ *be a real normed space. For* $f : X \to \mathbb{R}$*, the following statements are equivalent:*

(1) *There exists a nonempty closed convex set* $C \subseteq X$ *such that* $f = d_C$.

(2) *The following conditions hold:*

  (a) $\partial f(x) \cap B_{\|\cdot\|_*} \neq \emptyset$ *for every* $x \in X$,

  (b) $\eta_f(x^*) = f^*(x^*)$ *for every* $x^* \in B_{\|\cdot\|_*}$.

Proof. It is easy to prove that (a) of Corollary 2.4 implies (a) and (b) of Theorem 2.1. Conversely, if (a) and (b) of Theorem 2.1 hold, then, for every $x \in X$, we have

$$f(x) = f^{**}(x) = \sup_{x^* \in X^*} \{\langle x, x^* \rangle - f^*(x^*)\} = \sup_{x^* \in B_{\|\cdot\|_*}} \{\langle x, x^* \rangle - f^*(x^*)\};$$

hence, since the function $x^* \mapsto \langle x, x^* \rangle - f^*(x^*)$ is weak* upper semicontinuous, by the Banach-Alaoglu Theorem the latter supremum is attained. Clearly, the set of maximizers of that function is $\partial f(x) \cap B_{\|\cdot\|_*}$; therefore, this intersection is nonempty. $\square$

Remark 2.5. It is easy to see that the same examples of Remark 2.2 show that conditions (a) and (b) are independent. Indeed, from the fact that the function $f := \|\cdot\| \,\square\, \frac{1}{2}\|\cdot\|^2$ is 1-Lipschitz, one can easily prove that condition (a) holds, and for $f := 2\|\cdot\|$ one has $\partial f(x) \subset 2S_*$ for every $x \in X \setminus \{0\}$.

Corollary 2.6. *Let* $(X, \|\cdot\|)$ *be a real normed space. The mapping* $C \mapsto d_C$ *is a bijection from the set of nonempty closed convex subsets of* X *onto the set of functions* $f : X \to \mathbb{R}$ *satisfying* (a) *and* (b) *of Corollary* 2.4.
The inverse mapping is $f \mapsto f^{-1}(0)$.

3. The oriented distance function to a closed subset of a real Hilbert space

This section contains a characterization of the oriented distance function to a nonempty closed convex subset of $\mathbb{R}^n$.

Theorem 3.1. *Let* X *be a real Hilbert space. For* $f : X \to \mathbb{R}$*, the following statements are equivalent:*

(1) *There exists a nonempty closed convex set* $C \subseteq X$ *such that* $f = \Delta_C$.

(2) *The following conditions hold:*

  (a) $\partial f(x) \cap S_{\|\cdot\|_*} \neq \emptyset$ *for every* $x \in X$,

  (b) η_f *is convex and lsc.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds, and let $x \in X$. Since, by [3, Remark 4.2] (notice that the proof of Theorem 4.1 in [3] works in the general Hilbert space context), we have

$$\Delta_C(x) = \max_{x^* \in S_{\|\cdot\|_*}} \{\langle x, x^* \rangle - \delta_C^*(x^*)\}, \tag{1}$$

it is easy to prove that any maximizer in the latter expression belongs to the set $\partial \Delta_C(x) \cap S_{\|\cdot\|_*}$; therefore, this intersection is nonempty. This proves (a). To prove (b), we will use the equality $\Delta_C = \mu_C \Box \|\cdot\|$; here $\mu_C : X \to \mathbb{R} \cup \{+\infty\}$ is the function defined by

$$\mu_C(x) := \begin{cases} -d_{X \setminus C}(x) & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

Taking conjugates, we obtain $(\Delta_C)^* = (\mu_C)^* + \|\cdot\|^* = (\mu_C)^* + \delta_{B_{\|\cdot\|_*}}$. Hence, setting $f := \Delta_C$, for $x^* \neq 0$ we have

$$\begin{aligned} \eta_f(x^*) &= \|x^*\| f^*\left(\frac{x^*}{\|x^*\|}\right) = \|x^*\| (\mu_C)^* \left(\frac{x^*}{\|x^*\|}\right) \\ &= \|x^*\| \sup_{x \in X} \left\{ \left\langle x, \frac{x^*}{\|x^*\|} \right\rangle - \mu_C(x) \right\} \\ &= \|x^*\| \sup_{x \in C} \left\{ \left\langle x, \frac{x^*}{\|x^*\|} \right\rangle + d_{X \setminus C}(x) \right\} \\ &= \sup_{x \in C} \left\{ \langle x, x^* \rangle + \|x^*\| d_{X \setminus C}(x) \right\}. \end{aligned}$$

Taking into account that $\eta_f(0) = 0$, the latter expression shows that η_f is the pointwise supremum of a collection of sublinear continuous functions and is therefore sublinear and lsc. This proves (b).

(2) $\Rightarrow$ (1) Assume that (2) holds. By (a), for every $x \in X$ one has

$$\begin{aligned} f(x) &= \max_{x^* \in S_{\|\cdot\|_*}} \{\langle x, x^* \rangle - f^*(x^*)\} = \max_{x^* \in X \setminus \{0\}} \left\{ \left\langle x, \frac{x^*}{\|x^*\|} \right\rangle - f^*\left(\frac{x^*}{\|x^*\|}\right) \right\} \\ &= \frac{1}{\|x^*\|} \max_{x^* \in X \setminus \{0\}} \left\{ \langle x, x^* \rangle - \|x^*\| f^*\left(\frac{x^*}{\|x^*\|}\right) \right\} \\ &= \frac{1}{\|x^*\|} \max_{x^* \in X \setminus \{0\}} \{\langle x, x^* \rangle - \eta_f(x^*)\}. \end{aligned}$$

Since, by (b), η_f is sublinear and lsc, we have $\eta_f = \delta_C^*$ for some nonempty closed convex set $C \subseteq X$. We thus obtain

$$\begin{aligned} f(x) &= \frac{1}{\|x^*\|} \max_{x^* \in X \setminus \{0\}} \{\langle x, x^* \rangle - \delta_C^*(x^*)\} \\ &= \max_{x^* \in X \setminus \{0\}} \left\{ \left\langle x, \frac{x^*}{\|x^*\|} \right\rangle - \delta_C^*\left(\frac{x^*}{\|x^*\|}\right) \right\} = \max_{x^* \in S_{\|\cdot\|_*}} \{\langle x, x^* \rangle - \delta_C^*(x^*)\} \\ &= \Delta_C(x), \end{aligned}$$

the latter equality with $\Delta_C(x)$ being due to the displayed formula (1). This proves property (1) in the statement. $\Box$

Remark 3.2. Properties (a) and (b) are independent. Indeed, for $f := \|\cdot\| + 1$, one has

$$\partial f(x) = \begin{cases} \left\{ \frac{x}{\|x\|} \right\} & \text{if } x \neq 0, \\ B_* & \text{if } x = 0, \end{cases}$$

so f satisfies (a), but not (b), since $\eta_f = -\|\cdot\|$; on the other hand, the function $f := 2\|\cdot\|$ satisfies (b), because $\eta_f \equiv 0$, but not (a), because, as observed in Remark 2.5, one has $\partial f(x) \subset 2S_*$ for every $x \in \mathbb{R}^n \setminus \{0\}$.

Corollary 3.3. *Let X be a real Hilbert space. The mapping $C \mapsto \Delta_C$ is a bijection from the set of nonempty closed convex subsets of X onto the set of functions $f : X \to \mathbb{R}$ satisfying* (a) *and* (b) *of Theorem* 3.1.
The inverse mapping is $f \mapsto f^{-1}((-\infty, 0])$.

4. The farthest distance function to a compact subset of $\mathbb{R}^n$

In this last section, a characterization of the farthest distance function (corresponding to a norm $\|\cdot\|$ less than or equal to the Euclidean norm $\|\cdot\|_2$) to a compact subset of $\mathbb{R}^n$ is provided. The class of norms considered is rather large as it includes, e.g., all p-norms with $p \geq 2$.

Theorem 4.1. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ less than or equal to $\|\cdot\|_2$. For $f : \mathbb{R}^n \to \mathbb{R}$, the following statements are equivalent:*

(1) *There exists a nonempty compact convex set $C \subseteq \mathbb{R}^n$ such that $f = F_C^{\|\cdot\|}$.*

(2) *The following conditions hold:*
   (a) *$\partial f(x) \cap S_{\|\cdot\|_*} \neq \emptyset$ for every $x \in \mathbb{R}^n$,*
   (b) *η_f is finite-valued and concave.*

Proof. (1) ⇒ (2) Assume that (1) holds and take any $x \in \mathbb{R}^n$. Choose $c \in C$ such that

$$\|x - c\| = F_C^{\|\cdot\|}(x).$$

It is easy to prove that
$$\partial \|\cdot - c\|(x) \cap S_{\|\cdot\|_*} \neq \emptyset. \tag{2}$$

Since $\|\cdot - c\| \leq F_C^{\|\cdot\|}$ and these two functions coincide at x, we have

$$\partial \|\cdot - c\|(x) \subseteq \partial F_C^{\|\cdot\|}(x).$$

Hence, in view of (2), property (a) holds for $f := F_C^{\|\cdot\|}$. To prove that this function also satisfies (b), it will suffice to prove that $\eta_{F_C^{\|\cdot\|}} = \tau_C$. Observe first that these two functions agree at the origin: $\eta_{F_C^{\|\cdot\|}}(0) = 0 = \tau_C(0)$. Since both of them are positively homogeneous of degree 1, we only need to prove that they coincide on $S_{\|\cdot\|_*}$, which amounts to saying that $\left(F_C^{\|\cdot\|}\right)^*$ and τ_C coincide on $S_{\|\cdot\|_*}$. Let $x^* \in S_{\|\cdot\|_*}$. For every $x \in \mathbb{R}^n$ and $y \in C$, we have

$$F_C^{\|\cdot\|}(x) \geq \|y - x\| \geq \langle x - y, x^* \rangle.$$

Hence $\langle x, x^* \rangle - F_C^{\|\cdot\|}(x) \leq \langle y, x^* \rangle$; taking $\sup_{x \in \mathbb{R}^n}$ in the left hand side of this inequality and $\min_{y \in C}$ in the right hand side, we get $\left(F_C^{\|\cdot\|}\right)^*(x^*) \leq \tau_C(x^*)$.

To prove the opposite inequality, take $\epsilon > 0$ and $c \in C$ such that

$$\langle c, x^* \rangle = \tau_C(x^*).$$

We claim that the family of open sets $\{int\, B(c + \lambda x^*, \lambda + \epsilon)\}_{\lambda>0}$ covers C. Indeed, for $y \in C$, it is easy to check that

$$y \in intB(c + \lambda x^*, \lambda + \epsilon) \text{ for every } \lambda > \max\left\{0, \frac{\|y-c\|_2^2 - \epsilon^2}{2\epsilon}\right\}.$$

Therefore, since C is compact and the sets $intB(c + \lambda x^*, \lambda + \epsilon)$ depend increasingly (in the sense of inclusion) on λ, for some $\lambda_0 > 0$ we have

$$C \subseteq intB(c + \lambda_0 x^*, \lambda_0 + \epsilon),$$

that is, $F_C^{\|\cdot\|}(c + \lambda_0 x^*) < \lambda_0 + \epsilon$. Hence

$$\begin{aligned}\left(F_C^{\|\cdot\|}\right)^*(x^*) &\geq \langle c + \lambda_0 x^*, x^* \rangle - \left(F_C^{\|\cdot\|}\right)^*(c + \lambda_0 x^*) \\ &\geq \langle c, x^* \rangle + \lambda_0 - F_C^{\|\cdot\|}(c + \lambda_0 x^*) > \tau_C(x^*) - \epsilon.\end{aligned}$$

Since ϵ is an arbitrary positive number, it follows that $\left(F_C^{\|\cdot\|}\right)^*(x^*) \geq \tau_C(x^*)$, which ends the proof of (b).

(2) $\Rightarrow$ (1) Assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies (a) and (b). Clearly, the set

$$\Gamma_f := \{c \in \mathbb{R}^n : \|x - c\| \leq f(x) \ \forall x \in \mathbb{R}^n\} \tag{3}$$

is convex and compact. To prove that it is nonempty, observe that, for any $c \in \mathbb{R}^n$, the following equivalences hold:

$$\begin{aligned} c \in \Gamma_f &\Leftrightarrow \langle x - c, x^* \rangle \leq f(x) \,\forall (x^*, x) \in S_{\|\cdot\|_*} \times \mathbb{R}^n \\ &\Leftrightarrow f^*(x^*) \leq \langle c, x^* \rangle \,\forall x^* \in S_{\|\cdot\|_*} \Leftrightarrow \eta_f(x^*) \leq \langle c, x^* \rangle \,\forall x^* \in \mathbb{R}^n \\ &\Leftrightarrow c \in -\partial(-\eta_f)(0).\end{aligned}$$

Thus, $\Gamma_f = -\partial(-\eta_f)(0) \neq \emptyset$, since the subdifferential of a finite-valued convex function on $\mathbb{R}^n$ is nonempty everywhere. We will now prove that $F_{\Gamma_r}^{\|\cdot\|} = f$. For every $x^* \in \mathbb{R}^n$, in view of (3) we have

$$\begin{aligned}\tau_{\Gamma_f}(x^*) &= \min_{c \in -\partial(-\eta_f)(0)} \langle c, x^* \rangle = -\max_{c' \in \partial(-\eta_f)(0)} \langle c', x^* \rangle = -(-\eta_f)'(0, x^*) \\ &= \eta_f'(0, x^*) = \eta_f(x^*),\end{aligned}$$

the latter equality being a consequence of the positive homogeneity of η_f.

We thus have

$$\begin{aligned} F_{\Gamma_r}^{\|\cdot\|}(x) &= \max_{c \in \Gamma_f} \max_{x^* \in S_{\|\cdot\|_*}} \langle x - c, x^* \rangle = \max_{x^* \in S_{\|\cdot\|_*}} \left\{ \langle x, x^* \rangle - \min_{c \in \Gamma_f} \langle c, x^* \rangle \right\} \\ &= \max_{x^* \in S_{\|\cdot\|_*}} \left\{\langle x, x^* \rangle - \tau_{\Gamma_f}(x^*)\right\} = \max_{x^* \in S_{\|\cdot\|_*}} \left\{\langle x, x^* \rangle - \eta_f(x^*)\right\} \\ &= \max_{x^* \in S_{\|\cdot\|_*}} \left\{\langle x, x^* \rangle - f^*(x^*)\right\} = f(x),\end{aligned}$$

the latter equality following from (a). $\square$

Remark 4.2. Properties (a) and (b) are independent. Indeed, for $f := \langle \cdot, x_0^* \rangle$, with $x_0^* \in S_{\|\cdot\|_*}$, one has (a), because $\partial f(x) = \{x_0^*\}$ for every $x \in \mathbb{R}^n$, but not (b), since η_f is not finite-valued; on the other hand, the function $f := 2\|\cdot\|$, with $\|\cdot\|$ denoting the Euclidean norm, satisfies (b), because $\eta_f \equiv 0$, but not (a), because

$$\partial f(x) = \left\{ 2 \frac{x}{\|x\|} \right\} \subseteq 2S^{n-1} = 2S_{\|\cdot\|_*}$$

for every $x \in \mathbb{R}^n \setminus \{0\}$.

The similarity between the characterizations given in Theorems 3.1 and 4.1 is remarkable. It is interesting to see that the only essential difference between them is that, in the case of the oriented distance function, which is obtained by minimizing individual distances, the function η_f is convex, whereas for the farthest distance function, which is defined by maximizing individual distances, η_f is concave.

Corollary 4.3. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ less than or equal to $\|\cdot\|_2$. The mapping $C \mapsto F_C^{\|\cdot\|}$ is a bijection from the set of nonempty compact convex subsets of $\mathbb{R}^n$ onto the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying* (a) *and* (b) *of Theorem* 4.1.
The inverse mapping is $f \mapsto \Gamma_f$, with Γ_f given by (3).

Acknowledgment. I am very grateful to an anonymous referee for his/her careful reading of the manuscript, suggesting many important corrections that have allowed me to significantly polish and improve the presentation.

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