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Nonlinear Sharp Minimum and the Stability of a Local Minimum on Metric Spaces

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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We put the emphasis on nonlinear (weak) sharp minimizers with respect to a gauge function. This concept plays an important role in both theoretical and numerical aspects of optimization. In the last part of the contribution we study the stability (in some appropriate sense) of local/global minimizers of an objective function f perturbed to $f + g$ by a function g belonging to a suitable class of Lipschitz functions defined on metric spaces.

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1. Introduction

This work is in the stream of variational and non-smooth analysis which is a field where Professor Jean-Paul Penot has exercised his talents as a fine analyst. We would like through this contribution to express our gratitude to him for the important role he has played in our life as researchers.

The notion of sharp/weak minima, or the closely related error bound property plays an important role in the theory of mathematical optimization. In the theoretical aspect, it is an essential ingredient in the study of the stability of optimization and variational problems. In the algorithmic aspect, this notion is a crucial condition to guarantee the convergence of certain algorithms which use for example the penalty method, the proximal method and the augmented Lagrangian method (see, e.g., [1, 2, 5–7, 11, 12, 24] and the references given therein). In [24], a relationship between the (Höderian) sharp minima and the (Höderian) tilt stability in the Banach setting was studied. This latter notion of tilt stability was introduced by Poliquin and Rockafellar [22] in finite dimension, and later studied extensively by others in Banach spaces (see, Mordukhovich & Nghia [17], Zheng & Ng [24] and references therein).

In [2], in the metric framework, the notion of steep slope was used to study nonlinear (local) error bounds. By an approach based on a change of metric, the authors

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established a slope characterization of the nonlinear error bound with respect to a convex gauge function, and some consequences related to nonlinear weak sharp minima in Banach spaces were given.

The first purpose of this paper is to investigate the weak sharp minima in the metric setting for which the gauge functions are not necessarily convex. By a direct approach using the Ekeland variational principle and some estimates of the strong slope of quotient functions, we establish local and global characterizations of the weak sharp minima. Secondly, we introduce extensively the notion of nonlinear tilt/ weak tilt stability of local minimizers in the metric setting. Using an established slope characterization of nonlinear sharp/weak minima, we investigate some characterizations of these nonlinear stabilities.

Let us introduce some notations and definitions. Throughout the text, the notation employed is fairly standard. Let $(\mathbb{X}, \mathbf{d})$ be a complete metric space. As usual, for $x \in \mathbb{X}$, $r > 0$, $\mathbb{B}(x,r)$, $\mathbb{B}[x,r]$ stands for the open and closed balls centered at x with radius r, respectively. If $x \in \mathbb{X}$, we define the distance from x to a subset $C \subseteq \mathbb{X}$ by $\mathbf{d}(x,C) := \inf_{y\in C} \mathbf{d}(x,y)$. Given an extended real-valued function $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, we note as usual $\operatorname{Dom} f := \{x \in \mathbb{X} : f(x) < +\infty\}$. If A and B are nonempty subsets of $(\mathbb{X}, \mathbf{d})$, we define the *Hausdorff distance* between them by

$$\mathbf{H}_{\mathbf{d}}(A,B) := \inf\{\epsilon > 0 : B \subset S_{\mathbf{d}}(A,\epsilon) \quad \text{and} \quad A \subset S_{\mathbf{d}}B,\epsilon)\},$$

where,

$$S_{\mathbf{d}}(A,\epsilon) := \{x \in X : \mathbf{d}(x,A) < \epsilon\}, \text{ (resp. } \ S_{\mathbf{d}}(B,\epsilon) := \{x \in X : \mathbf{d}(x,B) < \epsilon\}).$$

Note that the Hausdorff distance between nonempty subsets of $\mathbb{X}$ is the uniform distance between their associated distance functionals:

$$\mathbf{H}_{\mathbf{d}}(A,B) = \sup_{x\in\mathbb{X}} |\mathbf{d}(x,A) - (\mathbf{d}(x,B)|.$$

2. Nonlinear weak sharp minimizers

In connection with the problem of minimizing an extended real-valued function $f : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ defined on a metric space $\mathbb{X}$ endowed with a metric $\mathbf{d}$, we consider a point $\bar{x}$, a closed ball $\mathbb{B}[x,r]$ and we define the set

$$S := S(f,\bar{x},r) := \operatorname{argmin}\{f(x) : \ x \in \mathbb{B}[\bar{x},r]\}.$$

The notion of *sharp minimizer* of f is well known and very useful in optimization theory. We say that a point $\bar{x}$ is a *sharp minimizer (also called strong isolated local minimizer)* of f if there exist positive constants κ and δ such that

$$\kappa\mathbf{d}(x,\bar{x}) \le f(x) - f(\bar{x}) \quad \text{for all} \quad x \in \mathbb{B}[\bar{x},\delta]. \tag{1}$$

Let us observe that if $\bar{x}$ is a sharp minimizer of f, then it is necessarily an isolated local minimizer of f as clearly $f(\bar{x})$ is strictly less than $f(x)$ for all $x \in \mathbb{B}[\bar{x},\delta] \setminus \{\bar{x}\}$. Note that (1) is a rather strong condition since obviously a smooth function does not have any sharp minimizer.

When in (1) $\mathbf{d}(x,\bar{x})$ is replaced by $\mathbf{d}(x,\bar{x})^q$ for some $q>1$, we obtain the concept of *q-order minimizer* (also called *growth condition of order* q, (see [4,14]) that is fundamental when studying sensitivity and quantitative stability analysis for optimization problems: for some $\kappa,\delta\in(0,+\infty)$ it holds,

$$\kappa\mathbf{d}(x,\bar{x})^q\le f(x)-f(\bar{x})\quad\text{for all}\quad x\in\mathbb{B}[\bar{x},\delta].\tag{2}$$

Clearly, when $q<1$, then (2) is weaker that (1). The case $q=2$ (often called quadratic growth) is particularly interesting, since when f is a C^2 function, $\bar{x}$ is a 2-order minimizer of f if and only if the gradient $\nabla f(\bar{x})=0$ and the Hessian $\nabla^2 f(\bar{x})$ is positively definite.

Motivated by the sensibility analysis and the convergence analysis of a wide range of optimization algorithms, Ferris [11], introduced in his PhD thesis the terminology of *weak sharp minima.* A point $\bar{x}\in\operatorname{Dom} f$ is called a *(local) weak sharp minimizer* of f, if there exist $\tau,r,\delta>0$ such that $f(\bar{x})=\inf_{x\in\mathbb{B}(\bar{x},r)}f(x)$ and that

$$\tau\mathbf{d}(x,S)\le f(x)-f(\bar{x}),\qquad\text{for all }\ x\in\mathbb{B}(\bar{x},\delta).\tag{3}$$

Note that when $S(f,\bar{x},r)=\{\bar{x}\}$ then $\bar{x}$ is a *sharp minimizer* of f, as defined before. Concerning this notion, the reader is referred for instance to Burke and Deng [5,6], Klatte [16] and the references therein.

The extension of the notion of (weak) sharp minimizer to Hölderian (weak) sharp minimizer by replacing $\mathbf{d}(x,S(f,\bar{x},r))$ in the left hand of the inequality (3) by $\mathbf{d}(x,S(f,\bar{x},r))^p$ with some order $p>1$ was considered by several authors (e.g., see [24] and references given therein). Our objective in this paper is to consider the case of nonlinear (weak) sharp minimizers with respect to a *gauge function,* that is with respect to a function $\beta:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\beta(0)=0$, and $\beta(t)>0$ for all $t>0$. This approach consists of replacing in the left term of (3), $\mathbf{d}(x,S)$ by $\beta(\mathbf{d}(x,S))$:

$$\tau\beta(\mathbf{d}(x,S))\le f(x)-f(\bar{x}),\qquad\text{for all }\ x\in\mathbb{B}(\bar{x},\delta).\tag{4}$$

Such a point $\bar{x}$ satisfying (4) will be called *β-weak sharp minimizer* of the function f.

For the purpose of this study we need to use the concept of *strong slope* $|\nabla f|(x)$ of a function f at $x\in\operatorname{Dom} f$. This quantity introduced by De Giorgi, Marino & Tosques [8] to extend steepest descent curves to metric spaces has proved to be a very useful tool for analyzing local properties of real-valued functions (see [1,20]). It is defined by

$$|\nabla f|(x):=\begin{cases}0 & \text{if } x \text{ is a local minimum point of } f\\ \displaystyle\limsup_{y\to x,\,y\neq x}\frac{[f(x)-f(y)]}{\mathbf{d}(x,y)} & \text{otherwise.}\end{cases}$$

For $x\notin\operatorname{Dom} f$, we set $|\nabla f|(x)=+\infty$.

When f is a convex function defined on a Banach space and x is not a minimum point, then according to Ioffe [15, Poposition 3.8] (see also Azé & Corvelec [1, Proposition 3.2])

$$|\nabla f|(x)=\sup_{y\neq x}\frac{[f(x)-f(y)]}{\mathbf{d}(x,y)}\quad\text{ and }\quad|\nabla f|(x)=\mathbf{d}(0,\partial f(x)),$$

where $\partial f(x)$ stands for the convex subdifferential of f at x. Note that when $\mathbb{X}$ is a normed space and f is Fréchet differentiable at $\bar{x}$, then $|\nabla f|(\bar{x}) = \|Df(\bar{x})\|$, where $Df(\bar{x})$ is the Fréchet derivative of f at $\bar{x}$. Since in this work, the reference space on which we work is a complete metric space, we don't need to recall the notion of subdifferentials. For those readers interested, the connection between the notions of strong slope and subdifferentiability in Banach spaces can be found in some publications related to nonsmooth analysis as for instance in the books by Penot [21] and Thibault [23]. The following slope composite formula with a straightforward proof is useful. In the case when the derivative of the outer function is positive, it was proved in ([3, Lemma 4.1])

Lemma 2.1. *Let f be a function defined on $\mathbb{X}$ and for given $x \in \operatorname{Dom} f$, assume that f is continuous at x. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be differentiable at $f(x)$. Then*

(i) *If $\varphi'(f(x)) = 0$ and $|\nabla f|(x)$ is finite then $|\nabla \varphi \circ f|(x) = 0$;*

(ii) *If $\varphi'(f(x)) > 0$ then $|\nabla \varphi \circ f|(x) = \varphi'(f(x))|\nabla f|(x)$;*

(iii) *If $\varphi'(f(x)) < 0$ then $|\nabla \varphi \circ f|(x) = -\varphi'(f(x))|\nabla(-f)|(x)$.*

The next lemma gives useful estimates of the strong slope of the product and quotient functions in terms of the ones of the component functions (see [**?**]).

Lemma 2.2. *Let $f, g : \mathbb{X} \longrightarrow \mathbb{R} \cup \{\infty\}$ be lower semicontinuous (lsc for short) extended real-valued functions and let $x \in \operatorname{Dom} f \cap \operatorname{Dom} g$ satisfying $f(x) \geqslant 0$ and $g(x) \geq 0$. Suppose in addition that g is continuous at x. Under these assumptions, one has*

$$|\nabla(fg)|(x) \leqslant |\nabla f|(x)g(x) + f(x)|\nabla g|(x). \tag{5}$$

As a result, if $g(x) > 0$, then

$$\left|\nabla\left(\frac{f}{g}\right)\right|(x) \geqslant \frac{|\nabla|f(x)}{g(x)} - \frac{f(x)}{[g(x)]^2}|\nabla|g(x). \tag{6}$$

Proof. If x is a local minimizer of fg, then (5) holds trivially. Otherwise, pick a sequence (x_n) converging to x such that

$$\lim_{n\to\infty} \frac{f(x)g(x) - f(x_n)g(x_n)}{d(x, x_n)} = |\nabla|(fg)(x).$$

Then, taking into account the relation

$$\frac{f(x)g(x) - f(x_n)g(x_n)}{d(x, x_n)} \leq f(x)\frac{g(x) - g(x_n)}{d(x, x_n)} + g(x_n)\frac{f(x) - f(x_n)}{d(x, x_n)},$$

letting n tend to infinity and using the continuity of g at x, one has

$$\begin{aligned}|\nabla(fg)|(x) &\leq f(x)\limsup_{n\to\infty}\frac{g(x) - g(x_n)}{d(x, x_n)} + \limsup_{n\to\infty} g(x_n)\frac{f(x) - f(x_n)}{d(x, x_n)} \\ &= |\nabla f|(x)g(x) + f(x)|\nabla g|(x).\end{aligned}$$

Applying (5) to the two functions f/g and g, one obtains

$$|\nabla f|(x) = |\nabla(f/g)g|(x) \leq |\nabla|(f/g)(x)g(x) + f(x)/g(x)|\nabla g|(x)$$

and now (6) follows immediately by dividing the two parts of the inequality by $g(x) > 0$. □

In what follows, we consider a differentiable gauge function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and

$$\limsup_{t\to 0^+} \frac{t|\beta'(t)|}{\beta(t)} < +\infty. \tag{7}$$

A standard example of such a gauge function is given by the family of functions defined by $t \to \beta(t) := t^p$, for $p \geq 1$. Obviously, (7) holds for this class of gauges and furthermore, all assumptions on gauge functions required in the sequel are satisfied immediately for this class.

Theorem 2.3. *Let $\mathbb{X}$ be a complete metric space. Assume $\bar{x}$ is a minimizer of f on $\mathbb{B}[\bar{x}, r]$, with some $r > 0$. For $S := S(f, \bar{x}, r)$, and $f^* = \min_{x\in\mathbb{B}[\bar{x},r]} f(x)$, if*

$$\liminf \left\{ \frac{|\nabla f|(x)\mathbf{d}(x, S)}{\beta(\mathbf{d}(x, S))} \;\middle|\; x \to \bar{x},\ \frac{f(x) - f^*}{\beta(\mathbf{d}(x, S))} \to 0 \right\} > 0 \tag{8}$$

then $\bar{x}$ is a β-weak sharp minimizer of f.

Proof. Without loss of generality, we may assume $f(\bar{x}) = 0$. We shall prove the theorem by the contradiction. Suppose to the contrary that $\bar{x}$ is not a β-weak sharp minimizer of f. It means that there is a sequence (x_n) with $x_n \neq \bar{x}$ for all $n \in \mathbb{N}$, converging to $\bar{x}$, such that $\lim_{n\to\infty} \frac{f(x_n)}{\beta(\mathbf{d}(x_n, S))} = 0$. Consider the lower semicontinuous functions defined on $\mathbb{X}$ by

$$g(x) := \begin{cases} \dfrac{\beta(\mathbf{d}(x, S)}{\mathbf{d}(x, S)}, & \text{if } x \notin S, \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(x) := \begin{cases} \dfrac{f(x)}{g(x)} = \dfrac{f(x)\mathbf{d}(x, S)}{\beta(\mathbf{d}(x, S))}, & \text{if } x \in \mathbb{B}[\bar{x}, r], \\ +\infty & \text{otherwise.} \end{cases}$$

By setting $\varepsilon_n = \dfrac{f(x_n)}{\beta(\mathbf{d}(x_n, S))}$, one has $h(x_n) \leq \inf_{x\in\mathbb{X}} h(x) + \varepsilon_n \mathbf{d}(x_n, S)$.

Hence by virtue of the Ekeland variational principle, one can find for each n, a point $z_n \in \mathbb{B}[x_n, \mathbf{d}(x_n, S)/2]$ such that $h(z_n) \leq h(x_n)$, and

$$h(z_n) \leq h(x) + 2\varepsilon_n \mathbf{d}(x, z_n), \quad \forall x \in X. \tag{9}$$

As a result, $|\nabla h|(z_n) \leq 2\varepsilon_n$, for all $n \in \mathbb{N}$. Noticing that we have $z_n \notin S$, since $z_n \in \mathbb{B}[x_n, \mathbf{d}(x_n, S)/2]$, this implies that $g(z_n) > 0$. As $h(z_n) \leq h(x_n)$, one has

$$\lim_{n\to\infty} \frac{f(z_n)}{\beta(\mathbf{d}(z_n, S))} = 0,$$

since $\dfrac{f(x_n)}{\beta(\mathbf{d}(x_n, S))} \to 0$, and

$$\frac{f(z_n)}{\beta(\mathbf{d}(z_n, S))} \leq \frac{f(x_n)\mathbf{d}(x_n, S)}{\beta(\mathbf{d}(x_n, S))\mathbf{d}(z_n, S)} \leq \frac{2f(x_n)}{\beta(\mathbf{d}(x_n, S))},$$

where the second inequality is due to the fact that

$$\mathbf{d}(z_n, S) \geq \mathbf{d}(x_n, S) - \mathbf{d}(z_n, x_n) \geq \mathbf{d}(x_n, S)/2.$$

In view of Lemma 2.2, one has the estimate

$$|\nabla h|(z_n) \geqslant \frac{|\nabla f|(z_n)}{g(z_n)} - \frac{f(z_n)}{[g(z_n)]^2}|\nabla g|(z_n). \tag{10}$$

Using $\varphi(t) = \beta(t)/t$, $t > 0$, let us define δ_n by setting

$$\begin{cases} \delta_n = 1, & \text{if } \varphi'(\mathbf{d}(z_n, S)) \geq 0, \\ \delta_n = -1, & \text{otherwise.} \end{cases}$$

Next, from Lemma 2.1, applied to $g = \varphi \circ \mathbf{d}(\cdot, S)$ we derive,

$$|\nabla g|(z_n) \ = |\varphi'(\mathbf{d}(z_n, S))||\delta_n|\nabla \mathbf{d}(\cdot, S)|(z_n)| \leq |\varphi'(\mathbf{d}(z_n, S))|.$$

Therefore, one has the following estimate for the second term of the right hand side of (10):

$$0 \leq \frac{f(z_n)}{[g(z_n)]^2}|\nabla g|(z_n) \leq \frac{f(z_n)}{\beta(\mathbf{d}(z_n, S))}\frac{\mathbf{d}(z_n, S)|\varphi'(\mathbf{d}(z_n, S))|}{\varphi(\mathbf{d}(z_n, S))}.$$

Taking into account this estimate, the facts that $\dfrac{f(z_n)}{\beta(\mathbf{d}(z_n, S))} \to 0$, and the sequence

$$\frac{\mathbf{d}(z_n, S)|\varphi'(\mathbf{d}(z_n, S))|}{\varphi(\mathbf{d}(z_n, S))} = \frac{|\mathbf{d}(z_n, S)\beta'(\mathbf{d}(z_n, S)) - \beta(\mathbf{d}(z_n, S))|}{\beta(\mathbf{d}(z_n, S))}$$

is bounded by assumption (7), one derives that $\lim_{n\to\infty} \dfrac{f(z_n)}{[g(z_n)]^2}|\nabla g|(z_n) = 0$. Thus since $|\nabla h|(z_n) \leq 2\varepsilon_n$, $|\nabla h|(z_n) \to 0$, in view of (10), one obtains

$$\lim_{n\to\infty} \frac{|\nabla f|(z_n)}{g(z_n)} = \lim_{n\to\infty} \frac{|\nabla f|(z_n)\mathbf{d}(z_n, S)}{\beta(\mathbf{d}(z_n, S))} = 0,$$

which contradicts (8). □

Since $\mathbf{d}(x, S) \leq \mathbf{d}(x, \bar{x})$, the previous theorem yields immediately the following corollary:

Corollary 2.4. *Under the assumptions of the previous Theorem* 2.3*, but instead of* (8)*, assume that*

$$\begin{cases} \liminf \left\{ \dfrac{|\nabla f|(x)\mathbf{d}(x, S)}{\beta(\mathbf{d}(x, S))} \;\middle|\; x \to \bar{x},\ \dfrac{f(x) - f^*}{\beta(\mathbf{d}(x, \bar{x}))} \to 0 \right\} > 0 \\ \text{and } \beta \text{ is a nondecreasing function.} \end{cases} \tag{11}$$

Then $\bar{x}$ is a β-weak sharp minimizer of f.

When β is a convex function, then obviously $\beta'(t) \geq \beta(t)/t$, for all $t > 0$. Hence one obtains the following corollary that is closely related to [3, Theorem 4.2]. However the sufficient condition stated here is a bit weaker.

Corollary 2.5. *Assume in addition that β is a convex function. If*

$$\liminf \left\{ \frac{|\nabla f|(x)}{\beta'(\mathbf{d}(x,S))} \;\middle|\; x \to \bar{x},\ \frac{f(x)-f^*}{\beta(\mathbf{d}(x,S))} \to 0 \right\} > 0, \tag{12}$$

then $\bar{x}$ is a β-weak sharp minimizer of f.

The following theorem gives a slope characterization for $\bar{x}$ to be a weak sharp minimum over a ball $B[\bar{x}, r]$ with a fixed radius $r > 0$, even if $\bar{x}$ is not necessarily assumed to be a minimum.

Theorem 2.6. *Let f be a lower semicontinuous function defined on $\mathbb{X}$ and assume $\bar{x} \in \mathrm{Dom}\, f$. For a given $r > 0$, suppose that $S := \{z \in B[\bar{x}, r] : f(z) = \min_{x \in B[x,r]} f(x)\}$ is nonempty and define $f^* := \min_{x \in \mathbb{B}[\bar{x},r]} f(x)$. Let β be a gauge function such that for any $T > 0$,*

$$\sup_{t \in]0,T]} \frac{t|\beta'(t)|}{\beta(t)} < +\infty. \tag{13}$$

Assume in addition to the assumptions of Theorem 2.3,, there exist reals $\kappa, \alpha > 0$ and $\sigma > r$ such that

$$\kappa\beta(\mathbf{d}(x,S)) \le |\nabla f|(x)\mathbf{d}(x,S),\ \ \forall x \in B[\bar{x},\sigma],\ f(x) - f^* \le \alpha\beta(\mathbf{d}(x,S)), \tag{14}$$

then there exists $\tau > 0$ such that

$$\tau\beta(\mathbf{d}(x,S)) \le f(x) - f^*,\ \ \forall x \in B[\bar{x}, r]. \tag{15}$$

Proof. Set $\sup_{t \in]0,2\sigma]} \frac{t|\beta'(t)|}{\beta(t)} = M$, and without loss of generality, assume that $\min_{x \in B[\bar{x},r]} f(x) = 0$. Consider the lower semicontinuous functions defined on $\mathbb{X}$:

$$g(x) := \begin{cases} \dfrac{\beta(\mathbf{d}(x,S)}{\mathbf{d}(x,S)} & \text{if } x \notin S, \\ 0 & \text{otherwise,} \end{cases}$$

and
$$h(x) := \begin{cases} \dfrac{f(x)}{g(x)} = \dfrac{f(x)\mathbf{d}(x,S)}{\beta(\mathbf{d}(x,S))}, & \text{if } x \in B[\bar{x},\sigma], \\ +\infty & \text{otherwise.} \end{cases}$$

For a given $x \in B[\bar{x}, r]$, and for $\varepsilon = \frac{f(x)}{\beta(\mathbf{d}(x,S))}$, one has

$$h(x) \le \inf_{u \in X} h(u) + \varepsilon\mathbf{d}(x,S).$$

Pick $\lambda > 0$ with $2r\lambda + \delta < \sigma$. By the Ekeland variational principle, there exists a point $z \in B[x, \lambda\mathbf{d}(x,S)]$ such that $h(z) \le h(x)$, and

$$h(z) \le h(u) + \frac{\varepsilon}{\lambda}\mathbf{d}(u,z), \quad \forall u \in X. \tag{16}$$

As $\mathbf{d}(x,S) \le \mathbf{d}(x,\bar{x}) + \mathbf{d}(\bar{x},S) < 2r$ and $\mathbf{d}(z,x) \le \lambda\mathbf{d}(x,S) < 2\lambda r$, we deduce that

$$\mathbf{d}(z,\bar{x}) \le \mathbf{d}(z,x) + \mathbf{d}(x,\bar{x}) < 2\lambda r + \delta < \sigma.$$

Hence (16) yields $|\nabla h|(z) \leq \varepsilon\lambda^{-1}$. As $z \in B[x, \mathbf{d}(x,S)\lambda]$, this implies that $z \notin S$, and therefore $g(z) > 0$. As $h(z) \leq h(x)$, one has

$$\frac{f(z)}{\beta(\mathbf{d}(z,S))} \leq \frac{f(x)\mathbf{d}(x,S)}{\beta(\mathbf{d}(x,S))\mathbf{d}(z,S)} \leq \frac{f(x)}{(1-\lambda)\beta(\mathbf{d}(x,S))}, \tag{17}$$

where the second inequality is due from

$$\mathbf{d}(z,S) \geq \mathbf{d}(x,S) - \mathbf{d}(z,x) \geq (1-\lambda)\mathbf{d}(x,S)/2.$$

Consider the case $\dfrac{f(x)}{\beta(\mathbf{d}(x,S))} < (1-\lambda)\alpha$. Then $\dfrac{f(z)}{\beta(\mathbf{d}(z,S))} < \alpha$.

In view of Lemma 2.2, one has the estimate

$$|\nabla h|(z) \geqslant \frac{|\nabla f|(z)}{g(z)} - \frac{f(z)}{[g(z)]^2}|\nabla g|(z). \tag{18}$$

Next, similarly to the proof of the preceding theorem, for $g = \varphi \circ \mathbf{d}(\cdot, S)$ with $\varphi(t) = \beta(t)/t$, $t > 0$, one has,

$$\nabla g|(z)\mathbf{d}(\cdot,S)|(z) \leq |\varphi'(\mathbf{d}(z,S))|.$$

Therefore, one has the estimate

$$0 \leq \frac{f(z)}{[g(z)]^2}|\nabla g|(z) \leq \frac{f(z)}{\beta(\mathbf{d}(z,S))}\frac{\mathbf{d}(z,S)|\varphi'(\mathbf{d}(z,S))|}{\varphi(\mathbf{d}(z,S))}.$$

Observing that $\mathbf{d}(z,S) \leq \mathbf{d}(z,\bar{x}) + \mathbf{d}(\bar{x},S) \leq 2\sigma$, we get

$$\frac{\mathbf{d}(z,S)|\varphi'(\mathbf{d}(z,S))|}{\varphi(\mathbf{d}(z,S))} = \frac{|\mathbf{d}(z,S)\beta'(\mathbf{d}(z,S)) - \beta(\mathbf{d}(z,S))|}{\beta(\mathbf{d}(z,S))} \leq M+1.$$

Since $|\nabla h|(z) \leq \varepsilon\lambda^{-1} = \dfrac{f(x)}{\lambda\beta(\mathbf{d}(x,S))}$, using relations (15), (17), by (18), one obtains

$$\begin{aligned}\kappa \leq \frac{f(z)\mathbf{d}(z,S)}{\beta(\mathbf{d}(z,S))} &\leq |\nabla h|(z) + \frac{f(z)}{[g(z)]^2}|\nabla g|(z) \leq \frac{f(x)}{\lambda\beta(\mathbf{d}(x,S))} + \frac{(M+1)f(z)}{\lambda\beta(\mathbf{d}(z,S))} \\ &\leq \frac{f(x)}{\lambda\beta(\mathbf{d}(x,S))} + \frac{(M+1)f(x)}{\lambda(1-\lambda)\beta(\mathbf{d}(x,S))}.\end{aligned}$$

Thus, $L\beta(\mathbf{d}(x,S)) \leq f(x) = f(x) - f^*$, where $L = \kappa\lambda(1-\lambda)(M+2-\lambda)^{-1}$.

As this inequality holds for

$$\frac{f(x)}{\beta(\mathbf{d}(x,S))} < (1-\lambda)\alpha,$$

one obtains the desired inequality:

$$\tau\beta(\mathbf{d}(x,S)) \leq f(x) - f^*, \quad \forall x \in B[\bar{x}, r], \ \tau := \min\{(1-\lambda)\alpha, L\}.$$

The proof is complete □

3. Stability of minimizers

This section focuses on an extension of tilt stability of local minimizers introduced in finite dimension by Poliquin and Rockafellar [22] in the general extended real-valued framework of unconstrained optimization. Stability (in some appropriate sense) of local/global minimizers of an objective function f perturbed to $f+g$ by a function g considered in a suitable class of functions, is one of the central questions in optimization and variational methods in which many authors have been involved.

A point $\bar{x}$ is said to be a *tilt-stable local minimum* of the function $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ if $f(\bar{x})$ is finite and there exists $\delta > 0$ such that the mapping

$$M : v \to \operatorname{argmin}_{\|x-\bar{x}\|\leq\delta}\{f(x) - f(\bar{x}) - \langle v, x-\bar{x}\rangle\}$$

is single-valued and Lipschitzian in some neighborhood of 0 with $M(0) = \{\bar{x}\}$. The connection between this notion of tilt stability and the classical picture of local optimality is as follows:

Proposition 3.1. [22, Proposition 1.2.] *For a C^2 function f verifying $\nabla(\bar{x}) = 0$, a point $\bar{x}$ is a tilt-stable local minimum of f if and only if $\nabla^2 f(\bar{x})$ is positive-definite.*

In [22], tilt stability has been characterized in finite dimensions in terms of second-order coderivatives for prox-regular functions and thereafter studied extensively. Notably, nonsmooth extensions of the Poliquin-Rockafellar classical second-order condition have been done (e.g., see Zhang & Ng [24], Mordukhovich and his co-authors [9, 10, 17–19] and references therein, as well as the comments 7.10 in the book by L. Thibault [23]).

In this section, we consider some (more general) stability of local minimizers with respect to a suitable class of Lipschitz functions defined on metric spaces. For a complete metric space $\mathbb{X}$, we consider the space of (global) Lipschitz functions on $\mathbb{X}$, denoted by $\mathcal{L}_{\mathbb{X}}$. It is well known that this space is a Banach space endowed with the norm defined as follows: pick a fixed point $x_0 \in \mathbb{X}$, for $g \in \mathcal{L}_{\mathbb{X}}$, define

$$\|g\| = |g(x_0)| + \sup_{x,y\in\mathbb{X},y\neq x} \frac{|g(y)-g(x)|}{\mathbf{d}(y,x)}.$$

Let $\mathcal{C} \subseteq \mathcal{L}_{\mathbb{X}}$, be a given subspace of $\mathcal{L}_{\mathbb{X}}$, the space of Lipschitz functions defined on $\mathbb{X}$, and let $\beta : \mathbb{R}_+ \to R_+$ be a gauge function as defined in the precedent section. Let us suppose that an *additional assumption* holds:

($\mathscr{H}$) $\lim_{t\to 0^+} \beta(t)/t = 0$; $\gamma(t) := \beta(t)/t$ is *increasing* on $]0, +\infty)$, and there exists a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(0) = 0$, and

$$\delta\gamma(t) \leq \gamma(\rho(\delta)t, \quad \forall \delta, t \geq 0.$$

Given the pair $(\mathcal{C}, \beta)$, next let us introduce some notions of stability. The first one called the $(\mathcal{C}, \beta)$-*(weak) tilt stability*, follows the terminology by Poliquin & Rockefallar.

Definition 3.2. Let f be a function defined on $\mathbb{X}$. For any $g \in \mathcal{C}$ we define

$$M(g) := M(f,g,r) = \{z \in \mathbb{B}[\bar{x},r] : \quad f(z)+g(z) = \min_{x\in\mathbb{B}[\bar{x},r]} f(x)+g(x)\}.$$

Given a local minimizer $\bar{x} \in \operatorname{Dom} f$ of f, $\bar{x}$ is said to be a $(\mathcal{C},\beta)$-*weak tilt stable minimizer* of f, if there are $r,\delta,\kappa > 0$ such that for any $g \in \mathcal{C}$ with $\|g\| \le \delta$,

$$\beta(\mathbf{H}_{\mathbb{X}}(M(g),M(h))) \le \kappa\|g-h\|, \quad \forall g,h \in \mathcal{C}, \text{ such that } \quad \|g\|,\|h\| \le \delta,$$

where, $\mathbf{H}_{\mathbb{X}}$ denotes the Hausdorff distance between subsets of $\mathbb{X}$.

Definition 3.3. Assuming $f,\bar{x}$ as in the precedent definition, $\bar{x}$ is said to be a $(\mathcal{C},\beta)$-*weak sharp stable minimizer* of f, if there is $r,\delta,\tau > 0$ such that for any $g \in \mathcal{C}$ with $\|g\| \le \delta$,

$$\tau\beta(\mathbf{d}(x,M(g))) \le (f+g)(x) - \inf_{\mathbb{B}[x,r]}(f+g), \ \ \forall x \in \mathbb{B}[\bar{x},r].$$

In the preceding two definitions, if $M(g)$ is required to be a singleton for all $g \in \mathcal{C}$, with $\|g\| \le \delta$, then we speak about $(\mathcal{C},\beta)$-*tilt stable minimizer* and $(\mathcal{C},\beta)$-*sharp stable minimizer*, respectively. The first result gives the relationships between the above two properties of stability.

Theorem 3.4. *Let f be a function defined on $\mathbb{X}$, and let $\bar{x} \in \operatorname{Dom} f$ be a local minimizer of f. Given a gauge function β, define*

$$\gamma(t) = \beta(t)/t, \quad \textit{if} \quad t > 0, \qquad \textit{and} \quad \gamma(0) = 0.$$

(i) *If $\bar{x}$ is a $(\mathcal{C},\beta)$-weak sharp stable minimizer of f, then it is a $(\mathcal{C},\gamma)$-weak tilt stable minimizer of f.*

(ii) *Conversely, assume in addition that the subspace $\mathcal{C}$ contains the distance functions $\mathbf{d}(\cdot,z)$, for all $z \in \mathbb{X}$, and ($\mathscr{H}$) holds. If $\bar{x}$ is a $(\mathcal{C},\gamma)$-weak tilt stable minimizer of f, then it is $(\mathcal{C},\beta)$-weak sharp stable minimizer of f.*

Proof. For (i), let r,τ,δ be as in Definition 3.3. For $g,h \in \mathcal{C}$, with $\|g\|,\|h\| \le \delta$, for any $x_h \in M(h)$, one has

$$\tau\beta(\mathbf{d}(x_h,M(g))) \le (f+g)(x_h) - \inf_{\mathbb{B}[x,r]}(f+g).$$

For any $\varepsilon > 0$, picking $x_g \in M(g)$ such that $\mathbf{d}(x_h,x_g) \le (1+\varepsilon)\mathbf{d}(x_h,M(g))$, then one has

$$\begin{aligned}(1+\varepsilon)^{-1}\tau\mathbf{d}(x_h,x_g) &\le \tau\beta(\mathbf{d}(x_h,M(g)))\\ &\le (1+\varepsilon)(f+g)(x_h) - (f+g)(x_g)\\ &= (f+h)(x_h) - (f+h)(x_g) + (g-h)(x_h) - (g-h)(x_g)\\ &\le |(g-h)(x_h) - (g-h)(x_g)| \le \|g-h\|\mathbf{d}(x_h,x_g),\end{aligned}$$

where the last inequality is due to the fact that x_h is a minimizer of $f+h$ on $\mathbb{B}[\bar{x},r]$.

Therefore, as $\varepsilon > 0$ is arbitrarily small, for $\kappa = \tau^{-1}$ we have,

$$\gamma(\mathbf{d}(x_h, M(g))) \leq \kappa\|g - h\|, \ \ \forall x_h \in M(h).$$

By changing the roles between g and h, this implies that

$$\gamma(\mathbf{H}_{\mathbb{X}}(M(h), M(g))) \leq \kappa\|g - h\|, \ \ \forall h, g \in \mathcal{C}, \ \|g\|, \|h\| \leq \delta,$$

and (i) is proved.

For (ii), let r, τ, δ as in Definition 3.2. Assume to the contrary that $\bar{x}$ is not a $(\mathcal{C}, \beta)$-weak sharp stable minimizer of f. Pick parameters $K, \eta > 0$ such that

$$\eta + \frac{K\gamma(2r)}{\eta} < \delta; \ \ \eta + \rho(K/(\tau\eta)) < 1.$$

Then we can find $g \in \mathcal{C}$ with $\|g\| \leq \delta$, $x_1 \in \mathbb{B}[\bar{x}, r]$ such that

$$f(x_1) + g(x_1) < \min_{x \in \mathbb{B}[x,r]} (f(x) + g(x)) + K\beta(\mathbf{d}(x_1, M(g))).$$

So $x_1 \notin M(g)$, and by applying the Ekeland variational principle to the function $f + g$, there exists $z \in \mathbb{X}$ such that $\mathbf{d}(z, x_1) \leq \eta\mathbf{d}(x_1, M_g)$, and

$$(f + g)(z) \leq (f + g)(x) + K\eta^{-1}\gamma(\mathbf{d}(x_1, M(g)))\mathbf{d}(z, x), \ \ \forall x \in \mathbb{B}[\bar{x}, r].$$

This shows that z is a minimum of the function $f + h$, with

$$f := g + K\eta^{-1}\gamma(\mathbf{d}(x_1, M(g)))\mathbf{d}(z, \cdot),$$

which belongs to $\mathcal{C}$ and

$$\|h\| \leq \|g\| + K\eta^{-1}\gamma(\mathbf{d}(x_1, M(g))) \leq \eta + K\eta^{-1}\gamma(2r) < \delta.$$

Hence,
$$\begin{aligned}\gamma(\mathbf{d}(z, M(g))) &\leq \tau^{-1}\|h - g\| \leq K\eta^{-1}\tau^{-1}\gamma(\mathbf{d}(x_1, M(g))) \\ &\leq \gamma\left(\rho(K\eta^{-1}\tau^{-1})\mathbf{d}(x_1, M(g))\right),\end{aligned}$$

where, the second inequality is due to $(\mathscr{H})$. Hence, since γ is increasing, it implies $\mathbf{d}(z, M(g)) \leq \rho(K\eta^{-1}\tau^{-1})\mathbf{d}(x_1, M(g))$. Thus

$$\mathbf{d}(x_1, M(g)) \leq \mathbf{d}(x_1, z) + \mathbf{d}(z, M(g)) \leq \left(\eta + \rho(K\eta^{-1}\tau^{-1})\mathbf{d}(x_1, M(g))\right) < \mathbf{d}(x_1, M(g)),$$

a contradiction. The proof is complete. □

Obviously from the definition, the statements of the above theorem are also valid for the $(\mathcal{C}, \beta)$-sharp stable minimizer and the $(\mathcal{C}, \beta)$-sharp stable minimizer instead of the respectively weak notions.

From Theorem 2.6 in the preceding section, using the strong slopes, we obtain a sufficient condition for the $(\mathcal{C}, \beta)$-weak sharp stability.

Theorem 3.5. *Suppose that the gauge function β satisfies condition* (13) *in Theorem* 2.6. *Assume $\bar{x}$ is a local minimizer of f. If there are $r, \delta, \kappa > 0$ and $\sigma > r$ such that for all $g \in \mathcal{C}$, $\|g\| \le \delta$,*

$$\kappa\beta(\mathbf{d}(x, M(g))) \le |\nabla(f+g)|(x)\mathbf{d}(x, M(g)), \ \ \forall x \in \mathbb{B}[\bar{x}, \sigma],$$

then $\bar{x}$ is a $(\mathcal{C}, \beta)$-weak sharp stable minimizer of f.

The last theorem provides a slope characterization of $(\mathcal{C}, \beta)$-sharp stability/tilt stability.

Theorem 3.6. *Suppose that the gauge function β verifies condition* (13) *given in Theorem* 2.6 *and that $\bar{x}$ is a local minimizer of f. If there are $\varepsilon, \kappa > 0$ such that for all $g \in \mathcal{C}$, $\|g\| \le \varepsilon$, there exists $x_g \in \mathbb{X}$ such that*

$$\kappa\beta(\mathbf{d}(x, x_g)) \le |\nabla(f+g)|(x)\mathbf{d}(x, x_g), \ \ \forall x \in \mathbb{B}[\bar{x}, \varepsilon], \tag{19}$$

then $\bar{x}$ is a $(\mathcal{C}, \beta)$-sharp stable minimizer of f. As a result, $\bar{x}$ is a $(\mathcal{C}, \beta)$-tilt stable minimizer.

Proof. In view of Theorems 2.6 and 3.4, it suffices to show that there exist $r \in]0, \varepsilon[$, $\delta \in]0, \varepsilon]$, such that for all $g \in \mathcal{C}$, $\|g\| \le \delta$, the point x_g verifying (19) is a (unique) minimum of $f + g$ over $\mathbb{B}[\bar{x}, r]$. Indeed, firstly regarding (19) with $g = 0$, one has $x_{g=0} = \bar{x}$, so by Theorem 2.3, there are $\tau > 0$, $0 < \eta < \varepsilon/2$ such that

$$\tau\beta(\mathbf{d}(x, \bar{x})) \le f(x) - f(\bar{x}), \ \ \forall x \in \mathbb{B}[\bar{x}, \eta]. \tag{20}$$

Pick $0 < \delta < \eta/2$ such that $\tau\gamma(\eta/4) > 2\delta$, and $\eta/4 < r < \eta/2$, and a sequence of positive reals $(\delta_n) \to 0$ with $\delta_n < \delta$ for all $n \in \mathbb{N}$. For $g \in \mathcal{C}$, with $\|g\| \le \delta$, pick $x_n \in \mathbb{B}[\bar{x}, r]$ such that

$$(f+g)(x_n) \le \inf_{x \in \mathbb{B}[\bar{x}, r]} f(x) + g(x) + \delta_n^2.$$

By the Ekeland variational principle, we can find $z_n \in \mathbb{B}[x_n, \delta_n] \cap \mathbb{B}[\bar{x}, r]$ such that

$$(f+g)(z_n) \le f(x) + g(x) + \delta_n \mathbf{d}(x, z_n), \ \ \forall x \in \mathbb{B}[\bar{x}, r]. \tag{21}$$

As $z_n \in \mathbb{B}[\bar{x}, r] \subseteq \mathbb{B}[\bar{x}, \eta]$, due to (20), one has

$$\tau\beta(\mathbf{d}(z_n, \bar{x})) \le f(z_n) - f(\bar{x}) \le (\delta + \delta_n)\mathbf{d}(z_n, \bar{x}).$$

Thus $\tau\gamma(\mathbf{d}(z_n, \bar{x})) \le 2\delta$, and since γ is increasing and $\tau\gamma(\eta/4) > 2\delta$, one has $\mathbf{d}(z_n, \bar{x}) < \eta/4 < r$. Thus by (21), $|\nabla(f+g)|(z_n) \le \delta_n$. Hence,

$$\kappa\beta(z_n, x_g) \le |\nabla(f+g)|(x)\mathbf{d}(z_n, x_g) \le \delta_n \mathbf{d}(z_n, x_g),$$

or $\kappa\gamma(z_n, x_g) \le \delta_n$, which implies that (z_n) converges to x_g. By (21), letting $n \to \infty$, this shows that x_g is a minimum of $f + g$ on $\mathbb{B}[\bar{x}, r]$. □

4. Concluding remarks and perspectives

The present work has aimed to study some slope characterizations of nonlinear sharp/weak minima in the metric framework. We introduced an extensive notion of tilt/weak tilt stability of local minimizers and established some characterizations of these stabilities by using the strong slope in metric spaces.

The study of the nonlinear tilt stability in relationship with the regular properties of subdifferential operators in Banach spaces and some applications to the convergence analysis of some optimization algorithms are the important issues for future researches.

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