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Critical Values of Multilinear Forms under Conic Constraints

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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Let $\Phi : \Pi_{i=1}^r E_i \to \mathbb{R}$ be a multilinear form on the Cartesian product of finitely many Euclidean vector spaces. We suppose that each factor E_i is equipped with its own closed convex cone K_i. We analyze the concept of critical point and critical value of Φ when each argument of this function is restricted to a normalization constraint and a conic constraint. Our study encompasses the theory of cone-constrained singular values of bilinear and trilinear forms.

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1. Introduction

A finite dimensional multilinear variational problem (MVP) is a variational problem involving a multilinear form $\Phi : \Pi_{i=1}^r E_i \to \mathbb{R}$ on a Cartesian product of finitely many Euclidean vector spaces. The multilinear form Φ plays the role of an objective function if the MVP under consideration is an optimization problem; think for instance of the maximization problem

$$\|\Phi\|_{\rm op} = \max_{\|x_1\|=1,\dots,\|x_r\|=1} \Phi(x_1,\dots,x_r) \tag{1}$$

that characterizes the operator norm of Φ. By positive homogeneity reasons, it is natural to restrict Φ to the multisphere $S = \Pi_{i=1}^r S_{E_i}$, where S_E denotes the unit sphere of a Euclidean vector space E. In the present work, each x_i is of unit length and satisfies a constraint of conic type. The study we are undertaking is motivated by the fact that many MVPs arising in practice can be cast in the maximization format

$$\max_{\substack{x_1\in K_1,\dots,x_r\in K_r\\ \|x_1\|=1,\dots,\|x_r\|=1}} \Phi(x_1\dots,x_r)\,, \tag{2}$$

where K_i is a nonzero closed convex cone in E_i. The decision vector x_i is taken from $K_i \cap S_{E_i}$, a set that we call a *cone-sphere intersection.* A cone-sphere intersection (or, alternatively, a spherical section of a cone) is a special type of nonconvex compact

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set. The feasible set of (2) is a Cartesian product of r cone-sphere intersections. Sometimes the MVP is cast in the minimization format

$$\min_{\substack{x_1\in K_1,\ldots,x_r\in K_r\\ \|x_1\|=1,\ldots,\|x_r\|=1}} \Phi(x_1\ldots,x_r)\,, \tag{3}$$

but, of course, one can pass from minimization to maximization, and viceversa. In order to avoid repetitions in the exposition, we give the priority to minimization problems. There are different goals for consideration in the analysis of a problem like (3). The first idea that comes to mind is to compute the minimum value of (3) and find a feasible point $x^* = (x_1^*,\ldots,x_r^*)$ attaining such a minimum value. A global solution x^* always exists, but it may not be unique. Since (3) is a nonconvex minimization problem, it may be of interest to search also for local solutions and corresponding local minimal values. More generally, it may be of interest to search for critical points and corresponding critical values. This work deals mainly with the last issue and much of our attention is devoted to the task of counting the number of critical values. The reason why we focus on the critical values of (3) is that such mathematical objects can be viewed as singular values of Φ in the presence of conic constraints.

1.1. On singular values of unconstrained multilinear forms

For the sake of the exposition, we recall some basic material of the classical theory of singular values of multilinear forms. The expression introduced in (1) is sometimes called the spectral norm of Φ, the reason for this name is that $\|\Phi\|_{\rm op}$ is equal to the largest singular value of Φ. By definition, cf. Lim [11, Section 3], a singular value of a multilinear form $\Phi:\Pi_{i=1}^r E_i\to\mathbb{R}$ is a critical value of the Rayleigh quotient

$$R_\Phi(\xi)=\frac{\Phi(\xi_1,\ldots,\xi_r)}{\|\xi_1\|\ldots\|\xi_r\|}\,.$$

Let Ξ be the open subset of $\Pi_{i=1}^r E_i$ on which the denominator of R_Φ is nonzero. Since R_Φ is differentiable on Ξ, the set of singular values of Φ is given by

$$\Sigma(\Phi)=\{R_\Phi(\xi):\,\xi\in\Xi \text{ such that } \nabla R_\Phi(\xi)=0\}\,.$$

Proposition 1.1 proposes two alternative ways of characterizing the elements of $\Sigma(\Phi)$. In what follows, $\mathbb{N}_r=\{1,\ldots,r\}$ and $\nabla_i\Phi$ is the partial gradient of Φ with respect to the i-th argument. The intrinsic gradient of Φ (relative to the multisphere S) is the map $\vec{\nabla}\Phi:S\to\Pi_{i=1}^r E_i$ given by $\vec{\nabla}\Phi(x)=\pi_x\nabla\Phi(x)$, where $\pi_x:\Pi_{i=1}^r E_i\to\Pi_{i=1}^r E_i$ is the orthogonal projector onto

$$T_x(S)=\{h\in\Pi_{i=1}^r E_i:\langle x_i,h_i\rangle=0 \text{ for all } i\in\mathbb{N}_r\},$$

the tangent space to S at x.

Proposition 1.1. *Let $\Phi:\Pi_{i=1}^r E_i\to\mathbb{R}$ be a multilinear form. For $\sigma\in\mathbb{R}$, the following conditions are equivalent:*

(a) *σ is a singular value of Φ.*

(b) *There exists a tuple $x = (x_1, \ldots, x_r)$ in S satisfying the system of equations*

$$\nabla_i \Phi(x) = \sigma x_i \quad \text{for all } i \in \mathbb{N}_r. \tag{4}$$

(c) $\sigma = \Phi(x)$ *for some* $x \in S$ *such that* $\vec{\nabla}\Phi(x) = 0$.

Proof. The case $r = 1$ is easy. Therefore we assume that $r \geq 2$. The equivalence (a) $\Leftrightarrow$ (b) is obtained by working out the criticality condition $\nabla R_\Phi(\xi) = 0$ and exploiting the multilinearity of Φ. The details are omitted. For proving the equivalence (b) $\Leftrightarrow$ (c), we rely on two fundamental facts. Firstly, since Φ is a multilinear form, each partial gradient $\nabla_i \Phi$ is well defined and

$$\langle \nabla_i \Phi(x), h_i \rangle = \Phi(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_r) \tag{5}$$

for all $h_i \in E_i$. In particular,

$$\langle \nabla_i \Phi(x), x_i \rangle = \Phi(x) \quad \text{for all } i \in \mathbb{N}_r. \tag{6}$$

And, secondly, $T_x(S) = \Pi_{i=1}^r T_{x_i}(S_{E_i})$ for all $x \in S$, where

$$T_{x_i}(S_{E_i}) \;=\; \{h_i \in E_i : \langle x_i, h_i \rangle = 0\}$$

is the tangent space to S_{E_i} at x_i. Hence, for $x \in S$, we have

$$\begin{aligned} \vec{\nabla}\Phi(x) = 0 \;&\Leftrightarrow\; \pi_{x_i} \nabla_i \Phi(x) = 0 \;\text{ for all } i \in \mathbb{N}_r \\ &\Leftrightarrow\; \nabla_i \Phi(x) \in \mathbb{R}x_i \;\text{ for all } i \in \mathbb{N}_r\,, \end{aligned} \tag{7}$$

where $\pi_{x_i} : E_i \to E_i$ is the orthogonal projector onto $T_{x_i}(S_{E_i})$. But the combination of (6) and $\|x_i\| = 1$ shows that (7) is equivalent to writing (4) with $\sigma = \Phi(x)$. $\square$

The right-hand side of (5) is usually indicated with the shorter notation $\Phi(h_i\,; x_{-i})$, where x_{-i} is obtained from x by removing the component x_i. The model (4) is illustrated below with the help of two examples.

Example 1.2. A bilinear form $\Phi : E_1 \times E_2 \to \mathbb{R}$ can be represented in terms of a linear map $A \in \mathcal{L}(E_2, E_1)$ by writing $\Phi(u, v) = \langle u, Av \rangle$. The model (4) becomes the familiar system

$$\begin{cases} Av = \sigma u, \\ A^\top u = \sigma v, \end{cases}$$

where $A^\top \in \mathcal{L}(E_1, E_2)$ is the adjoint map of A. Hence, a singular value of a bilinear form Φ is just a singular value of the associated linear map A.

Example 1.3. A trilinear form $\Phi : E_1 \times E_2 \times E_3 \to \mathbb{R}$ can be represented with a bilinear map. By the Riesz representation theorem, there are three bilinear maps $B_1 : E_2 \times E_3 \to E_1$, $B_2 : E_1 \times E_3 \to E_2$, and $B_3 : E_1 \times E_2 \to E_3$, such that

$$\Phi(u, v, w) \;=\; \langle B_1(v, w), u \rangle \;=\; \langle B_2(u, w), v \rangle \;=\; \langle B_3(u, v), w \rangle$$

for all $(u, v, w) \in E_1 \times E_2 \times E_3$.

The B_i's are called the canonical bilinear maps induced by Φ. The model (4) becomes

$$\begin{cases} B_1(v,w) - \sigma u = 0, \\ B_2(u,w) - \sigma v = 0, \\ B_3(u,v) - \sigma w = 0. \end{cases} \tag{8}$$

The singular value problem (8) must be solved in tandem with the normalization conditions

$$\|u\|^2 - 1 = 0\,,\ \|v\|^2 - 1 = 0\,,\ \|w\|^2 - 1 = 0\,. \tag{9}$$

The smooth system (8)–(9) that can be handled for instance with Newton's method.

See Bernhardsson and Peetre [3] for a different treatment of the concept of singular value of a trilinear form. Example 1.3 can be easily extended to the case of a multilinear form $\Phi : \Pi_{i=1}^r E_i \to \mathbb{R}$. For $i \in \mathbb{N}_r$, consider the smaller dimensional Cartesian product $E_{-i} = \Pi_{j\in\mathbb{N}_r\setminus\{i\}} E_j$ obtained by dropping the factor E_i. Note that, for all $x \in \Pi_{i=1}^r E_i$, one has

$$\Phi(x) = \langle B_i(x_{-i}), x_i\rangle\,,$$

where $B_i : E_{-i} \to E_i$ is the multilinear map given by $B_i(x_{-i}) = \nabla_i\Phi(x)$. As shown in (5), the vector $\nabla_i\Phi(x)$ does not depend on x_i. Hence,

$$B_i(x_{-i}) = \sigma x_i \quad \text{for all } \ i \in \mathbb{N}_r \tag{10}$$

corresponds to a more explicit way of writing (4). Lim [11] proposes several interesting variants of the concept of singular value of a multilinear form, but we shall not follow his line of thought. What we want to do in this work is to incorporate constraints of conic type into the discussion. For pedagogical reasons, we focus first on the case $r = 1$ and then we proceed with the case of a general r.

2. The cone-constrained linear case

This section concerns the case $r = 1$. Let E be a Euclidean vector space and

$$\mathcal{C}(E) = \{K \subseteq E : K \text{ is a nonzero closed convex cone}\}.$$

Consider the minimization or maximization a linear form on a cone-sphere intersection, i.e.,

$$\min\{\langle y, x\rangle : x \in K \cap S_E\}, \tag{11}$$

$$\max\{\langle y, x\rangle : x \in K \cap S_E\}, \tag{12}$$

where $y \in E$ and $K \in \mathcal{C}(E)$. Such problems arise in various fields of applied mathematics. Concrete examples can be found for instance in Ginchev et al. [7, Proposition 1], Iusem and Seeger [9, p. 263], and Tropp [19, Definition 3.1]. Two illuminating examples are mentioned below.

Example 2.1. Tanenhaus [18, p. 506] considers the problem of finding the minimal distance from a given point $y \in E$ to a cone-sphere intersection:

$$\min\{\|y - x\| : x \in K \cap S_E\}. \tag{13}$$

Tanenhaus assumes that K is polyhedral, but such an assumption is not essential. Bauschke et al. [2, Theorem 8.1] and Ferreira et al. [5, Proposition 8] discuss the metric projection problem (13) without assuming polyhedrality. By developing $\|y-x\|^2$, we see that (13) has the same solution set as (12).

Example 2.2. Let $y \in E$ be a unit vector. The optimization problems

$$\max\{\arccos\langle y, x\rangle : x \in K \cap S_E\}, \tag{14}$$

$$\min\{\arccos\langle y, x\rangle : x \in K \cap S_E\} \tag{15}$$

have a clear interpretation: The first (respectively, second) problem is about finding a unit vector in K that forms a largest (respectively, smallest) angle with respect to y. By passing to cosines in (14) and (15), one gets (11) and (12), respectively.

Our work is not about algorithmic issues. Rather than solving numerically the optimization problems (11) or (12), we are interested in analyzing their critical points and corresponding sets of critical values, i.e.,

$$\begin{aligned} \mathrm{LCV}(y, K) &= \{\langle y, x\rangle : x \text{ is a critical point of } (11)\}, \\ \mathrm{UCV}(y, K) &= \{\langle y, x\rangle : x \text{ is a critical point of } (12)\}. \end{aligned}$$

The capital letters L and U are used to distinguish between minimization (lower extremization) and maximization (upper extremization). The next definition explains what we mean exactly by a critical point of a minimization problem

$$\min\{f(x) : x \in K \cap S_E\} \tag{16}$$

whose objective function $f : E \to \mathbb{R}$ is differentiable and whose feasible set $K \cap S_E$ is a cone-sphere intersection. The abstract problem (16) has been studied in Ferreira et al. [5, 6] under the additional assumption that K is pointed, in which case $K \cap S_E$ is a spherically convex set in the traditional geodesic sense, cf. [6, Proposition 2]. Recall that a closed convex cone is pointed if it does no contain a line. In what follows,

$$K^* = \{c \in E : \langle c, x\rangle \geq 0 \text{ for all } x \in K\}$$

is the dual cone of K. The intrinsic gradient of f (relative to the sphere S_E) is the map $\vec{\nabla} f : S_E \to E$ given by $\vec{\nabla} f(x) = \pi_x \nabla f(x)$, where

$$\begin{aligned} &\pi_x : E \to E \\ &\pi_x y = y - \langle y, x\rangle x \end{aligned}$$

is the orthogonal projector onto the tangent space $T_x(S_E)$.

Definition 2.3. Let $f : E \to \mathbb{R}$ be differentiable and $K \in \mathcal{C}(E)$. A *critical point* of (16) is a point $x \in K \cap S_E$ such that $\vec{\nabla} f(x) \in K^*$. The value of f at such x is called a *critical value* of (16).

Definition 2.3 introduces a notion of criticality that relies on the intrinsic gradient of f and the dual cone of K.

Another possibility is to consider an extrinsic approach based on the usual gradient of f and the Bouligand tangent cone $T_x(K \cap S_E)$. If Ω is a closed set containing x, then the *Bouligand tangent cone* to Ω at x is given by

$$T_x(\Omega) = \{h \in E : \exists \{h_k\} \to h \text{ and } \{t_k\} \downarrow 0 \text{ with } x + t_k h_k \in \Omega \text{ for all } k\}.$$

A *Bouligand critical point* of (16) is a point $x \in K \cap S_E$ such that

$$\langle \nabla f(x), h \rangle \geq 0 \quad \text{for all } \; h \in T_x(K \cap S_E).$$

As shown in the next proposition, both definitions of criticality coincide. We start by observing that $T_x(K \cap S_E)$ is equal to the closure of the convex cone

$$\pi_x(K) = \{h - \langle h, x \rangle \, x : h \in K\}.$$

Lemma 2.4. *Let $K \in \mathcal{C}(E)$ and $x \in K \cap S_E$. Then*

$$T_x(K \cap S_E) \; = \; \mathrm{cl}[\pi_x(K)] \, . \tag{17}$$

Proof. Since a Bouligand tangent cone is closed by construction, it suffices to check that

$$\pi_x(K) \; \subseteq \; T_x(K \cap S_E) \; \subseteq \; \mathrm{cl}[\pi_x(K)]. \tag{18}$$

For proving the first inclusion, we pick $w = \pi_x h$ with $h \in K$. Since the function $u \mapsto \Psi(u) = \|u\|^{-1} u$ is differentiable at x with differential given by $\Psi'(x) = \pi_x$, we have

$$\|x + d\|^{-1}(x + d) = x + \pi_x d + \|d\| \, \varepsilon(d)$$

with $\varepsilon(d) \to 0$ as $\|d\| \to 0$. Hence,

$$x + t(w + \|h\| \varepsilon(th)) \; = \; \|x + th\|^{-1}(x + th) \; \in K \cap S_E$$

for all $t > 0$ small enough. This shows that $w \in T_x(K \cap S_E)$. For proving the second inclusion in (18), we pick $h \in T_x(K \cap S_E)$. In such a case, $h \in T_x(K)$ and $\langle x, h \rangle = 0$. There are sequences $\{h_k\} \to h$ and $\{t_k\} \downarrow 0$ with $x + t_k h_k \in K$ for all k. Since π_x is a linear map and $\pi_x \, x = 0_E$, we get $t_k \, \pi_x h_k = \pi_x(x + t_k h_k) \in \pi_x(K)$. By dividing by t_k and passing to the limit, we get $\pi_x h \in \mathrm{cl}[\pi_x(K)]$. But we know $\pi_x h = h - \langle h, x \rangle x = h$. Hence, $h \in \mathrm{cl}[\pi_x(K)]$. $\square$

Formula (17) is likely to be known, but we were unable to find a suitable reference. Our proof will be extended later to the case of a Cartesian product of several cone-sphere intersections. In what follows, the symbol $\perp$ stands for orthogonality in a corresponding Euclidean vector space.

Proposition 2.5. *Let $f : E \to \mathbb{R}$ be differentiable and $K \in \mathcal{C}(E)$. Then the following conditions are equivalent:*

(a) *x is a Bouligand critical point of* (16).

(b) *x is a critical point of* (16).

(c) *$\|x\| = 1$ and there exists $\sigma \in \mathbb{R}$ such that $K \ni x \perp (\nabla f(x) - \sigma x) \in K^*$.*

Furthermore, criticality is a necessary condition for x to be a local solution to (16).

Proof. That Bouligand criticality is necessary for local minimality is part of the folklore of optimization theory, cf. Rockafellar and Wets [12, Theorem 6.12]. Condition (b) says that

$$x \in K \cap S_E\,,\ \nabla f(x) - \langle \nabla f(x), x\rangle x \in K^*,$$

but this is clearly equivalent to (c). Indeed, the scalar σ mentioned in (c) is linked to x by means of the relation $\sigma = \langle \nabla f(x), x\rangle$. It remains to show that (a) is equivalent to (b). We assume that $x \in K \cap S_E$, otherwise both conditions are false. Since π_x is self-adjoint, a passage to dual cones in (17) yields

$$[T_x(K \cap S_E)]^* = \{y \in E : \pi_x\, y \in K^*\}.$$

This shows that (a) and (b) are equivalent. □

The next corollary is obtained from Proposition 2.5 by choosing f as a linear form.

Corollary 2.6. *Let $K \in \mathcal{C}(E)$ and $y \in E$. Then the following conditions are equivalent:*

(a) $x \in K \cap S_E$ *and* $y \in [T_x(K \cap S_E)]^*$.

(b) $x \in K \cap S_E$ *and* $y - \langle y, x\rangle x \in K^*$.

Furthermore, these equivalent conditions are necessary for x to be a local solution to (11).

If "min" is changed by "max" in (16), then the associated concept of critical point is similar as in Definition 2.3, but the dual cone K^* must be changed by the polar cone $K^\circ = -K^*$. The sets

$$\begin{aligned} \mathrm{LCV}(y, K) &= \{\langle y, x\rangle : x \in K \cap S_E\,,\ y - \langle y, x\rangle x \in K^*\}, \\ \mathrm{UCV}(y, K) &= \{\langle y, x\rangle : x \in K \cap S_E\,,\ y - \langle y, x\rangle x \in K^\circ\} \end{aligned}$$

are equal if K is a linear subspace, but they are different in general. Anyhow, one can recover the second set from the first one by using the formula

$$\mathrm{UCV}(y, K) = -\mathrm{LCV}(-y, K). \tag{19}$$

The next lemma is recorded for subsequent use. Its proof is easy and, therefore, omitted. The notation $U(Q) = \{Uz : z \in Q\}$ stands for the image of a set Q under a linear map U.

Lemma 2.7. *Let $U : X \to Y$ be a linear isometry between two Euclidean vector spaces. If Q belongs to $\mathcal{C}(X)$, then $U(Q)$ belongs to $\mathcal{C}(Y)$ and, for all $y \in Y$,*

$$\mathrm{LCV}(y, U(Q)) = \mathrm{LCV}(U^\top y, Q). \tag{20}$$

For proceeding further with the exposition, we need to introduce some notation. Let $\mathcal{F}(K)$ be the set of nonzero faces of a cone $K \in \mathcal{C}(E)$. By definition, a face of K is a closed convex cone F, subset of K, such that $u \in K$, $v - u \in K$, and $v \in F$ imply

$u \in F$. The symbol span$F$ stands for the linear span of a face F and $\Pi^F : E \to E$ is the orthogonal projector onto span$F$. As shown in the next proposition, the set

$$\Gamma(y,K) = \left\{ \pm \|\Pi^F y\| : F \in \mathcal{F}(K) \right\} \tag{21}$$

captures all the elements of LCV(y,K) and UCV(y,K). The cost of evaluating $\Gamma(y,K)$ depends essentially on the number of faces of K and our ability to identify such faces.

Proposition 2.8. *Let $K \in \mathcal{C}(E)$ and $y \in E$. Then*

$$\mathrm{LCV}(y,K) \cup \mathrm{UCV}(y,K) \;\subseteq\; \Gamma(y,K). \tag{22}$$

Proof. Let y be a nonzero vector, otherwise the result is trivial. In view of (19), it suffices to check that LCV$(y,K) \subseteq \Gamma(y,K)$. Let $\sigma \in$ LCV(y,K). Hence, $\sigma = \langle y, x\rangle$ for some $x \in S_E$ such that

$$x \in K\,,\; y - \langle y,x\rangle x \in K^*. \tag{23}$$

There exists a nonzero face $F \in \mathcal{F}(K)$ such that $x \in \mathrm{ri}(F)$, where the acronym "ri" stands for relative interior. Such a face F is unique and corresponds to the intersection of all faces of K containing x. We claim that

$$|\langle y,x\rangle| = \|\Pi^F y\|. \tag{24}$$

We divide the proof of (24) into three steps.

Step 1: Let $K = E$, in which case $K^* = \{0_E\}$ and $F = E$. Conditions (23) and (24) become

$$y - \langle y,x\rangle x \;=\; 0_E\,, \tag{25}$$

$$|\langle y,x\rangle| \;=\; \|y\|\,, \tag{26}$$

respectively. It is clear that (25) implies (26).

Step 2: We now assume that $K = L$, where L is a nonzero linear subspace of E. Since $K^* = L^\perp$ and $F = L$, conditions (23) and (24) become

$$x \in L\,,\; y - \langle y,x\rangle x \in L^\perp, \tag{27}$$

$$|\langle y,x\rangle| = \|P_L y\|\,, \tag{28}$$

respectively. Here, $P_L : E \to E$ is the orthogonal projector onto L. By the orthogonal decomposition formula $y = P_L y + P_{L^\perp} y$, conditions (27) and (28) can be written as

$$x \in L\,,\; P_L y - \langle P_L y, x\rangle x = 0_L\,,$$

$$|\langle P_L y, x\rangle| = \|P_L y\|\,,$$

respectively. So, we are back to Step I with L as underlying Euclidean vector space.

Step 3: Finally, we assume that K is not a linear subspace. Recall that F is the face associated to x and span$F$ is the linear span of F. By using (23) and a similar argument as in Seeger and Torki [16, Theorem 3.4], we get

$$x \in \mathrm{span}F\,,\;\; y - \langle y,x\rangle x \in (\mathrm{span}F)^\perp\,.$$

So, we are back to Step 2 with $L = \mathrm{span}F$. □

2.1. Counting critical values in a polyhedral setting

If $K \in \mathcal{C}(E)$ is polyhedral, then, for all $y \in E$, problem (11) has finitely many critical values. Such an observation is a straightforward consequence of Proposition 2.8. As shown in the next theorem, one can be more precise and give an upper estimate for

$$\mathfrak{a}(K) = \sup_{y \in E} \operatorname{card}[\mathrm{LCV}(y, K)]\,. \tag{29}$$

The supremum in (29), which depends only on K, can be viewed as a uniform upper bound for the number of critical values of (11). Parenthetically, formula (19) implies that $\mathfrak{a}(K)$ does not change if one writes UCV instead of LCV. By economy of language, $\mathfrak{a}(K)$ is called the *critical-value capacity* of the cone K. Roughly speaking, $\mathfrak{a}(K)$ reflects the "capacity" of K for producing critical-values. In the proof of Theorem 2.9, we partition the set of nonzero faces of K into two subsets:

$$\begin{aligned} \mathcal{F}_{\mathrm{reg}}(K) &= \{F \in \mathcal{F}(K) : F \text{ is not a linear subspace}\}, \\ \mathcal{F}_{\mathrm{deg}}(K) &= \{F \in \mathcal{F}(K) : F \text{ is a linear subspace}\}. \end{aligned}$$

Each element of the first (respectively, second) subset is called a *regular* (respectively, *degenerate*) nonzero face of K. The number of nonzero faces of K is expressible as

$$\mathfrak{n}(F) = \mathfrak{n}_{\mathrm{reg}}(K) + \mathfrak{n}_{\mathrm{deg}}(K)$$

with $\mathfrak{n}_{\mathrm{reg}}(K) = \operatorname{card}[\mathcal{F}_{\mathrm{reg}}(K)]$ and

$$\mathfrak{n}_{\mathrm{deg}}(K) = \operatorname{card}[\mathcal{F}_{\mathrm{deg}}(K)] = \begin{cases} 0 & \text{if } K \text{ is pointed}\,, \\ 1 & \text{if } K \text{ is unpointed}\,. \end{cases}$$

Theorem 2.9. *Let $K \in \mathcal{C}(E)$ be polyhedral. Then*

$$\mathfrak{a}(K) \le \mathfrak{n}_{\mathrm{reg}}(K) + 2\,\mathfrak{n}_{\mathrm{deg}}(K)\,. \tag{30}$$

Proof. The choice $y = 0_E$ is uninteresting because $\mathrm{LCV}(0_E, K) = \{0\}$. We pick then a nonzero vector $y \in E$. For each $F \in \mathcal{F}(K)$, consider the possibly empty set

$$\mathrm{LCV}^F(y, K) = \{\langle y, x\rangle : x \text{ is a critical point of (11)}, \ x \in \mathrm{ri}(F)\}.$$

Such a set gathers the critical values of (11) obtained with a critical point admitting F as associated face. Clearly,

$$\operatorname{card}[\mathrm{LCV}(y, K)] \le \sum_{F \in \mathcal{F}_{\mathrm{reg}}(K)} \operatorname{card}[\mathrm{LCV}^F(y, K)] + \sum_{F \in \mathcal{F}_{\mathrm{deg}}(K)} \operatorname{card}[\mathrm{LCV}^F(y, K)]\,.$$

Whether F is regular or not, the proof of Proposition 2.8 shows that

$$\mathrm{LCV}^F(y, K) \subseteq \left\{\pm \|\Pi^F y\|\right\}. \tag{31}$$

Hence, $\mathrm{LCV}^F(y, K)$ has two elements at most. We claim that $\operatorname{card}[\mathrm{LCV}^F(y, K)] \le 1$ for all $F \in \mathcal{F}_{\mathrm{reg}}(K)$. Suppose that for some $F \in \mathcal{F}_{\mathrm{reg}}(K)$, the set $\mathrm{LCV}^F(y, K)$

contains at least two elements, say σ_0 and σ_1. Thanks to (31), problem (11) has corresponding critical points $x_0 \in \mathrm{ri}(F)$ and $x_1 \in \mathrm{ri}(F)$ of such that

$$\begin{aligned} \langle y, x_0 \rangle &= \sigma_0 = -\|\Pi^F y\|\,, \\ \langle y, x_1 \rangle &= \sigma_1 = \|\Pi^F y\|\,. \end{aligned}$$

Note that $\Pi^F y$ is nonzero, because $\sigma_0 \neq \sigma_1$.

Since $x_0, x_1 \in \mathrm{span}F$ and $y - \Pi^F y \in (\mathrm{span}F)^\perp$, we get $\langle y - \Pi^F y, x_0 \rangle = 0$ and $\langle y - \Pi^F y, x_1 \rangle = 0$. Hence, $\langle \Pi^F y, x_0 \rangle = -\|\Pi^F y\|$ and $\langle \Pi^F y, x_1 \rangle = \|\Pi^F y\|$. In such a case, $x_0 = -\|\Pi^F y\|^{-1} \Pi^F y$ and $x_1 = \|\Pi^F y\|^{-1} \Pi^F y$ are opposite unit vectors in $\mathrm{ri}(F)$ and, therefore, $0_E = x_0 + x_1 \in \mathrm{ri}(F)$. This implies that $F = \mathrm{span}F$, contradicting the fact that F is not a linear subspace. □

In principle, counting faces is easier than counting critical values. We do not claim that (30) is always a tight upper estimate of $\mathfrak{a}(K)$. The next proposition displays however a nontrivial example of polyhedral cone for which (30) holds as an equality.

Proposition 2.10. *Let $K \in \mathcal{C}(E)$ be a simplicial cone whose generators are mutually orthogonal. Then*

$$\mathfrak{a}(K) = \mathfrak{n}(K) = 2^{\dim(E)} - 1\,. \tag{32}$$

Proof. Let $n = \dim(E)$. The second equality in (32) is true because

$$K = \mathrm{cone}\{u_1, \ldots, u_n\}$$

is a polyhedral cone generated by a basis of E. Without loss of generality, the generators u_k's are taken of unit length. If the u_k's are mutually orthogonal, then $K = U(\mathbb{R}^n_+)$ for some linear isometry $U : \mathbb{R}^n \to E$.

We must construct a vector $y \in E$ such that

$$\mathrm{card}[\mathrm{LCV}(y, K)] = 2^n - 1. \tag{33}$$

Lemma 2.7 implies that $\mathrm{LCV}(Uz, K) = \mathrm{LCV}(z, \mathbb{R}^n_+)$ for all $z \in \mathbb{R}^n$. If the entries of z are positive, then a matter of computation yields

$$\mathrm{LCV}(z, \mathbb{R}^n_+) = \{s_J : \emptyset \neq J \subseteq \mathbb{N}_n\} \tag{34}$$

with $s_J = [\sum_{j \in J} z_j^2]^{1/2}$. The set (34) is of cardinality $2^n - 1$ if the s_J's are all distinct. This happens for instance if $z = (\tau, \tau^2, \ldots, \tau^n)^\top$, where τ is a positive transcendental number. If we choose z as just mentioned, then $y = Uz$ satisfies (33). □

We do not know if the first equality in (32) is true for an arbitrary simplicial cone. Entering into such a discussion would lead us too far away from our main concern. We leave as an open question the problem of identifying classes of polyhedral cones for which (30) holds as an equality.

2.2. Beyond polyhedrality

Counting critical values in a non-polyhedral setting is more subtle. If the cone K is non-polyhedral, then it is not clear beforehand whether $\mathfrak{a}(K)$ is finite or infinite. In fact, both cases are possible. Recall that a revolution cone in E is an axially symmetric cone of the form

$$\operatorname{Rev}(\vartheta, c) = \{x \in E : (\cos\vartheta)\|x\| \leq \langle c, x\rangle\}\,, \tag{35}$$

where $c \in S_E$ determines the symmetry axis and $0 < \vartheta < \pi/2$ is the half-aperture angle. The following result is implicit in Seeger and Sossa [14, Theorem 2.1].

Proposition 2.11. *Let K be a revolution cone in a Euclidean vector space of dimension at least 3. Then $\mathfrak{a}(K) = 3$.*

A revolution cone is a non-polyhedral cone with a lot of symmetry in it. This explains somehow why its critical-value capacity is so low. The next proposition shows that $\mathfrak{a}(K)$ may be infinite if K is a non-polyhedral cone with a rather nasty structure.

Proposition 2.12. *Let E be a Euclidean vector space of dimension at least 3. Then, for all nonzero vector $y \in E$, there exists a non-polyhedral cone $K \in \mathcal{C}(E)$ such that problem (11) has infinitely many critical values.*

Proof. Let E be of dimension n. Without loss of generality, we may assume that $y \in E$ is a unit vector. We must construct $K \in \mathcal{C}(E)$ such that $\operatorname{LCV}(y, K)$ is infinite.

Step 1: As a first step, we consider the three-dimensional unit vector $c = (0, 0, 1)^\top$ and construct a cone $P \in \mathcal{C}(\mathbb{R}^3)$ such that

$$\min_{\substack{u \in P \\ \|u\|=1}} \langle c, u\rangle \tag{36}$$

has infinitely many critical values. Let $\{\theta_k\}_{k\geq 1}$ be an increasing sequence in $[0, \pi/2]$ and $\{\gamma_k\}_{k\geq 1}$ be a decreasing sequence such that $\gamma_k > 1$ for all $k \geq 1$. Both sequences are convergent of course. Let $g_* = \lim_{k\to\infty} g_k$ with

$$g_k = [1 + \gamma_k^2]^{-1/2} \left(\gamma_k \cos\theta_k\,, \gamma_k \sin\theta_k\,, 1\right)^\top.$$

The g_k's are all different and lie on the upper half of the unit sphere of $\mathbb{R}^3$. In fact,

$$\langle c, g_k\rangle = [1 + \gamma_k^2]^{-1/2} \; > 0\,. \tag{37}$$

Let P be the smallest convex cone capturing the g_k's and the limit g_*. Such a convex cone P is closed, but non-polyhedral. Indeed, the ray generated by each g_k is a one-dimensional face of P. The dual cone of P is given by

$$P^* = \{z \in \mathbb{R}^3 : \langle g_\ell, z\rangle \geq 0 \;\text{ for all } \ell \geq 1\}.$$

Hence, each g_k is a critical point of (36) if and only if

$$\langle c, g_\ell\rangle - \langle c, g_k\rangle\langle g_\ell, g_k\rangle \geq 0 \quad \text{for all } \; k, \ell \geq 1. \tag{38}$$

We now explain how to built $\{\gamma_k\}_{k\geq 1}$ so as to obtain (38). By combining (37) and

$$\langle g_\ell, g_k\rangle \;=\; \frac{\gamma_\ell\gamma_k\cos(\theta_k-\theta_\ell)+1}{[1+\gamma_\ell^2]^{1/2}\,[1+\gamma_k^2]^{1/2}}\,,$$

we see that (38) can be written as

$$\cos(\theta_k-\theta_\ell)\leq \gamma_k/\gamma_\ell \quad \text{for all } k,\ell\geq 1. \tag{39}$$

Note that $\eta_k=\cos(\theta_{k+1}-\theta_k)$ belongs to $]0,1[$ for all $k\geq 1$. Let $\{\gamma_k\}_{k\geq 1}$ be initialized with $\gamma_1>1$ and constructed recursively with the rule

$$\max\{1,\delta_k\}\;<\;\gamma_{k+1}<\gamma_k\,, \tag{40}$$

where $\delta_k=\max_{1\leq j\leq k}\gamma_j\eta_j$. By proceeding as in [10, Proposition 1], one can prove by induction that $\max\{1,\delta_k\}<\gamma_k$. Hence, choosing γ_{k+1} as in (40) is always possible. We claim that with such a construction of $\{\gamma_k\}_{k\geq 1}$, the condition (39) is true. Since $\cos(\theta_k-\theta_\ell)\leq 1\leq \gamma_k/\gamma_\ell$ for $k\leq \ell$, we may assume that $1\leq \ell<k$. But in such a case,

$$\gamma_\ell\cos(\theta_k-\theta_\ell)\leq \gamma_\ell\,\eta_\ell\leq \delta_k<\gamma_k\,,$$

as needed. We have proven that $[1+\gamma_k^2]^{-1/2}\in \mathrm{LCV}(c,P)$ for all $k\geq 1$. Since the γ_k's are different, the set $\mathrm{LCV}(c,P)$ is infinite.

Step 2: In this step we pass from $\mathbb{R}^3$ to $\mathbb{R}^n$ by forming the Cartesian product $\tilde P=P\times\{0\}^{n-3}$ and the n-dimensional unit vector $\tilde c=(c,0,\ldots,0)$. Note that

$$\mathrm{LCV}(\tilde c,\tilde P)=\mathrm{LCV}(c,P)$$

is infinite. Finally, we consider the given unit vector $y\in E$ and construct a suitable $K\in\mathcal{C}(E)$. For doing this, we pick any linear isometry $U:\mathbb{R}^n\to E$ such that $y=U\tilde c$ and set $K=U(\tilde P)$. By using Lemma 2.7, we deduce that $\mathrm{LCV}(y,K)=\mathrm{LCV}(\tilde c,\tilde P)$ is infinite. Note that K is non-polyhedral because $\tilde P$ is non-polyhedral. □

3. The cone-constrained multilinear case

This section studies the set of critical values of (3) for the case $r\geq 2$. We adopt a notion of criticality in the same vein as in Definition 2.3. The notation $\vec\nabla_i\Phi(x)$ refers to the i-th component of the intrinsic gradient $\vec\nabla\Phi(x)$. A critical point of (3) is defined as a tuple $x=(x_1,\ldots,x_r)$ in S such that

$$x_i\in K_i\,,\;\; \vec\nabla_i\Phi(x)\in K_i^* \quad \text{for all } i\in\mathbb{N}_r. \tag{41}$$

The set of critical values of (3) is given by

$$\mathrm{LCV}(\Phi,K_1,\ldots,K_r)=\{\Phi(x): x \text{ is a critical point of } (3)\}.$$

The above set is nonempty and compact. By combining (5) and the fact that

$$\langle\vec\nabla_i\Phi(x),h_i\,\rangle\;=\;\langle\nabla_i\Phi(x),h_i\rangle-\langle\nabla_i\Phi(x),x_i\rangle\,\langle x_i,h_i\rangle$$

for all $h_i \in E_i$, we see that (41) amounts to saying that

$$\begin{cases} x_i \in K_i\,, \\ \Phi(h_i\,; x_{-i}) - \Phi(x)\langle x_i, h_i\rangle \geq 0 \quad \text{for all } \ h_i \in K_i \end{cases} \tag{42}$$

for all $i \in \mathbb{N}_r$. Since K_i is a cone, the variational inequality (42) can be written as a complementarity problem. This observation is further clarified in Proposition 3.1. An alternative way of defining a critical point of (3) is by relying on the usual gradient $\nabla\Phi(x)$ and the Bouligand tangent cone to

$$\Omega(K_1, \ldots, K_r) \ = \ \Pi_{i=1}^r (K_i \cap S_{E_i}). \tag{43}$$

A *Bouligand critical point* of (3) is a point $x \in \Omega(K_1, \ldots, K_r)$ such that

$$\langle \nabla\Phi(x), h\rangle \geq 0 \quad \text{for all } \ h \in T_x(\Omega(K_1, \ldots, K_r)). \tag{44}$$

The next result is an extension of Proposition 2.5.

Proposition 3.1. *For $x = (x_1, \ldots, x_r)$ in S, the following conditions are equivalent:*

(a) *x is a Bouligand critical point of* (3).

(b) *x is a critical point of* (3).

(c) *There exists $\sigma \in \mathbb{R}$ such that*

$$K_i \ni x_i \perp (\nabla_i\Phi(x) - \sigma x_i) \in K_i^* \quad \textit{for all } \ i \in \mathbb{N}_r. \tag{45}$$

Furthermore, criticality is a necessary condition for x to be a local solution to (3).

Proof. We just show that (a) $\Leftrightarrow$ (b), because everything else is clear. We claim that

$$T_x(\Omega(K_1, \ldots, K_r)) = \Pi_{i=1}^r \mathrm{cl}[\pi_{x_i}(K_i)]\,. \tag{46}$$

We start by proving the inclusion $\Pi_{i=1}^r W_i \subseteq T_x(\Omega)$, where $\Omega = \Omega(K_1, \ldots, K_r)$ and $W_i = \pi_{x_i}(K_i)$. Let $w = (w_1, \ldots, w_r) \in \Pi_{i=1}^r W_i$. For each $i \in \mathbb{N}_r$, there exists a vector $h_i \in K_i$ such that $w_i = \pi_{x_i} h_i$. Furthermore, for all $t > 0$ small enough, we can write

$$x_i + t(w_i + \|h_i\|\varepsilon_i(th_i)) \ = \ \|x_i + th_i\|^{-1}(x_i + th_i) \ \in K_i \cap S_{E_i}$$

with $\varepsilon_i(d_i) \to 0$ as $\|d_i\| \to 0$. Hence, $x + t(w + \varphi(t)) \in \Omega$, where

$$\varphi(t) = (\|h_1\|\varepsilon(th_1), \ldots, \|h_r\|\varepsilon(th_r))$$

is a vector in $\Pi_{i=1}^r E_i$ converging to zero as $t \downarrow 0$. This proves that $w \in T_x(\Omega)$. On the other hand, we get $T_x(\Omega) \subseteq \Pi_{i=1}^r \mathrm{cl}(W_i)$ by combining the obvious inclusion

$$T_x(\Pi_{i=1}^r (K_i \cap S_{E_i})) \ \subseteq \ \Pi_{i=1}^r T_x(K_i \cap S_{E_i})$$

and formula (17). Since each π_{x_i} is self-adjoint, a passage to dual cones in (46) yields

$$[T_x(\Omega(K_1, \ldots, K_r))]^* = \{y \in \Pi_{i=1}^r E_i : \pi_{x_i}\, y_i \in K_i^* \ \text{ for all } i \in \mathbb{N}_r\}.$$

This shows that (a) and (b) are equivalent. $\square$

The model (45) corresponds to a system of complementary problems whose unknown is a pair (x,σ) formed by a real number σ and a point x in the multisphere S. Unless Φ and the K_i's have a very simple structure, the analysis of (45) offers a number of mathematical challenges, both from a theoretical and numerical point of view. To start with, consider the minimization problem

$$\min_{\|x_1\|=1,\dots,\|x_r\|=1} \Phi(x_1,\dots,x_r)\,. \tag{47}$$

Already with $r=3$, solving (47) is far from being a trivial matter. Except for the normalization of each x_i, this problem is unconstrained in the sense that $K_i=E_i$ for all $i\in\mathbb{N}_r$. In such a context, the model (45) reduces to (4) and, therefore, the critical values of (47) are just the singular values of Φ.

The minimization problem (47) and the maximization counterpart (1) have the same critical values, namely, the singular values of Φ. In the literature devoted to multilinear spectral analysis, it is usually the maximization problem that is considered. Most authors work with a multilinear form $\Phi_T:\Pi_{i=1}^r\mathbb{R}^{d_i}\to\mathbb{R}$ induced by a tensor $T=(T_{k_1,\dots,k_r})$ of order r and size $d_1\times\ldots\times d_r$, i.e.,

$$\Phi_T(x)=\sum_{k_1=1}^{d_1}\ldots\sum_{k_r=1}^{d_r} T_{k_1,\dots,k_r}\, x_1^{k_1}\ldots x_r^{k_r}\,. \tag{48}$$

Here, the components of $x_i\in\mathbb{R}^{d_i}$ are indexed with the superscript $k_i\in\{1,\dots,d_i\}$. The maximization of (48) on the multisphere $\Pi_{i=1}^r S_{\mathbb{R}^{d_i}}$ is considered for instance in He et al. [8] and So [17]. It is asserted in [8, Proposition 3.2] that such a problem is NP-hard. Two important cone-constrained examples are presented below.

Example 3.2. As said before, a bilinear form $\Phi:E_1\times E_2\to\mathbb{R}$ can be represented in terms of a linear map $A\in\mathcal{L}(E_2,E_1)$ by writing $\Phi(u,v)=\langle u,Av\rangle$. Consider the minimization problem

$$\min_{\substack{u\in P,v\in Q\\ \|u\|=1,\,\|v\|=1}} \langle v,Au\rangle\,, \tag{49}$$

where $P\in\mathcal{C}(E_1)$ and $Q\in\mathcal{C}(E_2)$ are given cones. Seeger and Sossa [15] provide a long list of concrete problems that can be cast in the format (49). The model (45) takes here the special form

$$\begin{cases} P\ni u\perp(A\,v-\sigma u)\ \in P^*,\\ Q\ni v\perp(A^\top u-\sigma v)\in Q^*.\end{cases} \tag{50}$$

This couple of complementarity problems is at the origin of the theory of cone-constrained singular values of linear maps. In the terminology of [15], a *lower singular value* of A relative to (P,Q) is a real number σ such that (50) holds for some pair (u,v) of unit vectors. An *upper singular value* of A relative to (P,Q) is defined in a similar way: "min" is changed by "max" and, consistently, dual cones are changed by polar cones.

Example 3.3. Consider next the case of a minimization problem

$$\min_{\substack{u\in P,\ v\in Q,\ w\in R\\ \|u\|=1,\ \|v\|=1,\ \|w\|=1}} \Phi(u,v,w) \tag{51}$$

involving a trilinear form $\Phi : E_1 \times E_2 \times E_3 \to \mathbb{R}$ and cones $P \in \mathcal{C}(E_1)$, $Q \in \mathcal{C}(E_2)$, and $R \in \mathcal{C}(E_3)$. For finding the critical values of (51), we must solve the system of complementarity problems

$$\begin{cases} P \ni u \perp (B_1(v,w) - \sigma u) \in P^*, \\ Q \ni u \perp (B_2(u,w) - \sigma v) \in Q^*, \\ R \ni u \perp (B_3(u,v) - \sigma w) \in R^* \end{cases} \tag{52}$$

under the normalization conditions mentioned in (9). Here, B_1, B_2, and B_3, are the canonical bilinear maps induced by Φ, cf. Example 1.3. To the best of our knowledge, the cone-constrained trilinear case has not been treated in the literature. Neither Lim [11] nor Bernhardsson and Peetre [3] consider the possibility of handling constraints of conic type.

Example 3.3 can be extended from a trilinear form to a r-linear form $\Phi : \Pi_{i=1}^r E_i \to \mathbb{R}$. The obvious way of writing (52) is

$$K_i \ni x_i \perp (B_i(x_{-i}) - \sigma x_i) \in K_i^* \quad \text{for all } i \in \mathbb{N}_r, \tag{53}$$

where the B_i's are the $(r-1)$-linear maps induced by Φ.

Remark 3.4. From an algorithmic point of view, the handling of (53) is a difficult matter in general, and specially so if the K_i's are unstructured. A favorable situation occurs when, for each $i \in \mathbb{N}_r$, one can construct an easily computable complementarity function $\Upsilon_i : E_i \times E_i \to E_i$ for K_i. The reason is that Υ_i allows to reformulate (53) in the form

$$\Upsilon_i(x_i, B_i(x_{-i}) - \sigma x_i) = 0 \quad \text{for all } i \in \mathbb{N}_r\,, \tag{54}$$

and such a system of equations can be treated with a nonsmooth version of Newton's method, cf. Facchinei and Pang [4] for the general theory of complementarity functions.

We would like to point out that the number of critical values of the minimization problem (3) could be countable infinite or even uncountable. This fact is illustrated with the next example.

Example 3.5. Consider the particular case

$$E_1 = E_2 = \mathbb{R}^3\,, \;\; E_3 = \ldots = E_r = \mathbb{R}\,,$$
$$K_1 = K_2 = K\,, \;\; K_3 = \ldots = K_r = \mathbb{R}_+,$$
$$\Phi(x) = \langle x_1, x_2 \rangle (x_3 \ldots x_r),$$

where K is a nonzero closed convex cone in $\mathbb{R}^3$. Note that any feasible point x of problem (3) satisfies $x_3 = \ldots = x_r = 1$. Hence, problem (3) reduces to

$$\min_{\substack{x_1, x_2 \in K \\ \|x_1\|=1,\, \|x_2\|=1}} \langle x_1, x_2 \rangle\,,$$

i.e., one searches for a pair of unit vectors in K that forms a largest possible angle. If K has countably infinite (respectively, uncountably many) critical angles, then (3) has countably infinite (respectively, uncountably many) critical values. See Iusem and Seeger [10] for the construction of both types of cone K and for the definition of a critical angle of a cone.

3.1. A partially constrained trilinear variational problem

We present as a toy example a MVP in which some decision vectors are cone-constrained and some others are free. The criticality conditions are expressed in terms of a hybrid system involving equations and complementarity problems. Let the space $\mathbb{M}_{m,n}$ of real matrices of size $m \times n$ be equipped with the Frobenius inner product. The operator norm of $M \in \mathbb{M}_{m,n}$ is defined as

$$\|M\|_{\mathrm{op}} \;=\; \max_{\|v\|=1} \|Mv\| \,.$$

This norm is different of course from the Frobenius norm. Given $A_1, \ldots, A_s \in \mathbb{M}_{m,n}$, we wish to find a nonnegative unit vector $w \in \mathbb{R}^s$ such that the operator norm of the linear combination $A(w) = \sum_{k=1}^{s} w_k A_k$ is as large as possible. The maximization problem under consideration is

$$\max_{\substack{\|w\|=1\\ w\geq 0}} \|A(w)\|_{\mathrm{op}} = \max_{\substack{\|w\|=1\\ w\geq 0}} \max_{\|u\|=1,\, \|v\|=1} \langle u, A(w)v\rangle = \max_{\substack{\|u\|=1,\, \|v\|=1\\ \|w\|=1,\, w\geq 0}} \sum_{k=1}^{s} w_k \langle u, A_k v\rangle \,.$$

The later formulation can be converted into a minimization problem

$$\min_{\substack{\|u\|=1,\, \|v\|=1\\ \|w\|=1,\, w\geq 0}} \underbrace{\sum_{k=1}^{s} w_k \langle u, C_k v\rangle}_{\Phi(u,v,w)} \tag{55}$$

with $C_k = -A_k$. We are in the context of (51) with $E_1 = \mathbb{R}^m$, $E_2 = \mathbb{R}^n$, $E_3 = \mathbb{R}^s$, $P = \mathbb{R}^m$, $Q = \mathbb{R}^m$, and $R = \mathbb{R}^s_+$. By taking partial gradients of Φ, we get

$$\begin{aligned}
\nabla_1 \Phi(u,v,w) = B_1(v,w) \; &= \; \sum_{k=1}^{s} w_k C_k v \,,\\
\nabla_2 \Phi(u,v,w) = B_2(u,w) \; &= \; \sum_{k=1}^{s} w_k C_k^{\top} u \,,\\
\nabla_3 \Phi(u,v,w) = B_3(u,v) \; &= \; (\langle u, C_1 v\rangle, \ldots, \langle u, C_s v\rangle)^{\top} \,.
\end{aligned}$$

Since u and v are not cone-constrained, the first two complementarity problems in (52) yield a system of equations, namely,

$$\begin{cases} \sum_{k=1}^{s} w_k C_k v - \sigma u \;= 0, \\ \sum_{k=1}^{s} w_k C_k^{\top} u - \sigma v = 0. \end{cases} \tag{56}$$

Note that (56) is a classical singular value problem for a matrix $\sum_{k=1}^{s} w_k C_k$ depending on a vector parameter w.

On the other hand, the last complementarity problem in (52) becomes

$$\text{for } k = 1,\dots,s \qquad \begin{cases} w_k \geq 0, \\ \langle u, C_k v\rangle - \sigma w_k \geq 0 \\ w_k(\langle u, C_k v\rangle - \sigma w_k) = 0. \end{cases} \tag{57}$$

The hybrid system (56)–(57) must be solved under the normalization conditions mentioned in (9). We make use of the Fischer-Burmeister complementarity function $\Upsilon(a,b) = a + b - \sqrt{a^2+b^2}$, which allows us to reformulate (57) as a system of equations, namely,

$$\Upsilon(w_k, \langle u, C_k v\rangle - \sigma w_k) = 0 \qquad \text{for } k = 1,\dots,s. \tag{58}$$

The function Υ is not differentiable, but it is strongly semismooth. Hence, the equations listed in (56), (58), and (9), can be handled with a semismooth version of Newton's method. See Adly and Seeger [1] for the implementation of such an approach in the context of a cone-constrained eigenvalue problem. For the sake of bibliographical completeness, we mention that (55) can also be written in tensor format

$$\min_{\substack{\|u\|=1,\, \|v\|=1 \\ \|w\|=1,\, w\geq 0}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} T_{i,j,k} u_i v_j w_k\,, \tag{59}$$

where $T_{i,j,k}$ is the (i,j)-entry of the matrix C_k. See Zhang et al. [20, Section 4] for a discussion on algorithmic aspects concerning problem (59).

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