Journal of Convex Analysis
Volume 31 (2024), No. 4, 1245–1272

Compactly Locally Uniformly Convex Functions

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

Received: February 21, 2024
Accepted: March 13, 2024

We introduce compactly locally uniformly convex functions and present their properties, as well as their relations with other classes of convex functions.

Keywords: Locally uniformly convex function, compactly locally uniformly convex function, nearly strongly convex function, locally nearly uniformly convex function, locally nearly uniformly smooth function.

2010 Mathematics Subject Classification: Primary 26B25, 46G05, 49N15, 58C20; secondary 46B20, 54C50.

1. Introduction

Locally uniformly convex functions, which are analogous to locally uniformly convex norms, are known in the literature (see [8, 27]). A notion called compactly locally uniformly rotund, which is a generalization of local uniform convexity (or rotundity), is introduced by Vlasov ([24]) in 1967 and this notion has been well studied (see [10, 11, 13, 18, 19, 22, 25]):

Definition 1.1. A Banach space X is said to be *CLUR (compactly locally uniformly rotund)* if (x_n) has a convergent subsequence whenever $x, x_n \in X$, $\|x_n\| = \|x\| = 1$ and $\|x_n + x\| \to 2$. $\square$

In this paper, we introduce compactly locally uniformly convex functions which are analogous to compactly locally uniformly convex norms and study their properties.

It is known that uniformly convex functions (see [1, 6, 8, 26, 27]) have been characterized in terms of smoothness properties of their conjugates. In Section 3, compactly locally uniformly convex functions and locally uniformly convex functions are characterized in terms of sub-differentials of their conjugates. These characterizations are used in Sections 4 and 5. Characterizations of compactly locally uniformly convex functions in terms of some convexity properties of functions are presented in Section 4. Relationship between compactly locally uniformly convex functions and

ISSN 0944-6532 / \$ 2.50

various other properties of convex functions are investigated in Section 5. Most of the properties and results which are presented in this paper have their counterparts in the geometry of Banach spaces (see [2, 3, 5, 13, 14, 15, 16, 17, 18, 22]).

2. Preliminaries

Throughout the paper we assume that X is a nontrivial real Banach space. The *topological dual* X^* of X is endowed with its dual norm $\|x^*\| := \sup\{x^*(x) \mid \|x\| \le 1\}$; the *bidual* of X is $X^{**} := (X^*)^*$. It is well known that X^* is a Banach space. The *weak* topology of X (resp. X^*) is $w := \sigma(X, X^*)$ (resp. $w := \sigma(X^*, X^{**})$), while the *weak-star* topology of X^* (resp. X^{**}) is $w^* := \sigma(X^*, X)$ (resp. $w^* := \sigma(X^{**}, X^*)$); hence, on X^* and X^{**} one has three topologies, namely $\tau_{\|\cdot\|}$ (strong or norm), w (weak) and w^* (weak*) with $w \subset w^* \subset \tau_{\|\cdot\|}$. Of course $w = w^*$ if and only if (iff for short) X is reflexive and $w = \tau_{\|\cdot\|}$ iff $\dim X < \infty$. For $x \in X$, $x^* \in X^*$ (and $x^{**} \in X^{**}$) we set $\langle x, x^* \rangle := x^*(x)$ (and $\langle x^*, x^{**} \rangle := x^{**}(x^*)$). The *open unit ball*, the *closed unit ball* and the *unit sphere* of X are

$$B_X := \{x \in X \mid \|x\| < 1\}, \quad U_X := \{x \in X \mid \|x\| \le 1\}, \quad S_X := \{x \in X \mid \|x\| = 1\},$$

respectively. It is also well known that $F_X : X \to X^{**}$ defined by $F_X(x) := x^{**}$, where $x^{**}(x^*) := \langle x, x^* \rangle$ for $x^* \in X^*$ (and $x \in X$), is a linear isometry (of course, by definition, X is *reflexive* when F_X is surjective). In the sequel, we identify $x \in X$ with $F_X(x)$ and X with $F_X(X)$, when $\langle x^*, x \rangle := \langle x^*, F_X(x) \rangle$ for $x \in X$ and $x^* \in X^*$; hence X *is a closed* (because X is complete) *linear subspace of* X^{**}. The *closure* of $A \subset X$ with respect to (wrt for short) the norm topology is denoted by $\overline{A}$; for $A \subset X$ or $A \subset X^{**}$, $\operatorname{cl} A$ denotes the (norm) $\|\cdot\|$-*closure* of A in X^{**}, $\operatorname{cl}_{w^*} A$ denotes the w^*-*closure* of A in X^{**} and $\operatorname{diam} A$ the *diameter* of A; clearly, $\operatorname{diam} A = \operatorname{diam} \overline{A}$. Having $\emptyset \ne A, B \subset X$ and $x \in X$, we set $A + B := \{a + b \mid a \in A,\, b \in B\}$, $A + \emptyset := \emptyset$ and $x + A := \{x\} + A$; then, for $\emptyset \ne \Gamma \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$, $\Gamma A := \{\gamma a \mid \gamma \in \Gamma,\, a \in A\}$ and $\alpha A := \{\alpha\} A$.

The *domain* of the function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ is the set defined by $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$; f is *proper* if $\operatorname{dom} f \ne \emptyset$ and $f(x) \ne -\infty$ for all $x \in X$; f is *convex* if its *epigraph* $\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \le t\}$ is convex. The set of all proper convex functions on X is denoted by $\Lambda(X)$, while the set of those $f \in \Lambda(X)$ which are also lower semi-continuous (lsc for short) on X is denoted by $\Gamma(X)$, $\overline{\mathbb{R}}$ being endowed with its usual topology.

Having $f \in \Lambda(X)$, $\overline{x} \in \operatorname{dom} f$ and $u \in X$, the mapping

$$]0, \infty[=: \mathbb{P} \ni t \mapsto t^{-1}\big[f(\overline{x} + tu) - f(\overline{x})\big] \in \mathbb{R} \cup \{\infty\}$$

is nondecreasing, and so

$$f'_+(\overline{x}, u) := \lim_{t \downarrow 0} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \inf_{t \in]0, \delta[} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \in \overline{\mathbb{R}} \quad \forall \delta \in]0, \infty]; \qquad (1)$$

$f'_+(\overline{x}, \cdot)$ is called the *directional derivative* of f at $\overline{x}$.

Clearly, $\operatorname{dom} f'_+(\overline{x}, \cdot) = \mathbb{R}_+ (\operatorname{dom} f - \overline{x})$, where $\mathbb{R}_+ := [0, \infty[$. It is well known (see, e.g., [27, Th. 2.1.13]) that $f'_+(\overline{x}, \cdot)$ is a sublinear (that is, subadditive, positively homogeneous and 0 at $0 \in X$) function for $f \in \Lambda(X)$ and $\overline{x} \in \operatorname{dom} f$; hence $f'_+(\overline{x}, \cdot)$ is convex.

The function $f \in \Lambda(X)$ is *directionally Fréchet differentiable at* $\overline{x} \in \operatorname{dom} f$ if $f'_+(\overline{x}, u) \in \mathbb{R}$ for every $u \in X$ and the limit in (1) is uniform on S_X; clearly, in such a case, $\overline{x} \in \operatorname{int}(\operatorname{dom} f)$.

The *conjugate* of $f : X \to \overline{\mathbb{R}}$ is the function

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) \mid x \in \operatorname{dom} f\} = -\inf_X (f - x^*)$$

with $\sup \emptyset := -\infty$; clearly, f^* is convex and w^*-lsc. It is known (see, e.g., [9, Lemma 4.1]) that $f^* \in \Gamma(X^*)$, and $f^{**}|_X = f$ if $f \in \Gamma(X)$, where $f^{**} := (f^*)^*$. For $\varepsilon \in \mathbb{R}_+$, the ε-*subdifferential* of $f : X \to \overline{\mathbb{R}}$ *at* $x \in X$ is defined by

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid \forall x' \in X : \langle x' - x, x^* \rangle \leq f(x') - f(x) + \varepsilon\}$$

if $f(x) \in \mathbb{R}$ and $\partial_\varepsilon f(x) := \emptyset$ otherwise; $\partial_\varepsilon f(x)$ is a convex and w^*-closed subset of X^*, and so $\partial_\varepsilon f(x)$ is $\|\cdot\|$-closed. Moreover,

$$0 \leq \delta < \varepsilon < \infty \Rightarrow \partial_\delta f(x) \subset \partial_\varepsilon f(x) \text{ and } \textstyle\bigcap_{\varepsilon > \delta} \partial_\varepsilon f(x) = \partial_\delta f(x). \tag{2}$$

The set $\partial f(x) := \partial_0 f(x)$ is called the *subdifferential* of f *at* x. It is known (see, e.g., [27, Th. 2.4.4(iii)]) that $\partial_\varepsilon f(x) \neq \emptyset$ for $\varepsilon \in \mathbb{P}$ and $x^* \in \partial_\varepsilon f(x) \Leftrightarrow x \in \partial_\varepsilon f^*(x^*)$ for every $\varepsilon \in \mathbb{R}_+$ whenever $f \in \Gamma(X)$ and $x \in \operatorname{dom} f$. Clearly, for $x \in X$ with $f(x) \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$ one has

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x, x^* \rangle + \varepsilon, \quad x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x, x^* \rangle .$$

The *indicator function* $\iota_E : Y \to \overline{\mathbb{R}}$ of $E \subset Y$ $(\neq \emptyset)$ is defined by $\iota_E(y) := 0$ if $y \in E$ and $\iota_E(y) := \infty$ if $y \in Y \setminus E$; clearly, for $A \subset X$, ι_A is convex (resp., lsc) iff A is convex (resp., closed). The *normal cone* of the convex set $A \subset X$ at $x \in A$ is the set $N_A(x) := \partial \iota_A(x)$.

Having the proper function $f : X \to \overline{\mathbb{R}}$, it is clear that

$$\begin{aligned} 0 \in \partial f(x_0) \Leftrightarrow x_0 \in \arg\min f := \{x \in X \mid \forall x' \in X : f(x) \leq f(x')\}, \\ x^* \in \partial f(x_0) \Leftrightarrow x_0 \in \arg\min (f - x^*). \end{aligned} \tag{3}$$

It is obvious that $\arg\min f$ is convex if $f \in \Lambda(X)$; moreover, $\arg\min f$ is at most a singleton if f is *strictly convex*, that is, $f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$ for all $x, x' \in \operatorname{dom} f$ with $x \neq x'$ and $\lambda \in]0, 1[$.

Clearly, $\partial_\varepsilon f^*(x^*)$ is a w^*-closed and convex subset of X^{**}; in particular, $\partial_\varepsilon f^*(x^*)$ is $\|\cdot\|$-closed. One sets

$$\partial_\varepsilon^X f^*(x^*) := X \cap \partial_\varepsilon f^*(x^*), \quad \partial^X f^*(x^*) := \partial_0^X f^*(x^*) = \arg\min (f - x^*); \tag{4}$$

of course, $\partial_\varepsilon^X f^*(x^*)$ is a $\|\cdot\|$-closed and convex subset of X. Moreover, $\partial_\varepsilon^X f^*(x^*) \neq \emptyset$ for $f \in \Gamma(X)$, $x^* \in \operatorname{dom} f^*$ and $\varepsilon \in \mathbb{P}$, but one might have $\partial^X f^*(x^*) = \emptyset \neq \partial f^*(x^*)$.

We recall the definition of the measure of non-compactness which is used in the sequel.

For $\emptyset \neq A \subset X$ and $\varepsilon > 0$, one sets $B_\varepsilon(A) := \{x \in X \mid d(x, A) < \varepsilon\}$ $(= A + \varepsilon B_X)$, where $d(x, A) := \inf\{\|x - a\| \mid a \in A\}$. The *Hausdorff index* or *measure of non-compactness* of $\emptyset \neq A \subset X$ is

$$\alpha(A) := \inf\{\delta > 0 \mid \exists F \in \mathcal{F}(X) : A \subset B_\delta(F)\} \in [0, \infty],$$

where $\mathcal{F}(X) := \{F \mid \emptyset \neq F \subset X,\ \operatorname{card} F < \infty\}$. In the next result we recall some (easy to prove) properties of the mapping α (see, e.g., [4, Lems. 5.9, 5.10]).

Proposition 2.1. *Let A, B be nonempty subsets of X and $x \in X$. Then the following assertions hold:* (i) $\alpha(A) \leq \operatorname{diam} A$; (ii) $\alpha(A) < \infty \Leftrightarrow A$ *is bounded;* (iii) $\alpha(A) = \alpha(\overline{A})$; (iv) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$; (v) $\max\{\alpha(A), \alpha(B)\} \leq \alpha(A+B) \leq \alpha(A) + \alpha(B)$; (vi) $\alpha(x + A) = \alpha(A)$.

In the sequel, all the sequences are indexed by $\mathbb{N}^*$, the set of positive integer; having a sequence (x_n), (x_{n_k}) represents a subsequence of (x_n), that is, $n_k < n_{k+1}$ for all $k \in \mathbb{N}^*$. The norm convergence of a sequence is denoted by "$\to$", while the weak and weak* convergences are denoted by "$\to^w$" and "$\to^{w^*}$", respectively.

Having a sequence (A_n) of nonempty subsets of X and $\emptyset \neq A \subset X$, $A_n \to^{H^+} A$ if for every $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that $A_n \subset B_\varepsilon(A)$ for all $n \geq n_\varepsilon$ (H^+ stands for upper Hausdorff); observe that $A_n \to^{H^+} A \Leftrightarrow \overline{A_n} \to^{H^+} A \Leftrightarrow A_n \to^{H^+} \overline{A}$. By $(a_n) \subset (A_n)$ one means that $a_n \in A_n$ for all $n \in \mathbb{N}^*$.

The following result is an immediate consequence of [9, Prop. 2.2, Cor. 2.3]), the above observation on H^+ and Proposition 2.1.

Proposition 2.2. *Let (C_n) be a sequence of nonempty subsets of X satisfying the condition $C_{n+1} \subset C_n$ for all n and let $C := \cap_{n=1}^\infty \overline{C_n}$. Consider the following conditions:*

(i) *every sequence $(x_n) \subset (C_n)$ has a convergent subsequence;*

(ii) *C is nonempty, compact and $C_n \to^{H^+} C$;*

(iii) *there exists a nonempty compact set $E \subset X$ such that $C_n \to^{H^+} E$;*

(iv) $\alpha(C_n) \to 0$;

(v) $\operatorname{diam} C_n \to 0$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (v). *If C is a singleton, then* (iv) $\Leftrightarrow$ (v).

3. Characterizations of CLUR functions in terms of subdifferentials

The definition of local uniform convexity at a point of a convex function is known; we introduce the notion of compactly locally uniformly convex function as a generalization of such functions.

Definition 3.1. Consider $f \in \Lambda(X)$ and $x_0 \in \operatorname{dom} f$. The function f is said to be:

(i) *locally uniformly convex* or *locally uniformly rotund* (*LUR* for short) at x_0 if $x_n \to x_0$ whenever $(x_n) \subset \operatorname{dom} f$ and

$$\tfrac{1}{2} f(x_n) + \tfrac{1}{2} f(x_0) - f(\tfrac{1}{2} x_n + \tfrac{1}{2} x_0) \to 0; \tag{5}$$

(ii) *compactly locally uniformly convex* or *compactly locally uniformly rotund* (*CLUR* for short) at x_0 if the sequence (x_n) has a convergent subsequence whenever $(x_n) \subset \operatorname{dom} f$ and (5) holds;

(iii) f is *LUR* (resp., *CLUR*) if f is LUR (resp., CLUR) at each point of $\operatorname{dom} f$. □

The above definition of a LUR function at a point is given in [8, p. 239]; notice that f is LUR at x_0 iff f is uniformly convex at x_0 in the sense of [26, Def. 2.1] (see also [27, p. 201]).

Remark 3.2. (i) Notice that f is LUR at x_0 if and only if the next condition holds: (i′) any sequence $(x_n) \subset \operatorname{dom} f$ verifying (5) has a subsequence (x_{n_k}) such that $x_{n_k} \to x_0$. Indeed, assume that (i′) holds, but f is not LUR at x_0; then there exists a sequence $(x_n) \subset \operatorname{dom} f$ verifying (5) such that $x_n \not\to x_0$. Hence, there exists $\varepsilon > 0$ and an increasing sequence $(n_k) \subset \mathbb{N}^*$ such that $\|x_{n_k} - x_0\| \geq \varepsilon$ for all $k \geq 1$. Consider the sequence (x'_k) defined by $x'_k := x_{n_k}$ for $k \geq 1$; clearly (x'_k) verifies (5), and so, by (i′), it has a subsequence (x'_{k_p}) converging to x_0, a contradiction because $\left\|x_{n_{k_p}} - x_0\right\| \geq \varepsilon$ for $p \geq 1$.

(ii) It is worth observing that f is LUR at x_0 if and only if $x_n \to x_0$ whenever $(x_n) \subset \operatorname{dom} f$ is bounded and (5) holds (see [8, 5.3.1, p. 246]). □

Remark 3.3. Notice that a necessary condition for f to be CLUR at x_0 is that we have for all $(x_n) \subset \operatorname{dom} f$:

$$\delta_n := \tfrac{1}{2} f(x_n) + \tfrac{1}{2} f(x_0) - f(\tfrac{1}{2} x_n + \tfrac{1}{2} x_0) \to 0 \quad \Rightarrow \quad (x_n) \text{ is bounded;} \tag{6}$$

in particular, if $f : X \to \overline{\mathbb{R}}$ is sublinear and $\operatorname{dom} f \neq \{0\}$, then f is not CLUR at each $x \in \operatorname{dom} f$. Indeed if there exists an unbounded sequence $(x_n) \subset \operatorname{dom} f$ such that $\delta_n \to 0$, then there is a subsequence (x_{n_k}) of (x_n) with $\|x_{n_k}\| \to \infty$. Taking $x'_k := x_{n_k}$, one has $\delta'_k := \delta_{n_k} \to 0$, and so (x'_k) verifies (5), but it has not any convergent subsequence. □

We will see that f is LUR iff it is strictly convex and CLUR when f is a continuous function on $\operatorname{dom} f$. In this section, we characterize LUR and CLUR functions in terms of subdifferentials of their conjugates.

The function $f \in \Lambda(X)$ is *strictly convex at* $x_0 \in \operatorname{dom} f$ if

$$f(\tfrac{1}{2} x_0 + \tfrac{1}{2} x) < \tfrac{1}{2} f(x_0) + \tfrac{1}{2} f(x)$$

for all $x \in \operatorname{dom} f \setminus \{x_0\}$ (see [8, p. 239]). Hence, if f is not strictly convex at x_0 then there exists $x_1 \in \operatorname{dom} f \setminus \{x_0\}$ such that $f(\frac{1}{2} x_0 + \frac{1}{2} x_1) = \frac{1}{2} f(x_0) + \frac{1}{2} f(x_1)$; then $f(x_\lambda) = (1-\lambda) f(x_0) + \lambda f(x_1)$ for all $\lambda \in [0,1]$, where $x_\lambda := (1-\lambda) x_0 + \lambda x_1$, by [27, Th. 2.1.5 (i)] applied for $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$, $\varphi(t) := f\big((1-t)x_0 + t x_1\big)$ and $0 =: t_1 < t_2 := 1$. Hence f is strictly convex at $x_0 \in \operatorname{dom} f$ iff $f(x_\lambda) < \lambda f(x_0) + (1-\lambda) f(x_1)$ for all $\lambda \in \,]0,1[$ and $x_1 \in \operatorname{dom} f \setminus \{x_0\}$.

Having $f \in \Lambda(X)$, $x_0 \in \operatorname{dom} f$ and $\delta \in \mathbb{R}_+$, consider the set

$$C_\delta f(x_0) := \left\{ y \in \operatorname{dom} f \mid f\left(\tfrac{1}{2} x_0 + \tfrac{1}{2} y\right) \geq \tfrac{1}{2} f(x_0) + \tfrac{1}{2} f(y) - \delta \right\}. \tag{7}$$

Clearly we have

$$0 \leq \delta < \varepsilon < \infty \Rightarrow x_0 \in C_\delta f(x_0) \subset C_\varepsilon f(x_0) \quad \text{and} \quad C_\delta f(x_0) = \cap_{\varepsilon > \delta} C_\varepsilon f(x_0); \tag{8}$$

moreover, one has

$$f \text{ is strictly convex at } x_0 \quad \Leftrightarrow \quad C_0 f(x_0) = \{x_0\}. \tag{9}$$

Lemma 3.4. *Assume that $f \in \Gamma(X)$ is continuous at $x_0 \in \operatorname{dom} f$ and consider $(\delta_n) \subset \mathbb{R}_+$ with $\delta_n \to \delta$ as well as $(x_n) \subset \big(C_{\delta_n} f(x_0)\big)$. If $x_n \to y$, then $y \in C_\delta f(x_0)$. Consequently, $C_\delta f(x_0)$ is closed for every $\delta \in \mathbb{R}_+$; moreover, if we have $\delta = 0$, then $f(x_n) \to f(y) = 2f(\frac{1}{2} x_0 + \frac{1}{2} y) - f(x_0)$.*

Proof. Since f is continuous at x_0, f is continuous on $\operatorname{int}(\operatorname{dom} f)$ (see, e.g., [27, Th. 2.2.9]). Because $(x_n) \subset \operatorname{dom} f$, $y \in \operatorname{cl}(\operatorname{dom} f)$, and so $\overline{x} := \frac{1}{2}x_0 + \frac{1}{2}y \in \operatorname{int}(\operatorname{dom} f)$. Because $\overline{x}_n := \frac{1}{2}x_0 + \frac{1}{2}x_n \to \overline{x}$ and f is continuous at $\overline{x}$, we have that $f(\overline{x}_n) \to f(\overline{x})$. Because $x_n \in C_{\delta_n} f(x_0)$, one has $f(x_n) \le 2f(\overline{x}_n) - f(x_0) + 2\delta_n$ for $n \ge 1$, and so

$$\begin{aligned} \tfrac{1}{2} f(y) &\le \tfrac{1}{2} \liminf f(x_n) \le \tfrac{1}{2} \limsup f(x_n) \le \limsup \left(f(\overline{x}_n) - \tfrac{1}{2} f(x_0) + \delta_n\right) \\ &= f(\overline{x}) - \tfrac{1}{2} f(x_0) + \delta = f\left(\tfrac{1}{2}x_0 + \tfrac{1}{2}y\right) - \tfrac{1}{2} f(x_0) + \delta, \end{aligned} \tag{10}$$

whence $y \in C_\delta f(x_0)$. If $\delta = 0$ one gets $\frac{1}{2}f(y) + \frac{1}{2}f(x_0) \le f\left(\frac{1}{2}x_0 + \frac{1}{2}y\right)$, and so $\frac{1}{2}f(y) + \frac{1}{2}f(x_0) = f\left(\frac{1}{2}x_0 + \frac{1}{2}y\right)$ because f is convex; hence the inequalities in (10) become equalities, and so $\lim f(x_n) = f(y) = 2f\left(\frac{1}{2}x_0 + \frac{1}{2}y\right) - f(x_0)$. □

Based on Lemma 3.4 and (8), as well as on Remark 3.2 and (9) for its second assertion, one obtains the next result; we omit its easy proof.

Corollary 3.5. *Assume that $f \in \Gamma(X)$ is continuous at $x_0 \in \operatorname{dom} f$; define $C_\delta := C_\delta f(x_0)$ for $\delta \in \mathbb{R}_+$. Then the following assertions hold:*

(i) *f is CLUR at x_0 $\Leftrightarrow$ $[\forall (x_n) \subset (C_{1/n}) : \exists (x_{n_k}) \to x \in C_0]$.*

(ii) *f is LUR at x_0 $\Leftrightarrow$ $[\forall (x_n) \subset (C_{1/n})$, (x_n) bounded: $x_n \to x_0]$*
$\Leftrightarrow [\forall (x_n) \subset (C_{1/n}) : x_n \to x_0] \Leftrightarrow [\forall (x_n) \subset (C_{1/n}) : \exists (x_{n_k}) \to x_0]$
$\Leftrightarrow$ [f is CLUR and strictly convex at x_0].

Lemma 3.6. *Assume that $f \in \Lambda(X)$; consider $x_0, x_1 \in X$ and define for $\lambda \in [0,1]$: $x_\lambda := (1-\lambda)x_0 + \lambda x_1$. The following assertions hold:*

(i) *if $\lambda \in]0,1[$, $x_\lambda \in \arg\min f$ and $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$, then $x_0, x_1 \in \arg\min f$;*

(ii) *if $x^* \in \partial f(x_0) \cap \partial f(x_1)$, then $x^* \in \partial f(x_\lambda)$ and $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$ for every $\lambda \in [0,1]$;*

(iii) *if $x_0, x_1 \in \operatorname{dom} f$ and $\lambda_0 \in]0,1[$ are such that $f(x_{\lambda_0}) = (1-\lambda_0)f(x_0) + \lambda_0 f(x_1)$, then $\partial f(x_\lambda) = \partial f(x_0) \cap \partial f(x_1)$ for all $\lambda \in]0,1[$.*

Proof. (i) Assume that $x_\lambda \in \arg\min f$ is such that $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$; then $f(x_\lambda) \le f(x_0)$ and $f(x_\lambda) \le f(x_1)$; assuming that (at least) one of the inequalities is strict, multiplying the first inequality by $1-\lambda$ (> 0) and the second one by λ (> 0), then summing them side-by-side, one gets the contradiction $f(x_\lambda) < (1-\lambda)f(x_0) + \lambda f(x_1)$.

(ii) Assume that $x^* \in \partial f(x_0) \cap \partial f(x_1)$. One has $x_0, x_1 \in \arg\min(f - x^*)$ because of (3). Having in view that $f - x^* \in \Lambda(X)$, $\arg\min(f - x^*)$ is convex, and so $x_\lambda \in \arg\min(f - x^*)$, or, equivalently, $x^* \in \partial f(x_\lambda)$ by (3).

(iii) Let $x_0, x_1 \in \operatorname{dom} f$ and $\lambda_0 \in]0,1[$ be such that

$$f(x_{\lambda_0}) = (1-\lambda_0)f(x_0) + \lambda_0 f(x_1);$$

as observed after the definition of strict convexity of f at a point, we conclude $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$ for every $\lambda \in [0,1]$. Fix $\lambda \in]0,1[$ and take $x^* \in \partial f(x_\lambda)$; then $x_\lambda \in \arg\min(f - x^*)$ by (3). Using (i), one gets $x_0, x_1 \in \arg\min(f - x^*)$, and so $x^* \in \partial f(x_0) \cap \partial f(x_1)$. Hence $\partial f(x_\lambda) \subset \partial f(x_0) \cap \partial f(x_1)$. As the reverse inclusion is true by (ii), the conclusion follows. □

Lemma 3.7. *Let $f \in \Lambda(X)$, $x_0 \in \operatorname{dom} f$, and consider the following conditions:*

(i) *f is strictly convex at x_0;*

(ii) *$\partial f(x) \cap \partial f(x_0) = \emptyset$ for all $x \in \operatorname{dom} f \setminus \{x_0\}$;*

(iii) *$\partial f(x) \cap \partial f(x_0) = \emptyset$ for all $x \in \operatorname{int}(\operatorname{dom} f) \setminus \{x_0\}$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*; moreover, if f is continuous at x_0, then* (iii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) Assume that for $x \in \operatorname{dom} f \setminus \{x_0\}$ there exists $x^* \in \partial f(x) \cap \partial f(x_0)$. Then, using Lemma 3.6(ii) for $x_1 := x$, $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$ for every $\lambda \in [0,1]$ (where, as usual, $x_\lambda := (1-\lambda)x_0 + \lambda x_1$ for $\lambda \in [0,1]$), and so f is not strictly convex at x_0.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) Assume that f is continuous at x_0, but (i) is not true. Then there exists $x_1 \in \operatorname{dom} f \setminus \{x_0\}$ such that $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$ for every $\lambda \in [0,1]$. Using Lemma 3.6(iii) we get $\partial f(x_\lambda) = \partial f(x_0) \cap \partial f(x_1)$ for all $\lambda \in]0,1[$. Because f is continuous at x_0, $x_0 \in \operatorname{int}(\operatorname{dom} f)$ and f is continuous on $\operatorname{int}(\operatorname{dom} f)$, and so $\partial f(x) \neq \emptyset$ for every $x \in \operatorname{int}(\operatorname{dom} f)$. It follows that $x_{1/2} \in \operatorname{int}(\operatorname{dom} f) \setminus \{x_0\}$, and so $\emptyset \neq \partial f(x_{1/2}) \subset \partial f(x_0)$, contradicting (iii). □

Corollary 3.8. *Let $f \in \Gamma(X)$. The following assertions hold:*

(i) *Assume that f is strictly convex at $x_0 \in \operatorname{dom} f$ and $x^* \in \partial f(x_0)$; then $\partial^X f^*(x^*) = \{x_0\}$.*

(ii) *Assume that f is continuous on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{card}\left(\partial^X f^*(x^*)\right) \leq 1$ for every $x^* \in \operatorname{dom} f^*$; then f is strictly convex on $\operatorname{int}(\operatorname{dom} f)$.*

Proof. Observe that $f^{**}|_X = f$ because $f \in \Gamma(X)$.

(i) Clearly, $x_0 \in \partial^X f^*(x^*)$. Take $x \in \partial^X f^*(x^*)$; then

$$\langle x, x^* \rangle = \langle x^*, x \rangle = f^*(x^*) + f^{**}(x) = f^*(x^*) + f(x),$$

and so $x^* \in \partial f(x)$, whence $\partial f(x) \cap \partial f(x_0) \neq \emptyset$. Because f is strictly convex at x_0, one gets $x = x_0$ using the implication (i) $\Rightarrow$ (ii) from Lemma 3.7.

(ii) Take $x_0 \in \operatorname{int}(\operatorname{dom} f)$; by hypothesis, f is continuous at x_0. Assuming that f is not strictly convex at x_0, using the equivalence (iii) $\Leftrightarrow$ (ii) from Lemma 3.7, there exist $x \in \operatorname{int}(\operatorname{dom} f) \setminus \{x_0\}$ and $x^* \in \partial f(x) \cap \partial f(x_0)$. Hence $x_0, x \in \partial f^*(x^*)$, contradicting the fact that $\operatorname{card}\left(\partial^X f^*(x^*)\right) \leq 1$. □

In the sequel, $f \in \Gamma(X)$ and $x_0 \in \operatorname{dom} f$ are given; supplementary conditions on f and/or x_0 will be mentioned when needed.

To the pair (f, x_0) and $\delta \in \mathbb{R}_+$ we associate the sets

$$A_\delta f(x_0) := A_\delta, \quad A_\delta^X f(x_0) := A_\delta^X, \quad E_\delta f(x_0) := E_\delta, \quad E_\delta^X f(x_0) := E_\delta^X,$$

where

$$A_\delta := \cup\{\partial_\delta f^*(x^*) \mid x^* \in \partial_\delta f(x_0) \cap \operatorname{Im} \partial f\} \subset \cup\{\partial_\delta f^*(x^*) \mid x^* \in \partial_\delta f(x_0)\} =: E_\delta,$$
$$A_\delta^X := X \cap A_\delta, \quad E_\delta^X := X \cap E_\delta.$$

Having in view (8) with $C_\delta f(x_0) =: C_\delta$ and (2), one obviously has

$$0 \le \delta < \varepsilon < \infty \Rightarrow \left[C_\delta \subset C_\varepsilon \wedge A_\delta \subset A_\varepsilon \wedge A_\delta^X \subset A_\varepsilon^X \wedge E_\delta \subset E_\varepsilon \wedge E_\delta^X \subset E_\varepsilon^X\right]. \quad (11)$$

Observe that

$$A_0 = E_0 = \partial f^*\big(\partial f(x_0)\big), \quad A_0^X = E_0^X = \{x \in X \mid \partial f(x) \cap \partial f(x_0) \neq \emptyset\};$$

hence $A_0 \neq \emptyset \Leftrightarrow A_0^X \neq \emptyset \Leftrightarrow \partial f(x_0) \neq \emptyset \Leftrightarrow x_0 \in A_0$.

In the sequel, in the proofs, we omit $f(x_0)$ when using the sets introduced above.

Lemma 3.9. *Let $\delta \ge 0$. The following assertions hold.*

(i) $A_\delta^X f(x_0) \subset E_\delta^X f(x_0) \subset C_\delta f(x_0)$.

(ii) *Assume that f is continuous at x_0. Then $C_{\delta/2} f(x_0) \subset A_\delta^X f(x_0)$;*

$$\text{consequently} \quad C_{\delta/2} f(x_0) \subset A_\delta^X f(x_0) \subset \overline{A_\delta^X f(x_0)} \subset C_\delta f(x_0),$$
$$C_0 f(x_0) = A_0^X f(x_0) = \overline{A_0^X f(x_0)}, \quad (12)$$

and the ones in which A is replaced by E. Consequently,

$$A_0^X f(x_0) = \textstyle\bigcap_{\delta>0} A_\delta^X f(x_0) = \bigcap_{\delta>0} \overline{A_0^X f(x_0)} = \bigcap_{\delta>0} C_\delta f(x_0) = C_0 f(x_0). \quad (13)$$

Proof. (i) The first inclusion is obvious. Take $y \in E_\delta^X$; then $y \in \partial_\delta^X f^*(x^*)$ with $x^* \in \partial_\delta f(x_0)$. Because $y \in \operatorname{dom} f$ and $x^* \in \operatorname{dom} f^*$ one has

$$0 \le \alpha := f(x_0) + f^*(x^*) - \langle x_0, x^* \rangle, \quad 0 \le \beta := f(y) + f^*(x^*) - \langle y, x^* \rangle, \quad (14)$$

and $\alpha, \beta \le \delta$; adding side-by-side the inequalities above, one gets

$$\tfrac{1}{2} f(x_0) + \tfrac{1}{2} f(y) + \left(f^*(x^*) - \left\langle \tfrac{1}{2}(x_0 + y), x^* \right\rangle\right) = \tfrac{1}{2}(\alpha + \beta) \le \delta. \quad (15)$$

Because $\left\langle \frac{1}{2}(x_0 + y), x^* \right\rangle - f^*(x^*) - f(\frac{1}{2}x_0 + \frac{1}{2}y) \le 0$ by the Young–Fenchel inequality, one obtains that $\delta \ge \frac{1}{2} f(x_0) + \frac{1}{2} f(y) - f(\frac{1}{2}x_0 + \frac{1}{2}y)$ using (15), and so $y \in C_\delta$ by (7).

(ii) Note first that $x_0 \in \operatorname{int}(\operatorname{dom} f)$ and f is subdifferentiable on $\operatorname{int}(\operatorname{dom} f)$. Take $y \in C_{\delta/2}$ and set $\overline{x} := \frac{1}{2}x_0 + \frac{1}{2}y$; then $\overline{x} \in \operatorname{int}(\operatorname{dom} f)$, and so there exists $x^* \in \partial f(\overline{x})$.

Then $\quad \left\langle \frac{1}{2}x_0 + \frac{1}{2}y, x^* \right\rangle = f\left(\frac{1}{2}x_0 + \frac{1}{2}y\right) + f^*(x^*) \ge \frac{1}{2} f(x_0) + \frac{1}{2} f(y) - \frac{1}{2}\delta + f^*(x^*)$,

and so $\quad \delta \ge f(x_0) + f(y) + 2f^*(x^*) - \langle x_0 + y, x^* \rangle = \alpha + \beta$,

α and β being defined in (14). Because $\alpha, \beta \ge 0$, it follows that $\alpha, \beta \in [0, \delta]$, whence $y \in \partial_\delta^X f^*(x^*)$ and $x^* \in \partial_\delta f(x_0) \cap \operatorname{Im} \partial f$, and so $y \in A_\delta^X$. Hence $C_{\delta/2} \subset A_\delta^X$. The inclusions and equalities in (12) follow from (i) and Lemma 3.4. □

Having in view (11), for every sequence $(\delta_n)_{n\ge1} \subset \mathbb{P}$ with $\delta_n \to 0$, one has

$$\lim_{0<\delta\to0} \alpha\big(A_\delta^X f(x_0)\big) = 0 \Leftrightarrow \alpha\big(A_{\delta_n}^X f(x_0)\big) \to 0, \quad (16)$$

$$\lim_{0<\delta\to0} \alpha\big(A_\delta f(x_0)\big) = 0 \Leftrightarrow \alpha\big(A_{\delta_n} f(x_0)\big) \to 0, \quad (17)$$

$$\lim_{0<\delta\to0} \alpha\big(C_\delta f(x_0)\big) = 0 \Leftrightarrow \alpha\big(C_{\delta_n} f(x_0)\big) \to 0. \quad (18)$$

Theorem 3.10. *Let f be continuous at x_0. The following assertions are equivalent:*

(i) *f is CLUR at x_0;*

(ii) *every sequence $(x_n) \subset \big(C_{1/n}f(x_0)\big)$ has a subsequence converging to an element of $C_0 f(x_0)$;*

(iii) $\alpha\big(C_{1/n}f(x_0)\big) \to 0$*;*

(iv) $\alpha\big(A^X_{1/n}f(x_0)\big) \to 0$*;*

(v) *$A^X_0 f(x_0)$ is nonempty, compact and $A^X_{1/n}f(x_0) \to^{H^+} A^X_0 f(x_0)$;*

(vi) *$A_0 f(x_0)$ is a compact subset of X and $A_{1/n}f(x_0) \to^{H^+} A_0 f(x_0)$;*

(vii) $\alpha\big(A_{1/n}f(x_0)\big) \to 0$.

Proof. One has:

(i) $\Leftrightarrow$ (ii) by Corollary 3.5(i);

(ii) $\Leftrightarrow$ (iii) by the equivalence (i) $\Leftrightarrow$ (iv) from Proposition 2.2;

(iii) $\Leftrightarrow$ (iv) by the inclusions in (12) from Lemma 3.9, (16), (18) and Proposition 2.1(iv);

(iv) $\Leftrightarrow$ (v) by the equalities in (13) and the equivalence (iv) $\Leftrightarrow$ (ii) from Proposition 2.2;

(vi) $\Leftrightarrow$ (vii) by the equivalence (iv) $\Leftrightarrow$ (iii) from Proposition 2.2;

(vii) $\Rightarrow$ (iv) by Proposition 2.1 (iv) because $A^X_\delta \subset A_\delta$ for $\delta \in \mathbb{R}_+$.

(v) $\Rightarrow$ (vi) Fix $\varepsilon > 0$; there exists $n_\varepsilon \in \mathbb{N}^*$ such that $A^X_{1/n_\varepsilon} \subset A^X_0 + \varepsilon B_X$ by hypothesis,

and so $\qquad (A^X_0 \subset)\ A^X_{1/n_\varepsilon} \subset A^X_0 + \varepsilon B_X \subset A^X_0 + \varepsilon B_{X^{**}} \subset A^X_0 + \varepsilon U_{X^{**}}$.

Consider $y^{**} \in A_{1/n_\varepsilon}$; then there exists $x^* \in \partial_{1/n_\varepsilon} f(x_0) \cap \operatorname{Im}\partial f$ such that we have $y^{**} \in \partial_{1/n_\varepsilon} f^*(x^*)$. Hence $x_0 \in \partial^X_{1/n_\varepsilon} f^*(x^*) \subset A^X_{1/n_\varepsilon} \subset A^X_0 + \varepsilon U_{X^{**}}$. Because A^X_0 is compact (in X by hypothesis), A^X_0 is also compact in X^{**} (hence w^*-compact), and so $A^X_0 + \varepsilon U_{X^{**}}$ is w^*-compact and $\|\cdot\|$-closed in X^{**}. By [8, Prop. 4.2] (which asserts that $\partial_{\varepsilon'} f^*(x^*) = \operatorname{cl}_{w^*}\left(\partial^X_{\varepsilon'} f^*(x^*)\right)$ for all $x^* \in \operatorname{dom} f^*$ and $\varepsilon' \in \mathbb{P}$), one has

$$\partial_{1/n_\varepsilon} f^*(x^*) = \operatorname{cl}_{w^*}\left(\partial^X_{1/n_\varepsilon} f^*(x^*)\right) \subset A^X_0 + \varepsilon U_{X^{**}},$$

and so $y^{**} \in A^X_0 + \varepsilon U_{X^{**}}$. It follows that $A_{1/n_\varepsilon} \subset A^X_0 + \varepsilon U_{X^{**}}$, whence

$$\operatorname{cl} A_{1/n_\varepsilon} \subset A^X_0 + \varepsilon U_{X^{**}} \subset A_0 + \varepsilon U_{X^{**}}. \tag{19}$$

Therefore, $\cap_{n\geq 1} \operatorname{cl} A_{1/n} \subset A^X_0 + \varepsilon U_{X^{**}}$ for every $\varepsilon > 0$, and so

$$\operatorname{cl} A_0 \subset \cap_{\delta>0} \operatorname{cl} A_\delta = \cap_{n\geq 1} \operatorname{cl} A_{1/n} \subset \cap_{\varepsilon>0}(A^X_0 + \varepsilon U_{X^{**}}) = \operatorname{cl} A^X_0 = A^X_0 \subset A_0.$$

It follows that
$$A_0 = A^X_0 = \cap_{n\geq 1} \operatorname{cl} A_{1/n}. \tag{20}$$

From (19) and (20) one obtains that (vi) holds. $\square$

Corollary 3.11. *Let f be continuous at x_0. The following assertions are equivalent:*

(i) *f is LUR at x_0;*

(ii) $\operatorname{diam}\left(C_{1/n}f(x_0)\right) \to 0$;

(iii) $\operatorname{diam}\left(A^X_{1/n}f(x_0)\right) \to 0$;

(iv) $A^X_{1/n}f(x_0) \to^{H^+} \{x_0\}$;

(v) $A_{1/n}f(x_0) \to^{H^+} \{x_0\}$;

(vi) $\operatorname{diam}\left(A_{1/n}f(x_0)\right) \to 0$.

Proof. Given $y_0 \in F_{n+1} \subset F_n \subset X$ for $n \geq 1$, one has $\operatorname{diam} F_n \to 0 \Leftrightarrow F_n \to^{H^+} \{y_0\}$. Using this remark, the definitions of the sets A_δ, A^X_δ, C_δ, (12) and Corollary 3.5, one has

$$\text{(vi)} \Leftrightarrow \text{(v)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(ii)} \Rightarrow [C_0 = \{x_0\} \text{ and } f \text{ is CLUR at } x_0] \Leftrightarrow \text{(i)}.$$

Assume that f is CLUR at x_0 and $C_0 = \{x_0\}$; then the assertions (iii)–(vii) of Theorem 3.10 hold, and so $A_{1/n} \to^{H^+} A_0 = A^X_0$, whence $A_{1/n} \to^{H^+} \{x_0\}$ because $A^X_0 = C_0$ by Lemma 3.9. Hence (v) holds, and so the proof is complete. □

Example 3.12. (i) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 - 1$ if $|x| > 1$ and by $g(x) = 0$ if $|x| \leq 1$. The function g is not strictly convex and hence it is not LUR at all points of $[-1, 1]$. Elementary calculations show that g is LUR at x iff $|x| > 1$ and CLUR for $|x| \leq 1$.

(ii) As observed in Remark 3.3, the function $f = \|\cdot\|$ is not CLUR at each $x \in X$, but $(X, \|\cdot\|)$ is CLUR at $x_0 \in S_X$ i.e., (x_n) has a convergent subsequence whenever $(x_n) \subset S_X$ and $\|x_n + x_0\| \to 2$ iff $g := \|\cdot\|^2$ is CLUR at x_0.

First observe that, for x_0, $x_n \in X$ $(n \geq 1)$, one has

$$\tfrac{1}{2}\|x_n\|^2 + \tfrac{1}{2}\|x_0\|^2 - \left\|\tfrac{1}{2}x_n + \tfrac{1}{2}x_0\right\|^2 \to 0 \Rightarrow \left[\, \|x_n\| \to \|x_0\| \wedge \|x_n + x_0\| \to 2\|x_0\| \,\right]. \quad (21)$$

Indeed, denoting by δ_n the expression from the left hand side (lhs) of (21), one has

$$2\|x_0\|^2 + 2\|x_n\|^2 = 4\delta_n + \|x_0 + x_n\|^2 \leq 4\delta_n + \|x_0\|^2 + 2\|x_0\|\|x_n\| + \|x_n\|^2,$$

and so $(\|x_n\| - \|x_0\|)^2 \leq 4\delta_n$ and $\|x_n + x_0\|^2 = 2\left(\|x_n\|^2 + \|x_0\|^2\right) - 4\delta_n$, whence $\|x_n\| \to \|x_0\|$ and $\|x_n + x_0\| \to 2\|x_0\|$ because $\delta_n \to 0$.

Assume that g is CLUR at $x_0 \in S_X$ and take $(x_n) \subset S_X$ such that $\|x_n + x_0\| \to 2$. Then,

$$\delta_n := \tfrac{1}{2}g(x_n) + \tfrac{1}{2}g(x_0) - g\left(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0\right) = 1 - \tfrac{1}{4}\|x_n + x_0\|^2 \to 0, \quad (22)$$

and so (x_n) has a convergent subsequence because g is CLUR at x_0. Hence $(X, \|\cdot\|)$ is CLUR at x_0. Conversely, assume that $(X, \|\cdot\|)$ is CLUR at x_0 and take $(x_n) \subset X$ such that $\delta_n \to 0$, where δ_n is defined in (22); because δ_n has the same expression as the one in the lhs of (21), from (21) one gets $\alpha_n := \|x_n\| \to \|x_0\| = 1$ and $\|x_n + x_0\| \to 2$. We (may) assume that $\alpha_n > 0$, and so $x'_n := \alpha_n^{-1}x_n \in S_X$ and $x_n = \alpha_n x'_n$. Because $\left|\|x_n + x_0\| - \|x'_n + x_0\|\right| \leq \|x_n - x'_n\| = |\alpha_n - 1| \to 0$, one obtains that $\|x'_n + x_0\| \to 2$. As $(X, \|\cdot\|)$ is CLUR at x_0, (x'_n) has a convergent subsequence (x'_{n_k}) and so the sequence $(x_{n_k}) = (\alpha_{n_k}x'_{n_k})$ is convergent, too. Hence g is CLUR at x_0.

The next result completes [9, Th. 5.19] in the sense that one removes its general hypothesis $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ and adds new characterizations; also, it will be used in the next sections. In order to see better the relations with [9, Th. 5.19], and having in view that (2) is valid also for f replaced by f^*, observe that for every sequence $(\delta_n) \subset \mathbb{P}$ with $\delta_n \to 0$, one has

$$\lim_{0<\delta\to 0} \alpha\big(\partial_\delta f^*(x^*)\big) = 0 \Leftrightarrow \alpha\big(\partial_{\delta_n} f^*(x^*)\big) \to 0, \tag{23}$$

$$\lim_{0<\delta\to 0} \alpha\left(\partial^X_\delta f^*(x^*)\right) = 0 \Leftrightarrow \alpha\left(\partial^X_{\delta_n} f^*(x^*)\right) \to 0, \tag{24}$$

$$\partial f^*(x^*) = \cap_{n\geq 1}\partial_{\delta_n} f^*(x^*), \quad \partial^X f^*(x^*) = \cap_{n\geq 1}\partial^X_{\delta_n} f^*(x^*). \tag{25}$$

Proposition 3.13. *Consider $x^* \in \operatorname{dom} f^*$. The following assertions are equivalent:*

(i) $\lim_{0<\delta\to 0} \alpha\left(\partial^X_\delta f^*(x^*)\right) = 0$;

(ii) $\lim_{0<\delta\to 0} \alpha\big(\partial_\delta f^*(x^*)\big) = 0$;

(iii) *$\partial^X f^*(x^*)$ is nonempty, compact and $\partial^X_{1/n} f^*(x^*) \to^{H^+} \partial^X f^*(x^*)$;*

(iv) *$\partial f^*(x^*)$ is nonempty, compact and $\partial_{1/n} f^*(x^*) \to^{H^+} \partial f^*(x^*)$ in X^{**};*

(v) *$\partial f^*(x^*)$ is a nonempty compact subset of X and $\partial_{1/n} f^*(x^*) \to^{H^+} \partial f^*(x^*)$ (in X^{**});*

(vi) *if $(x_n) \subset \operatorname{dom} f$ is such that $(f - x^*)(x_n) \to \inf(f - x^*)$, then (x_n) has a convergent subsequence;*

(vii) *$\partial^X f^*(x^*)$ is nonempty, compact and $d\big(x_n, \partial^X f^*(x^*)\big) \to 0$ whenever $(x_n) \subset \operatorname{dom} f$ is such that $(f - x^*)(x_n) \to \inf(f - x^*)$;*

(viii) *$\partial^X f^*(x^*)$ is nonempty, compact and f^* is directionally Fréchet differentiable at x^*.*

Consequently, if one of the conditions (i)–(viii) *holds, then $\emptyset \neq \partial f^*(x^*) \subset X$.*

Proof. (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follow immediately by (23), (24) and the equivalence (ii) $\Leftrightarrow$ (iv) of Proposition 2.2; (ii) $\Rightarrow$ (i) is obvious because $\partial^X_\delta f^*(x^*) \subset \partial_\delta f^*(x^*)$ for $\delta \geq 0$; (v) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (iii) are obvious.

(i) $\Rightarrow$ (v) Set $S_n := \partial^X_{1/n} f^*(x^*)$ for $n \geq 1$ and $S_0 := \partial^X f^*(x^*)$; then $S_0 = \cap_{n\geq 1} S_n$ by (25). As observed in Section 2, $\emptyset \neq S_{n+1} \subset S_n = \overline{S_n} = \operatorname{cl} S_n$ for $n \geq 1$; moreover, $\alpha(S_n) \to 0$ by the present hypothesis. Using the implication (iv) $\Rightarrow$ (ii) from Proposition 2.2, it follows that S_0 is a nonempty compact subset of X and for every $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that

$$S_{n_\varepsilon} \subset S_0 + \varepsilon B_X \subset S_0 + \varepsilon B_{X^{**}} \subset S_0 + \varepsilon U_{X^{**}}. \tag{26}$$

Because S_0 is convex and compact (also) in X^{**} and $\varepsilon U_{X^{**}}$ is $\|\cdot\|$-closed and w^*-compact, $S_0 + \varepsilon U_{X^{**}}$ is $\|\cdot\|$-closed and w^*-compact in X^{**}. Using now [8, Prop. 4.2] and (26) one gets

$$\partial_{1/n_\varepsilon} f^*(x^*) = \operatorname{cl}_{w^*}\left(\partial^X_{1/n_\varepsilon} f^*(x^*)\right) = \operatorname{cl}_{w^*} S_{n_\varepsilon} \subset S_0 + \varepsilon U_{X^{**}}, \tag{27}$$

and so $S_0 \subset S := \cap_{n\geq 1}\partial_{1/n} f^*(x^*) \subset S_0 + \varepsilon U_{X^{**}}$ for every $\varepsilon > 0$, whence

$$S_0 \subset S \subset \cap_{\varepsilon>0}(S_0 + \varepsilon U_{X^{**}}) = \operatorname{cl} S_0 = S_0.$$

Because $S_0 = \partial^X f^*(x^*)$ and $S = \partial f^*(x^*)$ by (25), we get $S_0 = S = \partial f^*(x^*)$; this, together with (27), shows that (v) holds.

Set $g := f - x^*$ and $\mu := \inf g = -f^*(x^*)$; hence $(S_0 =)\partial^X f^*(x^*) = \arg\min g$ by (4).

(i) $\Rightarrow$ (vi) Consider $(x_n) \subset \operatorname{dom} f$ such that $f(x_n) - \langle x_n, x^*\rangle \to \inf(f - x^*)$. Take $(0 \le)\ \delta_n := f(x_n) + f^*(x^*) - \langle x_n, x^*\rangle \to 0$ and set $\varepsilon_n := \delta_n + 1/n$; then $0 < \varepsilon_n \to 0$ and $x_n \in \partial^X_{\varepsilon_n} f^*(x^*)$ for $n \ge 1$. Passing to a subsequence if necessary, we (may) assume that (ε_n) is decreasing. By (i) one obtains that $\alpha\left(\partial^X_{\varepsilon_n} f^*(x^*)\right) \to 0$, and so (x_n) has a convergent subsequence by the implication (iv) $\Rightarrow$ (i) of Proposition 2.2.

(vi) $\Rightarrow$ (vii) Clearly, $\mu := \inf g = -f^*(x^*) \in \mathbb{R}$, and so there exists $(x_n) \subset \operatorname{dom} f$ such that $g(x_n) \to \mu$; hence $\delta_n := f(x_n) + f^*(x^*) - \langle x_n, x^*\rangle \to 0$, and so there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x \in X$. Because $\mu \le g(x_{n_k})\ (= \delta_{n_k} + \mu)$ for $k \ge 1$ and g is lsc (f being so), one gets

$$\mu \le g(x) \le \liminf g(x_{n_k}) = \liminf(\delta_{n_k} + \mu) = 0 + \mu = \mu;$$

consequently, $g(x) = \mu$, whence $x \in \arg\min g = S_0$.

Take $(x_n) \subset \operatorname{dom} f = \operatorname{dom} g$ with $g(x_n) \to \mu$. Assume that $d(x_n, S_0) \not\to 0$; then there exists a subsequence $(x'_k) := (x_{n_k})$ of (x_n) and $\varepsilon > 0$ such that $d(x'_k, S_0) \ge \varepsilon$ for all $k \ge 1$. Invoking the argument in the preceding paragraph for (x'_k) instead of (x_n), passing to a subsequence, we (may) assume that $x'_k \to x' \in S_0$, and so $(0 =)$ $d(x', S_0) \ge \varepsilon > 0$; this contradiction shows that the conclusion holds.

(vii) $\Rightarrow$ (viii) The present hypothesis shows that condition (ii) of Theorem 3.4.3 in [27] is verified, and so, by the equivalence (ii) $\Leftrightarrow$ (iv) from the same result, there exists an lsc convex function $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+ := [0, \infty]$ such that $\lim_{0<t\to 0} t^{-1}\varphi(t) = 0$ and

$$g^*(y^*) \le g^*(0) + \iota^*_{S_0}(y^*) + \varphi(\|y^*\|) \quad \forall y^* \in X^*.$$

For $y^* \in X^*$ and $x \in S_0\ \left(= \partial^X f^*(x^*)\right)$, one has the equations $g^*(y^*) = f^*(x^* + y^*)$, $f(x) + f^*(x^*) = \langle x, x^*\rangle$ and the inequalities $f(x) + f^*(x^* + y^*) \ge \langle x, x^* + y^*\rangle$, and so $\langle x, y^*\rangle \le f^*(x^* + y^*) - f^*(x^*)$, whence

$$(\sup\nolimits_{x\in S_0} \langle x, y^*\rangle =)\ \iota^*_{S_0}(y^*) \le f^*(x^*+y^*) - f^*(x^*) \le \iota^*_{S_0}(y^*) + \varphi(\|y^*\|)\ \forall y^* \in X^*. \quad (28)$$

Because $\lim_{0<t\to 0} t^{-1}\varphi(t) = 0$, φ is bounded on $[0, t_0]$ for some $t_0 > 0$, and so $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Replacing $y^* \in X^*$ by ty^* with $t > 0$ and $y^* \in S_{X^*}$ in (28), one gets

$$\iota^*_{S_0}(y^*) \le \frac{f^*(x^* + ty^*) - f^*(x^*)}{t} \le \iota^*_{S_0}(y^*) + t^{-1}\varphi(t) \quad \forall y^* \in S_{X^*},\ t > 0,$$

and so $f^{*\prime}_+(x^*, y^*) := \lim_{0<t\to 0} t^{-1}[f^*(x^* + ty^*) - f^*(x^*)] = \iota^*_{S_0}(y^*)$, the limit being uniform wrt $y^* \in S_{X^*}$. Therefore, f^* is directionally Fréchet differentiable at x^*, and so (viii) holds.

(viii) $\Rightarrow$ (i) Because the very definition of the directional Fréchet differentiability at x^* implies that $x^* \in \operatorname{int}(\operatorname{dom} f^*)$, the conclusion is provided by the implication (i) $\Rightarrow$ (ix) from [9, Th. 5.19]. □

Remark 3.14. Notice that the equivalence of conditions (i), (ii) and (viii) from Proposition 3.13 are established in [9, Th. 5.19] under the supplementary hypothesis that $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Proposition 3.13 shows also that $\partial f^*(x^*)$ is a subset of X if one of the conditions (i)–(x) holds in [9, Th. 5.19]. To Proposition 3.13 one might add other (equivalent) conditions, similar to those in [9, Th. 5.19]. □

Corollary 3.15. *Consider $x^* \in \operatorname{dom} f^*$. The following assertions are equivalent:*

(i) $\lim_{0<\delta\to 0} \operatorname{diam}\left(\partial_\delta^X f^*(x^*)\right) = 0$;

(ii) $\lim_{0<\delta\to 0} \operatorname{diam}\left(\partial_\delta f^*(x^*)\right) = 0$;

(iii) *there exists $x \in X$ such that $\partial_{1/n}^X f^*(x^*) \to^{H^+} \{x\}$;*

(iv) *there exists $x^{**} \in X^{**}$ such that $\partial_{1/n} f^*(x^*) \to^{H^+} \{x^{**}\}$;*

(v) *there exists $x \in X$ such that $\partial_{1/n} f^*(x^*) \to^{H^+} \{x\}$ (in X^{**});*

(vi) *if $(x_n) \subset \operatorname{dom} f$ is such that $(f - x^*)(x_n) \to \inf(f - x^*)$, then (x_n) is convergent;*

(vii) *there exists $x \in \partial^X f^*(x^*)$ such that $x_n \to x$ whenever $(x_n) \subset \operatorname{dom} f$ is such that $(f - x^*)(x_n) \to \inf(f - x^*)$;*

(viii) *f^* is Fréchet differentiable at x^*.*

Consequently, if one of the conditions (i)–(viii) *holds, then $\nabla f^*(x^*) \in X$.*

Proof. Similarly to the proof of Corollary 3.11, the implications (v) $\Rightarrow$ (iv) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) $\Leftrightarrow$ (iii) are (almost) obvious.

(i) $\Rightarrow$ (v) Obviously, one has $\lim_{0<\delta\to 0} \alpha\left(\partial_\delta^X f^*(x^*)\right) = 0$, and so $\partial f^*(x^*)$ is a nonempty compact subset of X and $\partial_{1/n} f^*(x^*) \to^{H^+} \partial f^*(x^*)$ in X^{**} by the implication (i) $\Rightarrow$ (v) of Proposition 3.13, whence $\partial_{1/n}^X f^*(x^*) \to^{H^+} \partial f^*(x^*)$ in X^{**}. Because $\partial f^*(x^*) = X \cap \partial f^*(x^*) = \partial^X f^*(x^*) \subset \partial_{1/n}^X f^*(x^*)$ for $n \geq 1$, one has $\operatorname{diam}\left(\partial f^*(x^*)\right) = 0$, and so $\partial f^*(x^*) = \{x\}$ for some $x \in X$. Hence (v) holds.

Therefore, the conditions (i)–(v) are equivalent and there exists $x \in X$ such that $\partial f^*(x^*) = \{x\}$ if one of these conditions is verified.

For proving the implications (i) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (i), it is useful to consider $g := f - x^*$, $S_0 := \partial^X f^*(x^*) = \operatorname{arg\,min} g$ and $\mu := -f^*(x^*)$, to observe that the corresponding hypotheses from Proposition 3.13 are verified, and each of them implies that $\partial^X f^*(x^*)$ is a singleton $\{x\}$ with $x \in X$; applying then Proposition 3.13 one gets the respective conclusion. For example, in the proof of (viii) $\Rightarrow$ (i) one obtains that $f_+^{*\prime}(x^*, y^*) = \langle x, y^* \rangle$ for every $y^* \in X^*$; because f^* is directionally Fréchet differentiable, f^* is Fréchet differentiable with $\nabla f^*(x^*) = x$. □

Changing Proposition 3.13 by Corollary 3.15 and [9, Th. 5.19] by [9, Th. 5.20] in Remark 3.14, the obtained statement remains valid.

4. Convexity properties of CLUR functions

Also throughout this section $f \in \Gamma(X)$ and $x_0 \in \operatorname{dom} f$; additional hypothesis will be mentioned when needed.

In this section we introduce some notions which are analogous to known notions in geometry of Banach spaces and relate CLUR with these notions.

As observed in (3), $x^* \in \partial f(x_0)$ iff $f - x^*$ attains its minimum at x_0; this fact is relevant to the following definitions.

Definition 4.1. The function f is said to be:

(i) *strongly convex at* x_0, if $\partial f(x_0) \neq \emptyset$ and the sequence (x_n) converges whenever $(f - x^*)(x_n) \to (f - x^*)(x_0)$ and $x^* \in \partial f(x_0)$;

(ii) *nearly strongly convex at* x_0, if $\partial f(x_0) \neq \emptyset$ and (x_n) has a convergent subsequence whenever $(f - x^*)(x_n) \to (f - x^*)(x_0)$ and $x^* \in \partial f(x_0)$;

(iii) *U-convex at* x_0 if for every sequence $(x_n) \subset \operatorname{dom} f$ satisfying

$$\tfrac{1}{2}f(x_n) + \tfrac{1}{2}f(x_0) - f(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0) \to 0$$

there exists $x^* \in \partial f(x_0)$ such that $(f - x^*)(x_n) \to (f - x^*)(x_0)$;

(iv) *nearly U-convex at* x_0 if for every sequence $(x_n) \subset \operatorname{dom} f$ satisfying

$$\tfrac{1}{2}f(x_n) + \tfrac{1}{2}f(x_0) - f(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0) \to 0$$

there exists $x^* \in \partial f(x_0)$ and a subsequence (x_{n_k}) of (x_n) such that we have $(f - x^*)(x_{n_k}) \to (f - x^*)(x_0)$;

(v) f is *(nearly) strongly convex* or *(nearly) U-convex on* $A \subset \operatorname{dom} f$ if f is *(nearly) strongly convex* or *(nearly) U-convex at each* $x \in A$, respectively. □

Remark 4.2. The notions of strong convexity, near strong convexity and U-convexity defined above are analogous to the notions of strong convexity, near strong convexity and local U-convexity defined in the geometry of Banach spaces:

(i) The space $(X, \|\cdot\|)$ is said to be *strongly convex* (resp., *nearly strongly convex*) *at* $x_0 \in S_X$ (see [9, 13, 15, 28]) if (x_n) converges (resp., has a convergent subsequence) whenever $(x_n) \subset S_X$, $x^* \in S_{X^*}$, $\langle x_n, x^* \rangle \to 1$ and $\langle x_0, x^* \rangle = 1$.

(ii) The space $(X, \|\cdot\|)$ is said to be *locally U-convex at* $x_0 \in S_X$ (see [13]), if for every sequence $(x_n) \subset S_X$ satisfying $\|x_n + x_0\| \to 2$, there exists $x^* \in S_{X^*}$ such that $\langle x_0, x^* \rangle = 1$ and $\langle x_n, x^* \rangle \to 1$. □

Remark 4.3. Notice that condition (vi) in Corollary 3.15 for $x^* \in \operatorname{dom} f^*$ and in Definition 4.1(i) [for a fixed $x^* \in \partial f(x_0)$ ($\subset \operatorname{dom} f^*$)] is saying that, in the problem *(P) minimize* $g(x) := f(x) - \langle x, x^* \rangle$ *s.t.* $x \in X$, any minimizing sequence for *(P)* is convergent; this is nothing else than saying that *(P) is well-posed in the sense of Tikhonov* (see, e.g., [12, p. 1] and [8, 5.2.2]); moreover, the condition (vi) in Proposition 3.13 for $x^* \in \operatorname{dom} f^*$ and in Definition 4.1(ii) [for a fixed $x^* \in \partial f(x_0)$] is equivalent to the fact that *(P) is well-posed in the generalized sense* (see, e.g., [12, p. 24]).

Remark 4.4. (i) The notion of strong convexity of a function is not entirely new (see, e.g., [8, Th. 5.2.3]).

(ii) Notice that, for $x^* \in \partial f(x_0)$ one has $(f - x^*)(x_0) = -f^*(x^*) = \inf(f - x^*)$ and $\arg\min(f - x^*) = \partial^X f^*(x^*)$, as seen in (3) and (4).

(iii) Let f be strongly convex at x_0 and take $x^* \in \partial f(x_0)$. Then $\partial^X f^*(x^*) = \{x_0\}$ and $x_n \to x_0$ whenever $(x_n) \subset \operatorname{dom} f$ and $(f - x^*)(x_n) \to (f - x^*)(x_0)$. Indeed, besides the already taken sequence (x_n) consider $x \in \partial^X f^*(x^*)$ and (x'_n) defined by $x'_{3n-2} := x_0$, $x'_{3n-1} := x$ and $x'_{3n} := x_n$ for $n \geq 1$. Having in view (ii) and the choice of (x_n), one has $(f - x^*)(x'_n) \to (f - x^*)(x_0)$ because

$$(f - x^*)(x'_{3n}) = (f - x^*)(x_n) \to -f^*(x^*) = (f - x^*)(x'_{3n-2}) = (f - x^*)(x'_{3n-1}) \ \ \forall n \geq 1.$$

Since f is strongly convex at x_0, there exists $x' \in X$ such that $x'_n \to x'$; as $x'_{3n-1} \to x$ and $x'_{3n-2} \to x_0$, one gets $x' = x_0 = x$, whence $x_n \to x_0$ and $\partial^X f^*(x^*) = \{x_0\}$.

(iv) Let f be nearly strongly convex at x_0; take $x^* \in \partial f(x_0)$ and $(x_n) \subset \operatorname{dom} f$ such that $(f - x^*)(x_n) \to (f - x^*)(x_0)$. If a subsequence (x_{n_k}) of (x_n) converges to $x \in X$ (such (x_{n_k}) exists!), then $x^* \in \partial f(x)$, and so $x \in \partial^X f^*(x^*)$. Indeed, take (x_{n_k}) as above; then

$$(f - x^*)(x) \leq \liminf (f - x^*)(x_{n_k}) = \lim (f - x^*)(x_{n_k}) = (f - x^*)(x_0) = -f^*(x^*)$$

by the lower semicontinuity of f and (ii), whence $x^* \in \partial f(x)$.

(v) From (iii) and (iv) one clearly has that [f is strongly convex at x_0] if and only if [f is nearly strongly convex at x_0 and $\partial^X f^*(x^*) = \{x_0\}$ for all $x^* \in \partial f(x_0)$].

(vi) If f is strictly convex and nearly strongly convex at x_0, then f is strongly convex at x_0. Indeed, by Corollary 3.8(i), $\partial^X f^*(x^*) = \{x_0\}$ for all $x^* \in \partial f(x_0)$ when f is strictly convex at x_0. So, the conclusion follows from (v). □

Proposition 4.5. *Assume that* $\operatorname{dom} f = \operatorname{dom} \partial f$ *and* f *is strongly convex on* $\operatorname{dom} f$*; then* f *is strictly convex.*

Proof. Assume that f is not strictly convex at $x_0 \in \operatorname{dom} f$; then there exists $x_1 \in \operatorname{dom} f \setminus \{x_0\}$ such that $\frac{1}{2} f(x_0) + \frac{1}{2} f(x_1) = f\left(x_{1/2}\right)$, where $x_{1/2} := \frac{1}{2} x_0 + \frac{1}{2} x_1$ ($\in \operatorname{dom} \partial f$). Take $x^* \in \partial f(x_{1/2})$, and so $x_{1/2} \in \arg\min (f - x^*) = \partial^X f^*(x^*) =: S$. Using Lemma 3.6, one has $x_0, x_1 \in S$, contradicting the fact that f is strongly convex at $x_{1/2}$. □

The following result characterizes the near strong convexity of f at a point in terms of the sub-differentiability of f^*.

Proposition 4.6. *Assume that* $\partial f(x_0) \neq \emptyset$*. Consider the following conditions:*

(i) *f is nearly strongly convex at x_0;*

(ii) *$\alpha\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$ for every $x^* \in \partial f(x_0)$;*

(iii) *$\alpha\big(\partial_{1/n} f^*(x^*)\big) \to 0$ for every $x^* \in \partial f(x_0)$;*

(iv) *$\partial^X f^*(x^*)$ is compact and f^* is directionally Fréchet differentiable at each $x^* \in \partial f(x_0)$;*

(v) *$\partial f^*(x^*)$ is a nonempty compact subset of X for every $x^* \in \partial f(x_0)$.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

Proof. Because $(f - x^*)(x_0) = -f^*(x^*) = \inf (f - x^*)$, (i) holds iff condition (vi) from Proposition 3.13 is verified for every $x^* \in \partial f(x_0)$. Having in view (23), (24) and the equivalence of the conditions (i), (ii), (vi), (viii) from Proposition 3.13 for any $x^* \in \operatorname{dom} f^*$ (as proved in that result) one gets (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv); the implication (iv) $\Rightarrow$ (v) follows from the equivalence (v) $\Leftrightarrow$ (vi) from Proposition 3.13. □

Strong convexity is characterized as follows.

Proposition 4.7. *Assume that $\partial f(x_0) \neq \emptyset$. The following conditions are equivalent:*

(i) *f is strongly convex at x_0;*

(ii) $\operatorname{diam}\left(\partial^X_{1/n} f^*(x^*)\right) \to 0$ *for every $x^* \in \partial f(x_0)$;*

(iii) $\operatorname{diam}\left(\partial_{1/n} f^*(x^*)\right) \to 0$ *for every $x^* \in \partial f(x_0)$;*

(iv) *f^* is Fréchet differentiable at every $x^* \in \partial f(x_0)$.*

Consequently, if one of the conditions (i)–(iv) *holds, we have $\nabla f^*(x^*) = x_0$ for every $x^* \in \partial f(x_0)$.*

Proof. Notice that (i) holds iff condition (vi) from Corollary 3.15 is verified for every $x^* \in \partial f(x_0)$. Using now Corollary 3.15 instead of Proposition 3.13 in the proof of Proposition 4.6, one obtains the equivalences of conditions (i)–(iv). □

Example 4.8. (i) Let $(X, \|\cdot\|)$ and $f := \|\cdot\|$; f is not nearly strongly convex, and so it is not strongly convex, at any $x \in X$. Indeed, take first $x_0 \in X \setminus \{0\}$. Then $\partial f(x_0) = \{x^* \in S_{X^*} \mid \langle x_0, x^*\rangle = \|x_0\|\}$ $(\neq \emptyset)$ (see, e.g., [27, Cor. 2.4.16]); taking $x^* \in \partial f(x_0)$, one has $\mathbb{R}_+ x_0 \subset \arg\min(f - x^*)$, and so f is not nearly strongly convex at x_0. Take now $x := 0$, and consider $x_0 \in S_X$ and $x^* \in S_{X^*}$ with $\langle x_0, x^*\rangle = 1$; then $x^* \in \partial f(x)$ $(= U_{X^*})$ and, as seen above, $\mathbb{R}_+ x_0 \subset \arg\min(f - x^*)$. Hence f is not nearly strongly convex at 0.

(ii) Assume that $(X, \|\cdot\|)$ is reflexive and KK, that is, $[(x_n) \subset X, x \in X, \|x_n\| \to \|x\|$ and $x_n \to^w x] \Rightarrow x_n \to x$. Then the space $(X, \|\cdot\|)$ is nearly strongly convex at all points of S_X (see [18, 21]); see Remark 4.2 for the definition. We show that the function $f := \frac{1}{2}\|\cdot\|^2$ is nearly strongly convex on X.

Let $x_0 \in X$; then (see, e.g., [27, p. 230])

$$\partial f(x_0) = \Phi_X(x_0) := \{x^* \in X^* \mid \langle x_0, x^*\rangle = \|x_0\|^2 = \|x^*\|^2\}.$$

Clearly Φ_X is homogeneous, and so we (may) take $x_0 \in \{0\} \cup S_X$. Consider first $x_0 = 0$ and take $x^* \in \partial f(x_0)$ $(= \{0\})$; hence $f - x^* = f$. Taking $(x_n) \subset X$ with $f(x_n) = \frac{1}{2}\|x_n\|^2 \to \inf f = 0$, we get $x_n \to 0 = x_0$, and so f is strongly convex at 0; hence f is also nearly strongly convex at x_0.

Take now $x_0 \in S_X$ and $x^* \in \Phi_X(x_0)$. Consider $(x_n) \subset X$ such that

$$f(x_n) - \langle x_n, x^*\rangle \to f(x_0) - \langle x_0, x^*\rangle = -\tfrac{1}{2},$$

or, equivalently, $\delta_n := \frac{1}{2}\|x_n\|^2 - \langle x_n, x^*\rangle + \frac{1}{2} \to 0$. Then

$$(\|x_n\| - 1)^2 = \|x_n\|^2 - 2\|x_n\| + 1 \leq \|x_n\|^2 - 2\langle x_n, x^*\rangle + 1 = 2\delta_n \quad \forall n \geq 1, \tag{29}$$

and so $(\|x_n\| - 1)^2 \leq 2\delta_n$, which implies

$$\|x_n\| \to 1 \quad \text{and} \quad \langle x_n, x^*\rangle \to 1, \tag{30}$$

using the last equality in (29) for the second limit. Hence (x_n) is bounded, and so, because X is reflexive, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to^w x \in X$. Replacing x_n by x_{n_k} in (30), one gets $\|x\| \leq 1$ and $1 = \langle x, x^*\rangle \leq \|x\|$, and so $\|x\| = 1$. Consequently, $\|x_{n_k}\| \to 1 = \|x\|$ and $x_{n_k} \to^w x$, and so $x_{n_k} \to x$ because X is KK. Hence f is nearly strongly convex at x_0. Since $x_0 \in S_X$ is arbitrary, f is nearly strongly convex on X.

If $(X, \|\cdot\|)$ is also strictly convex, then $(X, \|\cdot\|)$ is strongly convex at all points of S_X (see [16, 21]); see Remark 4.2 for the definition. Moreover, f is strictly convex at every $x \in S_X$. By Remark 4.4(vi), f is strongly convex at every $x \in S_X$. Since we have already shown that f is strongly convex at 0, f is strongly convex on X. □

In the following result, the fact that f is CLUR at a point is characterized in terms of near strong convexity and near U-convexity.

Proposition 4.9. *Let f be continuous at x_0. The following assertions are equivalent:*

(i) *f is CLUR at x_0;*

(ii) *f is nearly strongly convex at x_0 and nearly U-convex at x_0.*

Proof. (i) $\Rightarrow$ (ii) Consider $x^* \in \partial f(x_0)$; because f is CLUR at x_0, by Theorem 3.10, one has $(0 \le \alpha\big(\partial_\delta^X f^*(x^*)\big) \le)\ \alpha\big(A_\delta^X f(x_0)\big) \to 0$ for $0 < \delta \to 0$, and so f is nearly strongly convex at x_0 by Proposition 4.6.

Take $(x_n) \subset \operatorname{dom} f$ with $(0 \le)\ \delta_n := \frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \to 0$; hence $x_n \in C_{\delta_n} f(x_0) \subset A_{2\delta_n}^X f(x_0)$ by Lemma 3.9. Because f is CLUR at x_0, by Theorem 3.10 (taking also into account (16) and (18)), and the implication (iv) $\Rightarrow$ (i) in Proposition 2.2, (x_n) has a subsequence (x_{n_k}) converging to $y \in A_0^X f(x_0)$, and so there exists $x^* \in \partial f(x_0) \cap \partial f(y)$, whence $(f - x^*)(y) = -f^*(x^*) = (f - x^*)(x_0)$. By the continuity of f at x_0 one has $(f - x^*)(x_{n_k}) \to (f - x^*)(x_0)$, and so f is nearly U-convex at x_0.

(ii) $\Rightarrow$ (i) It is sufficient to prove that $\alpha\left(C_{1/n}\right) \to 0$. For this, take $(x_n) \subset (C_{1/n})$, that is, $(0 \le)\ \delta_n := \frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \le 1/n$ for $n \ge 1$. Because $\delta_n \to 0$ and f is nearly U-convex at x_0, there exist $x^* \in \partial f(x_0)$ and a subsequence (x_{n_k}) of (x_n) such that $(f - x^*)(x_{n_k}) \to (f - x^*)(x_0)$. Because f is nearly strongly convex at x_0, (x_{n_k}) has a convergent subsequence, which, of course, is also a convergent subsequence of (x_n). Using the implication (i) $\Rightarrow$ (iv) in Proposition 2.2 one gets $\alpha(C_{1/n}) \to 0$, and so f is CLUR at x_0 by Theorem 3.10. □

Example 4.10. (i) By Proposition 4.9, the function g defined in Example 3.12(i) is nearly strongly convex at all points of $\mathbb{R}$. As the function is not strictly convex, it is not strongly convex at some points of $\mathbb{R}$.

(ii) We now present an example of a function which is nearly strongly convex at all points of X but it is not CLUR at a point in S_X. Let $X = \mathbb{R} \times \ell_2$ be endowed with the norm defined by $\|(\alpha, x)\| := |\alpha| + \|x\|_2$, where $\|\cdot\|_2$ denote the usual norm of ℓ_2. It is clear that X is reflexive; moreover, X is also KK (see also [18, Ex. 4.3]). Indeed, take $\xi_n := (\alpha_n, x_n)$ and $\xi := (\alpha, x)$ from X such that $\|\xi_n\| \to \|\xi\|$ and $\xi_n \to^w \xi$; then $|\alpha_n| + \|x_n\|_2 \to |\alpha| + \|x\|_2$ and $\alpha_n \to \alpha$, $x_n \to^w x$, whence $\|x_n\|_2 \to \|x\|_2$. It follows that $\|x_n - x\|_2^2 = \|x_n\|_2^2 - 2\langle x_n, x\rangle + \|x\|_2^2 \to \|x\|_2^2 - 2\langle x, x\rangle + \|x\|_2^2 = 0$, and so $x_n \to x$, whence $\xi_n \to \xi$. Therefore, by Example 4.8(ii), the function $f := \frac{1}{2}\|\cdot\|^2$ is nearly strongly convex on X. Consider $\xi_1 = (1, 0)$ and $\xi_n := (0, e_n)$ for $n \ge 2$, where (e_n) is the usual hilbertian basis of ℓ_2. Then

$$\forall n \ge 2\ :\ \tfrac{1}{2}f(\xi_1) + \tfrac{1}{2}f(\xi_n) - f(\tfrac{1}{2}\xi_1 + \tfrac{1}{2}\xi_n) = \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{2}(\tfrac{1}{2} + \tfrac{1}{2}) = 0\ \ (\to 0),$$

but, clearly, (ξ_n) does not have a $\|\cdot\|$-convergent subsequence. Hence f is not CLUR at ξ_1.

(iii) We now construct an example of a function which is U-convex at all points of X but it is not CLUR at a point in $X \setminus \{0\}$. First, let us define a norm $\|\cdot\|_W$ on ℓ_2 as follows. For $x = (x_1, x_2, x_3, \cdots, x_n, \cdots) \in \ell_2$, set $x' := (0, x_2, x_3, \cdots, x_n, \cdots)$ and define

$$\|x\|_S := \max\{|x_1|, \|x'\|_2\}, \quad \|x\|_W := \left(\|x\|_S^2 + \|(\gamma_1 x_1, \gamma_2 x_2, \gamma_3 x_3, ..., \gamma_n x_n, ...)\|_2^2\right)^{1/2},$$

where $\gamma_n := 1/n$ for $n \geq 1$.

Let $X = (\ell_2, \|\cdot\|_W)$. It is shown in [23] that X is weakly LUR, that is, for x_0, $x_n \in S_X$ with $n \geq 1$, $\|x_n + x_0\| \to 2 \Rightarrow x_n \to^w x_0$. Consider the function $g := \|\cdot\|_W^2$ and the usual hilbertian basis (e_n) of ℓ_2. Consider $x \in X$ and take $(x_n) \subset X$ such that

$$\tfrac{1}{2}g(x_n) + \tfrac{1}{2}g(x_0) - g\left(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0\right) = \tfrac{1}{2}\|x_n\|_W^2 + \tfrac{1}{2}\|x_0\|_W^2 - \|\tfrac{1}{2}x_n + \tfrac{1}{2}x_0\|_W^2 \to 0.$$

Using (21) one gets $\alpha_n := \|x_n\|_W \to \|x_0\|_W =: \alpha_0$ and $\|x_n + x_0\|_W \to 2\|x_0\|_W$. If $x_0 = 0$ one has $x_n \to 0$ and $x^* := 0 \in \partial g(x_0)$, whence, $(g - x^*)(x_n) \to (g - x^*)(x_0)$; hence g is U-convex at x_0. Assume now that $x_0 \neq 0$; because $\alpha_n \to \alpha_0 > 0$, we (may) assume that $\alpha_n > 0$ for $n \geq 1$. Taking $x_0' := \alpha_0^{-1}x_0$ ($\in S_X$) and $x_n' := \alpha_n^{-1}x_n$ ($\in S_X$), one clearly has $\|x_n'\|_W \to \|x_0'\|_W$. Using similar arguments to those from the end of Example 3.12, one obtains that $\|x_n' + x_0'\|_W \to 2$. Because X is weakly LUR, $x_n' \to^w x_0'$, and so $x_n = \alpha_n x_n' \to^w \alpha_0 x_0' = x_0$. Take $x^* \in \partial g(x_0)$ ($\neq \emptyset$ because g is continuous); then

$$(g - x^*)(x_n) = \|x_n\|_W^2 - \langle x_n, x^*\rangle \to \|x_0\|_W^2 - \langle x_0, x^*\rangle = (g - x^*)(x_0).$$

This shows that g is U-convex at x_0; hence g is U-convex on X. Observe that taking $x_0 := e_1$ and $x_n := e_1 + e_n$ for $n \geq 2$, one has

$$\tfrac{1}{2}g(x_n) + \tfrac{1}{2}g(x_0) - g\left(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0\right) = \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot(2 + \gamma_n^2) - (2 + \tfrac{1}{4}\gamma_n^2) = \tfrac{1}{4}n^{-2} \to 0,$$

but (x_n) does not have any convergent subsequence. Therefore, f is not CLUR at e_1.

(iv) We now illustrate that nearly U-convexity is weaker than U-convexity. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \max\{|x|, 2|x| - 1\}$. We show that f is CLUR at $x_0 := 0$, and so f is nearly U-convex at 0 by Proposition 4.9, but f is not U-convex at 0. Clearly f is convex and $\partial f(0) = [-1, 1]$. In order to show that f is CLUR at 0 it is sufficient to show that C_δ is compact for some $\delta > 0$, and so f is CLUR by Theorem 3.10. Indeed, for $\delta \in]0, 1/2[$ one has

$$x \in C_\delta \setminus [-2, 2] \Rightarrow \tfrac{1}{2}(2|x| - 1) - (2|\tfrac{1}{2}x| - 1) \leq \delta \Leftrightarrow \tfrac{1}{2} \leq \delta;$$

hence $C_\delta \subset [-2, 2]$ for $\delta \in]0, 1/2[$.

Consider now the sequence (x_n) with $x_n := (-1)^n \frac{n+2}{n+1}$ for $n \geq 1$; then

$$\tfrac{1}{2}f(x_n) + \tfrac{1}{2}f(0) - f(\tfrac{1}{2}x_n + \tfrac{1}{2}\cdot 0) = \tfrac{1}{2}\left(2\,\frac{n+2}{n+1} - 1\right) - \tfrac{1}{2}\,\frac{n+2}{n+1} = \frac{1}{2(n+1)} \to 0,$$

while for $u := x^* \in \partial f(0) = [-1, 1]$, one has

$$\gamma_n := (f - x^*)(x_n) - (f - x^*)(0) = 2\,\frac{n+2}{n+1} - 1 - u(-1)^n \frac{n+2}{n+1}.$$

Hence, $\gamma_n \to 1$ for $u = 0$ and (γ_n) has no limit for $u \neq 0$, and so f is not U-convex at 0.

In the following result LUR at a point is characterized in terms of strong convexity.

Proposition 4.11. *Assume that f is continuous at x_0. The following assertions are equivalent:*

(i) *f is LUR at x_0;*

(ii) *f is strongly convex at x_0 and U-convex at x_0.*

Proof. The proof is similar to that of Proposition 4.9, using now Corollary 3.11 and Proposition 4.7 instead of Theorem 3.10 and Proposition 4.6, respectively. □

In the following example we construct a function which is strongly convex at all points of X and it is not LUR at an element of X.

Example 4.12. We use the norm $\|\cdot\|_A$ on ℓ^2 discussed in [23]. The usual norm of ℓ^2 is denoted by $\|\cdot\|_2$. As in Example 4.10(iii) one take $\gamma_n := 1/n$ for $n \geq 1$, (e_n) the hilbertian basis of ℓ_2, and one sets $x' = (0, x_2, x_3, \cdots)$ for $x := (x_1, x_2, x_3, \cdots) \in \ell^2$; this time one considers

$$\|x\|_A := \left[(|x_1| + \|x'\|_2)^2 + \|(\gamma_2 x_2, \gamma_3 x_3, \cdots)\|_2^2\right]^{1/2}.$$

It is shown in [23] that $\|\cdot\|_A$ is equivalent to $\|\cdot\|_2$ and, moreover, that $X := (\ell_2, \|\cdot\|_A)$ is KK and strictly convex.

Let $g := \|\cdot\|_A^2$ and $x \in X$; then g is strictly convex. We first show that g is not LUR. Take $x_0 := e_1$ and $x_n := e_n$ for $n \geq 2$; then $\|x_0\|_A = 1$, $\|x_n\|_A = (1+\gamma_n^2)^{1/2} \to 1$ and $\|x_0 + x_n\|_A = (4 + \gamma_n^2)^{1/2} \to 2$. Therefore

$$\tfrac{1}{2}g(x_n) + \tfrac{1}{2}g(x_0) - g(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0) = \tfrac{1}{2} + \tfrac{1}{2}(1+\gamma_n^2) - \tfrac{1}{4}(4+\gamma_n^2) \to 0,$$

but (x_n) does not converge to x_0.

Since the space X is reflexive, strictly convex and KK, by Example 4.8(ii), g $(= 2f)$ is strongly convex at every point of X. □

Remark 3.2(ii) and the proof of Proposition 4.11 suggest the use of the following weaker notion of U-convexity for verifying the LUR property.

Definition 4.13. The function f is called *boundedly U-convex at* $x_0 \in \operatorname{dom} f$ if for every bounded sequence $(x_n) \subset \operatorname{dom} f$ satisfying $\frac{1}{2}f(x_n)+\frac{1}{2}f(x_0)-f(\frac{1}{2}x_n+\frac{1}{2}x_0) \to 0$ there exists $x^* \in \partial f(x_0)$ such that $(f - x^*)(x_n) \to (f - x^*)(x_0)$.

In the following result, we present a sufficient condition for bounded U-convexity.

Proposition 4.14. *If f is Fréchet differentiable at x_0 then f is boundedly U-convex at x_0.*

Proof. Let f be Fréchet differentiable at $x_0 \in \operatorname{dom} f$ and

$$\delta_n := \tfrac{1}{2}f(x_n) + \tfrac{1}{2}f(x_0) - f\left(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0\right) \to 0$$

for the bounded sequence $(x_n) \subset \operatorname{dom} f$. Since f is Fréchet differentiable at x_0, $\partial f(x_0) = \{x_0^*\}$ for some $x_0^* \in X^*$. We show that $(f - x_0^*)(x_n) \to (f - x_0^*)(x_0)$.

By Lemma 3.9(ii), $x_n \in A^X_{\delta'_n} f(x_0)$ for $n \geq 1$, where $\delta'_n := 2\delta_n \to 0$. Therefore, there exists $x_n^* \in X^*$ such that $x_n \in \partial_{\delta'_n} f^*(x_n^*)$ and $x_0 \in \partial_{\delta'_n} f^*(x_n^*)$. By the implication

(i) $\Rightarrow$ (v) of [9, Theorem 3.8], $x_n^* \to x_0^*$. As $x_n^* \in \partial_{\delta_n'} f(x_n)$ and $x_n^* \in \partial_{\delta_n'} f(x_0)$, for $x \in X$ and $n \geq 1$ we have

$$\langle x, x_n^* \rangle - f(x) \leq \langle x_n, x_n^* \rangle - f(x_n) + \delta_n', \quad \langle x, x_n^* \rangle - f(x) \leq \langle x_0, x_n^* \rangle - f(x_0) + \delta_n'.$$

Replacing x by x_0 in the first inequality and by x_n in the second one, one gets

$$(f - x_n^*)(x_n) \leq (f - x_n^*)(x_0) + \delta_n', \quad (f - x_n^*)(x_0) \leq (f - x_n^*)(x_n) + \delta_n' \quad \forall n \geq 1,$$

and so
$$(f - x_n^*)(x_n) - (f - x_n^*)(x_0) \to 0. \tag{31}$$

Because $x_n^* \to x_0^*$ and (x_n) is bounded, one clearly has

$$\langle x_n, x_n^* - x_0^* \rangle \to 0 \quad \text{and} \quad (f - x_n^*)(x_0) \to (f - x_0^*)(x_0),$$

which, together with (31), give $(f - x_0^*)(x_n) \to (f - x_0^*)(x_0)$. Therefore, f is boundedly U-convex at x_0. □

An easy consequence of Proposition 4.14 is the next result.

Corollary 4.15. *If $f := \|\cdot\|$ is Fréchet differentiable at $x_0 \in X$, then the function $g := \|\cdot\|^2$ is U-convex at x_0.*

Proof. Note that g is Fréchet differentiable at x_0 as the composition of two Fréchet differentiable functions. Let $(x_n) \subset X$ be such that

$$\tfrac{1}{2} g(x_n) + \tfrac{1}{2} g(x_0) - g(\tfrac{1}{2} x_n + \tfrac{1}{2} x_0) = \tfrac{1}{2} \|x_n\|^2 + \tfrac{1}{2} \|x_0\|^2 - \|\tfrac{1}{2} x_n + \tfrac{1}{2} x_0\|^2 \to 0.$$

Using the implication (21), proved in Example 3.12(ii), one gets the boundedness of (x_n). The conclusion follows from Proposition 4.14. □

5. Relationships with other properties

As in the previous sections, $f \in \Gamma(X)$ and $x_0 \in \operatorname{dom} f$; additional hypothesis will be mentioned when needed.

We first present the Kadec-Klee property for convex functions and relate it with CLUR. It is known that $(X, \|\cdot\|)$ is KK if it is CLUR (see [18, Th. 3.1]).

Definition 5.1. We say that f is *KK* at x_0 if $\partial f(x_0) \neq \emptyset$ and $x_n \to x_0$ whenever $x_n \to^w x_0$ is such that $f(x_n) \to f(x_0)$; f is *KK on $A \subset \operatorname{dom} f$* if f is KK at every $x \in A$. □

Notice that in [7] one says that f has the *Kadec property* if $x_n \to x$ whenever $x_n \to^w x \in \operatorname{dom} f$ and $f(x_n) \to f(x)$. Of course, if f is KK at x_0 and $\alpha \in \mathbb{P}$, then αf is KK at x_0. Moreover, it is worth observing that

$$(X, \|\cdot\|) \text{ is KK} \;\Leftrightarrow\; \|\cdot\| \text{ is KK on } S_X \text{ (on } X) \;\Leftrightarrow\; \|\cdot\|^2 \text{ is KK on } S_X \text{ (on } X). \tag{32}$$

Proposition 5.2. *Assume that f is CLUR at x_0; then f is KK at x_0.*

Proof. Because f is CLUR at x_0, $A_0^X f(x_0) \neq \emptyset$ by Theorem 3.10, and so $\partial f(x_0) \neq \emptyset$. Take $(x_n) \subset \operatorname{dom} f$ such that $x_n \to^w x_0$ and $f(x_n) \to f(x_0)$. It is sufficient to show that (x_n) has a subsequence $\|\cdot\|$-converging to x_0.

For this, set $\delta_n := \frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \geq 0$. Then

$$\begin{aligned} f(x_0) &= f(\tfrac{1}{2}x_0 + \tfrac{1}{2}x_0) \leq \liminf f(\tfrac{1}{2}x_n + \tfrac{1}{2}x_0) = \liminf \big(\tfrac{1}{2}f(x_n) + \tfrac{1}{2}f(x_0) - \delta_n\big) \\ &= \tfrac{1}{2}\lim f(x_n) + \tfrac{1}{2}f(x_0) - \limsup \delta_n = f(x_0) - \limsup \delta_n, \end{aligned}$$

and so $\limsup \delta_n \leq 0$; hence $\delta_n \to 0$. Because f is CLUR at x_0, there exists a subsequence (x_{n_k}) of (x_n) converging to $x \in X$; therefore, $x_{n_k} \to^w x$, and so $x = x_0$ because the w-limit is unique. □

A function which is KK need not be CLUR (see, e.g., Example 4.10(ii)).

We now present two notions called locally nearly uniformly smooth (LNUS) and locally nearly uniformly convex (LNUC). We relate CLUR with LNUS and LNUC.

The space X is said to be *locally nearly uniformly smooth (LNUS)* (see, e.g., [2, 3, 5]) if

$$\alpha\big(\{x^* \in U_{X^*} \mid \langle x, x^*\rangle \geq 1 - 1/n\}\big) \to 0 \quad \forall x \in S_X.$$

Note that, for $x \in S_X$ and $n \geq 1$, $\{x^* \in U_{X^*} \mid \langle x, x^*\rangle \geq 1 - 1/n\} = \partial_{1/n} \|\cdot\| (x)$.

We now generalize the notion recalled above to convex functions.

Definition 5.3. The function f is said to be *LNUS at* x_0 if $\alpha\big(\partial_{1/n} f(x_0)\big) \to 0$; f is *LNUS on* $A \subset \operatorname{dom} f$ if $\alpha\big(\partial_{1/n} f(x)\big) \to 0$ for every $x \in A$.

Remark 5.4. (i) If f is LNUS at x_0 then $\partial f(x_0)$ is non-empty and compact by the previous definition and Proposition 2.2, which also implies that $x_0 \in \operatorname{qi}(\operatorname{dom} f)$. This is due to the fact that $\partial f(x_0) = \partial f(x_0) + N(\operatorname{dom} f, x_0)$ and that for $A \subset X$ a convex set, its *quasi interior* is $\operatorname{qi} A := \big\{a \in A \mid N(A, a) = \{0\}\big\}$ with $N(A, a) := \partial\iota_A(a)$; if $\operatorname{int} A \neq \emptyset$, then $\operatorname{qi} A = \operatorname{int} A$.

(ii) Observe that X is LNUS if and only if the function $f := \|\cdot\|$ is LNUS on S_X.

(iii) f is directionally Fréchet differentiable at x_0 and $\partial f(x_0)$ is compact if and only if f is continuous at x_0 and LNUS at x_0 (see [9, Theorem 3.7])

(iv) f is Fréchet differentiable at x_0 if and only if f is continuous at x_0, LNUS at x_0 and $\partial f(x_0)$ is a singleton (see [9, Theorem 3.8]). □

Having in view that $\partial \|\cdot\| (0) = U_{X^*}$ and $(U_{X^*} \supset)\ \partial \|\cdot\| (x)$ is closed, it follows that $[f := \|\cdot\|$ is LNUS on $X] \Leftrightarrow \dim X < \infty$; of course, f is not Gateaux (and so not Fréchet) differentiable at 0.

The space X is said to be *LNUC* (see [3, Def. 1]) if

$$\alpha\big(\{x \in U_X \mid \langle x, x^*\rangle \geq 1 - 1/n\}\big) \to 0 \quad \forall x^* \in S_{X^*}.$$

Note that, for $x^* \in S_{X^*}$ and $n \geq 1$, $\{x \in U_X \mid \langle x, x^*\rangle \geq 1 - 1/n\} = \partial^X_{1/n} \|\cdot\| (x^*)$. Of course, if X is LNUC then for every $x^* \in S_{X^*}$ there exists $x \in S_X$ such that $\langle x, x^*\rangle = 1$.

The LNUC property is referred as "drop property" in [2, 17, 20].

We first relate LNUC with near strong convexity. The relation between the LNUC property and nearly strongly convex Banach spaces are listed in the following known result.

Proposition 5.5. *The following statements are equivalent:*

(i) *X is LNUC;*

(ii) *X is reflexive and KK;*

(iii) *X is reflexive and nearly strongly convex on S_X.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is proved in [17, 20] (see [9, 21] for alternative proofs). The equivalence (i) $\Leftrightarrow$ (iii) follows easily from the definitions. $\square$

Definition 5.6. We say that f is *locally nearly uniformly convex (LNUC) with respect to* $x^* \in \operatorname{dom} f^*$ if $\alpha\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$; f is *LNUC with respect to* $\mathcal{A} \subset \operatorname{dom} f^*$ if $\alpha\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$ for every $x^* \in \mathcal{A}$. $\square$

Remark 5.7. Having in view the relations (24) and (25), f is LNUC with respect to $x^* \in \operatorname{dom} f^*$ if and only if one of the conditions (i)–(viii) from Proposition 3.13 is verified; in particular, if f is LNUC with respect to $x^* \in \operatorname{dom} f^*$ then $\partial f^*(x^*)$ is compact and $\partial^X f^*(x^*) = \partial f^*(x^*) \neq \emptyset$. $\square$

We now see some relations between LNUC, near strong convexity and KK property of functions.

Proposition 5.8. *Assume that $\partial f(x_0) \neq \emptyset$. Consider the following statements:*

(i) *f is nearly strongly convex at x_0;*

(ii) *f is LNUC with respect to $\partial f(x_0)$;*

(iii) *f is KK at x_0.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Leftrightarrow$ (ii) Follows from the equivalence (i) $\Leftrightarrow$ (ii) in Proposition 4.6.

(i) $\Rightarrow$ (iii) Suppose $(x_n) \subset \operatorname{dom} f$ is such that $x_n \to^w x_0$ and $f(x_n) \to f(x_0)$. Let $x^* \in \partial f(x_0)$. Then $(f - x^*)(x_n) \to (f - x^*)(x_0)$. By the definition of near strong convexity, (x_n) has a convergent subsequence; in fact, $x_n \to x_0$ because $x_n \to^w x_0$. This shows that f is KK at x_0. $\square$

Corollary 5.9. *Consider the following statements:*

(i) *X is LNUC;*

(ii) *the function ι_{U_X} is LNUC with respect to S_{X^*};*

(iii) *the function $f := \frac{1}{2}\|\cdot\|^2$ is LNUC with respect to S_{X^*}.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). *If X is reflexive then* (iii) $\Rightarrow$ (i).

Proof. (i) $\Leftrightarrow$ (ii) This equivalence follows directly from the definitions.

(i) $\Rightarrow$ (iii) By Proposition 5.5, X is reflexive and KK, and so f is nearly strongly convex, as shown in Example 4.8(ii). Let $x_0^* \in S_{X^*}$; since X is LNUC, there exists $x_0 \in S_X$ such that $\langle x_0, x_0^*\rangle = 1$ as already observed. Since $x_0^* \in \partial\|\cdot\|(x_0)$ we obtain $x_0^* \in \partial f(x_0)$. Hence f is LNUC wrt x_0^* by Proposition 5.8.

(iii) $\Rightarrow$ (i) Assume that X is reflexive. Let $x_0 \in S_X$; because $\emptyset \neq \partial f(x_0) \subset S_{X^*}$, condition (ii) of Proposition 5.8 holds, and so f is KK at x_0. As $x_0 \in S_X$ is arbitrary, X is KK by (32), and so X is LNUC by Proposition 5.5. $\square$

In the case of strong convexity we have the following characterizations besides those of Proposition 4.7.

Proposition 5.10. *Assume that $\partial f(x_0) \neq \emptyset$. Consider the following statements:*

(i) *f is strongly convex at x_0;*

(ii) *f is LNUC with respect to $\partial f(x_0)$ and $\partial^X f^*(x^*) = \{x_0\}$ for all $x^* \in \partial f(x_0)$;*

(iii) *f is KK at x_0 and $\partial^X f^*(x^*) = \{x_0\}$ for all $x^* \in \partial f(x_0)$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). *If $\partial f(x_0) \subset \operatorname{int}(\operatorname{dom} f^*)$ and X is reflexive then* (iii) $\Rightarrow$ (i).

Proof. Having in view Proposition 5.8, (ii) is equivalent to the fact that f is nearly strongly convex at x_0 and $\partial^X f^*(x^*) = \{x_0\}$ for all $x^* \in \partial f(x_0)$, and so (ii) $\Leftrightarrow$ (i) by Remark 4.4(v).

(ii) $\Rightarrow$ (iii) This is an immediate consequence of Proposition 5.8.

(iii) $\Rightarrow$ (i) Suppose that $\partial f(x_0) \subset \operatorname{int}(\operatorname{dom} f^*)$ and X is reflexive. In order to apply Proposition 4.7, we show that $\operatorname{diam}\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$ for every $x^* \in \partial f(x_0)$. Let $x^* \in \partial f(x_0)$ and take $(x_n) \subset \partial^X_{1/n} f^*(x^*)$. Because $x^* \in \operatorname{int}(\operatorname{dom} f^*)$, f^* is continuous at x^* (being lsc), and so $\partial_\delta f^*(x^*)$ is w^*-compact (hence $\|\cdot\|$-bounded) for $\delta \in \mathbb{R}_+$; hence (x_n) is bounded. Since X is reflexive, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to^w y_0$ for some $y_0 \in X$. Since $x_n \in \partial^X_{1/n} f^*(x^*)$ one directly has $f(x_n) + f^*(x^*) \le \langle x_n, x^* \rangle + 1/n$ for all n; hence

$$
\begin{aligned}
f(y_0) &\le \liminf f(x_{n_k}) \le \limsup f(x_{n_k}) \le \lim \big(\langle x_{n_k}, x^* \rangle + 1/n_k - f^*(x^*)\big) \\
&= \langle y_0, x^* \rangle - f^*(x^*).
\end{aligned}
$$

Therefore, $f(y_0) = \langle y_0, x^* \rangle - f^*(x^*)$, whence $x^* \in \partial f(y_0)$ and $f(x_{n_k}) \to f(y_0)$. As $\partial^X f^*(x^*) = \{x_0\}$, one has $x_0 = y_0$. Since f is KK at x_0, $x_{n_k} \to x_0$; in fact, $x_n \to x_0$. This shows that $\operatorname{diam}\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$, and so (i) holds by Proposition 4.7. $\square$

The relationships between the concepts of LNUC and LNUS are established in the following result. We will see that the main results of [2] can be obtained as consequences of the following theorem.

Theorem 5.11. *Besides x_0 consider $x_0^* \in \operatorname{dom} f^*$. The following assertions hold:*

(i) *f is LNUC with respect to x_0^* if and only if f^* is LNUS at x_0^*;*

(ii) *f is LNUS at x_0 if and only if f^* is LNUC with respect to x_0.*

Proof. (i) Taking into account (23) and (24), one has

$$
\alpha\big(\partial^X_{1/n} f^*(x_0^*)\big) \to 0 \;\Leftrightarrow\; \alpha\big(\partial_{1/n} f^*(x_0^*)\big) \to 0
$$

by the equivalence (i) $\Leftrightarrow$ (ii) from Proposition 3.13. Having in view Definitions 5.3 and 5.6, one obtains that f^* is LNUS at $x_0^* \in \operatorname{dom} f^*$ iff f is LNUC wrt x_0^*.

(ii) By Definition 5.6, f^* is LNUC wrt $x^{**} \in \operatorname{dom}(f^*)^*$ if $\alpha\big(\partial^{X^*}_{1/n} f^{**}(x^{**})\big) \to 0$. In the present case $x^{**} := x_0$; having in view that $f^{**}|_X = f$ because $f \in \Gamma(X)$ (see, e.g., [9, Lem. 4.1]), one has

$$
\begin{aligned}
\partial^{X^*}_{1/n} f^{**}(x_0) &= \{x^* \in X^* \mid f^*(x^*) + f^{**}(x_0) \le \langle x^*, x_0 \rangle + 1/n\} \\
&= \{x^* \in X^* \mid f^*(x^*) + f(x_0) \le \langle x_0, x^* \rangle + 1/n\} = \partial_{1/n} f(x_0).
\end{aligned}
$$

Therefore, f^* is LNUC wrt x_0 iff f is LNUS at x_0, that is, the conclusion holds. $\square$

Corollary 5.12. ([2, Ths. 3, 4]) *The following assertions hold:*

(i) *X is LNUC if and only if X^* is LNUS;*

(ii) *X^* is LNUC if and only if X is reflexive and LNUS.*

Proof. To be more clear, in the proof, the dual norms on X^* and X^{**} are denoted by $\|\cdot\|_{X^*}$ and $\|\cdot\|_{X^{**}}$, respectively.

(i) Consider $f := \iota_{U_X}$ ($\in \Gamma(X)$); notice that $f^* = \|\cdot\|_{X^*}$, and so $S_{X^*} \subset \operatorname{dom} f^* = X^*$. Applying Theorem 5.11(i) one gets

$$X \text{ is LNUC } \Leftrightarrow \ f \text{ is LNUC wrt } S_{X^*} \ \Leftrightarrow \ f^* \text{ is LNUS on } S_{X^*} \ \Leftrightarrow \ X^* \text{ is LNUS.}$$

(ii) If X^* is LNUC then X^* is reflexive by Proposition 5.5, and so X is reflexive (being complete); so assume that X is reflexive, and so $X^{**} = X$ and $S_X = S_{X^{**}}$. Consider $f = \|\cdot\|$ ($\in \Gamma(X)$); then $f^* = \iota_{U_{X^*}}$ and $f^{**} = f$. Using Theorem 5.11 (ii), one gets

$$X \text{ is LNUS} \Leftrightarrow f \text{ is LNUS on } S_X \ \Leftrightarrow \ f^* \text{ is LNUC wrt } S_X \ (= S_{X^{**}}) \Leftrightarrow X^* \text{ is LNUC,}$$

that is, the conclusion holds. □

Definition 5.13. The function f is said to be *strongly convex with respect to* $x^* \in \operatorname{dom} f^*$ if $\operatorname{diam}\left(\partial^X_{1/n} f^*(x^*)\right) \to 0$; f is said to be *strongly convex with respect to* $\mathcal{A} \subset \operatorname{dom} f^*$ if $\operatorname{diam}\left(\partial^X_{1/n} f^*(x^*)\right) \to 0$ for every $x^* \in \mathcal{A}$. □

Assertion (i) of the following result is provided by the equivalence (i) ⇔ (viii) from Corollary 3.15, while assertion (ii) is the equivalence (i) ⇔ (ii) from Proposition 4.7.

Proposition 5.14. (i) *Let $x_0^* \in \operatorname{dom} f^*$; then f is strongly convex with respect to x_0^* if and only if f^* is Fréchet differentiable at x_0^*. Moreover, if one of the conditions holds, $\nabla f^*(x_0^*) \in X$.*

(ii) *Assume that $\partial f(x_0) \neq \emptyset$; then f is strongly convex with respect to $\partial f(x_0)$ if and only if f is strongly convex at x_0.*

If one takes $f := \iota_{U_X}$ and applies Corollary 5.14(i) for each $x^* \in S_{X^*}$ ($\subset \operatorname{dom} f^* = X^*$), one obtains Smulian's theorem (see [16, Th. 5.6.9]).

Corollary 5.15. *If f^* is strongly convex with respect to x_0 then f is Fréchet differentiable at x_0. The converse is true if X is reflexive.*

Proof. Assume that f^* is strongly convex wrt x_0 ($\in \operatorname{dom} f \subset \operatorname{dom} f^{**}$). By Proposition 5.14, f^{**} is Fréchet differentiable at x_0 and $x^* := \nabla f^{**}(x_0) \in X^*$, and so, clearly, f is Fréchet differentiable at x_0 with $\nabla f(x_0) = x^*$.

Assume now that X is reflexive and f is Fréchet differentiable at x_0 ($\in \operatorname{dom} f$). Set $Y := X^*$ (and so $Y^* = X$), and consider $g := f^*$ ($\in \Gamma(Y)$); hence g^* ($= f$) is Fréchet differentiable at $x_0 \in \operatorname{dom} g^* = \operatorname{dom} f \subset Y^*$. By Proposition 5.14(i), one obtains that f^* ($= g$) is strongly convex wrt x_0. □

If one takes $f := \|\cdot\|$ and applies Corollary 5.15 for each $x \in S_X$ ($\subset \operatorname{dom} f$), one obtains [16, Cor. 5.6.10].

Theorem 5.16. *Assume that f is continuous and LNUS at x_0. Then the following assertions hold:*

(i) *f is CLUR at x_0 if and only if f is nearly strongly convex at x_0 and condition (6) holds;*

(ii) *f is LUR at x_0 if and only if f is strongly convex at x_0.*

Proof. (i) Assume first that f is CLUR at x_0. Then f is nearly strongly convex at x_0 by Proposition 4.9; moreover, condition (6) holds by Remark 3.3.

Conversely, assume that f is nearly strongly convex at x_0 and condition (6) holds. Take $(x_n) \subset (C_{1/n})$, and so $(0 \leq)\ \gamma_n := \frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f\left(\frac{1}{2}x_n + \frac{1}{2}x_0\right) \leq 1/n$, whence $\gamma_n \to 0$. Then (x_n) is bounded because condition (6) is verified, and so there exists $\mu \in \mathbb{R}_+$ such that $\|x_n\| \leq \mu$ for $n \geq 1$. Using Lemma 3.9(ii), one has that $(x_n) \subset \left(A^X_{2/n}\right)$. By the definition of A^X_δ, for each $n \geq 1$, there exists $x^*_n \in \partial_{2/n} f(x_n) \cap \partial_{2/n} f(x_0)$, and so

$$f(x_n) + f^*(x^*_n) \leq \langle x_n, x^*_n\rangle + 2/n, \quad f(x_0) + f^*(x^*_n) \leq \langle x_0, x^*_n\rangle + 2/n \quad \forall n \geq 1.$$

Because $(x^*_n) \subset \big(\partial_{2/n} f(x_0)\big)$ and f is LNUS at x_0 (that is, $\alpha\big(\partial_{1/n} f(x_0)\big) \to 0$), there exists a subsequence $(x^*_{n_k})$ of (x^*_n) converging to $x^*_0 \in \partial f(x_0)$ $(\subset \operatorname{dom} f^*)$ by Proposition 2.2. Set $\beta_k := f^*(x^*_{n_k}) - f^*(x^*_0)$ for $k \geq 1$. On one hand, one has $\liminf \beta_k = \liminf f^*(x^*_{n_k}) - f^*(x^*_0) \geq f^*(x^*_0) - f^*(x^*_0) = 0$ because f^* is lsc at x^*_0, and so $\liminf \beta_k \geq 0$. One the other hand

$$\begin{aligned} 0 \leq f(x_{n_k}) + f^*(x^*_0) - \langle x_{n_k}, x^*_0\rangle &= f(x_{n_k}) + f^*(x^*_{n_k}) - \langle x_{n_k}, x^*_0\rangle - \beta_k \\ &\leq \left\langle x_{n_k}, x^*_{n_k} - x^*_0\right\rangle + 2/n_k - \beta_k \leq \mu \left\|x^*_{n_k} - x^*_0\right\| + 2/n_k - \beta_k =: \delta_k \end{aligned}$$

for $k \geq 1$; hence $\limsup \beta_k \leq 0$ because $x^*_{n_k} \to x^*_0$, and so $\delta_k \to 0$. Therefore, $(f - x^*_0)(x_{n_k}) \to -f^*(x^*_0) = (f - x^*_0)(x_0)$. Since f is nearly strongly convex at x_0, (x_{n_k}) has a convergent subsequence $(x_{n_{k_p}})$; as $(x_{n_{k_p}})$ is also a subsequence of (x_n), it follows that f is CLUR at x_0.

(ii) Assume first that f is LUR at x_0; then f is strongly convex at x_0 by Proposition 4.11. Conversely, assume that f is strongly convex at x_0. Having in view Corollary 3.5(ii), it is sufficient to prove that any bounded sequence $(x_n) \subset (C_{1/n})$ has a subsequence (x_{n_k}) converging to x_0. The proof is exactly the same as in the proof of the implication "$\Leftarrow$" in (i) until the place where one gets that $(f - x^*_0)(x_{n_k}) \to -f^*(x^*_0) = (f - x^*_0)(x_0)$. Now, because f is strongly convex at x_0, the sequence (x_{n_k}) converges to x_0 by Remark 4.4(iii), and so $x_n \to x_0$ because x_0 is the only $\|\cdot\|$-limit point of (x_n). □

Corollary 5.17. *Assume that f is continuous at x_0, f^* is LNUC with respect to x_0 and condition (6) is verified. Then the following assertions hold:*

(i) *f is nearly strongly convex at x_0 if and only if f is CLUR at x_0;*

(ii) *f is LNUC with respect to $\partial f(x_0)$ if and only if f is CLUR at x_0;*

(iii) *if f is strongly convex with respect $\partial f(x_0)$ then f is LUR at x_0.*

Proof. Because f^* is LNUC wrt x_0, f is LNUS at x_0 by Theorem 5.11(ii).

(i) Having in view the previous observation, this is nothing else but assertion (i) from Theorem 5.16.

(ii) Because f is nearly strongly convex at x_0 iff f is LNUC wrt $\partial f(x_0)$ by Proposition 5.8, the conclusion follows.

(iii) By Corollary 5.14(ii), one has that f is strongly convex at x_0, and so the conclusion follows from Theorem 5.16(ii). □

The following result, which is a counterpart of Theorem 3.5 from [18], is immediate from Corollary 5.17.

Corollary 5.18. *Assume that $A := \operatorname{int}(\operatorname{dom} f)$ is nonempty, f^* is LNUC with respect to A and condition* (6) *is verified for $x \in A$. Then*

(i) *f is nearly strongly convex on A if and only if f is CLUR on A;*

(ii) *if f is LNUC with respect to* $\operatorname{dom} f^*$ *then f is CLUR on A;*

(iii) *if f is strongly convex with respect to* $\operatorname{dom} f^*$ *then f is LUR on A.*

The following result is a counterpart of Theorem 3.3 from [18]. Compare the following result with Theorem 5.11(ii).

Proposition 5.19. *Assume that $x_0^* \in \partial f(x_0)$. The following assertions hold:*

(i) *If f^* is CLUR at x_0^*, then f is LNUS at x_0; in particular, $\partial f(x_0)$ is compact.*

(ii) *If f is continuous at x_0 and f^* is LUR at x_0^*, then f is Fréchet differentiable x_0 and $\nabla f(x_0) = x_0^*$.*

Proof. (i) Assume that f^* is CLUR at x_0^*, and consider $(x_n^*) \subset \big(\partial_{1/n} f(x_0)\big)$; then $x_0 \in \partial_{1/n} f^*(x_n^*) \cap \partial f^*(x_0^*)$ for $n \geq 1$. Therefore, for $y^* \in \operatorname{dom} f^*$ and $n \geq 1$, one has

$$\langle x_0, y^* - x_n^* \rangle \leq f^*(y^*) - f^*(x_n^*) + 1/n \quad \text{and} \quad \langle x_0, y^* - x_0^* \rangle \leq f^*(y^*) - f^*(x_0^*).$$

Hence, for $y^* := \frac{1}{2}x_n^* + \frac{1}{2}x_0^*$ ($\in \operatorname{dom} f^*$) and $\delta_n := \frac{1}{2}f^*(x_n^*) + \frac{1}{2}f^*(x_0^*) - f(\frac{1}{2}x_n^* + \frac{1}{2}x_0^*)$, one gets

$$\delta_n \leq \tfrac{1}{2}\big(f^*(y^*) - \langle x_0, y^* - x_n^* \rangle + \frac{1}{n}\big) + \tfrac{1}{2}\left(f^*(y^*) - \langle x_0, y^* - x_0^* \rangle\right) - f(\tfrac{1}{2}x_n^* + \tfrac{1}{2}x_0^*) = \frac{1}{2n},$$

and so $\delta_n \to 0$. Because f^* is CLUR at x_0^* and $\delta_n \to 0$, (x_n^*) has a convergent subsequence. By Proposition 2.2, $\alpha\big(\partial_{1/n} f(x_0)\big) \to 0$, and so f is LNUS at x_0 by Definition 5.3.

(ii) Assume now that f is continuous at x_0 and f^* is LUR at x_0^*. Take (x_n^*) and (δ_n) as in the proof of (i). Since $\delta_n \to 0$ and f^* is LUR at x_0^*, one has $x_n^* \to x_0^*$ by Definition 3.1(i). It follows that $\operatorname{diam}\big(\partial_{1/n} f(x_0)\big) \to 0$, and so $\lim_{0<\delta\to 0} \operatorname{diam}\big(\partial_\delta f(x_0)\big) = 0$. Having in view that f is continuous at x_0, one obtains that f is Fréchet differentiable at x_0 by the implication (ii) $\Rightarrow$ (i) of [9, Th. 3.8]. Because $x_0 \in \partial f^*(x^*)$, one also has $\nabla f(x_0) = x_0^*$. □

From Proposition 5.19(ii) one obtains [8, Prop. 5.3.6(b)].

Proposition 5.20. *Let $x^* \in \operatorname{dom} f^*$. Consider the following statements:*

(i) *f is LNUC with respect to x^*;*

(ii) *$\partial^X f^*(x^*) \neq \emptyset$ and f is KK on $\partial^X f^*(x^*)$.*

Then (i) $\Rightarrow$ (ii). *If $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ and X is reflexive, then* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) One observed in Remark 5.7 that $\partial^X f^*(x^*) \neq \emptyset$ when f is LNUC wrt x^*. Take $x \in \partial^X f^*(x^*)$ ($\Leftrightarrow x^* \in \partial f(x)$). Suppose $x_n \to^w x$ and

$f(x_n) \to f(x)$. Then $(f - x^*)(x_n) \to (f - x^*)(x) = -f^*(x^*) = \inf(f - x^*)$. Setting $\delta_n := (f - x^*)(x_n) + f^*(x^*) + 1/n > 0$, one has $\delta_n \to 0$ and $x_n \in \partial^X_{\delta_n} f^*(x^*)$ for $n \geq 1$. Since f is LNUC wrt to x^*, by (24) and Proposition 2.2, (x_n) has a convergent subsequence. In fact, $x_n \to x$ because $x_n \to^w x$, and so f is KK at x.

(ii) $\Rightarrow$ (i) Suppose X is reflexive and $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Take $(x_n) \subset \partial^X_{1/n} f^*(x^*)$. Because f^* is continuous at x^*, $\partial_\delta f^*(x^*)$ is w^*-compact for $\delta \in \mathbb{R}_+$; hence (x_n) is bounded. Since X is reflexive, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to^w y_0$ for some $y_0 \in X$. By repeating the argument involved in the proof of (iii) $\Rightarrow$ (i) of Proposition 5.10, we obtain that $f(x_{n_k}) \to f(y_0)$ and $y_0 \in \partial^X f^*(x^*)$. Since f is KK at y_0, $x_{n_k} \to y_0$. By Proposition 2.2, $\alpha\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$, and so (i) holds.

Proposition 5.21. *Let $\emptyset \neq \mathcal{A} \subset \operatorname{dom} f^*$ be such that $\mathcal{A} = \partial f\,((\partial f)^{-1}(\mathcal{A}))$. Then f is LNUC with respect to $\mathcal{A}$ if and only if $\partial^X f^*(x^*) \neq \emptyset$ for every $x^* \in \mathcal{A}$ and f is nearly strongly convex on the set $\bigcup\{\partial^X f^*(x^*) \mid x^* \in \mathcal{A}\}$.*

Proof. Set $A := \bigcup\{\partial^X f^*(x^*) \mid x^* \in \mathcal{A}\} = \{x \in X \mid \exists x^* \in \mathcal{A} : x^* \in \partial f(x)\} = (\partial f)^{-1}(\mathcal{A})$; hence $\mathcal{A} = (\partial f)(A)$.

Suppose f is LNUC wrt $\mathcal{A}$; then $\partial^X f^*(x^*) \neq \emptyset$ for every $x^* \in \mathcal{A}$ by Remark 5.7.

Consider $x \in A$. Then there exists $x^* \in \mathcal{A}$ such that $x^* \in \partial f(x)$; hence $\partial f(x) \neq \emptyset$ and $x \in (\partial f)^{-1}(x^*) \subset (\partial f)^{-1}(\mathcal{A})$, whence $\partial f(x) \subset \partial f\,((\partial f)^{-1}(\mathcal{A})) = \mathcal{A}$. Let $y^* \in \partial f(x)$; then $y^* \in \mathcal{A}$ because $\partial f(x) \subset \mathcal{A}$ as seen above. Since f in LNUC wrt y^* (as $y^* \in \mathcal{A}$), one has $\alpha\big(\partial^X_{1/n} f^*(y^*)\big) \to 0$. Because $\partial f(x) \neq \emptyset$ and $y^* \in \partial f(x)$ is arbitrary, f is nearly strongly convex at x by the implication (ii) $\Rightarrow$ (i) from Proposition 4.6. Since $x \in A$ is arbitrary, one has that f is nearly strongly convex on A.

Suppose now that f is nearly strongly convex on A and $\partial^X f^*(x^*) \neq \emptyset$ for every $x^* \in \mathcal{A}$ and consider $x^* \in \mathcal{A}$ $\big(= (\partial f)(A)\big)$; then there exists $x \in A$ such that $x^* \in \partial f(x)$. Because f is nearly strongly convex on A, one has that $\alpha\big(\partial^X_{1/n} f^*(x^*)\big) \to 0$ by the implication (i) $\Rightarrow$ (ii) from Proposition 4.6, and so f is LNUC wrt x^*. As $x^* \in \mathcal{A}$ is arbitrary, f is LNUC wrt $\mathcal{A}$. □

Remark 5.22. Notice that $\mathcal{A} := \operatorname{Im} \partial f$ satisfies the conditions assumed in Proposition 5.21. □

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