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General Formulation of the S-Procedure

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Dedicated to Jean-Paul Penot on the occasion of his 80th birthday.

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We extend the S-procedure formulation to the general perturbational duality frame and give necessary and sufficient conditions for its validity, pointing out the hidden part of convexity.

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1. Introduction

Let X be a nonempty set, Y a locally convex Hausdorff topological vector space and $F : X \times Y \to \overline{\mathbb{R}}$ an extended real-valued function. What we call generalized S-procedure, or S_F-procedure, is the extension of the relation

$$F(x, 0_Y) \geq 0 \qquad \text{for all } x \in X \tag{1}$$

by using continuous linear forms on Y, 0_Y denoting the zero element of Y. Namely, we consider the following property

$$\exists \lambda \in Y' \text{ such that } F(x,y) + \langle \lambda, y \rangle \geq 0, \ \forall (x,y) \in X \times Y \tag{2}$$

where Y' is the topological dual of Y, and $\langle \lambda, y \rangle := \lambda(y)$ is the standard bilinear form.

Definition 1.1. We say that the S_F-procedure is *valid* if implication (1)⇒(2) holds, or equivalently, (1)⇔(2). □

We note that the S_F-procedure is valid if and only if exactly one of the statements: (2) and Not(1) holds. The classical S-procedure is the particular case in which $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$ and F is defined by

$$F(x,y) = \begin{cases} f(x) & \text{if } g_i(x) \leq y_i, \ i = 1, \dots, p \\ +\infty & \text{else,} \end{cases}$$

where $y := (y_1, \dots, y_p) \in \mathbb{R}^p$, f and g_i, $i = 1, \dots, p$ are real-valued functions on $\mathbb{R}^n$.

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The S-procedure is valid if and only if the property below

$$g_i(x) \leq 0, \ i = 1, \dots, p \Longrightarrow f(x) \geq 0$$

implies, where $\mathbb{R}_+ := [0, +\infty[$,

$$\exists \alpha := (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \text{ s.t. } f(x) + \sum_{i=1}^{p} \alpha_i g_i(x) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

A rather general result for the validity of the S-procedure recently established in [3] Theorem 3, involves two conditions:

(i) (Slater's type condition) $\exists a \in \mathbb{R}^n$ such that $g_i(a) < 0, \ i = 1, \dots, p$.

(ii) $\exists\, Z \subset (\mathbb{R}_+)^{p+1}$ such that $(0_{\mathbb{R}^p}, 0) \in Z$ and the cone $\mathbb{R}_+\Big(Z + \bigcup_{x \in \mathbb{R}^n} \{(g_1(x), \dots, g_p(x), f(x))\}\Big)$ is convex.

The final Section 5 of this paper is devoted to some complements of this approach in terms of conic constraints.

1.1. Notation

For a set A in a locally convex Hausdorff space V, $\bar{A}$, $\operatorname{co} A$, $\overline{\operatorname{co}} A$ stand for the closure, the convex hull, the closed convex hull of A respectively. The positive hull of A is the set

$$\operatorname{pos}(A) := \Big\{ \sum_{i=1,\dots,m} \alpha_i a_i \ : \ m \geq 1, \ \alpha_i \geq 0, a_i \in A, i = 1, \dots, m \Big\}, \ \operatorname{pos}(\emptyset) = \{0_V\}.$$

It is the convex cone containing the origin generated by A. We set $\overline{\operatorname{pos}}(A) := \overline{\operatorname{pos}(A)}$.

Given another set B in V, one says that B is closed regarding A (resp. closed convex regarding A) if $\overline{B} \cap A = B \cap A$ (resp. $\overline{\operatorname{co}}(B) \cap A = B \cap A$). These concepts are important for the duality in optimization (see for instance [1], [6]). We denote by δ_A the *indicator function* of $A \subset V$ ($\delta_A(v) = 0$ if $v \in A$, $\delta_A(v) = +\infty$ if $v \in V \setminus A$).

A function $k : V \to \overline{\mathbb{R}}$ is said to be *proper* if $\operatorname{dom} k := \{v \in V : \ k(v) < +\infty\}$ is nonempty and $k(v) \neq -\infty$ for all $v \in V$. The *epigraph* and the *strict epigraph* of k are denoted

$$\operatorname{epi} k := \{(v, r) \in V \times \mathbb{R} : \ k(v) \leq r\} \text{ and } \operatorname{epi}_s k := \{(v, r) \in V \times \mathbb{R} : \ k(v) < r\}$$

respectively. We denote by $[k \leq r] := \{v \in V : \ k(v) \leq r\}$ the sublevel of k at the level r. The conjugate of k is given by $k^*(v') := \sup_{v \in V} \{\langle v', v\rangle - h(v)\}, \ v' \in V'$. Here $\langle v', v\rangle := v'(v)$ denotes the value of the continuous linear form $v' \in V'$ at $v \in V$. For a function $\xi : V' \to \overline{\mathbb{R}}$ the *conjugate* is defined similarly by

$$\xi^*(v) = \sup_{v' \in V'} \{\langle v', v\rangle - \xi(v')\}, \ v \in V.$$

We denote by $k^{**} := (k^*)^*$ the *biconjugate* of k and by $\xi^{**} := (\xi^*)^*$ the biconjugate of ξ.

For the function $F : X \times Y \to \overline{\mathbb{R}}$ the projection of $\operatorname{epi} F$ on $Y \times \mathbb{R}$ is denoted by $\mathcal{F}$; more precisely: $\mathcal{F} := \{(y, r) \in Y \times \mathbb{R} \ : \ \exists x \in X, F(x, y) \leq r\}$.

We denote by D the projection of $\mathcal{F}$ on Y which is also the projection of epi F on Y. Throughout this paper, we assume that $\mathcal{F} \neq \emptyset$, which means $epiF \neq \emptyset$, $D \neq \emptyset$.

When X is a Hausdorff locally convex space (as Y), the *projection* of the epigraph of the conjugate F^* of F on $X' \times \mathbb{R}$ is denoted $\mathcal{F}^{\#}$, that is

$$\mathcal{F}^{\#} := \{(x', s) \in X' \times \mathbb{R} : \ \exists \mu \in Y', F^*(x', \mu) \leq s\}.$$

As usual, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ denotes the extended real line, $\mathbb{R}_- := -\mathbb{R}_+$, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$, $\mathbb{R}^*_+ := \mathbb{R}_+ \setminus \{0\}$, $\mathbb{R}^*_- := -\mathbb{R}^*_+$. The *relative interior* of a nonempty convex subset $A \subset \mathbb{R}^p$ is denoted by ri(A). A function q on the standard Euclidean space $\mathbb{R}^n$ is said to be a quadratic function if there exist an $n \times n$ symmetric matrix A, a vector $a \in \mathbb{R}^n$ and a real number r such that we have $q(x) = \langle Ax, x \rangle + \langle a, x \rangle + r, \ x \in \mathbb{R}^n$.

1.2. Main results

The purpose of our study is to develop conditions for the validity of the S_F-procedure in a general setting without Slater type condition and relaxing convexity as much as possible. This leads to a number of variants of the classical S-procedure. The main results of the paper are presented in Theorems 2.2, 3.1 and 4.1. Theorem 2.2 provides a characterization of the S_F-procedure in terms of the cone $\mathbb{R}_+\mathcal{F}$. It requires $\mathbb{R}_+\mathcal{F}$ to be closed convex regarding the set $\{0_Y\} \times \mathbb{R}_-$ or $\{(0_Y, -1)\}$ only, which is a minimal convexity argument for the validity of the S_F-procedure. A number of sufficient conditions are obtained from this result (Corollary 2.3, Remark 2.4, Propositions 2.5, 2.6). Theorem 3.1 deals with another characterization of the S_F-procedure involving the optimal value function

$$h(y) := \inf_{x \in X} F(x, y), \ y \in Y$$

or its conic hull h_c

$$h_c(y) := \inf_{\theta > 0} \theta h(y/\theta), \ y \in Y.$$

Links with subdifferentiability criteria are given in Corollaries 3.2, 3.3, Remark 3.4.

Assuming X is a locally convex Hausdorff space, Theorem 4.1 gives a dual characterization of the S_F-procedure by the w^*-closedness of the set $\mathcal{F}^{\#}$ regarding $\{(0_{X'}, 0)\}$ under the relaxed convex closedness assumption (14) on F. Note that if $\epsilon := h(0_Y) \in \mathbb{R}_+$, then the validity of the S_F-procedure is nothing, but the ϵ-stability introduced in [11] of the problem

$$\text{(P)} \qquad \inf_{x \in X} F(x, 0_Y)$$

whose Lagrangian dual problem is classically defined by (see [14],[16], ... for the connections between Lagrangian and perturbational duality):

$$\text{(D)} \qquad \sup_{\lambda \in Y'} \ \inf_{(x,y) \in X \times Y} \left(F(x, y) + \langle \lambda, y \rangle\right).$$

The fundamental strong duality formula

$$\inf(\mathrm{P}) = \max(\mathrm{D}) \tag{3}$$

is evidently a sufficient condition for the validity of the S_F-procedure.

When $\alpha := \inf(\mathrm{P})$ is finite, this relation (3) amounts to the validity of the S_{F_α}-procedure, where $F_\alpha(x,y) := F(x,y) - \alpha$. It is, however, far from being a necessary condition as it is shown in the following examples. Precise links between (3) and the generalized S-procedure are explained in Lemma 4.2, Proposition 4.3.

Example 1.2. Let $Y = \mathbb{R}$ and define

$$F(x,y) = \begin{cases} 0 & \text{if } (x,y) \in X \times \mathbb{R}^* \\ 1 & \text{if } (x,y) \in X \times \{0\} \end{cases}$$

where X is an arbitrary set. For $\lambda \in \mathbb{R}$ we have

$$\inf_{(x,y)\in X\times\mathbb{R}} (F(x,y) + \lambda y) = \begin{cases} -\infty & \text{if } \lambda \in \mathbb{R}^* \\ 0 & \text{if } \lambda = 0. \end{cases}$$

By this, (3) does not hold, while the S_F-procedure is valid.

A similar situation occurs when X is a linear space, say $X = \mathbb{R}$, and F is modified as

$$F(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \mathbb{R} \times \mathbb{R}^*_- \\ 1 & \text{if } (x,y) \in \mathbb{R} \times \{0\} \\ +\infty & \text{if } (x,y) \in \mathbb{R} \times \mathbb{R}^*_+. \end{cases}$$

It is clear that F is convex because the set

$$\operatorname{epi} F = \left(\mathbb{R} \times \mathbb{R}^*_-\right) \times \mathbb{R}_+ \bigcup (\mathbb{R} \times \{0\}) \times [1,+\infty[$$

is convex. Moreover,

$$\inf_{(x,y)\in\mathbb{R}^2} (F(x,y) + \lambda y) = \begin{cases} -\infty & \text{if } \lambda \in \mathbb{R}^*_+ \\ 0 & \text{if } \lambda \in \mathbb{R}_-. \end{cases}$$

Again, (3) does not hold, but the S_F-procedure is valid. □

Now we give some examples of what the S_F-procedure may be.

Example 1.3. S-Lemma ([4] [8] [15] [17] ...)

Let f, g be two quadratic functions on $X = \mathbb{R}^n$. Define

$$F(x,y) = \begin{cases} f(x) & \text{if } g(x) \le y \\ +\infty & \text{else} \end{cases} \quad , (x,y) \in \mathbb{R}^n \times \mathbb{R}.$$

We have $\mathcal{F} = \{(g(x), f(x)) : x \in \mathbb{R}^n\} + \mathbb{R}_+ \times \mathbb{R}_+$ which is a convex set by [7], see Assertion b of Theorem 4.19.

Assume that the Slater condition "$\exists\, a \in \mathbb{R}^n$ s.t. $g(a) < 0$" is satisfied. Applying [3] Theorem 3 with $X = \mathbb{R}^n$, $Y = \mathbb{R}$ and $Z = \mathbb{R}_+$ we obtain the equivalence

$$[g(x) \le 0 \Longrightarrow f(x) \ge 0] \Longleftrightarrow [\exists \lambda \in \mathbb{R}_+ \text{ s.t. } f(x) + \lambda g(x) \ge 0,\ \forall x \in \mathbb{R}^n],$$

which is precisely the S-Lemma ([18]).

In fact by [7] Theorem 4.19 and [3] Theorem 3 it can be checked that under the Slater condition one has

$$\operatorname{epi}(f+\delta_{[g\leq 0]})^* = \bigcup_{\lambda\geq 0} \operatorname{epi}(f+\lambda g)^* \neq \mathbb{R}^n\times\mathbb{R}$$

or, equivalently,

$$+\infty > \inf_{g(x)\leq 0}\left(f(x)-\langle x',x\rangle\right) = \max_{\lambda\geq 0}\inf_{x\in\mathbb{R}^n}\left(f(x)-\langle x',x\rangle+\lambda g(x)\right),\ \forall x'\in\mathbb{R}^n,$$

meaning that the quadratic optimization problem

$$\inf\{f(x) : g(x)\leq 0\}$$

has the stable strong duality property from which we go back to S-lemma by taking $x'=0_{\mathbb{R}^n}$. □

Example 1.4. Let X be a nonempty set, $f : X \to \mathbb{R}$ a real valued function, $G : X \to Y$ a map and C a nonempty subset of the Hausdorff locally convex space Y. The support function and the barrier cone of C are respectively denoted by $\sigma_C := \delta_C^*$ and $b(C) := \operatorname{dom}\sigma_C$. We consider the function

$$F(x,y) = \begin{cases} f(x) & \text{if } G(x)\in y-C \\ +\infty & \text{else }, \end{cases} \qquad (x,y)\in X\times Y.$$

One has $\mathcal{F} = H(X) + C\times\mathbb{R}_+$ where

$$H(x) := (G(x), f(x)) \text{ for } x\in X, \text{ and } H(X) := \{H(x) \ :\ x\in X\}.$$

Then, the S$_F$-procedure is valid if and only if the relation

$$G(x)\in -C \Longrightarrow f(x)\geq 0$$

entails $\exists\,\lambda\in -b(C)$ such that $\inf_{x\in X}\{f(x)+\langle\lambda, G(x)\rangle\}\geq \sigma_C(-\lambda)$.
If C is a cone then the line above reads

$$\exists\lambda\in C^+ \text{ such that } f(x)+\langle\lambda, G(x)\rangle\geq 0, \text{ for all } x\in X, \tag{4}$$

where C^+ is the positive dual cone of C:

$$C^+ = \{\lambda\in Y' :\ \langle\lambda, y\rangle\geq 0,\ \forall y\in C\}.$$ □

Example 1.5. Let $X = Y$ a locally convex Hausdorff space, and $f, g : X\to\overline{\mathbb{R}}$ two proper functions. Define $F(x,y) = f(x+y)+g(x)$. Then, the S$_F$-procedure is valid if and only if the inequality

$$f(x)+g(x)\geq 0 \text{ for all } x\in X$$

entails $\exists\,\lambda\in X'$ such that $f^*(\lambda)+g^*(-\lambda)\leq 0$.
Note that since the functions f, g are proper, their conjugates f^*, g^* do not take the value $-\infty$. Therefore, the sum $f^*(\lambda)+g^*(-\lambda)$ is well defined. □

Example 1.6. Let X, Y be locally convex Hausdorff spaces, $A : X \to Y$ a continuous linear map, and $k = Y \to \overline{\mathbb{R}}$ a proper function. Define

$$F(x, y) = k(y + Ax),\ (x, y) \in X \times Y.$$

For this function, the S_F-procedure is valid if and only if inequality

$$(k \circ A)(x) \geq 0, \text{ for all } x \in X$$

entails that $\exists\, \lambda \in Y'$ such that $A^T(\lambda) = 0_{X'}$ and $k^*(\lambda) \leq 0$, where A^T denotes the adjoint operator of A. □

Example 1.7. (Farkas Lemma) Let X, Y and A be as in Example 1.6. Let C a nonempty convex cone in Y, $b \in Y$, and $(a', t) \in X' \times \mathbb{R}$. Define

$$F(x, y) = \begin{cases} t - \langle a', x \rangle & \text{if } Ax \in y + b - C \\ +\infty & \text{else.} \end{cases}$$

The S_F-procedure is valid if and only if relation

$$Ax \in b - C \Longrightarrow \langle a', x \rangle \leq t$$

entails that $\exists\, \lambda \in C^+$ s.t. $a' = A^T(\lambda),\ t \geq \langle \lambda, b \rangle$.
This is the nonhomogeneous Farkas lemma statement. We have

$$\mathcal{F} = \bigcup_{x \in X} \{(Ax, -\langle a', x \rangle)\} + \{(-b, t)\} + C \times \mathbb{R}_+$$

$$\mathcal{F}^{\#} = \bigcup_{\lambda \in C^+} \{(A^T(\lambda), \langle \lambda, b \rangle)\} - \{(a', t)\} + \{0_{X'}\} \times \mathbb{R}_+$$

By Corollary 2.3 below the nonhomogeneous Farkas Lemma procedure is valid if and only if the cone $\mathbb{R}_+\mathcal{F}$ (which is convex since F and $\mathcal{F}$ are convex) is closed regarding $\{0_Y\} \times \mathbb{R}_-$.

Assuming that $A^{-1}(b - C) \neq \emptyset$ and the convex cone C is closed we have $F = F^{**}$ and by Theorem 4.1 below, the nonhomogenous Farkas Lemma procedure is valid if and only if

$$\bigcup_{\lambda \in C^+} \{(A^T(\lambda), \langle \lambda, b \rangle)\} + \{0_{X'}\} \times \mathbb{R}_+$$

(which is a convex cone) is w^*-closed regarding $\{(a', t)\}$. The homogenous Farkas Lemma is obtained for $(b, t) = (0_Y, 0)$. In such a case we have

$$\mathcal{F} = \bigcup_{x \in A} \{(Ax, -\langle a', x \rangle)\} + C \times \mathbb{R}_+ \quad \text{and} \quad \mathcal{F}^{\#} = A^T(C^+) \times \mathbb{R}_+$$

which are convex cones. From Corollary 2.3 and Theorem 4.1 below we get the following equivalences:

$$\begin{aligned} &[Ax \in -C \Longrightarrow \langle a', x \rangle \leq 0] \\ &\Longleftrightarrow [\{(Ax, -\langle a', x \rangle) : x \in X\} + C \times \mathbb{R}_+ \text{ is closed regarding } \{0_Y\} \times \mathbb{R}_-]. \end{aligned}$$

If in addition the convex cone C is closed we have

$$[Ax \in -C \Longrightarrow \langle a', x\rangle \leq 0] \Longleftrightarrow [A^T(C^+) \times \mathbb{R}_+ \text{ is } w^*\text{-closed regarding } \{(0'_X, 0)\}]$$
$$\Longleftrightarrow [A^T(C^+) \text{ is } w^*\text{-closed regarding } \{a'\}].$$

Various forms of Farkas type Lemmas can be found in [2], [5], [9], [10], ... □

2. S_F-procedure via separation of sets

Relation (1), the premises of the S_F-procedure, is nothing but the empty intersection

$$\mathcal{F} \cap \left(\{0_Y\} \times \mathbb{R}^*_-\right) = \emptyset,$$

saying that the two sets $\mathcal{F}$ and $\{0_Y\} \times \mathbb{R}^*_-$ are disjoint in the product space $Y \times \mathbb{R}$. Let us recall that two nonempty sets A and B in a locally convex Hausdorff space V are said to be separated if there is a nonzero $\mu \in V'$ such that

$$\langle \mu, a\rangle \leq \langle \mu, b\rangle \quad \forall (a, b) \in A \times B. \tag{5}$$

One says that A and B are *properly separated* if condition (5) holds and there is some $(\bar{a}, \bar{b}) \in A \times B$ such that $\langle \mu, \bar{a}\rangle < \langle \mu, \bar{b}\rangle$. Recall that A and B are properly separated if and only if $\overline{\mathrm{pos}}(A - B)$ is not a linear subspace of V ([13] Theorem III.1). Note that A and B are separated (resp. properly separated) if and only if $\overline{\mathrm{co}}A$ and $\overline{\mathrm{co}}B$ are separated (resp. properly separated). In the case $V = Y \times \mathbb{R}$, we say that A and B are non-vertically separated if there is some $(\mu, s) \in Y \times \mathbb{R}$ with $s \neq 0$ such that

$$\langle \mu, a\rangle + sr \leq \langle \mu, b\rangle + st \; \forall (a, r) \in A, \; \forall (b, t) \in B. \tag{6}$$

Next we characterize relation (2). Tractable conditions for (2) are given in Propositions 2.5, 2.6.

Proposition 2.1. *The following statements are equivalent:*

(i) (2) *holds.*

(ii) $\mathcal{F}$ *and* $\{0_Y\} \times \mathbb{R}_-$ *are non-vertically separated.*

(iii) $\mathcal{F}$ *and* $\{0_Y\} \times \mathbb{R}^*_-$ *are non-vertically separated.*

(iv) $\mathcal{F}$ *and* $\{(0_Y, 0)\}$ *are non-vertically separated.*

(v) $\overline{pos}(\mathcal{F}) \cap \left(\{0_Y\} \times \mathbb{R}^*_-\right) = \emptyset$.

(vi) $(0_Y, -1) \notin \overline{pos}(\mathcal{F})$.

Proof. Let us prove implication (i)⇒(ii) first. Under (i), there is some $\lambda \in Y'$ such that

$$-\langle \lambda, y\rangle - r \leq -\langle \lambda, y\rangle - F(x, y) \leq 0 \leq -t$$

for all $((x, y), r) \in \operatorname{epi} F$ and $t \in \mathbb{R}_-$. Taking $\mu = -\lambda$ and $s = -1$ we deduce that (6) holds for $A = \mathcal{F}$ and $B = \{0_Y\} \times \mathbb{R}_-$. Hence (ii) follows.

Implication (ii)⇒(iii) is obvious because $\{0_Y\} \times \mathbb{R}^*_- \subset \{0_Y\} \times \mathbb{R}_-$.

Implication (iii)⇒(iv) is derived from the continuity of the function $t \mapsto st$ at $t = 0$, which means that because (6) is true for $t < 0$, it is true for $t = 0$ too.

For implication (iv)⇒(v) assume that there is $(\mu, s) \in Y' \times \mathbb{R}^*$ such that

$$\langle \mu, y\rangle + st \leq 0 \text{ for all } (y, t) \in \mathcal{F}.$$

It is clear that this inequality remains true for any element $(y,t) \in \overline{\mathrm{pos}}(\mathcal{F})$ and that $s < 0$. Therefore

$$\langle \mu, y\rangle + st \leq 0 < \langle \mu, 0_Y\rangle + sp \ \forall (y,t) \in \overline{\mathrm{pos}}(\mathcal{F}),\ \forall p \in \mathbb{R}^*_-,$$

showing that (v) holds.

Implication (v)⇒(vi) is evident. We prove the final implication (vi)⇒(i). Since the set $\overline{\mathrm{pos}}(\mathcal{F})$ is closed convex, we may apply the Hahn-Banach separation Theorem to find a nonzero vector $(\mu, s) \in Y' \times \mathbb{R}$ such that

$$q\left(\langle \mu, y\rangle + sr\right) < \langle 0_Y, \mu\rangle - s = -s,\ \forall q \geq 0,\ \forall (y,r) \in \mathcal{F}.$$

In particular, $s < 0$ and $\langle \mu, y\rangle + sr \leq 0\ \forall (y,r) \in \mathcal{F}$.

Setting $\lambda = \mu/s$, we deduce $\langle \lambda, y\rangle + r \geq 0\ \forall (y,r) \in \mathcal{F}$, which is equivalent to (i). □

We are now able to give a characterization of the general S_F-procedure involving a weak condition on the convex closedness of the cone $\mathbb{R}_+\mathcal{F}$.

Theorem 2.2. *Let $F : X \times Y \to \overline{\mathbb{R}}$ with $\mathrm{dom}\, F \neq \emptyset$. The following statements are equivalent:*

(i) *The S_F-procedure is valid.*

(ii) *$\mathbb{R}_+\mathcal{F}$ is closed convex regarding $\{0_Y\} \times \mathbb{R}_-$.*

(iii) *$\mathbb{R}_+\mathcal{F}$ is closed convex regarding $\{(0_Y, -1)\}$.*

Proof. Assume (i). We prove (ii):

$$(0_Y, q) \in \overline{\mathrm{co}}(\mathbb{R}_+\mathcal{F}) \cap \{0_Y\} \times \mathbb{R}_- \text{ implies } (0_Y, q) \in \mathbb{R}_+\mathcal{F}.$$

We may consider the case $q < 0$ only. If (1) holds, then by (i), (2) holds too.

Relation (2) is equivalent to $\langle -\lambda, y\rangle \leq r$ for all $(y,r) \in \mathcal{F}$, which means that $\mathcal{F} \subset \mathrm{epi}(-\lambda)$. Because $\mathrm{epi}(-\lambda)$ is a closed convex cone we obtain

$$(0_Y, q) \in \overline{\mathrm{co}}(\mathbb{R}_+\mathcal{F}) = \overline{\mathrm{pos}}(\mathcal{F}) \subset \mathrm{epi}(-\lambda).$$

It follows that $0 = \langle -\lambda, 0_Y\rangle \leq q$, contradicting the fact that $q < 0$. By this we conclude that (1) does not holds, that is $F(a, 0_Y) < 0$ for some $a \in X$. Choose a small $\epsilon > 0$ such that $F(a, 0_Y) < \epsilon q < 0$. One has $(0_Y, \epsilon q) \in \mathcal{F}$, implying $(0_Y, q) \in \mathbb{R}_+\mathcal{F}$. This show that (ii) is true. Implication (ii)⇒(iii) is clear because $(0_Y, -1) \in \{0_Y\} \times \mathbb{R}_-$. For the final implication (iii)⇒(i), assume that (1) is true. Then $(0_Y, -1) \notin \mathbb{R}_+\mathcal{F}$. By (iii) we have

$$(0_Y, -1) \notin \overline{\mathrm{co}}(\mathbb{R}_+\mathcal{F}) = \overline{\mathrm{pos}}(\mathcal{F}),$$

and, by Proposition 2.1, (2) holds. This completes the proof. □

Here is a direct consequence of Theorem 2.2.

Corollary 2.3. *Assume the cone $\overline{\mathbb{R}_+\mathcal{F}}$ is convex, then the S_F-procedure is valid if and only if $\mathbb{R}_+\mathcal{F}$ is closed regarding $\{0_Y\} \times \mathbb{R}_-$. In particular, if the cone $\mathbb{R}_+\mathcal{F}$ is closed convex, the S_F-procedure is valid.*

Remark 2.4. Note that if $\mathcal{F}$ is convex, then $\mathbb{R}_+\mathcal{F}$ is convex and hence $\overline{\mathbb{R}_+\mathcal{F}}$ is convex. The converse implications are not true.

The next proposition gives equivalent conditions for the strong duality relation (3).

Proposition 2.5. *Assume that $\alpha := \inf(P) \neq +\infty$. The following statements are equivalent:*

(i) *Strong duality holds for (P).*

(ii) *For every real number $r \leq \alpha$, the cone $\mathbb{R}_+(\mathcal{F} - (0_Y, r))$ is closed convex regarding $\{0_Y\} \times \mathbb{R}_-$ (or $\{(0_Y, -1)\}$). If, in addition α is finite, then each one of the above statement is equivalent to*

(iii) *The cone $\mathbb{R}_+(\mathcal{F} - (0_Y, \alpha))$ is closed convex regarding $\{0_Y\} \times \mathbb{R}_-$ (or $\{(0_Y, -1)\}$).*

Proof. Assume (i). Let $r \leq \alpha$. There exists $\lambda \in Y'$ such that

$$\inf_{(x,y)\in X\times Y} (F(x,y) + \langle \lambda, y\rangle) \geq r,$$

which means $F_r := F - r$ satisfies (2). Then, the S_{F_r}-procedure is valid and, by Theorem 2.2, (ii) holds.

Conversely, assume (ii). Since the strong duality formula holds trivially for $\alpha = -\infty$, we may assume that $\alpha \in \mathbb{R}$. It follows from (ii) and Theorem 2.2 that the S_{F_α}-procedure is valid. Since $F_\alpha(x, 0_Y) \geq 0$ for all $x \in X$, there is $\lambda \in Y'$ such that

$$\inf_{(x,y)\in X\times Y} (F(x,y) + \langle \lambda, y\rangle) \geq \alpha.$$

This yields (i) because the opposite inequality is always true.

It is plain that (ii) implies (iii) when $\alpha \in \mathbb{R}$. The implication (iii)⇒(ii) is obtained as in the proof of implication (ii)⇒(i) above. □

We derive now some sufficient conditions for the validity of the S_F-procedure. Recall that D denotes the projection of $\mathcal{F}$ on Y.

Proposition 2.6. *Each of the following condition entails the validity of the S_F-procedure.*

(i) *$\overline{pos}(D)$ is a linear subspace but $\overline{pos}(\mathcal{F})$ is not.*

(ii) *$\overline{pos}(D) = Y$ and the sets $\mathcal{F}$ and $\{(0_Y, 0)\}$ are separated.*

Proof. Assume (i) first. Since $\overline{\text{pos}}(\mathcal{F})$ is not a linear space, the sets $\mathcal{F}$ and $\{0_Y, 0\}$ are properly separated. One can find a nonzero element $(\mu, s) \in Y' \times \mathbb{R}$ and $(\bar{y}, \bar{r}) \in \mathcal{F}$ such that

$$\langle \mu, y\rangle + sr \leq 0 \ \forall (y, r) \in \mathcal{F} \tag{7}$$

$$\langle \mu, \bar{y}\rangle + s\bar{r} < 0. \tag{8}$$

We show that $s \neq 0$. Indeed, if $s = 0$, then $\mu \neq 0_{Y'}$, and by (7), $-\mu \in D^+$. Moreover, since $\overline{\text{pos}}(D = (D^+)^+$ is a linear space, so is D^+. Hence $\mu \in D^+$ which implies $\langle \mu, \bar{y}\rangle \geq 0$ because $\bar{y} \in D$. This contradicts (8) and gives $s \neq 0$. By this, the sets $\mathcal{F}$ and $\{(0_Y, 0)\}$ are non-vertically separated. In view of Proposition 2.1, (2) holds and the S_F-procedure is valid.

Assume (ii). Again, there is some $(\mu, s) \in Y' \times \mathbb{R}$ such that (7) holds. We have $s \neq 0$ because otherwise $-\mu \in D^+$ is nonzero, contradicting the hypothesis that $\overline{\text{pos}}(D) = Y$. Therefore, the sets $\mathcal{F}$ and $\{(0_Y, 0)\}$ are non-vertically separated and we conclude as above that the S_F-procedure is valid. □

3. The S_F-procedure and the value function

Consider the optimal value function h associated with the function $F : X \times Y \to \overline{\mathbb{R}}$,

$$h(y) := \inf_{x \in X} F(x, y), \; y \in Y.$$

One has $\operatorname{dom} h = D$,

$$\operatorname{epi}_s h = \{(y, r) \in Y \times \mathbb{R} : \; h(y) < r\} \subset \mathcal{F} \subset \operatorname{epi} h = \{(y, r) \in Y \times \mathbb{R} : \; h(y) \leq r\},$$

$$h(y) = \inf \{r \in \mathbb{R} : \; (y, r) \in \mathcal{F}\}, \; y \in Y.$$

Recall that a function $k : Y \to \overline{\mathbb{R}}$ is said to be positively homogeneous if

$$k(ty) = tk(y), \; \forall (t, y) \in \mathbb{R}_+^* \times Y.$$

Note that $k(0_Y) \in \{-\infty, 0, +\infty\}$.

The conic hull h_c of h is the greatest positively homogeneous minorant of h. One has (see [12] §8.10)

$$h_c(y) = \inf_{\theta > 0} \theta h\left(\frac{y}{\theta}\right), \; y \in Y, \; \operatorname{dom} h_c = \mathbb{R}_+^* \operatorname{dom} h,$$

$$\operatorname{epi}_s h_c \subset \mathbb{R}_+ \mathcal{F} \subset \operatorname{epi} h_c$$

$$h_c(y) = \inf \{r \in \mathbb{R} : \; (y, r) \in \mathbb{R}_+ \mathcal{F}\}, \; y \in Y.$$

By this, h_c is convex whenever the cone $\mathbb{R}_+ \mathcal{F}$ is convex.

Given a function $g : Y \to \overline{\mathbb{R}}$, $a \in g^{-1}(\mathbb{R})$ and $\epsilon \geq 0$, let us recall that the ϵ-subdifferential of g at a is defined by

$$\partial^\epsilon g(a) = \{\mu \in Y' : \; g(y) \geq g(a) + \langle \mu, y - a \rangle - \epsilon, \; \forall y \in Y\}.$$

For $\epsilon = 0$ we note $\partial g(a) = \partial^0 g(a)$. The function g is said to be subdifferentiable at a if $\partial g(a) \neq \emptyset$. Of course, g may be not convex and subdifferentiable at a.

Theorem 3.1. *Assume* (1) *holds, that is* $\alpha := h(0_Y) \in \mathbb{R}_+ \cup \{+\infty\}$. *The following statements are equivalent:*

(i) *The S_F-procedure is valid.*

(ii) *h admits a continuous linear minorant.*

(iii) *The set $[h^* \leq 0]$ is nonempty.*

If in addition α is finite, then each of the above statements is equivalent to each of the statements below

(iv) $\partial^\alpha h(0_Y) \neq \emptyset$.

(v) $\partial h_c(0_Y) \neq \emptyset$.

Proof. Assume (i). There is $\lambda \in Y'$ such that

$$F(x, y) \geq \langle -\lambda, y \rangle \; \forall (x, y) \in X \times Y.$$

It follows that $h(y) \geq \langle -\lambda, y \rangle \; \forall y \in Y$, which means that $-\lambda$ is a continuous linear minorant of h.

The implication (ii)⇒(iii) is direct from the definition of h^*.

For the implication (iii)⇒(i) let $\mu \in [h^* \leq 0]$. We have

$$h(y) - \langle \mu, y \rangle \geq 0 \ \forall y \in Y \tag{9}$$

or equivalently $F(x,y) + \langle -\mu, y \rangle \geq 0 \ \forall (x,y) \in X \times Y$, which shows that (2) holds and the S_F-procedure is valid.

Finally assume $\alpha \in \mathbb{R}_+$. We prove first (iii)⇒(iv). Let $\mu \in [h^* \leq 0]$. In view of (9),

$$h(y) - h(0_Y) \geq \langle \mu, y - 0_Y \rangle - \alpha \ \forall y \in Y$$

by which $\mu \in \partial^\alpha h(0_Y)$.

For (iv)⇒(v) let $\mu \in \partial^\alpha h(0_Y)$. Since $h(0_Y)$ is finite, one has $h_c(0_Y) = 0$, and for every $y \in Y$,

$$h_c(y) - h_c(0_Y) = \inf_{\theta>0} \theta h(y/\theta) \geq \inf_{\theta>0} \theta \left(\langle \mu, y/\theta \rangle + h(0_Y) - \alpha \right) = \langle \mu, y \rangle.$$

By this, $\mu \in \partial h_c(0_Y)$.

To complete the proof we show (v)⇒(i). Let $\mu \in \partial h_c(0_Y)$. Then we obtain for every $(\theta, y) \in \mathbb{R}_+^* \times Y$ that $\theta h(y/\theta) \geq \langle \mu, y \rangle$. We deduce that $h(y) + \langle \lambda, y \rangle \geq 0$ for all $y \in Y$. This establishes (2) and the S_F-procedure is valid. □

Corollary 3.2. *Assume that $Y = \mathbb{R}^p$, the cone $\mathbb{R}_+\mathcal{F}$ is convex and $0_Y \in D \cap ri\,(\mathbb{R}_+ D)$. Then the S_F-procedure is valid.*

Proof. Assume (1). We have $h_c(0_Y) \in \{0; +\infty\}$ and in fact $h_c(0_Y) = 0$ since $0_Y \in D = \operatorname{dom} h$. Since $\mathbb{R}_+\mathcal{F}$ is convex, the function h_c is convex (in fact sublinear). Since $\operatorname{dom} h_c = \mathbb{R}_+^* D = \mathbb{R}_+ D$ we have $0_Y \in \operatorname{ri}(\operatorname{dom} h_c)$ and h_c is subdifferentiable at 0_Y. It remains to apply Theorem 3.1 to complete the proof. □

We wish to apply the S_F-procedure to obtain a criterion for the subdifferentiability of a function.

Corollary 3.3. *Let $g : Y \to \overline{\mathbb{R}}$ be an extended real-valued function on a locally convex Hausdorff space Y, and let $a \in g^{-1}(\mathbb{R})$. The following statements are equivalent.*

(i) *g is subdifferentiable at a.*

(ii) $\mathbb{R}_+ (\operatorname{epi} g - (a, g(a))))$ *is closed convex regarding the set* $\{0_Y\} \times \mathbb{R}_-$.

(iii) $\mathbb{R}_+(\operatorname{epi} g - (a, g(a)))$ *is closed convex regarding the set* $\{(0_Y, -1)\}$.

Proof. Let us choose an arbitrary nonempty set X and define

$$F(x,y) = g(y+a) - g(a), \ \forall (x,y) \in X \times Y.$$

One has $h(y) = g(y+a) - g(a)$, $\partial h(0_Y) = \partial g(a)$, $\mathcal{F} = \operatorname{epi} g - (a, g(a))$, $D = \operatorname{dom} g - a$, and $F(x, 0_Y) = 0$ for every $x \in X$. By Theorem 3.1, the S_F-procedure is valid if and only if $\partial g(a)$ is nonempty. It remains to apply Theorem 2.2 to achieve the proof. □

Remark 3.4. By Proposition 2.6 and Theorem 3.1 we obtain that each of the following conditions is sufficient for the subdiffentiability of $g : Y \to \overline{\mathbb{R}}$ at $a \in g^{-1}(\mathbb{R})$.

(i) $\overline{\operatorname{pos}}\,(\operatorname{dom} g - a)$ is a linear subspace, but $\overline{\operatorname{pos}}\,(\operatorname{epi} g - (a, g(a))$ is not.

(ii) $\overline{\operatorname{pos}}\,(\operatorname{dom} g - a) = Y$ and the sets $\operatorname{epi} g$ and $\{(a, g(a))\}$ are separated.

4. The S_F-procedure and the conjugate function of F

In this section we assume that X and Y are locally convex Hausdorff spaces. Recall that $\mathcal{F}^{\#}$ denotes the projection of the epigraph of the conjugate of F^* of F on $X' \times \mathbb{R}$. The relations (1) and (2) of S_F-procedure can respectively be written as

$$(F(.,0_Y))^* (0_{X'}) \leq 0 \tag{10}$$

$$\exists \mu \in Y' \text{ such that } F^*(0_{X'}, \mu) \leq 0 \tag{11}$$

The dual value function $\varphi : X' \to \overline{\mathbb{R}}$ is defined by

$$\varphi(x') := \inf_{\mu \in X'} F^*(x', \mu), \;\; x' \in X'.$$

It has the following properties:

$$\operatorname{epi}_s \varphi \subset \mathcal{F}^{\#} \subset \operatorname{epi} \varphi$$

$$\varphi(x') = \inf\{s \in \mathbb{R} \;:\; (x', s) \in \mathcal{F}^{\#}\}.$$

Since $\mathcal{F}^{\#}$ is convex, the dual value function φ is convex ant its conjugate satisfies

$$\varphi^*(x) = F^{**}(x, 0_Y), \;\; x \in X. \tag{12}$$

Let the w^*-lower semicontinuous hull of φ denoted by $\overline{\varphi}$. Since φ^{**} is w^*-lower semicontinuous one has $\varphi^{**} \leq \overline{\varphi}$ and, moreover,

$$\overline{\mathcal{F}^{\#}} = \operatorname{epi} \overline{\varphi} \subset \operatorname{epi} \varphi^{**} = \operatorname{epi} \left(F^{**}(.,0_Y)\right)^* \subset \operatorname{epi} \left(F(.,0_Y)\right)^*. \tag{13}$$

If $\operatorname{dom} F^{**}(.,0_Y) \neq \emptyset$ then φ admits a w^*-continuous affine minorant and by this $\varphi^{**} = \overline{\varphi}$.

Theorem 4.1. *Let $F : X \times Y \to \overline{\mathbb{R}}$ be given. Consider the following statements*

(i) *The S_F-procedure is valid.*

(ii) *The $S_{F^{**}}$-procedure is valid.*

(iii) *The set $\mathcal{F}^{\#}$ is w^*-closed regarding $\{(0_{X'}, 0)\}$.*

We have the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)*. If, in addition*

$$\inf_{x \in X} F(x, 0_Y) = \inf_{x \in X} F^{**}(x, 0_Y) \neq +\infty, \tag{14}$$

then the above statements are equivalent.

Proof. For the implication (i)$\Rightarrow$(ii), let (1) holds for F^{**} replacing F. Then (1) holds for F too because $F^{**} \leq F$. By (i) and (11) there is some $\lambda \in Y'$ such that

$$F^*(0_X, -\lambda) \leq 0.$$

Since $F^* = (F^{**})^*$ we deduce that for all $(x, y) \in X \times Y$,

$$0 \leq -\left(F^{**}\right)^* (0_X, -\lambda) \leq F^{**}(x, y) - \langle 0_X, x\rangle + \langle \lambda, y\rangle = F^{**}(x, y) + \langle \lambda, y\rangle$$

which gives (2) for F^{**} replacing F.

For the implication (ii)⇒(iii), let $(0_X, 0) \in \overline{\mathcal{F}^\#}$. By (13), $(0_X, 0) \in \operatorname{epi}(F^{**}(., 0_Y))^*$. In view of (ii) there exists $\lambda \in Y'$ such that

$$0 \leq -(F^{**})^*(0_{X'}, -\lambda) = -F^*(0_{X'}, -\lambda),$$

which means that $(0_{X'}, 0) \in \mathcal{F}^\#$ and (iii) holds.

It remains to prove that under (14) the implication (iii)⇒(i) holds. Assume (10) holds. By (12), (14) we have

$$(0_{X'}, 0) \in \operatorname{epi}(F^{**}(., 0_Y))^* = \operatorname{epi}\varphi^{**},\ 0_X \in \operatorname{dom}\varphi^* \neq \emptyset.$$

We then have $\varphi^{**} = \overline{\varphi}$ and $(0_{X'}, 0) \in \operatorname{epi}\overline{\varphi} = \overline{\mathcal{F}^\#}$. By (iii) we obtain $(0_{X'}, 0) \in \mathcal{F}^\#$. The proof is complete. □

Let us derive conditions for strong duality by using the S_F-procedure. Remember that the strong duality formula can be stated in the form

$$\inf_{x\in X} F(x, 0_Y) = \max_{\mu\in Y'} -F^*(0_{X'}, \mu) \in \overline{\mathbb{R}}. \tag{15}$$

Lemma 4.2. *Let $F : X \times Y \to \overline{\mathbb{R}}$. If* (15) *holds, then $\mathcal{F}^\#$ is w^*-closed regarding $0_{X'} \times \mathbb{R}$.*

Proof. Let $(0_{X'}, s) \in \overline{\mathcal{F}^\#}$. By (13) we have $(0_{X'}, s) \in \operatorname{epi}(F(., 0_Y))^*$, that means

$$\inf_{x\in X} F(x, 0_Y) = -(F(., 0_Y))^*(0_{X'}) \geq -s$$

By (15) there exists $\mu \in Y'$ such that $-F^*(0_{X'}, \mu) \geq -s$, that is $(0_{X'}, 0) \in \mathcal{F}^\#$. □

Proposition 4.3. *Assume that $\alpha := \inf_{x\in X} F(x, 0_Y) = \inf_{x\in X} F^{**}(x, 0_Y) \in \mathbb{R}$.*
Then, the following statements are equivalent:

(i) $\alpha = \max_{\mu\in Y'} -F^*(0_{X'}, \mu)$.

(ii) *$\mathcal{F}^\#$ is w^*-closed regarding $\{0_{X'}\} \times \mathbb{R}$.*

(iii) *$\mathcal{F}^\#$ is w^*-closed regarding $\{(0_{X'}, -\alpha)\}$.*

Proof. (i)⇒(ii) holds by Lemma 4.2 and (ii)⇒(iii) is obvious. Let us prove that (iii)⇒(i). Define $F_\alpha(x, y) := F(x, y) - \alpha$ and denote by $(\mathcal{F}_\alpha)^\#$ the projection of $\operatorname{epi}(F_\alpha)^*$ on $X' \times \mathbb{R}$. We have $(\mathcal{F}_\alpha)^\# = \mathcal{F}^\# - \{(0_{X'}, \alpha)\}$. By (iii) $(\mathcal{F}_\alpha)^\#$ is w^*-closed regarding $\{(0_{X'}, 0)\}$. Since $\inf_{x\in X} F_\alpha(x, 0_Y) = \inf_{x\in X}(F_\alpha)^{**}(x, 0_Y) = 0$, Theorem 4.1 says that the S_{F_α}-procedure is valid. Since $\mathcal{F}_\alpha(x, 0_Y) \geq 0$ for all $x \in X$, it ensues that there exists $\bar{\mu} \in Y'$ such that

$$\alpha \leq -F_\alpha^*(0_{X'}, \bar{\mu}) = -F^*(0_{X'}, \bar{\mu}) - \alpha.$$

Finally, $0 \leq -F^*(0_{X'}, \bar{\mu}) \leq \sup_{\mu\in Y'} -F^*(0_{X'}, \mu) \leq \alpha$, and we are done. □

5. S-procedure involving conic constraints

To conclude the paper let us go back to Example 1.5 for giving some complements to [3] Theorem 3. We denote by 0_p the zero element of $\mathbb{R}^p$.

Proposition 5.1. *Let X be a set, $f : X \to \mathbb{R}$, $G : X \to \mathbb{R}^p$ two functions and C a convex cone in $\mathbb{R}^p$. Assume that there exists a set Z such that*

$$(0_p, 0) \in Z \subset C \times \mathbb{R}_+,$$

$$\mathbb{R}_+ \left(\{(G(x), f(x)) : \ x \in X\} + Z \right) \ \textit{is convex},$$

$$pos\,(G(X)) + C = \mathbb{R}^p.$$

Then, the following properties are equivalent:

(i) $G(x) \in -C \Longrightarrow f(x) \geq 0$.

(ii) $\exists \lambda \in C^+$ *such that* $f(x) + \langle \lambda, G(x) \rangle \geq 0$ *for all* $x \in X$.

Proof. It suffices to prove that (i)⇒(ii). Assume (i). Since C is a convex cone and $Z \subset C \times \mathbb{R}_+$ we have

$$\mathbb{R}_+(H(X) + Z) \cap (-C) \times \mathbb{R}^*_- = \emptyset,$$

where $H(x) := (G(x), f(x))$, $x \in X$. The disjoint convex cones $\mathbb{R}_+ \left(H(X) + Z\right)$ and $(-C) \times \mathbb{R}^*_-$ are separated in $\mathbb{R}^p \times \mathbb{R}$: there exists a nonzero vector $(\mu, s) \in \mathbb{R}^p \times \mathbb{R}$ such that

$$p := \sup_{\substack{\theta \geq 0, x \in X \\ (u,t) \in Z}} \theta \left(\langle \mu, G(x) + u \rangle + s(f(x) + t) \right) \leq q := \inf_{u \in C, t > 0} - \left(\langle \mu, u \rangle + st \right).$$

Since $(0_p, 0) \in Z$ we have

$$\sup_{\theta \geq 0, x \in X} \theta \left(\langle \mu, G(x) \rangle + sf(x) \right) \leq p \leq q = -\left(\delta^*_C(\mu) + \delta^*_{\mathbb{R}^*_+}(s) \right).$$

It follows that $\mu \in C^-$, $s \leq 0$ and

$$\langle \mu, G(x) \rangle + sf(x) \leq 0, \text{ for all } x \in X. \tag{16}$$

Observe that $s \neq 0$ because otherwise

$$\begin{aligned} 0_p \neq \mu \in (G(X))^- \cap \ C^- &= (G(X) \cup C)^- = (\mathrm{pos}(G(X) \cup C))^- \\ &= (\mathrm{pos}(G(X)) + C)^- = (\mathbb{R}^p)^- = \{0_p\} \end{aligned}$$

which is a contradiction. Now, $s < 0$ and setting $\lambda := \mu/s$, we obtain that $\lambda \in C^+$ and $\langle \lambda, G(x) \rangle + f(x) \geq 0$ for all $x \in X$, that is (ii). □

The condition $\mathrm{pos}(G(X)) + C = \mathbb{R}^p$ is in particular satisfied if $G(X)$ meets the topological interior of $-C$. More generally we have:

Proposition 5.2. *Let X be a set, Y a locally convex Hausdorff space, $f : X \to \mathbb{R}$, $G : X \to Y$, and C a convex cone in Y. Assume that there exists a set Z such that*

$$(0_Y, 0) \in Z \subset C \times \mathbb{R}_+,$$

$$\mathbb{R}_+ \left(\{(G(x), f(x)) : x \in X\} + Z\right) \textit{ is convex,}$$

$$\exists a \in X \textit{ such that } G(a) \in -int\ C.$$

Then, the following properties are equivalent:

(i) $G(x) \in -C \Longrightarrow f(x) \geq 0.$

(ii) $\exists \lambda \in C^+$ *such that* $f(x) + \langle \lambda, G(x) \rangle \geq 0$ *for all* $x \in X$.

Proof. Assume (i). Since the interior of $(-C) \times \mathbb{R}^*_-$ is nonvoid, the disjoint convex cones $\mathbb{R}_+ (H(X) + Z)$ and $(-C) \times \mathbb{R}^*_-$ are separated, that means (16) holds with $\mu \in C^-$ and $s \leq 0$. As in the proof of Proposition 5.1, it remains to check that $s \neq 0$. If not μ is nonzero and belongs to $(G(X))^- \cap\ C^-$. Since $G(a) \in -int\ C$, there is a neighborhood U of 0_Y such that $G(a) + U \subset -C$ and we have

$$0 \leq \langle \mu, G(a) + u \rangle = \langle \mu, G(a) + \langle \mu, u \rangle \leq \langle \mu, u \rangle \text{ for all } u \in U,$$

which is impossible. □

The last proposition below does not assume any Slater type condition.

Proposition 5.3. *Let X be a set, Y a locally convex Hausdorff space, $f : X \to \mathbb{R}$, $G : X \to Y$ and C a subset of Y. Assume that there exists a set Z such that*

$$(0_Y, 0) \in Z \subset C \times \mathbb{R}_+,$$

$$\mathbb{R}_+(H(X) + Z) \textit{ is closed convex regarding } \{0_Y\} \times \mathbb{R}^*_-.$$

Then the implication below holds

$$\{G(x) \in -C \Longrightarrow f(x) \geq 0\} \Longrightarrow \{\exists \nu \in Y' \textit{ s.t. } f(x) + \langle \nu, G(x) \rangle \geq 0 \ \textit{ for all } x \in X\}.$$

Proof. Assume that $\inf\{f(x) : \ G(x) \in -C\} \geq 0$. Since $Z \subset C \times \mathbb{R}_+$ we have $(0_Y, -1) \notin \mathbb{R}_+(H(X){+}Z)$. Since $\mathbb{R}_+(H(X){+}Z)$ is closed convex regarding $\{0_Y\}{\times}\mathbb{R}^*_-$ we have $(0_Y, -1) \notin \overline{\text{pos}}(H(X){+}Z)$. By the Hahn-Banach Separation Theorem, there exists a nonzero vector $(\mu, s) \in Y' \times \mathbb{R}$ such that

$$-s > \sup_{\substack{\theta \geq 0, x \in X \\ (u,t) \in Z}} \theta \left(\langle \mu, G(x) + u \rangle + s \left(f(x) + t\right)\right).$$

Since $(0_Y, 0) \in Z$, it follows that $\langle \mu, G(x) \rangle + sf(x) \leq 0$ for all $x \in X$ and that $s < 0$. Setting $\nu = \mu/s$ we are done. □

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