Journal of Convex Analysis
Volume 32 (2025), No. 1, 9–24

Coderivative of the Metric Projection of Strictly Hadamard Differentiable Sets in Euclidean Spaces

Laila Alsharief, Abderrahim Jourani*
Institut de Mathématiques de Bourgogne, UMR 5584 CNRS,
Université de Bourgogne, Dijon, France
laila.alsharief@u-bourgogne.fr, abderrahim.jourani@u-bourgogne.fr

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: October 12, 2024
Accepted: October 14, 2024

This paper focuses on the study of the coderivative of the metric projection mapping in finite dimensional spaces. It presents estimates for the Dini and Mordukhovich coderivatives for two specific classes of convex sets: strictly Hadamard differentiable sets and sets satisfying the polyhedral property.

Keywords: Projection mapping, subdifferential, coderivative, strict Hadamard differentiability, polyhedricity.

2020 Mathematics Subject Classification: 46C05, 46G05, 47H99, 49J52, 58C20.

1. Introduction

Given a Hilbert space X (or even a Euclidean space) and a closed convex set $C \subset X$, then each point $x \in X$ has a unique projection onto C denoted by $P_C(x)$. It is well known that the mapping P_C is $1-$Lipschitz, though it is not necessarily differentiable at points outside of C. In [10], Kruskal constructed a convex subset of $\mathbb{R}^3$ where the associated metric projection fails to have a one-sided directional derivative at some points in $\mathbb{R}^3$. In subsequent works by Haraux [7] and Mignot [12], the authors independently demonstrated that the directional derivative of the metric projection exists every where when the set C satisfies a polyhedral property (see the last section for further details).

One of the earliest results on the smoothness of the projection mapping was established by Holmes in 1973 [8]. He showed that if C is a closed convex body in the Hilbert space X, and if for some $\bar{x} \in X$ the boundary $\mathrm{bd}(C)$ of C is of class $\mathcal{C}^{k+1}$ near the boundary point $P_C(\bar{x})$, for some $k \geq 0$, then P_C is of class $\mathcal{C}^k$ on a neighborhood of the ray normal to C at $P_C(\bar{x})$, and in particular, in a neighborhood of the point $\bar{x}$.

This work was partially supported by the FEDER-BFC and EIPHI Graduate School (contract ANR-17-EURE-0002)

* Corresponding author.

ISSN 0944-6532 / \$ 2.50

In 1982, Fitzpatrick and Phelps [6] demonstrated that if C is a closed convex body in a Hilbert space X, then the following assertions are equivalent:

1. $\mathrm{bd}(C)$ is a $\mathcal{C}^{k+1}-$submanifold at any of its points.
2. P_C is of class $\mathcal{C}^k$ on $X\backslash C$ and for every $x \in X\backslash C$ the derivative operator $DP_C(x)$ restricted to $(x - P_C(x))^\perp$ is invertible as an operator from $(x - P_C(x))^\perp$ onto $(x - P_C(x))^\perp$.
3. There exists an open set U containing C such that P_C is of class $\mathcal{C}^k$ on $U\backslash C$ and for every $x \in U\backslash C$, the derivative operator $DP_C(x)$ restricted to $(x - P_C(x))^\perp$ is surjective onto $(x - P_C(x))^\perp$.

Since then, several results on the smoothness of the metric projection have been obtained in the convex setting (see [15], [20], [18], [19] and the references therein). One of the earliest results on smoothness of the metric projection onto nonconvex bodies was obtained in [4], where several characterizations in the spirit of those in [6] were obtained for prox-regular sets in Hilbert spaces. For further results in this direction, we refer to the recent books [18] [19] and the references therein.

All of the aforementioned results concern sets whose boundaries are submanifolds of class $\mathcal{C}^{k+1}$ in Hilbert spaces. In [9], the authors provided different characterizations in Banach spaces of the strict differentiability of the boundaries of epi-Lipschitzian sets in terms of Clarke's tangent cone as a key tool [2].

In this paper, we exploit this characterization to provide estimates for the coderivative of the metric projection mapping for convex bodies that satisfy the strict Hadamard differentiability property in finite dimensional spaces. The main results, presented below, are given for the Dini-subdifferential and the Mordukhovich subdifferential of the mapping $\omega_p : \mathbb{R}^d \to \mathbb{R}$, defined for some $p \in \mathbb{R}^d$ as:

$$\omega_p(x) = \langle p, P_C(x)\rangle.$$

Due to the Lipschitzness property of P_C, it follows from ([13]) that the Mordukhovich subdifferential of ω_p at x coincides with the coderivative of P_C at p (see Section 3 for more details).

The paper is organized as follows: Section 2 presents the essential tools from nonsmooth analysis that are required for our study. The main results are given in Section 3, where they are established for two classes of sets: the first one focuses on strictly Hadamard differentiable sets, while the second one involves sets that satisfy a polyhedral property.

2. Preliminaries

In order to make the paper as short as possible, some definitions and the complete wording of the results will not be repeated here, and as needed, will be referenced to [13] and [14]. Unless otherwise stated, we shall assume that X is an Euclidean space, say $\mathbb{R}^d$, equiped with a scalar product $\langle\cdot,\cdot\rangle$ with associated norm denoted by $\|\cdot\|$ to which we associate the distance function $d(\cdot, C)$ to a set C. The closed unit ball of X is denoted by $\mathbb{B}$. We write $x\xrightarrow{f}x_0$, and $x\xrightarrow{C}x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in C$, respectively.

Let $F : C \rightrightarrows Y$ be a set-valued mapping between two normed spaces X and Y, where $C \subset X$ is a set. Then the *limit inferior*, also called *inner limit*, of F at $\bar{x}$ is the set

$$\begin{aligned}\liminf_{x \xrightarrow{C} \bar{x}} F(x) &= \{y \in Y : \forall x_n \xrightarrow{C} \bar{x},\ \exists y_n \to y;\ y_n \in F(x_n),\ \forall n\} \\ &= \{y \in Y : \forall r > 0,\ \exists \delta > 0;\ \forall x \in B(\bar{x}, \delta) \cap C,\ F(x) \cap B(y, r) \neq \emptyset\}\end{aligned}$$

and the *limit superior*, also called *outer limit*, of F at $\bar{x}$ is the set

$$\limsup_{x \xrightarrow{C} \bar{x}} F(x) = \{y \in Y : \exists x_n \xrightarrow{C} \bar{x},\ \exists y_n \to y;\ y_n \in F(x_n),\ \forall n\}.$$

The set-valued mapping F is said to be *lower semicontinuous* (resp. *upper semicontinuous*) at $\bar{x}$ if

$$F(\bar{x}) = \liminf_{x \xrightarrow{C} \bar{x}} F(x) \quad (\text{resp. } F(\bar{x}) = \limsup_{x \xrightarrow{C} \bar{x}} F(x)).$$

Furthermore, we say that the limit of F at $\bar{x}$ exists if the outer and inner limit sets are equal. In this case, we say that F is continuous at $\bar{x}$, and write

$$F(\bar{x}) = \lim_{x \xrightarrow{C} \bar{x}} F(x).$$

Let f be an extended-real-valued function on $\mathbb{R}^d$. The *limiting Fréchet subdifferential* (also called the *Mordukhovich subdifferential*) of f at x is the set

$$\partial f(x) = \limsup_{x' \xrightarrow{f} x} \partial^F f(x'),$$

where $$\partial^F f(x') = \{x^* \in \mathbb{R}^d : \liminf_{h \to 0} \frac{f(x' + h) - f(x') - \langle x^*, h\rangle}{\| h \|} \geq 0\}$$

is the Fréchet subdifferential of f at x'.

The *limiting Fréchet normal cone* to a closed set $S \subset \mathbb{R}^d$ at a point $x \in S$ is given by $N(S, x) = \partial \delta_S(x)$, where δ_S denotes the indicator function of S.

If f is an extended-real-valued function on $\mathbb{R}^d$, the function

$$f^-(x, h) = \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(x + tu) - f(x)}{t}$$

is the *lower Dini directional derivative* of f at x. To this directional derivative, it is associated the so called Dini-subdifferential defined as

$$\partial^- f(x) = \{x^* \in \mathbb{R}^d : \langle x^*, h\rangle \leq f^-(x, h)\, \forall h \in \mathbb{R}^d\}.$$

It turns out that this subdifferential coincides with the Fréchet subdifferential and so that

$$\partial f(x) = \limsup_{x' \xrightarrow{f} x} \partial^- f(x').$$

Note that when f is locally Lipschitz at x, then

$$f^-(x, h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

We say that a locally Lipchitzian function f is *directionally differentiable* at x, if the following limit exists for all $h \in \mathbb{R}^d$

$$f'(x,h) = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}.$$

If f is locally Lipschitz at x, the function

$$f^0(x,h) = \limsup_{\substack{u \to x \\ t \downarrow 0}} t^{-1}(f(u+th) - f(u))$$

is the *Clarke's directional derivative* of f at x. The Clarke's subdifferential is then defined by

$$\partial_C f(x) = \{x^* \in \mathbb{R}^d : `\langle x^*, h\rangle \leq f^0(x,h) \, \forall h \in \mathbb{R}^d\}.$$

In this case, the Clarke subdifferential and the limiting Fréchet subdifferential are connected via the following formula:

$$\partial_C f(x) = \mathrm{co}\partial f(x).$$

The *tangent cone* $T(C,x)$ to a closed convex set $C \subset X$ at $x \in C$ is defined by

$$T(C,x) = \overline{T_r(C,x)},$$

where $\overline{A}$ is the closure of the set A and $T_r(C,x)$ denotes the radial cone to C at $x \in C$, which is given by $T_r(C,x) =]0,+\infty[(C-x)$. Equivalently,

$$T(C,x) = \{h \in X : \forall t_n \to 0^+, \, \exists h_n \to h; \, x + t_n h_n \in C, \text{ for } n \text{ sufficiently large}\}.$$

To the set C, we associate, at each $x \in X$, the cone

$$\mathcal{W}_C(x) = T(C, P_C(x)) \cap (x - P_C(x))^\perp, \tag{1}$$

where $P_C(x)$ denotes the projection of x onto C and

$$w^\perp = \{y \in X : \langle w, y\rangle = 0\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its topological dual X^*. Note that we always have

Lemma 2.1. *Let $C \subset \mathbb{R}^d$ be a closed convex set. Then*

$$T(C, P_C(\bar{x})) = \liminf_{x \to \bar{x}} T(C, P_C(x)), \quad \forall \bar{x} \in \mathbb{R}^d$$

or equivalently the set-valued mapping $x \rightrightarrows T(C, P_C(x))$ is lower semicontinuous. Here P_C denotes the projection mapping onto C.

Proof. Cornet [3] has found a topological connection between the Clarke's tangent cone $T_c(D,x)$ and the contingent cone $K(D,x)$ to D. He has shown that if $D \subset \mathbb{R}^d$ is any closed set then

$$T_c(D, \bar{d}) = \liminf_{d \to \bar{d}, \, d \in D} K(D,d) \quad \forall \bar{d} \in D.$$

Since C is convex, then $\quad T(C,x) = T_c(C,x) = K(C,x) \;\; \forall x \in C$

and hence $\quad T(C, P_C(\bar{x})) = \liminf_{\substack{x \to P_C(\bar{x}) \\ x \in C}} T(C,x).$

Using the definition of the "liminf", we obtain the desired result. $\square$

To conclude this preliminary section, we state the following result concerning the lower semicontinuity of the following set-valued mapping $\mathcal{V}_C : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ associated to a closed convex set $C \subset \mathbb{R}^d$ that has a nonempty interior. This mapping is defined by

$$\mathcal{V}_C(x) = T(C, P_C(x)) \cap (-T(C, P_C(x))). \tag{2}$$

Lemma 2.2. *Let $C \subset \mathbb{R}^d$ be a closed convex set with nonempty interior. Then the set-valued mapping $\mathcal{V}_C$ is lower semicontinuous and*

$$\mathcal{V}_C^0(\bar{x}) = \limsup_{x \to \bar{x}} \mathcal{V}_C^0(x) \, \forall \bar{x} \in \mathbb{R}^d.$$

where K^0 denotes the negative polar of the cone K, that is,

$$K^0 = \{x^* \in \mathbb{R}^d : \langle x^*, h \rangle \leq 0 \, \forall h \in K\}.$$

Proof. Lemma 2.1 ensures the lower semicontinuity of $x \rightrightarrows T(C, P_C(x))$. As $\mathrm{int} C \neq \emptyset$, then $\mathrm{int} T(C, P_C(\bar{x})) \neq \emptyset$ and Theorem B in [11] asserts that $\mathcal{V}_C$ is lower semicontinuous at $\bar{x}$. □

Remark 2.3. Note that, since $\mathrm{int} C \neq \emptyset$, the set C is epi-Lipschitz. Then, by [5], the Clarke's tangent cone to the boundary $\mathrm{bd} C$ of the convex set C at $x \in \mathrm{bd} C$, is given by

$$T_C(\mathrm{bd} C, x) = T(C, x) \cap (-T(C, x)) = \mathcal{V}_C(x).$$

Thus, for all $x \notin \mathrm{int} C$, $\mathcal{V}_C(x) = T_C(\mathrm{bd} C, P_C(x))$. However, these two sets differ when $x \in \mathrm{int} C$. □

3. Computing the coderivative of the metric projection mapping

In this section, we provide estimates for the coderivative of the metric projection mapping. We will consider two situations: the first one concerns convex polyhedral sets while the second one treats the case where the set is convex strictly Hadamard differentiable and epi-Lipschitzian.

For a closed convex set $C \subset \mathbb{R}^d$, the *coderivative* of the metric projection mapping P_C at $x \in \mathbb{R}^d$ is defined as a set-valued mapping $D^* P_C(x) : \mathbb{R}^d \mapsto \mathbb{R}^d$ defined by

$$D^* P_C(x)(p) = \{x^* \in \mathbb{R}^d : \quad (x^*, -p) \in N(Gr P_C, (x, P_C(x)))\}.$$

This can also be written equivalently as

$$D^* P_C(x)(p) = \partial \omega_p(x),$$

where, for a fixed $p \in \mathbb{R}^d$, $w_p : \mathbb{R}^d \mapsto \mathbb{R}$ is the Lipschitz function defined by

$$\omega_p(x) = \langle p, P_C(x) \rangle.$$

Estimating the coderivative of P_C is therefore equivalent to computing the subdifferential of ω_p.

Let us begin with some remarks regarding the subdifferential of ω_p. Since

$$\partial \omega_p(x_0) = \limsup_{x \to x_0} \partial^- \omega_p(x),$$

we may wonder whether we can restrict ourselves to elements of C, that is

$$\partial\omega_p(x_0) = \limsup_{x \xrightarrow{C} x_0} \partial^-\omega_p(x),$$

whenever $x_0 \in C$. Unfortunately, the answer is negative, as shown in the following example.

Example 3.1. Consider the set $C := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1,\ x_3 = 0\}$ and $x_0 = (0, 1, 0)$. Then the following inclusion may be strict

$$\limsup_{x \xrightarrow{C} x_0} \partial^-\omega_p(x) \subset \limsup_{x \to x_0} \partial^-\omega_p(x).$$

Indeed, let $p = (1, 1, 0)$ and $x := (x_1, x_2, x_3) \in \mathbb{R}^3$. The following assertions hold:

(1) if $x_1^2 + x_2^2 > 1$, then

$$\partial^-\omega_p(x) = \left\{ \left(\frac{p_1(x_1^2 + x_2^2) - x_1(x_1 + x_2)}{(x_1^2 + x_2^2)^{\frac{3}{2}}},\ \frac{p_2(x_1^2 + x_2^2) - x_2(x_1 + x_2)}{(x_1^2 + x_2^2)^{\frac{3}{2}}}, 0 \right) \right\};$$

(2) if $x_1^2 + x_2^2 < 1$, then $\partial^-\omega_p(x) = \{p\}$; and

(3) if $x_1^2 + x_2^2 = 1$, then $\partial^-\omega_p(x) = \{\emptyset\}$.

So that $\limsup_{x \xrightarrow{C} x_0} \partial^-\omega_p(x) = \{p\}$ while $\limsup_{x \to x_0} \partial^-\omega_p(x) = \{p, (1, 0, 0)\}$.

Remark 3.2. Even if C is strictly differentiable with nonempty interior, the following equality fails to hold

$$\partial\omega_p(x_0) = \limsup_{x \xrightarrow{C} x_0} \partial^-\omega_p(x).$$

To see this, it is enough to take $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, $\bar{x} = (1, 0)$ and $p = (1, 2)$ to obtain

$$\limsup_{x \xrightarrow{C} x_0} \partial^-\omega_p(x) = \{p\} \text{ whereas } \partial\omega_p(x_0) = \{0, p\}. \qquad \square$$

3.1. Convex strictly Hadamard differentiable sets

In this section, we compute the coderivative of the metric projection mapping of convex strictly Hadamard differentiable and epi-Lipschitzian sets in finite dimensional space. To facilitate this, we will first provide the definition of epi-Lipschitzness in normed spaces. Our approach is based on recent results obtained in [9], which characterize the Hadamard differentiability of epi-Lipschitz sets in general Banach spaces.

Let C be a subset of a normed space X. According to [16], C is said to be epi-Lipschitzian at a point $\bar{x} \in C$ in a direction $\bar{v} \neq 0$ if there exists $\gamma > 0$ such that the following inclusion holds:

$$C \cap B(\bar{x}, \gamma) + tB(\bar{v}, \gamma) \subset C \quad \forall\, t \in]0, \gamma[,$$

which is equivalent to stating (see [16], [17]) that C can locally be represented as the epigraph of a locally Lipschitz function f whenever $\bar{x} \in \mathrm{bd}(C)$. We denote this local representation of C near $\bar{x}$ as follows:

$$\textit{We call } f \textit{ a local representation of } C \textit{ near } \bar{x}. \tag{3}$$

We say that C is epi-Lipschitz if it is epi-Lipschitz at each of its points.

An epi-Lipschitz set C at $\bar{x} \in \mathrm{bd}(C)$ is called strictly Hadamard differentiable at $\bar{x}$ if its local representation f is strictly Hadamard differentiable at that point.

For closed convex sets, we have the following fundamental characterization.

Lemma 3.3. *Let C be a nonempty closed convex set in a normed space X. Then, the following assertions are equivalent:*

(1) *C is epi-Lipschitz;*

(2) *C has a nonempty interior.*

The following theorem provides a characterization of strictly differentiable sets in Banach spaces.

Theorem 3.4. ([9]) *Let C be a closed subset of the real Banach space X which is epi-Lipschitzian at $\bar{x} \in \mathrm{bd}\, C$. Then C is strictly Hadamard differentiable at $\bar{x}$ if and only if $T_c(\mathrm{bd}\, C, \bar{x})$ is a closed hyperplane.*

We derive the following corollary.

Corollary 3.5. *Let C be a closed convex subset of some Hilbert space X which is strictly Hadamard differentiable with nonempty interior. Then*

(1) *if $x \notin \mathrm{int} C$, there exists a unit vector $w_x \in X$ such that*

$$T(C, P_C(x)) = \{w \in X : \langle w_x, w \rangle \leq 0\},\ N(C, P_C(x)) = \mathbb{R}_+ w_x,\ \mathcal{V}_C(x) = \{w_x\}^\perp,$$

where $\{w_x\}^\perp = \{w \in X : \langle w, w_x \rangle = 0\}$, with $w_x = \frac{x - P_C(x)}{\|x - P_C(x)\|}$ whenever $x \notin C$.

(2) *Moreover for all $x \in X$, we have*

$$\mathcal{W}_C(x) = \begin{cases} T(C, x) & \textit{if } x \in C \\ \mathcal{V}_C(x) & \textit{if } x \notin C \end{cases} \tag{4}$$

and the set-valued mapping $x \rightrightarrows T(C, P_C(x))$ is continuous at any $\bar{x} \notin C$.

Proof. We establish only the last statement on the continuity of the set-valued mapping $x \rightrightarrows T(C, P_C(x))$. So let $\bar{x} \notin C$. Since, by Lemma 2.1, this set-valued mapping is lower semicontinuous, it is enough to show that it is upper semicontinuous at $\bar{x}$. So let $w \in \limsup_{x \to \bar{x}} T(C, P_C(x))$. Then, there exist sequences $w_n \to w$ and $x_n \to \bar{x}$ such that $w_n \in T(C, P_C(x_n))$ for all n. Then,

$$\langle x_n - P_C(x_n), w_n \rangle \leq 0 \text{ for all } n.$$

By passing to the limit, we get $\langle \bar{x} - P_C(\bar{x}), w \leq 0$.

As $N(C, P_C(x)) = \mathbb{R}_+(\bar{x} - P_C(\bar{x}))$, then $w \in T(C, P_C(\bar{x}))$, which shows that

$$\limsup_{x \to \bar{x}} T(C, P_C(x)) \subset T(C, P_C(\bar{x})).$$

As the reverse inclusion is always satisfied, the set-valued mapping $x \rightrightarrows T(C, P_C(x))$ is upper semicontinous at $\bar{x}$, and the proof is completed. $\square$

In the following result, we will give an (upper) estimate of the lower Dini-derivative of the function ω_p as well as its Dini-subdifferential. Before its statement, we have to say that the first equality was established by Zarontonello [20], but in order to keep the paper as self-contained as possible, we will provide direct arguments.

Theorem 3.6. *Let $p \in \mathbb{R}^d$ and let $C \subset \mathbb{R}^d$ be a closed convex.*

(1) *For all $x \in C$, the function ω_p is directionnally differentiable at x with directional derivative satisfying*

$$\omega'_p(x, w) = \langle p, P_{\mathcal{W}_C(x)}(w) \rangle \quad \forall w \in \mathbb{R}^d$$

and hence $\partial^- \omega_p(x) = \{x^* \in \mathbb{R}^d : \langle x^*, w\rangle \le \langle p, P_{\mathcal{W}_C(x)}(w)\rangle \, \forall w \in \mathbb{R}^d\}$.

Moreover $\qquad \partial^- \omega_p(x) = (p + N(C, x)) \cap T(C, x).$

(2) *Suppose that C is strictly Hadamard differentiable with nonempty interior and let $x \notin C$. Then $\mathcal{W}_C(x) = \{w_x\}^\perp$, with $w_x = \frac{x - P_C(x)}{\|x - P_C(x)\|}$, and for all $w \in \mathbb{R}^d$ there exists $v \in \mathcal{W}_C(x)$ such that*

$$\langle w - v, v\rangle \ge 0, \quad \omega^-_p(x, w) = \langle p, v\rangle.$$

Moreover $\quad \omega^-_p(x, w) \le \frac{1}{2}\big(\langle p, P_{\mathcal{W}_C(x)}(w)\rangle + \|P_{\mathcal{W}_C(x)}(w)\| \cdot \|P_{\mathcal{W}_C(x)}(p)\|\big)$.
Consequently,

$$\partial^- \omega_p(x) \subset \left\{ \begin{matrix} x^* \in \mathbb{R}^d \,:\, \forall w \in \mathbb{R}^d \\ \langle x^*, w\rangle \le \frac{1}{2}\big(\langle p, P_{\mathcal{W}_C(x)}(w)\rangle + \|P_{\mathcal{W}_C(x)}(w)\| \cdot \|P_{\mathcal{W}_C(x)}(p)\|\big) \end{matrix} \right\}$$
$$= \frac{1}{2}\Big(P_{\mathcal{W}_C(x)}(p) + \|P_{\mathcal{W}_C(x)}(p)\|\mathbb{B}) \cap \mathcal{W}_C(x).$$

Proof. Let us recall that relation (4) asserts that

$$\mathcal{W}_C(x) = \begin{cases} T(C, x), & \text{if } x \in C \\ \mathcal{V}_C(x), & \text{if } x \notin C. \end{cases}$$

We will use $T(C, x)$ and $\mathcal{V}_C(x)$ instead of $\mathcal{W}_C(x)$ in the first and second items, respectively.

(1) We will prove that for a given $w \in \mathbb{R}^d$, we have

$$\omega^-_p(x, w) = \langle p, P_{\mathcal{W}_C(x)}(w)\rangle = \limsup_{t \to 0^+} \frac{\omega_p(x + tw) - \omega_p(x)}{t}.$$

We start by estabilishing the first equality $\omega^-_p(x, w) = \langle p, P_{\mathcal{W}_C(x)}(w)\rangle$.

Let $w \in \mathbb{R}^d$ and and take any sequence $s_n \mapsto 0^+$ such that

$$\omega_p^-(x,w) = \lim_{n\to+\infty} \langle\, p,\ \frac{P_C(x+s_nw)-x}{s_n}\rangle.$$

Consider the sequence (v_n) defined by $v_n := (P_C(x+s_nw)-x)/s_n$. We may assume that $v_n \to v$. So we have $v \in T(C,x)$. We will show that $v = P_{T(C,x)}(w)$. Since $P_C(x+s_nw) = x+s_nv_n$, then

$$\langle x+s_nw-(x+s_nv_n), z-(x+s_nv_n)\rangle \le 0 \ \ \forall z \in C.$$

So that for all $y \in T(C,x)$, there exists a sequence (y_n) converging to y such that $x+s_ny_n \in C$, for all sufficiently large n, and hence

$$\langle w-v_n, y_n-v_n\rangle \le 0.$$

So passing to the limit, we get $\langle w-v, y-v\rangle \le 0$ for all $y \in T(C,x)$ and hence $v = P_{T(C,x)}(w)$. Similar arguments lead to the following equality

$$\limsup_{s\to 0^+} \frac{\omega_p(x+sw)-\omega_p(x)}{s} = \langle\, p, P_{\mathcal{W}_C(x)}(w)\rangle.$$

To establish the last equality of this item, it is enough to show that the following one holds

$$\mathcal{D} := \{x^* \in \mathbb{R}^d :\ \langle x^*, w\rangle \le \langle\, p, P_{T(C,x)}(w)\rangle\ \forall w \in \mathbb{R}^d\} = (p+N(C,x)) \cap T(C,x).$$

The inclusion $\mathcal{D} \subset (p+N(C,x)) \cap T(C,x)$ is obvious. Let us prove the reverse one. So let $x^* \in (p+N(C,x)) \cap T(C,x)$. Then $x^* \in T(C,x)$ and, by Corollary 3.5, there exists $\lambda \ge 0$ such that $x^* = p+\lambda w_x$. By Moreau decomposition, each element w in $\mathbb{R}^d$ can be decomposed on $T(C,x)$ and $N(C,x)$ as follows: $w = P_{T(C,x)}(w)+P_{N(C,x)}(w)$ with $\langle P_{T(C,x)}(w), P_{N(C,x)}(w)\rangle = 0$. As $x^* \in T(C,x)$, then

$$\langle x^*, w\rangle \le \langle x^*, P_{T(C,x)}(w)\rangle = \langle\, p+\lambda w_x, P_{T(C,x)}(w)\rangle \le \langle\, p, P_{T(C,x)}(w)\rangle,$$

and as w is arbitrary in $\mathbb{R}^d$, this shows that $x^* \in \mathcal{D}$.

(2) Let $x \notin C$ and $w \in \mathbb{R}^d$. Now let $s_n \mapsto 0^+$ be such that

$$\omega_p^-(x,w) = \lim_{n\to+\infty} \langle\, p,\ \frac{P_C(x+s_nw)-P_C(x)}{s_n}\rangle.$$

Consider the sequence (v_n), defined by $v_n := (P_C(x+s_nw)-P_C(x))/s_n$, which we assume converging to $v \in T(C, P_C(x))$ satisfying

$$\omega_p^-(x,w) = \langle\, p, v\rangle.$$

By the definition of v_n and the convexity of C, we get $v_n \in T(C, P_C(x))$ and

$$\langle x+s_nw-P_C(x)-s_nv_n, z-P_C(x)-s_nv_n\rangle \le 0,\ \forall z \in C.$$

Taking $z = P_C(x)$, we get

$$\langle x + s_n w - P_C(x) - s_n v_n, -v_n\rangle \leq 0. \tag{5}$$

As $v_n \in T(C, P_C(x))$, we have $\langle x - P_C(x), v_n\rangle \leq 0$, and hence

$$\langle w - v_n, -v_n\rangle \leq 0.$$

Next, we take the limit to obtain the desired inequality $\langle w - v, v\rangle \geq 0$. Now, by passing to the limit in (5), we obtain $\langle x - P_C(x), -v\rangle \leq 0$. Since $v \in T(C, P_C(x))$, we conclude that $\langle x - P_C(x), v\rangle = 0$. By Corollary 3.5, this guarantees that $v \in \mathcal{V}_C(x)$. Thus, we have

$$\omega_p^-(x, w) = \langle p, v\rangle \leq \sup\{\langle p, y\rangle : y \in \mathcal{V}_C(x),\ \|y\|^2 \leq \langle w, y\rangle\}.$$

If either $p \in (\mathcal{V}_C(x))^\perp$ or $w \in (\mathcal{V}_C(x))^\perp$, then it follows that

$$\omega_p^-(x, w) = \max\{\langle p, y\rangle : y \in \mathcal{V}_C(x),\ \|y\|^2 \leq \langle w, y\rangle\} = 0.$$

Now, let us assume that neither $p \in (\mathcal{V}_C(x))^\perp$ nor $w \in (\mathcal{V}_C(x))^\perp$, and consider the following optimization problem:

$$\max\{\langle p, y\rangle : y \in \mathcal{V}_C(x),\ \|y\|^2 \leq \langle w, y\rangle\}. \tag{6}$$

Set $\bar{w} = P_{\mathcal{V}_C(x)}(w)$ and $\bar{p} = P_{\mathcal{V}_C(x)}(p)$. Thus, solving the problem (6) is equivalent to solve the following optimization problem:

$$\max\{\langle \bar{p}, y\rangle : y \in \mathcal{V}_C(x),\ \|y\|^2 \leq \langle \bar{w}, y\rangle\}. \tag{7}$$

Both problems yield the same optimal value and have the same set of solutions. It is clear that each problem has at least one solution, denoted by $\bar{y}$. Note that the Slater qualification condition is satisfied with $y = \frac{1}{2}\bar{w}$ as $\frac{1}{2}\bar{w} \in \mathcal{V}_C(x)$ and $\frac{1}{4}\|\bar{w}\|^2 < \frac{1}{2}\|\bar{w}\|^2 \leq \langle \frac{1}{2}\bar{w}, \bar{w}\rangle$). Consequently, the necessary optimality conditions ensure the existence of a multiplier $\lambda \geq 0$ such that

$$\bar{p} + \lambda(\bar{w} - 2\bar{y}) \in (\mathcal{V}_C(x))^\perp,\ \lambda(\|\bar{y}\|^2 - \langle \bar{w}, \bar{y}\rangle) = 0.$$

Since $\bar{p} + \lambda(\bar{w} - 2\bar{y}) \in \mathcal{V}_C(x) \cap \mathcal{V}_C(x))^\perp$, it follows that

$$\bar{p} + \lambda(\bar{w} - 2\bar{y}) = 0.$$

Given that $p \notin (\mathcal{V}_C(x))^\perp$, we have $\lambda > 0$. Consequently, it follows that

$$\|\bar{y}\|^2 = \langle \bar{w}, \bar{y}\rangle \text{ and } \bar{y} = \frac{1}{2}\Big(\frac{1}{\lambda}\bar{p} + \bar{w}\Big).$$

By combining these two equalities, we obtain

$$\lambda = \frac{\|\bar{p}\|}{\|\bar{w}\|} \quad \text{and } \bar{y} = \frac{1}{2}\Big(\frac{\|\bar{w}\|}{\|\bar{p}\|}\bar{p} + \bar{w}\Big).$$

Thus, the proof is complete. □

The following set will be used in the following theorem

$$\mathcal{D}_C(\bar{x}) := \left\{ \begin{array}{l} x^* \in \mathbb{R}^d \, : \, \forall w \in \mathbb{R}^d \\ \langle x^*, w\rangle \le \frac{1}{2}\big(\langle\, p, P_{\mathcal{W}_C(\bar{x})}(w)\rangle + \|P_{\mathcal{W}_C(\bar{x})}(w)\| \cdot \|P_{\mathcal{W}_C(\bar{x})}(p)\|\big) \end{array} \right\}, \tag{8}$$

where $\mathcal{W}_C(x)$ is defined in (1).

Theorem 3.7. *Suppose that $C \subset \mathbb{R}^d$ is a closed convex set with nonempty interior and strictly Hadamard differentiable. Let $p \in \mathbb{R}^d \backslash \{0\}$ and $\bar{x} \in \mathbb{R}^d$. Then,*

(1) *for $\bar{x} \in \mathrm{bd}(C)$,* $$\limsup_{\substack{x \to \bar{x} \\ x \in int(C)}} \partial^- \omega_p(x) = \{p\}$$

$$\limsup_{\substack{x \to \bar{x} \\ x \in \mathrm{bd}(C)}} \partial^- \omega_p(x) = \partial^- \omega_p(\bar{x}) = \big\{x^* \in \mathbb{R}^d : \langle x^*, w\rangle \le \langle\, p, P_{\mathcal{W}_C(\bar{x})}(w)\rangle \; \forall w \in \mathbb{R}^d\big\}$$

$$\limsup_{x \to \bar{x},\, x \notin C} \partial^- \omega_p(x) \subset \mathcal{D}_C(\bar{x});$$

(2) *if $\bar{x} \in \mathbb{R}^d$, then*

$$\partial\omega_p(\bar{x}) \subset \mathcal{D}_C(\bar{x}) \subset \mathcal{W}_C(\bar{x}) \quad and \quad \partial_C\omega_p(\bar{x}) \subset \mathcal{D}_C(\bar{x}) \subset \mathcal{W}_C(\bar{x}).$$

We start by establishing the following result.

Lemma 3.8. *Let $w \in \mathbb{R}^d$. Then, under the assumptions of Theorem 3.7, we have*

(1) *for any $\bar{x} \notin int(C)$ and any sequence $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d \backslash int(C)$ converging to $\bar{x}$ such that the sequence $(P_{T(C,P_C(x_k))}(w))_{k\in\mathbb{N}}$ converges to some v, one has $v = P_{T(C,P_C(\bar{x}))}(w)$,*

(2) *for any $\bar{x} \in \mathbb{R}^d$ and any sequence $(x_k)_{k\in\mathbb{N}}$ converging to $\bar{x}$ such that the sequence $(P_{\mathcal{V}_C(x_k)}(w))_{k\in\mathbb{N}}$ converges to some v, one has $v = P_{\mathcal{V}_C(\bar{x})}(w)$.*

Proof. The key to establishing the two results lies in the lower semicontinuity of the set-valued mappings $T(C, P_C(\cdot))$ and $\mathcal{V}_C(\cdot)$ (see Lemmas 2.1 and 2.2).

We will focus on proving only the first result. Let us consider the case where $\bar{x} \in \mathrm{bd}\, C$. By definition of the projection, we have

$$\langle w - v_k, y - v_k\rangle \le 0, \; \forall y \in T(C, P_C(x_k)),$$

where $v_k = P_{T(C,P_C(x_k))}(w)$. Lemma 2.1 guarantees that for all $y \in T(C, P_C(\bar{x}))$, there exists a sequence $(y_k)_{k\in\mathbb{N}}$ such that $y_k \in T(C, P_C(x_k))$ for all k, and

$$\lim_{k\to+\infty} y_k = y.$$

Therefore, $\langle w - v_k, y_k - v_k\rangle \le 0$.

Taking the limit as k approaches infinity, we obtain $\langle w - v, y - v\rangle \le 0$.

It remains to show that $v \in T(C, P_C(\bar{x}))$.

Given that $P_C(\bar{x})$ and $P_C(x_k)$ are both on the boundary $\mathrm{bd}(C)$, and C is strictly Hadamard differentiable, Corollary 3.5 ensures the existence of unit vectors $w_{\bar{x}}$ and w_{x_k} such that

$$T(C, P_C(\bar{x})) = \{w \in \mathbb{R}^d : \langle w_{\bar{x}}, w\rangle \le 0\}, \quad N(C, P_C(\bar{x})) = \mathbb{R}_+ w_{\bar{x}}$$

and $\quad T(C, P_C(x_k)) = \{w \in \mathbb{R}^d : \langle w_{x_k}, w\rangle \leq 0\}, \quad N(C, P_C(x_k)) = \mathbb{R}_+ w_{x_k}.$

Now, extracting a subsequence if necessary, we may assume that $(w_{x_k})_k$ converges to some element v_0. Since

$$\limsup_{k\to+\infty} N(C, P_C(x_k)) \subset N(C, P_C(\bar{x})),$$

it follows that $v_0 = w_{\bar{x}}$, as v_0 lies in the normal cone $N(C, P_C(\bar{x})) = \mathbb{R}_+ w_{\bar{x}}$ and has a unit norm ($\|v_0\| = 1$). Consequently, we have

$$\langle v, w_{\bar{x}}\rangle \leq 0,$$

which confirms that $v \in T(C, P_C(\bar{x}))$ and the proof is complete. □

Remark 3.9. (1) The smoothness of C plays a central role in this lemma. To see this, take $C = \mathbb{R}_+ \times \mathbb{R}_+$ and $x_k = (0, \frac{1}{k+1}) \in \operatorname{bd} C$. Then $P_{T(C,x_k)}(-1,-1) = (0,-1)$ while $P_{T(C,(0,0))}(-1,-1) = (0,0)$.

(2) The smoothness of C is not sufficient to ensure the result of the lemma. It is enough to take

$$C = \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 + y^2 \leq 1,\ z = 0\}.$$

Let $x_k = (1 - \frac{1}{k+1}, 0, 0) \in \operatorname{bd} C$. Then we obtain $P_{T(C,x_k)}(1,1,1) = (1,1,0)$ and $P_{T(C,(1,0,0))}(1,1,1) = (0,1,0)$. This shows the necessity of the interiority assumption.

Proof of Theorem 3.7. (1) Let $\bar{x} \in \operatorname{bd}(C)$. The first equality $\partial^- \omega_p(\bar{x}) = \{p\}$ follows from the fact that

$$\omega_p(x) = \langle p, x\rangle \quad \forall x \in int(C).$$

Let us establish the second one and pick $x^* \in \limsup\limits_{\substack{x\to\bar{x}\\ x\in int(C)}} \partial^- \omega_p(x)$.

Then there are sequences $x_n^* \to x^*$ and $x_n \to \bar{x}$ such that

$$x_n^* \in \partial^- \omega_p(x_n) \quad \text{ for } n \text{ sufficiently large.}$$

Theorem 3.6 ensures that

$$\langle x_n^*, w\rangle \leq \langle p, P_{\mathcal{W}_C(x_n)}(w)\rangle\, \forall w \in \mathbb{R}^d$$

or equivalently, by Corollary 3.5,

$$\langle x_n^*, w\rangle \leq \langle p, P_{T(C,x_n)}(w)\rangle\, \forall w \in \mathbb{R}^d. \tag{9}$$

So let $w \in \mathbb{R}^d$. Since the sequence $(P_{T(C,x_n)}(w))_n$ is bounded, we can extract a subsequence, if necessary, such that this sequence converges to some $v \in \mathbb{R}^d$. Applying Lemma 3.8, we find that

$$v = P_{T(C,\bar{x})}(w).$$

Consequently, by passing to the limit in relation (9), we obtain

$$\langle x^*, w\rangle \leq \langle p, P_{P_{T(C,\bar{x})}(w)}(w)\rangle\, \forall w \in \mathbb{R}^d.$$

The proof of the third inclusion can be established similarly by employing Theorem 3.6 along with the second item of Lemma 3.8.

(2) For the last item, it suffices to prove that for all $\bar{x} \in \mathbb{R}^d$, the following inequality holds:

$$\limsup_{x \to \bar{x}} \omega_p^-(x, w) \le \frac{1}{2}\big(\langle p, P_{\mathcal{W}_C(\bar{x})}(w)\rangle + \|P_{\mathcal{W}_C(\bar{x})}(w)\| \cdot \|P_{\mathcal{W}_C(\bar{x})}(p)\|\big) \;\forall w \in \mathbb{R}^d.$$

It is noteworthy that the limit $\limsup_{x \to \bar{x}} \omega_p^-(x, w)$ is equivalent to the Clarke directional derivative of ω_p at $\bar{x}$ in the direction w. We will consider two cases:

Case 1. $\bar{x} \in C$.

Fix $w \in \mathbb{R}^d$ and choose $x_n \to \bar{x}$ such that

$$\limsup_{x \to \bar{x}} \omega_p^-(x, w) = \lim_{n \to +\infty} \omega_p^-(x_n, w).$$

If $x_n \in C$ for infinitely many n, then

$$\omega_p^-(x_n, w) = \langle p, P_{\mathcal{W}_C(x_n)}(w)\rangle = \langle p, P_{T(C,x_n)}(w)\rangle.$$

We may now proceed as in the first item to show the existence of a subsequence of $(P_{T(C,x_n)}(w))$ that converges to some $v = P_{T(C,\bar{x})}(w)$. This convergence implies that

$$\limsup_{x \to \bar{x}} \omega_p^-(x, w) = \langle p, P_{T(C,\bar{x})}(w)\rangle.$$

Let (p_1, p_2) be a Moreau decomposition of p, meaning that we have $p = p_1 + p_2$ with $p_1 = P_{T(C,\bar{x})}(p)$ and $p_2 = P_{N(C,\bar{x})}(p)$, $\langle p_1, p_2\rangle = 0$. Then we have

$$\begin{aligned}
\langle p, P_{T(C,\bar{x})}(w)\rangle &= \frac{1}{2}\Big(\langle p, P_{T(C,\bar{x})}(w)\rangle + \langle p, P_{T(C,\bar{x})}(w)\rangle\Big) \\
&= \frac{1}{2}\Big(\langle p, P_{T(C,\bar{x})}(w)\rangle + \langle p_1 + p_2, P_{T(C,\bar{x})}(w)\rangle\Big) \\
&\le \frac{1}{2}\Big(\langle p, P_{T(C,\bar{x})}(w)\rangle + \langle p_1, P_{T(C,\bar{x})}(w)\rangle\Big) \\
&\le \frac{1}{2}\Big(\langle p, P_{T(C,\bar{x})}(w)\rangle + \|p_1\| \cdot \|P_{T(C,\bar{x})}(w)\|\Big) \\
&= \frac{1}{2}\Big(\langle p, P_{\mathcal{W}_C(\bar{x})}(w)\rangle + \|p_1\| \cdot \|P_{\mathcal{W}_C(\bar{x})}(w)\|\Big)
\end{aligned}$$

Now, suppose that $x_n \notin C$ for infinitely many n. The same reasoning will lead to the desired result by using the second item of Lemma 3.8.

Case 2. $\bar{x} \notin C$.

The proof follows a similar approach to the first case, using the second items of Theorem 3.6 and Lemma 3.8. □

3.2. Polyhedral convex sets

In the first subsection of this section, we focus on the coderivative of the metric projection mapping. In particular, we have shown that for any closed convex set $C \subset \mathbb{R}^d$ and $x \in C$, the following holds:

$$\omega_p^-(x, w) = \langle p, P_{T(C,x)}(w)\rangle \quad \forall w \in \mathbb{R}^d.$$

The question arises: Can we express this directional derivative in a similar form when $x \notin C$? The answer is YES in a special class of sets satisfying the so called polyhedral property (see below).

To explore this, we first provide the following definition by Mignot [12] in Hilbert spaces and all the results below attributed to Mignot are established in a Hilbert space. Since we are interested in the finite dimension, we will cite them in these spaces.

Definition 3.10. A function $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ has a *conical differential* at $x \in \mathbb{R}^d$ equal to Q if

(1) if Q is positively homogeneous and

(2) $\forall y \in \mathbb{R}^d$ and all $t > 0$ we have

$$F(x+ty) = F(x) + tQ(y) + o(t,y) \tag{10}$$

where $\lim_{t\to 0^+} \frac{o(t,y)}{t} = 0$ and this limit is uniform for y in a compact set in $\mathbb{R}^d$.

Theorem 3.11. (Mignot [12]) *Let $x \in \mathbb{R}^d$.*

(1) *Suppose that*

$$\forall z \in \mathcal{W}_C(x), \quad P_C(x+tz) = P_C(x) + tz + o(t,z). \tag{11}$$

Then P_C has a conical differential at x equal to $P_{\mathcal{W}_C(x)}$.

(2) *Suppose that for all z in a some dense set of $\mathcal{W}_C(x)$ the following equality holds*

$$P_C(x+tz) = P_C(x) + tz + o(t,z). \tag{12}$$

Then P_C has a conical differential at x equal to $P_{\mathcal{W}_C(x)}$.

(3) *Then for all $z \in T_r(C, P_C(x)) \cap (x - P_C(x))^{\perp}$ the following equality holds for all $t > 0$ sufficiently small*

$$P_C(x+tz) = P_C(x) + tz. \tag{13}$$

Following Haraux [7], a set C has the polyhedral property at $x \in H$ if

$$\mathcal{W}_C(x) = \overline{T_r(C, P_C(x)) \cap (x - P_C(x))^{\perp}}.$$

Example 3.12. Each polyhedral closed set in H has the polyhedral property.

So that the following result holds.

Theorem 3.13. (Mignot[12]) *Let $C \subset \mathbb{R}^d$ be a closed convex set.*

(1) *If C has the polyhedral property at $x \in \mathbb{R}^d$, then P_C has a conical differential at x equal to $P_{\mathcal{W}_C(x)}$.*

(2) *If the mapping P_C has a conical differential at $x \in \mathbb{R}^d$ equal to $P_{\mathcal{W}_C(x)}$, then it admits the same differential at any point of the segment $]P_C(x), x]$*

The following result was established in [1].

Theorem 3.14. *Let C be a closed polyhedral subset of $\mathbb{R}^d$, and $\bar{x}, p \in \mathbb{R}^d$. Then,*

(1) *the mapping $x \mapsto P_C(x)$ is directionally differentiable at $\bar{x}$ and satisfies*

$$P'_C(\bar{x}, h) := \lim_{s \to 0^+} \frac{P_C(\bar{x} + sh) - P_C(\bar{x})}{s} = P_{\mathcal{W}_C(\bar{x})}(h), \quad \forall h \in \mathbb{R}^d,$$

$$\partial^- \omega_p(\bar{x}) = \Big(p + \mathcal{W}_C^0(\bar{x})\Big) \cap \mathcal{W}_C(\bar{x})$$

and

$$\partial \omega_p(\bar{x}) = \limsup_{x \to \bar{x}} \Big((p + \mathcal{W}_C(x)^0) \cap \mathcal{W}_C(x)\Big).$$

(2) *if the set-valued mapping $x \rightrightarrows \mathcal{W}_C(x)$ is lower semicontinuous at $\bar{x}$, then*

$$\partial \omega_p(\bar{x}) \subset (p + \mathcal{W}_C^0(\bar{x})) \cap \limsup_{x \to \bar{x}} \mathcal{W}_C(x),$$

and if in addition $\mathcal{W}_C$ is upper semicontinuous at $\bar{x}$, then

$$\partial \omega_p(\bar{x}) = (p + \mathcal{W}_C^0(\bar{x})) \cap \mathcal{W}_C(\bar{x});$$

(3) *if $int(C) \neq \emptyset$ and $\bar{x} \notin C$, then the set-valued mapping $\mathcal{W}_C$ is lower semicontinuous at $\bar{x}$,*

$$\partial \omega_p(\bar{x}) \subset (p + \mathcal{W}_C^0(\bar{x})) \cap \Big(\bar{x} - P_C(\bar{x})\Big)^{\perp} \cap \|p\|\mathbb{B},$$

and

$$\begin{aligned} T_c(bd(C), P_C(\bar{x})) &= T(C, P_C(\bar{x})) \cap (-T(C, P_C(\bar{x}))) \\ &\subset (\bar{x} - P_C(\bar{x}))^{\perp} \cap T(C, P_C(\bar{x})). \end{aligned} \tag{14}$$

(4) *for all $x \in \mathbb{R}^d$ we have $\mathcal{V}_C(x) \subset \mathcal{W}_C(x)$, where $\mathcal{V}_C(x)$ is defined in (2);*

(5) *if $int(C) \neq \emptyset$, then $\partial w_p(\bar{x}) \subset \Big(p + \mathcal{V}_C^0(\bar{x})\Big) \cap \|p\|\mathbb{B}\, \forall \bar{x} \in \mathbb{R}^d$.*

References

[1] L. Alsharief, A. Jourani: *Optimal control of polyhedral implicit sweeping processes involving the coderivative of the metric projection mapping*, submitted (2023).

[2] F. H. Clarke: *Optimisation and Nonsmooth Analysis*, Wiley, New York (1983).

[3] B. Cornet: *Regular Properties of Tangent and Normal Cones*, Thesis, University Paris-Dauphine (1980).

[4] R. Correa, D. Salas, L. Thibault: *Smoothness of the metric projection onto nonconvex bodies inHilbert spaces*, J. Math. Analysis Appl. 457 (2018) 1307–1332.

[5] M. O. Czarnecki, L. Thibault: *Sublevel representations of epi-Lipschitz sets and other properties*, Math. Programming Ser. B 168 (2018) 555–569.

[6] S. Fitzpatrick, R. R. Phelps: *Differentiability of the metric projection in Hilbert space*, Trans. Amer. Math. Soc. 170 (1982) 483–501.

[7] A. Haraux: *How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities*, J. Math. Soc. Japan 29 (1977) 615–631.

[8] R. B. Holmes: *Smoothness of certain metric projections on Hilbert space*, Trans. Amer. Math. Soc. 184 (1973) 87–100.

[9] A. Jourani, M. Séne: *Geometric characterizations of the strict Hadamard differentiability of sets*, Pure Appl. Funct. Analysis 6 (2021) 1333–1346.

[10] J. Kruskal: *Two convex counterexamples: a discontinuous envelope function and a nondifferentiable nearest-point mapping*, Proc. Amer. Math. Soc. 23 (1969) 697–703.

[11] A. Lechicki, A. Spakowski: *A note on intersection of lower semicontinous multifunctions*, Proc. Amer. Math. Soc. 95 (1985) 119–122.

[12] F. Mignot: *Contrôle dans les inéquations variationelles elliptiques*, J. Funct. Analysis 22 (1976) 130–185.

[13] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. I: Basic Theory*, updated reprint, Springer, New York (2013).

[14] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. II: Applications*, updated reprint, Springer, New York (2013).

[15] D. Noll: *Directional differentiability of the metric projection in Hilbert space*, Pacific J. Math. 170/2 (1995) 567–592.

[16] R. T. Rockafellar: *Clarke's tangent cones and the boundaries of closed sets in* $\mathbb{R}^n$, Nonlinear Analysis, Theory Methods Appl. 3 (1979) 145–154.

[17] R. T. Rockafellar: *Generalized direction derivatives and subgradients of nonconvex functions*, Canad. J. Math. 32 (1980) 257–280.

[18] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. Part I: General Theory*, World Scientific, Singapore (2023).

[19] L. Thibault: *Unilateral Variational Analysis in Banach Spaces. Part II: Special Classes of Functions and Sets*, World Scientific, Singapore (2023).

[20] E. H. Zarantonello: *Projections on convex sets in Hilbert space and spectral theory*, in: *Contributions to Nonlinear Functional Analysis*, Publ. No. 27, Math. Res. Center of the University of Wisconsin at Madison (1971) 237–424.