Journal of Convex Analysis
Volume 32 (2025), No. 1, 71–89

Quasi-Equilibrium Problems with Non-Self Constraint Maps in Topological Spaces. Necessary and Sufficient Conditions for the Existence of Solutions

Didier Aussel
Lab. PROMES, UPR CNRS 8521, Université de Perpignan, France
aussel@univ-perp.fr

John Cotrina
Universidad del Pacífico, Lima, Perú
cotrina_je@up.edu.pe

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: September 6, 2024
Accepted: October 12, 2024

Quasi-equilibrium problems represent a general framework covering, in many situations, quasi-variational inequalities, complementarity problems and generalized Nash equilibrium problems. In this work, we provide necessary and sufficient conditions guaranteeing the existence of a new kind of solution for quasi-equilibrium problems defined on Hausdorff topological spaces with non-self constraint maps. The main machineries for proving our results are a Tian's fixed point theorem for the convex case, and the finite intersection property for the non-convex case. Furthermore, quasi-variational inequality problems and generalized Nash equilibrium problems are considered as applications.

Keywords: Quasi-equilibrium problems, quasi-variational inequalities, generalized Nash equilibrium problems, non-self maps, finite intersection property.

2020 Mathematics Subject Classification: 47J20, 49J53, 91A10.

1. Introduction

Given a subset X of a Hausdorff topological space E, a bifunction $f : E \times E \to \mathbb{R}$ and a constraint set-valued map $K : X \rightrightarrows E$, the *quasi-equilibrium problem* (QEP in short) consists in finding

$$x \in X \text{ such that } x \in K(x) \text{ and } f(x, y) \geq 0, \text{ for all } y \in K(x). \tag{1}$$

This kind of problem has been proved to be extremely useful, for example in economics to be able to take into account that the amount of available and shared good is finite. It is well known that many problems such as quasi-variational inequality, generalized Nash equilibrium problems, among others, can be reformulated as quasi-equilibrium problems. There is an extended literature (see, for instance, [8, 15, 16, 31, 22, 4, 24] and their references therein) when the constraint map is constant and equal to X, i.e. $K(x) = X$ for all $x \in X$, which corresponds to

ISSN 0944-6532 / \$ 2.50

the classical *equilibrium problem* (EP in short) considered by Blum and Oettli in [11]. Several existence results have been established when the constraint map is a self-map, i.e. $K(X) \subset X$, see, for instance, [3, 25, 17, 19, 23].

Nevertheless the classical concepts of solution for quasi-equilibrium problems failed to make sense as soon as the constraint map is a non-self map, that is when $K(X)$ is not a subset of X. This troubling situation occurs in many applications, for example as presented in [5] for the modeling of electricity market biddings. Thus an adapted concept of solution, called *projected solution*, has been proposed in [5] for quasi-variational inequalities and later for quasi-equilibrium problems in [26], both on finite dimensional spaces. Let us recall that $\hat{x} \in X \subset \mathbb{R}^n$ is a *projected solution* of the QEP if, there exists $\hat{z} \in \mathbb{R}^n$ such that the following two conditions hold:

1. $\|\hat{x} - \hat{z}\| \leq \|x - \hat{z}\|$, for all $x \in X$, and
2. $\hat{z} \in K(\hat{x})$ and $f(\hat{z}, y) \geq 0$, for all $y \in K(\hat{x})$;

where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$. Recently in [1, 6, 13, 18] existence results for quasi-equilibrium problems on Banach spaces have been proved. However, this new concept of solution was based on a projection process, which requires a certain structure on the involved space. Thus, our aim in this work is to introduce, in topological spaces context, a new notion of solution covering the concept of projected solution and to establish some conditions characterizing the existence of such solutions for quasi-equilibrium problems.

The paper is organized as follows. First, in Section 3, we define the new concept of solution for quasi-equilibrium problems and make the our notation more precise. Then our necessary and sufficient conditions are established in Section 4, first in the case of a convex set X and then, when X is not assumed to be convex. Finally, in Section 5 we consider quasi-variational inequality problems and generalized Nash equilibrium problems as applications.

2. Notations and preliminaries

All along the paper E stands for a topological space and X and Y for two non-empty subsets of E. For any set-valued map $T : X \rightrightarrows Y$ the domain of T, the graph of T and the set of fixed points of T will be denoted respectively by $\mathrm{Dom}(T)$, $\mathrm{Graph}(T)$ and $\mathrm{Fix}(T)$. The map $T : X \rightrightarrows Y$ is said to be:

- *closed* if $\mathrm{Graph}(T) := \{(x, y) \in X \times Y : \ y \in T(x)\}$ is a closed subset of $X \times Y$,
- *lower semicontinuous* (LSC for short) if, for each $x \in X$ and each open set V such that $T(x) \cap V \neq \emptyset$, there exists a neighbourhood U_x of x such that $T(u) \cap V \neq \emptyset$ for all $u \in U_x$;
- *upper semicontinuous* (USC for short) if, for each $x \in X$ and any neighbourhood V of $T(x)$, there exists a neighbourhood U_x of x such that $T(u) \subset V$ for all $u \in U_x$;
- *transfer closed valued* if for every $x_0 \in X$, $y_0 \notin T(x_0)$ implies that there exists a point $x \in X$ such that $y_0 \notin \mathrm{cl}(T(x))$, where $\mathrm{cl}(T(x))$ denotes the closure of $T(x)$.
- *with open fibres* if, for each $y \in Y$, the set $T^{-1}(y) := \{x \in X : \ y \in T(x)\}$ is open in E.

As shown in [20, 35], a set-valued map T is transfer closed valued if, and only if,

$$\bigcap_{x\in X} T(x) = \bigcap_{x\in X} \mathrm{cl}(T(x)).$$

Thus, if T is closed then it has closed values, which in turn implies that it is transfer closed valued. Note also that as soon as $\bigcap_{x\in X} \mathrm{cl}(T(x)) = \emptyset$ then the map T is transfer closed valued. It is also well-known that every open fibres valued map is lower semicontinuous.

In [34, Theorem 3], a characterization of the non-emptyness of the set of fixed point of a set-valued map has been established.

Theorem 2.1. ([34]) *Let X be a non-empty subset of a locally convex Hausdorff topological vector space E and $S : X \rightrightarrows E$ be a set-valued map. If S is USC with non-empty closed convex values, then the set* $\mathrm{Fix}(S)$ *is non-empty if, and only if, there exists a non-empty compact convex subset $C \subset X$ such that*

$$S(x) \cap C \neq \emptyset, \ \forall x \in C.$$

Let us recall that, given a subset X of a topological vector space E, a function $h : X \to \mathbb{R}$ is said to be

- *lower semicontinuous* if, for each $x \in X$ and $\lambda \in \mathbb{R}$ such that $h(x) > \lambda$, there exists a neighbourhood U_x of x such that $h(x') > \lambda$, for all $x' \in U_x$;
- *upper semicontinuous* if, $-h$ is lower semicontinuous;
- *quasiconvex* if, for any $x, y \in X$ and any $t \in [0, 1]$

$$h(tx + (1-t)y) \leq \max\{h(x), h(y)\}.$$

We denote by $S_h(\lambda) := \{x \in X : \ h(x) \leq \lambda\}$, the *sublevel set* at level λ of h. It is well known that a function is lower semicontinuous (respectively quasiconvex) if and only if every sublevel set is closed (respectively convex).

A function $h : X \to \mathbb{R}$ is said to be *pseudocontinuous* if, the sets $S_h(h(x))$ and $S_{-h}(-h(x))$ are closed for all $x \in X$.

Clearly, any continuous function is pseudocontinuous. However, the converse is not true in general. As a simple example (see [12]), the function $h : \mathbb{R} \to \mathbb{R}$ defined as

$$h(x) := \begin{cases} x-1, & x < 0 \\ 0, & x = 0 \\ x+1, & x > 0 \end{cases}$$

is pseudocontinuous, but it is not lower semicontinuous nor upper semicontinuous. The concept of pseudocontinuity was first introduced in [29]; for more details on this notion, we suggest referring to [21].

Moreover, given $X \subset E$, a bifunction $f : E \times E \to \mathbb{R}$ is said

- to be *pseudomonotone* on X if for any $x, y \in X$ the following implication holds

$$f(x, y) \geq 0 \ \Rightarrow \ f(y, x) \leq 0;$$

- to have the *finite intersection property* [24] on X if for any $x_1, x_2, \ldots, x_n \in X$, there exists $x \in X$ such that $\max\limits_{i\in\{1,2,\ldots,n\}} f(x_i, x) \leq 0$.

3. The new concept of solution

As recalled in Section 1, given a subset X of a Hausdorff topological space E, a bifunction $f : E \times E \to \mathbb{R}$ and a constraint set-valued map $K : X \rightrightarrows E$, the *quasi-equilibrium problem* (QEP in short) consists in finding

$$x \in X \text{ such that } x \in K(x) \text{ and } f(x,y) \geq 0, \text{ for all } y \in K(x). \tag{2}$$

The set of (classical) solutions of problem QEP will denoted by $\mathrm{QEP}(f,K)$. Clearly, if the constraint map is constant ($K(x) = C$, for any $x \in X$) then $\mathrm{QEP}(f,K)$ is the classical equilibrium problem, denoted by $\mathrm{EP}(f,C)$.

As observed in [5], this classical concept of solution makes sense when the constraint map K is a self-map, that is whenever $K(X) \subset X$. If K is non-self, then an alternative notion of solution, called *projected solution* has been introduced in [5] for quasi-variational inequalities and generalized Nash games and in [26] for quasi-equilibrium problems.

But the concept of projected solution is intrinsically linked to the existence of projection, a notion which does not exist in general in the context of topological spaces. This is why we introduce in this work the new concept of P*-solution* for quasi-equilibrium problems defined in topological spaces.

Definition 3.1. Let X be a subset of a topological space E, $f : E \times E \to \mathbb{R}$ be a bifunction and $K : X \rightrightarrows E$, $P : E \rightrightarrows X$ be two set-valued maps. A point $\hat{x} \in X$ is called a P*-solution* of the QEP if, there exists $\hat{z} \in E$ such that the two following conditions hold: $\hat{x} \in P(\hat{z})$ and $\hat{z} \in \mathrm{EP}(f, K(\hat{x}))$.

When E is a norm vector space and P stands for the projection onto a subset X, the concept of P-solution coincides with the one of *projected solution* defined in [5, 26]. Moreover, if $x \in P(x)$ on X, that is $\mathrm{Fix}(P) = X$, then any classical solution of QEP is a P-solution and if $P(x) = \{x\}$ then both solution concepts coincide.

Nevertheless, and even E is a norm vector space and P is the projection on X, one can face the situation where $\mathrm{QEP}(f,K) = \emptyset$ while the set of P-solutions $\mathrm{PQEP}(f,K)$ is nonempty. The simple example below enlighten this case.

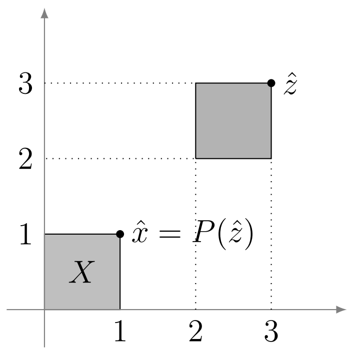

Figure 1: The set X and the images of K.

Example 3.2. Consider for instance $\mathbb{R}^2$ with the Euclidean norm $\|\cdot\|$, $X = [0,1]^2$, $K : X \rightrightarrows \mathbb{R}^2$ defined as $K(x) = [2,3]^2$ for any $x \in X$, P the projection onto X and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = \|x\| - \|y\|$. From Figure 1 we deduce that $\hat{x} = (1,1)$ is a P-solution, but it is not a classical solution of the QEP. $\square$

Nevertheless it is important to notice that, even if E is a norm vector space, the sets QEP and PQEP can be disjoint, that is classical solutions and P-solutions can be different. A simple example below illustrate this situation.

Example 3.3. Consider $X = [0, 2] \subset \mathbb{R}$, the maps $K : X \rightrightarrows \mathbb{R}$ and $P : \mathbb{R} \rightrightarrows X$ defined as

$$K(x) = [1, 2] \text{ for any } x \in X, \text{ and } P(x) = \{0\}, \text{ for any } x \in \mathbb{R};$$

and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $f(x, y) = x - y$, for any $x, y \in \mathbb{R}$. It is clear that $\mathrm{Fix}(K) = [1, 2]$, $\mathrm{QEP}(f, K) = \{2\}$ and $\mathrm{PQEP}(f, K) = \{0\}$. □

An alternative way to consider quasi-equilibrium problems is the so-called *Minty quasi-equilibrium problem* (MQEP in short), which consists in finding

$$x \in X \text{ such that } x \in K(x) \text{ and } f(y, x) \leq 0, \text{ for all } y \in K(x). \tag{3}$$

We will also denote by $\mathrm{MEP}(f, X)$ the solution set of the MQEP defined in (3) when the constraint map is constant and equal to X. It is not difficult to observe that $\mathrm{PQEP}(f, K) \subset \mathrm{PMQEP}(f, K)$ provided that f is pseudomonotone. In a similar way, if $\hat{f}$ is pseudomonotone, then due to (5) we have $\mathrm{PMQEP}(f, K) \subset \mathrm{PQEP}(f, K)$.

Similarly, $\hat{x}$ is called a *P-solution* of the MQEP defined in (3) if, there exists $\hat{z}$ such that the two following conditions hold:

$$\hat{x} \in P(\hat{z}) \text{ and } \hat{z} \in \mathrm{MEP}(f, K(\hat{x})).$$

We denote by $\mathrm{PQEP}(f, K)$ and $\mathrm{PMQEP}(f, K)$ the P-solution sets of the quasi-equilibrium problem (QEP) and the Minty quasi-equilibrium problem (MQEP), respectively.

Associated to the bifunction f, we consider $\hat{f}$ defined as

$$\hat{f}(x, y) := -f(y, x). \tag{4}$$

As immediate consequences of the definitions of P-solutions, one can make the following links between the different solution sets:

$$\mathrm{PQEP}(\hat{f}, K) = \mathrm{PMQEP}(f, K) \text{ and } \mathrm{PMQEP}(\hat{f}, K) = \mathrm{PQEP}(f, K) \tag{5}$$

We finish this subsection by giving sufficient conditions to guarantee the existence of P-solutions for quasi-equilibrium problems.

Theorem 3.4. *Let X be a non-empty, convex and closed subset of a locally convex Hausdorff topological vector space E. If the following hold:*

(1) *f is pseudocontinuous and vanishes on the diagonal of $E \times E$;*

(2) *for each $x \in X$, $f(x, \cdot)$ is quasiconvex;*

(3) *K is closed graph and LSC with convex, closed and nonempty values;*

(4) *$\mathrm{co}(K(X))$ is relatively compact;*

(5) *P is USC with compact, convex and nonempty values*

Then, there exists at least one P-solution for QEP.

Proof. Let $C = \overline{\mathrm{co}(K(X))}$, which is compact and convex. We also consider the map $S : X \times C \rightrightarrows C$ defined as

$$S(x, z) := \operatorname*{arg\,min}_{K(x)} f(z, \cdot).$$

By [14, Proposition 3.1], it is USC with compact and nonempty values. Moreover, since f is quasiconvex with respect to its second argument and K is convex-valued, S is convex-valued. Now, we define the map $D : X \times C \rightrightarrows X \times C$ by

$$D(x, z) := P(z) \times S(x, z).$$

Clearly, D is USC with convex, compact and nonempty values. Furthermore, the set $D(K \times C) \subset P(C) \times C$ is relatively compact. Thus, there exists $(\hat{x}, \hat{z})$, fixed point of D, due to Himmelberg's fixed point theorem [28]. This means $\hat{x} \in P(\hat{z})$ and $\hat{z} \in S(\hat{x}, \hat{z})$. Now, since f vanishes on the diagonal of $E \times E$ and $\hat{z} \in S(\hat{x}, \hat{z})$, we deduce that $\hat{x}$ is a P-solution. $\square$

The proof of Theorem 3.4 follows similar steps as in the alternative proof of [26, Corollary 3]. Moreover, in the context of projected solutions on Banach spaces, it is important to note that Theorem 3.4 is a consequence of [13, Theorem 4.1]. However, since lower semicontinuity of P is not required, Theorem 3.4 cannot be deduced from [26, Theorem 3].

4. Necessary and sufficient conditions for the existence of solutions

Having now a concept of solution, the P-solution, adapted to non-self quasi-equilibrium problems on topological spaces, our aim is not only to give some sufficient conditions for the existence of P-solutions but to characterize the existence of such P-solution for the different quasi-equilibrium problems considered in this paper.

Since the proof techniques differ significatively depending on whether both the constraint set-valued map K and the bifunction f associated to the QEP satisfy certain convexity assumptions or not, we divide this section into two subsections, each one dealing with one of these two cases.

4.1. Quasi-equilibrium problems with convex assumptions

So let us assume, all along this subsection, that the Hausdorff topological vector space E is locally convex, the set-valued map K has convex values and the bifunction f is quasiconvex with respect to its second argument.

Now, we consider the set-valued maps $R : X \rightrightarrows E$ and $W : X \times E \rightrightarrows X \times E$, respectively, defined as

$$R(x) := \mathrm{MEP}(f, K(x)) \tag{6}$$

$$W(x, z) := P(z) \times R(x). \tag{7}$$

As an immediate consequence of the definitions above, one has

$$\begin{aligned} & \bar{x} \in X \text{ is a } P\text{-solution of MQEP} \\ \Leftrightarrow\ & \text{there exists } \bar{z} \in E \text{ such that } \bar{x} \in P(\bar{z}) \text{ and } \bar{z} \in \mathrm{MEP}(f, K(\bar{x})) \\ \Leftrightarrow\ & \text{there exists } \bar{z} \in E \text{ such that } (\bar{x}, \bar{z}) \in \mathrm{Fix}(W) \end{aligned} \tag{8}$$

Taking into account this observation, we can now establish our first characterization of existence of solution for Minty-type quasi-equilibrium problem.

Theorem 4.1. *Let X be a non-empty, convex and closed subset of a locally convex Hausdorff topological vector space E, $f : E\times E \to \mathbb{R}$ be a bifunction and $K : X \rightrightarrows E$, $P : E \rightrightarrows X$ be two set-valued maps. If the following assumptions hold*

(a) *P is USC with convex closed and non-empty values,*

(b) *K is closed graph and LSC with convex closed and non-empty values,*

(c) *$K(X)$ is contained in a compact subset of E,*

(d) *the set $\{(x, y) \in E \times E : \ f(x, y) \leq 0\}$ is closed,*

(e) *for each $x \in X$, f has the finite intersection property on $K(x)$,*

(f) *for each $x \in X$, $f(x, \cdot)$ is quasiconvex;*

then the Minty quasi-equilibrium problem admits at least a P-solution if, and only if, there exists a non-empty compact convex subset $C \subset X \times E$ such that

$$\text{for all } (x, z) \in C, \quad W(x, z) \cap C \neq \emptyset.$$

Before proving Theorem 4.1, we recall the following result from [25, Proposition 2.2].

Proposition 4.2. *Let X be a subset of topological space E, $f : E \times E \to \mathbb{R}$ be a bifunction and $K : X \rightrightarrows E$ be a set-valued map. If K is closed and LSC, and the set $\{(x, y) \in E \times E : \ f(x, y) \leq 0\}$ is closed, then R is closed.*

Proof of Theorem 4.1. Combining Proposition 4.2 with assumptions (b) and (d) the set valued map $R : x \mapsto \mathrm{MEP}(f, K(x))$ has closed graph and thus also closed values. Together with hypothesis (c), R is thus USC. On the other hand since K is convex valued, and f is quasiconvex in its second argument, the values of map R are convex. Moreover, since f has the finite intersection property the map R is non-empty valued, due to [24, Lemma 1]. So finally since both maps P and R are USC with convex closed and non-empty values, so is W, and the conclusion follows from Theorem 2.1 and relation (8). □

As a direct consequence of Theorem 4.1 and of relations in (5) we can deduce the following characterization of the existence of P-solution for the quasi-equilibrium problem (QEP).

Corollary 4.3. *If, additionally to the assumptions of Theorem 4.1, f and $\hat{f}$ are pseudomonotone then, the quasi-equilibrium problem* (QEP) *admits at least a P-solution if, and only if, there exists a non-empty compact convex subset $C \subset X \times E$ such that*

$$W(x, z) \cap C \neq \emptyset, \qquad \text{for all } (x, z) \in C.$$

One can wonder under which assumptions the condition "there exists a non-empty compact convex subset $C \subset X \times E$ such that $W(x, z) \cap C \neq \emptyset$, for all $(x, z) \in C$" is satisfied, or, in other words, how Theorem 4.1 can help to obtain sufficient conditions guaranteeing for the existence of P-solutions for both quasi-equilibrium problem MQEP and QEP. Proposition below provides an answer to this natural question.

Proposition 4.4. *Let X be a convex, closed and non-empty subset of a locally convex Hausdorff topological space E, $f : E\times E \to \mathbb{R}$ be a bifunction and $K : X \rightrightarrows E$, $P : E \rightrightarrows X$ be two set-valued maps. If the following assumptions hold:*

(a) *P is non-empty valued,*

(b) *K is convex, compact and non-empty valued,*

(c) *for each $x \in X$ and each $z \in K(x)$, the set $\{y \in K(x) : \ f(z, y) \le 0\}$ is closed,*

(d) *for each $x \in X$, f has the finite intersection property on $K(x)$,*

then for each $(x, z) \in C = X \times \overline{\mathrm{co}(K(X))}$ we have $W(x, z) \cap C \neq \emptyset$.

Proof. Let us first observe that the set C is convex and non-empty. For each $(x, z) \in C$, thanks to [24, Lemma 1], we have that $R(x)$ is a non-empty subset of $K(X)$. Consequently, $W(x, z) \cap C \neq \emptyset$. □

Remark 4.5. (1) In Corollary 4.3 we can use the upper sign property of f instead the pseudomonotonicity of $\hat{f}$. Indeed, for any convex set X, $\mathrm{MEP}(f, X) \subset \mathrm{EP}(f, X)$ whenever f has the upper sign property, due to [3, Proposition 3.1]. This together with the pseudomonotonicity of f imply $\mathrm{PMQEP}(f, K) = \mathrm{PQEP}(f, K)$.

(2) Theorem 4.1 together with Proposition 4.4 imply [24, Theorem 2], provided that f has the upper sign property. □

4.2. Quasi-equilibrium problems without convex assumptions

Associated to the bifunction f and the set-valued maps K and P, we define the set-valued maps $S : X \times E \rightrightarrows X \times E$ and $U, L : E \rightrightarrows X \times E$, respectively, by

$$S(x, z) := P(z) \times K(x) \tag{9}$$

$$U(y) := \{(x, z) \in \mathrm{Fix}(S) : \ f(z, y) \ge 0 \ \vee \ x \notin K^{-1}(y)\} \tag{10}$$

$$L(y) := \{(x, z) \in \mathrm{Fix}(S) : \ f(y, z) \le 0 \ \vee \ x \notin K^{-1}(y)\}. \tag{11}$$

As an immediate consequence of definitions of the maps S, U and L, and of P-solution, one has

$$\begin{aligned} & \bar{x} \in X \text{ is a } P\text{-solution of QEP} \\ \Leftrightarrow\ & \text{there exists } \bar{z} \in E \text{ such that } (\bar{x}, \bar{z}) \in \mathrm{Fix}(S) \text{ and } \bar{z} \in \mathrm{EP}(f, K(\bar{x})) \\ \Leftrightarrow\ & \text{there exists } \bar{z} \in E \text{ such that } (\bar{x}, \bar{z}) \in \bigcap_{y\in E} U(y) \end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned} & \bar{x} \in X \text{ is a } P\text{-solution of MQEP} \\ \Leftrightarrow\ & \text{there exists } \bar{z} \in E \text{ such that } (\bar{x}, \bar{z}) \in \bigcap_{y\in E} L(y) \end{aligned} \tag{13}$$

Remark 4.6. Consider f a bifunction and $\hat{f}$ defined in (5). It is clear that the set-valued map L associated to $\hat{f}$ and K defined in (11) coincides with U associated to f and K defined in (10). □

Let us now state the main results of this subsection, that is theorems 4.7 and 4.9, in which we give necessary and sufficient conditions for the existence of P-solution, in terms of the set-valued maps U and S, defined in (10) and (9), respectively.

Theorem 4.7. *Assume that the set-valued map U defined in* (10) *is transfer closed valued. If* $\mathrm{cl}(U(y))$ *is compact for some* $y \in E$, *then the set* $\mathrm{PQEP}(f, K)$ *is non-empty if, and only if, the family of sets* $\{U(y)\}_{y\in E}$ *has the finite intersection property.*

Proof. From relation (12), the "only if" part is obvious. Thus, we only focus on the "if" part. It is clear that the family of closed sets $\{\mathrm{cl}(U(y))\}_{y\in E}$ has the finite intersection property. By the transfer closedness of U and the fact that $\mathrm{cl}(U(y))$ is compact for some $y \in E$, we have $\bigcap_{y\in E} U(y) \neq \emptyset$. The result follows from that relation (12). □

Since the transfer close property can appear to be quite technical, some sufficient conditions guaranteeing this property are provided in the following proposition.

Proposition 4.8. *Assume that X is closed, K has open fibres, f is upper semi-continuous in its first argument and* $\mathrm{Fix}(S)$ *is closed. Then the set-valued map U, defined in* (10), *is transfer closed valued.*

Proof. We first notice that for each $y \in E$

$$U(y) = \mathrm{Fix}(S) \bigcap (A(y) \cup B(y)),$$

where $A(y) = E \times \{z \in E : \ f(z, y) \geq 0\}$ and $B(y) = (E \setminus K^{-1}(y)) \times E$. Since f is upper semi-continuous in its first argument and K has open fibres, we deduce that the sets $A(y)$ and $B(y)$ are closed. The result follows. □

As a consequence of Theorem 4.7 and the relations in (5) we obtain the following.

Corollary 4.9. *Assume that the set-valued map L defined in* (11) *is transfer closed valued. If* $\mathrm{cl}(L(y))$ *is compact for some* $y \in E$, *then the set* $\mathrm{PMQEP}(f, K)$ *is non-empty if, and only if, the family of sets* $\{L(y)\}_{y\in E}$ *has the finite intersection property.*

Remark 4.10. If K is constant and $K(x) = X$, for all $x \in X$, and $P(x) = x$ for all $x \in E$, Corollary 4.9 generalizes [24, Lemma 4.1], where the authors assume the closedness of each set $L(y)$. □

Since the finite intersection property of the family $\{U(y)\}_{y\in X}$ plays a fundamental role in the characterization of existence of P-solution (Theorem 4.7) we end this section with a characterization of this property.

Proposition 4.11. *Let X be a subset of a topological space E, $f : E \times E \to \mathbb{R}$ be a bifunction, $K : X \rightrightarrows E$, $P : E \rightrightarrows X$ be two set-valued maps and U be the set-valued map defined in* (10). *The family of sets $\{U(y)\}_{y\in X}$ has the finite intersection property if, and only if, for any $y_1, \ y_2, \ldots, y_m \in X$ there exist $(x, z) \in \mathrm{Fix}(S)$ and $I \subset \{1, 2, \ldots, m\}$ such that*

$$\min_{i\in I} f(z, y_i) \geq 0 \quad \textit{and} \quad y_i \notin K(x), \ \forall i \notin I.$$

In order to prove the above result we need the following lemma for which we include a proof for sake of completeness.

Lemma 4.12. *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ and $\{B_\lambda\}_{\lambda\in\Lambda}$ be two families of sets. It holds*

$$\bigcap_{\lambda\in\Lambda}(A_\lambda \cup B_\lambda) = \bigcup_{I\subset\Lambda}\Big(\Big(\bigcap_{\lambda\in I} A_\lambda\Big)\cap\Big(\bigcap_{\lambda\notin I} B_\lambda\Big)\Big).$$

Proof. Let x be an element of $\bigcap_{\lambda\in\Lambda}(A_\lambda\cup B_\lambda)$, that means for each $\lambda\in\Lambda$, we have $x\in A_\lambda\cup B_\lambda$, which is equivalent to $x\in A_\lambda$ or $x\in B_\lambda$. We now define the set

$$I=\{\lambda\in\Lambda:\ x\in A_\lambda\}.$$

Clearly, $x\in B_\lambda$, for all $\lambda\in\Lambda\setminus I$. Hence, we deduce the inclusion "$\subset$".

Conversely, let x be an element of $\bigcup_{I\subset\Lambda}\big(\big(\bigcap_{\lambda\in I}A_\lambda\big)\cap\big(\bigcap_{\lambda\notin I}B_\lambda\big)\big)$, that means there exists $I\subset\Lambda$ such that

$$x\in\Big(\bigcap_{\lambda\in I}A_\lambda\Big)\cap\Big(\bigcap_{\lambda\notin I}B_\lambda\Big).$$

Thus, for each $\lambda\in I$ we have $x\in A_\lambda$, which in turn implies $x\in A_\lambda\cup B_\lambda$. Now, for any $\lambda\notin I$, we obtain $x\in B_\lambda$, and this implies $x\in A_\lambda\cup B_\lambda$. Thus, for every $\lambda\in\Lambda$ we see that $x\in A_\lambda\cup B_\lambda$. Therefore, the proof is complete. □

Proof of Proposition 4.11. Since, for each $y\in X$, we can decompose the set $U(y)$ as the union of the following sets $A(y)=\{(x,z)\in\mathrm{Fix}(S):\ f(z,x)\le 0\}$ and $B(y)=\{(x,z)\in\mathrm{Fix}(S):\ x\notin K^{-1}(y)\}$, i.e. $U(y)=A(y)\cup B(y)$.
The result follows from Lemma 4.12. □

5. Applications

As an application of the above results, in this section we give sufficient and necessary conditions in order to guarantee the existence of P-solutions for quasi-variational inequality problems and generalized Nash equilibrium problems.

5.1. Quasi-variational inequality problems

Let E be a topological vector space, E^* its topological dual and $\langle\cdot,\cdot\rangle$ the dual paring. Consider X a subset of E, $K:X\rightrightarrows E$, $P:E\rightrightarrows X$ and $T:E\rightrightarrows E^*$ three set-valued maps. In a similar way to [5, 26], a vector $\hat{x}\in X$ is said to be P-solution of the

- *Stampacchia quasi-variational inequality problem* if, there exist $\hat{z}\in K(\hat{x})$ and $\hat{z}^*\in T(\hat{z})$ such that the following hold

  (1) $\hat{x}\in P(\hat{z})$, (2) $\langle\hat{z}^*, w-\hat{z}\rangle\ge 0$, for all $w\in K(\hat{x})$.
- *Minty quasi-variational inequality problem* if, there exists $\hat{z}\in K(\hat{x})$ such that the following hold

  (1) $\hat{x}\in P(\hat{z})$, (2) $\langle w^*, \hat{z}-w\rangle\le 0$, for all $w\in K(\hat{x})$ and all $w^*\in T(w)$.

Given a set-valued map $T:E\rightrightarrows E^*$ with weak compact convex values, we consider the bifunction $f_T:E\times E\to\mathbb{R}$ defined as

$$f_T(x,y):=\sup_{x^*\in T(x)}\langle x^*, y-x\rangle \tag{14}$$

Clearly $f_T(x,\cdot)$ is lower semi-continuous and convex, for all $x \in E$. Moreover, $f_T = 0$ on the diagonal of $E \times E$. The key property of the bifunction f_T is given in the following lemma which is a direct consequence of Definition (14).

Lemma 5.1. *A vector $\hat{x} \in X$ is a P-solution of the (Minty) Stampacchia quasi-variational inequality problem associated to T, K and P if, and only if, it is a P-solution of the (M)QEP associated to f_T, K and P, where f_T is defined in* (14).

We recall that a set-valued map $T : E \rightrightarrows E^*$ is said to be:

- *pseudomonotone* if, for any $(x,x^*),(y,y^*) \in T$ the following implication holds
$$\langle x^*, y-x\rangle \geq 0 \quad \Rightarrow \quad \langle y^*, y-x\rangle \geq 0;$$
- *upper sign-continuous* at $x \in \mathrm{dom}(T)$ if, for any $v \in X$, the following implication holds:
$$\Big(\forall t \in]0,1[, \inf_{x_t^* \in T(x_t)} \langle x_t^*, v\rangle \geq 0\Big) \quad \Rightarrow \quad \sup_{x^* \in T(x)} \langle x^*, v\rangle \geq 0.$$

It is important to notice that, as for the classical solutions, if T is pseudomonotone, then any P-solution of the Stampacchia quasi-variational inequality problem is a P-solution of the Minty quasi-variational inequality problem. Reciprocally, the converse inclusion holds, when T is upper sign-continuous.

Theorem 5.2. *Let X be a non-empty and closed subset of a locally convex Hausdorff topological vector space E. Let $T : E \rightrightarrows E^*$ be a set-valued map with weak compact convex values, and $K : X \rightrightarrows E$, $P : E \rightrightarrows X$ be two set-valued maps. If the following assumptions hold*

(a) *P is USC with convex closed and non-empty values,*

(b) *K is closed graph and LSC with convex closed and non-empty values,*

(c) *$K(X)$ is contained in a compact subset of E,*

(d) *for each x and for any finite subset A of $K(x)$ there exists $z \in K(x)$ such that*
$$\langle a^*, z-a\rangle \leq 0, \ \forall a \in A, \ a^* \in T(a),$$

(e) *the set $\{(x,y) \in E \times E : \langle x^*, y-x\rangle \leq 0, \text{ for all } x^* \in T(x)\}$ is closed;*

then there exists a P-solution of the Minty quasi-variational inequality problem if, and only if, there exists a non-empty compact convex subset $C \subset X \times E$ such that
$$(P(z) \times M(x)) \bigcap C \neq \emptyset, \quad \text{for all } (x,z) \in C,$$
where $M(x) := \{w \in K(x) : \langle y^*, w-y\rangle \leq 0, \text{ for all } y^* \in T(y)\}$.

Proof. First, note that assumption (d) implies that f_T, defined in (14), has the finite intersection property, see e.g. [24, Proposition 12]. Moreover, it is not difficult to see that f_T satisfies the others assumptions of Theorem 4.1. Therefore, the result follows from Lemma 5.1. □

Whenever the set-valued map T satisfies some pseudomonotonicity assumptions, the above characterization also holds true for the P-solutions of the Stampacchia quasi-variational inequality.

Corollary 5.3. *If, additionally to the assumptions of Theorem* 5.2*, T and $-T$ are pseudomonotone, then the Stampacchia quasi-variational inequality problem admits at least a P-solution if, and only if, there exists a non-empty compact convex subset $C \subset X \times E$ such that*

$$(P(z) \times M(x)) \bigcap C \neq \emptyset, \quad \textit{for all } (x, z) \in C,$$

where $M(x) := \{w \in K(x) : \ \langle y^*, w - y \rangle \leq 0, \ \textit{for all } y^* \in T(y)\}$.

Proof. It is known that pseudomonotonicity of T implies that f_T is pseudomonotone. By [26, Proposition 1], $-f_T$ is pseudomonotone, provided $-T$ is pseudomonotone. Therefore, the result follows from Corollary 4.3 and Lemma 5.1. □

Remark 5.4. (1) An important instance of variational inequality problems where T and $-T$ are pseudomonotone was studied in [9] while such property has been analyzed in [7].

(2) In Corollary 5.3, it is possible to use the upper sign-continuity of T instead the pseudomonotonicity of $-T$.

(3) If P is the projection onto X, then [5, Theorem 3.2], [1, Theorem 4.1], [6, Theorem 5.1], [18, Theorem 3.1] and [26, Theorem 7] give sufficient conditions in order to guarantee the existence of P-solutions. □

5.2. Generalized Nash equilibrium problems

A generalized Nash equilibrium problem (GNEP) is a non cooperative game where a finite set of p players takes a strategic decision by solving a minimization/maximization problem parameterized by the decision of the other players. Each player ν controls the decision variable $x^\nu \in C_\nu$, where C_ν is a non-empty subset of a topological space E_ν. We denote by $x = (x^1, \dots, x^p) \in \prod_{\nu=1}^p C_\nu := C$ the vector formed by all these decision variables and by $x^{-\nu}$ the strategy vector of all the players but player ν. The set of all such vectors will be denoted by $C^{-\nu}$. Using classical notation, we sometimes write $(x^\nu, x^{-\nu})$ instead of x in order to emphasize the ν-th player's variable within x. Note that this is still the vector $x = (x^1, \dots, x^\nu, \dots, x^p)$, and the notation $(x^\nu, x^{-\nu})$ does not mean that the block components of x are reordered. Each player ν has an objective function $\theta_\nu : E = \prod_\nu^p E_\nu \to \mathbb{R}$ that depends on all player's strategies. Furthermore, each player's strategy must belong to a set identified by the set-valued map $K_\nu : C^{-\nu} \rightrightarrows C_\nu$ in the sense that the strategy space of player ν is $K_\nu(x^{-\nu})$, which depends on the rival players' strategies $x^{-\nu}$. Thus, given the strategy $x^{-\nu}$ of the rival players, the player ν chooses a strategy x^ν such that it solves the following optimization problem

$$\min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), \text{ subject to } x^\nu \in K_\nu(x^{-\nu}). \tag{15}$$

For any ν, the solution set of problem (15) is denoted by $\mathrm{Sol}_\nu(x^{-\nu})$. Thus, a *generalized Nash equilibrium* or *classical solution of the* GNEP is a vector $\hat{x}$ such that $\hat{x}^\nu \in \mathrm{Sol}_\nu(\hat{x}^{-\nu})$, for any $\nu \in J = \{1, 2, \dots, p\}$. The set of such generalized Nash equilibria will be here denoted by $\mathrm{GNEP}(\theta_\nu, K_\nu)$.

When, for every $\nu \in J$, the constraint map is constant and $K_\nu(x^{-\nu}) = C_\nu$ for all $x^{-\nu}$, the GNEP reduces to the well-known Nash equilibrium problem (NEP) [30].

Our aim here is to consider a more general situation where the constraint maps K_ν are possibly non-self maps. More precisely, for any $\nu \in J$, let $K_\nu : C^{-\nu} \rightrightarrows E_\nu$ and $P : E \rightrightarrows C$ be two set-valued maps. A point $\hat{x} \in C$ is called a *P-equilibrium* of the generalized Nash equilibrium problem if, there exists $\hat{z} \in E$ such that the following two conditions hold simultaneously

1. $\hat{x} \in P(\hat{z})$
2. for any $\nu \in J$, $\hat{z}^\nu \in K_\nu(\hat{x}^{-\nu})$ is a solution of the following optimization problem

$$\min_{z^\nu} \theta_\nu(z^\nu, \hat{z}^{-\nu}), \text{ subject to } z^\nu \in K_\nu(\hat{x}^{-\nu}).$$

Let us denote by $\mathrm{PGNEP}(\theta_\nu, K_\nu)$ the set of P-equilibria for the GNEP associated to the functions θ_ν, the constraint maps K_ν and the map P.

Obviously, if $x \in P(x)$ on C then any classical generalized Nash equilibrium is a P-equilibrium. It is also important to notice that when E is a finite dimensional space and P is the projection onto C, the concepts of P-equilibrium and projected solution defined in [5] for GNEPs coincide.

A somehow classical way to transform or reformulate the Generalized Nash equilibrium problem into a quasi-equilibrium problem (see e.g. [4, 32]) is to consider the Nikaidô-Isoda bifunction $f^{NI} : E \times E \to \mathbb{R}$ defined as

$$f^{NI}(x, y) := \sum_{\nu \in J} \{\theta_\nu(y^\nu, x^{-\nu}) - \theta_\nu(x^\nu, x^{-\nu})\}.$$

More recently and in order to be able to handle GNEP with quasiconvex objective function, the *star*-bifunction $f^* : E \times E \to \mathbb{R}$ has been defined in [22]

$$f^*(x, y) := \min_{\nu \in J} \left\{\theta_\nu(y^\nu, y^{-\nu}) - \theta_\nu(x^\nu, y^{-\nu})\right\}.$$

The relationship between the two previous bifunctions is given below

$$pf^* \leq f^{NI}.$$

Before stating necessary and sufficient conditions for the existence of P-equilibrium (see Theorem 5.7 below), let us precise the interrelations between the different P-solution concepts, where here and below the set-map K is defined as follows

$$K(x) := \prod_{\nu \in J} K_\nu(x^{-\nu}), \text{ for all } x \in C.$$

Lemma 5.5. *Using the above notations, the following interrelations hold:*

(i) *for the Stampacchia-type P-solution one has*

$$\mathrm{PQEP}(f^*, K) \subset \mathrm{PGNEP}(\theta_\nu, K_\nu) = \mathrm{PQEP}(f^{NI}, K).$$

(ii) *for the Minty-type P-solution one has*

$$\mathrm{PMQEP}(f^{NI}, K) \subset \mathrm{PGNEP}(\theta_\nu, K_\nu) \subset \mathrm{PMQEP}(f^*, K).$$

Proof. (i) The equality $\mathrm{PGNEP}(\theta_\nu, K_\nu) = \mathrm{PQEP}(f^{NI}, K)$ follows the same steps as in [26, Lemma 10]. Now assume that $\hat{x}$ is a P-solution of QEP associated to f^* and K.

Thus there exists $\hat{z} \in E$ such that $\hat{x} \in P(\hat{z})$ and $\hat{z} \in \mathrm{EP}(f^*, K(\hat{x}))$, that means $\hat{z} \in K(\hat{x})$ and

$$\min_{\nu=1,\cdots,N} \left\{\theta_\nu(y^\nu, y^{-\nu}) - \theta_\nu(\hat{z}^\nu, y^{-\nu})\right\} \geq 0, \text{ for all } y \in K(\hat{x})$$

i.e. for any ν, $\theta_\nu(y^\nu, y^{-\nu}) - \theta_\nu(\hat{z}^\nu, y^{-\nu}) \geq 0$, for all $y \in K(\hat{x})$. In particular, for each $y^\nu \in K_\nu(\hat{x}^{-\nu})$ we have $y = (y^\nu, \hat{z}^{-\nu}) \in K(\hat{x})$, which in turn implies

$$\theta_\nu(y^\nu, \hat{z}^{-\nu}) \geq \theta_\nu(\hat{z}^\nu, \hat{z}^{-\nu}).$$

Therefore, $\hat{x}$ is a P-equilibrium of the GNEP(θ_ν, K_ν).

(ii) Let $\hat{x}$ be a P-solution of the MQEP associated to f^{NI} and K. Therefore there exists $\hat{z} \in K(\hat{x})$ such that $\hat{x} \in P(\hat{z})$ and $\hat{z} \in \mathrm{MEP}(f^{NI}, K(\hat{x}))$. Now, for each $\nu_0 \in J$ and $y^{\nu_0} \in K_{\nu_0}(\hat{x}^{-\nu_0})$ we consider $y = (y^{\nu_0}, \hat{z}^{-\nu_0})$, which is an element of $K(\hat{x})$, and we have

$$f^{NI}(y, \hat{z}) = \sum_{\nu \in J} \theta_\nu(\hat{z}^\nu, y^{-\nu}) - \theta_\nu(y^\nu, y^{-\nu}) = \theta_{\nu_0}(\hat{z}^{\nu_0}, \hat{z}^{-\nu_0}) - \theta_{\nu_0}(y^{\nu_0}, \hat{z}^{-\nu_0}).$$

Since ν_0 was arbitrary, we conclude that $\hat{z}$ is a P-equilibrium of GNEP(θ_ν, K_ν) and the left side inclusion follows.

For the right side inclusion, let us assume that there exists $\hat{z} \in K(\hat{x})$ such that $\hat{x} \in P(\hat{z})$ and

$$\theta_\nu(\hat{z}) \leq \theta_\nu(y^\nu, \hat{z}^{-\nu}), \ \forall y^\nu \in K_\nu(\hat{x}^{-\nu}), \ \forall \nu \in J.$$

which implies $f^*(y, \hat{z}) \leq 0$. This means $\hat{z} \in \mathrm{MEP}(f^*, K(\hat{x}))$ and thus that $\hat{x}$ is a P-solution of the Minty-type QEP(f^*, K). □

Note that the left side inclusion in item (i) of the above lemma extends the similar approach developed in [22] for the NEP.

While the famous result of Arrow-Debreu [2] states the existence of equilibrium of a generalized Nash game assuming continuity of the objective function of the players (in both variables x^ν and $x^{-\nu}$), in the pioneering work of Reny [33, 27] (see also [10] and references therein) non cooperative games with discontinuous payoff have been also considered using the concept of *better reply secure game*. A game is said to be *better reply secure* if, for all couple $(x, \varphi) \in \mathrm{cl}\{(u, (\theta_\nu(u))_\nu) : u \in K\}$, if u is not a Nash equilibrium, then some player ν can secure (see [10, def. 2.3] for more details) a payoff strictly higher than φ_ν. In [33], the existence of Nash equilibrium is proved without continuity assumption on the objective functions θ_ν but the set K_ν are assumed to be compact and the θ_ν to be bounded. Our approach here is quite different since, first in the forthcoming result Theorem 5.7 we target generalized Nash equilibrium and provide with a characterization of the existence of such equilibrium, second our hypothesis don't involve any "pre-knowledge" on the set of Nash equilibrium (like in better reply secure assumption of [33]) and finally the set-valued maps K_ν are not assumed to be compact valued.

Let us now give a definition which will be useful to establish a characterization of the existence of P-equilibria for GNEPs.

Definition 5.6. For any ν, let C_ν be a non-empty subset of a topological space E_ν, $\theta_\nu : E \to \mathbb{R}$ be a function and $K_\nu : C^{-\nu} \rightrightarrows E_\nu$ be a set-valued map. Let Y be a subset of $C = \prod_\nu C_\nu$. A point (x, z) is said to be an *external P-equilibrium* of the generalized Nash equilibrium problem GNEP(θ_ν, K_ν) for the set Y if $(x, z) \in Fix(S)$ and, for any $y \in Y$,

$$y \in K(x) \quad \Longrightarrow \quad \theta_\nu(y^\nu, z^{-\nu}) \geq \theta_\nu(z), \text{ for all } \nu \in J, \tag{16}$$

where S is the set-valued map defined in (9). □

The terminology "external" is used here to emphasize that z is not assumed to be element of the subset Y. Clearly (x, z) is a P-equilibrium of GNEP(θ_ν, K_ν) if and only if (x, z) is an external P-equilibrium of GNEP(θ_ν, K_ν) for the set $C = \prod_\nu C_\nu$.

Theorem 5.7. *For any ν, let C_ν be a non-empty subset of a topological space E_ν, $\theta_\nu : E \to \mathbb{R}$ be a function and $K_\nu : C^{-\nu} \rightrightarrows E_\nu$ be a set-valued map. If the following two assumptions hold*

(i) *the set-valued map U defined as in* (10) *associated to f^{NI}, is transfer closed valued;*

(ii) *there exists $y \in C$ such that* $\text{cl}(U(y))$ *is compact;*

then there exists a P-equilibrium of the GNEP(θ_ν, K_ν) *if, and only if, for each finite subset Y of C, the* GNEP(θ_ν, K_ν) *admits an external P-equilibrium.*

Proof. Let us assume first that, for any finite subset Y of C, there exists some $(x, z) \in \text{Fix}(S)$ such that relation (16) holds, with S being the set-valued map defined in (9). Consequently, the family of sets $\{U(y)\}_{y \in C}$ has the finite intersection property. Thus, by Theorem 4.7, the set PQEP(f^{NI}, K) is nonempty. Therefore, the set GNEP(θ_ν, K_ν) is nonempty, due to Lemma 5.5, part (i).

To prove the reverse implication, let us assume that GNEP(θ_ν, K_ν) admits a P-solution $\bar{x}$. Thus there exists $\bar{z}$ such that $\bar{x} \in P(\bar{z})$ and $z \in K(\bar{x})$ with, for any ν, $\bar{z}_\nu$ being a solution of the optimization problem

$$\min_{z^\nu} \theta_\nu(z^\nu, \bar{z}^{-\nu}), \text{ subject to } z^\nu \in K_\nu(\bar{x}^{-\nu}).$$

This immediately implies that, for any finite subset Y of C, relation (16) holds with $(\bar{x}, \bar{z})$. □

The following result establishes sufficient conditions for assumption (i) of Theorem 5.7.

Proposition 5.8. *Assume that P has closed graph and for each ν the objective function θ_ν is continuous and the constraint map K_ν has open fibres and closed graph. If U is the map defined in* (10) *associated to f^{NI}, then it is transfer closed valued.*

Proof. It is clear that the map K has a closed graph, and since P also has closed graph, we consequently deduce that the map S has closed graph. Thus, Fix(S) is closed too. We also observe that the bifunction f^{NI} is upper semi-continuous in its first argument, as each objective function is continuous. Furthermore, since each K_ν has open fibres, the map K shares the same property. The result follows from Proposition 4.8. □

As a direct consequence of Theorem 5.7 we recover the following result.

Corollary 5.9. ([31, Theorem 7.1]) *For any ν , let C_ν be a non-empty and compact subset of a topological space E_ν and $\theta_\nu : E \to \mathbb{R}$ be a function. If f^{NI} is upper semicontinuous in its first variable, then there exists a solution of the NEP if, and only if, for any $y_1, y_2, \ldots, y_m \in C$ there exists $x \in C$ such that*

$$\theta_\nu(y_i^\nu, x^{-\nu}) \geq \theta_\nu(x), \text{ for any } i \text{ and any } \nu.$$

Proof. Let us consider that each constraint map is given as $K_\nu(x^{-\nu}) = C_\nu$, i.e. it is constant-valued, and the map P is defined as

$$P(x) = \begin{cases} \{x\}, & x \in C \\ C, & \text{otherwise.} \end{cases}$$

Thus, we note that $(x, z) \in P(z) \times K(x)$ implies $x = z$. Consequently, the set-valued map U defined in (10) is reduced as follows

$$U(y) = \{x \in C : \ f^{NI}(x, y) \geq 0\}.$$

Since f^{NI} is upper semicontinuous with respect to its first argument, U is closed-valued, which in turn implies that it is transfer closed valued. Moreover, since each set C_ν is compact, we deduce that U is compact-valued. Then the result follows from a direct application of Theorem 5.7. □

While Theorem 5.7 is based on a reformulation of the non cooperative game into an equilibrium problem involving the Nikaidô-Isoda function, we will now state sufficient conditions for the existence of P-equilibrium for GNEPs but articulating the approach on the concept of *star*-function.

Theorem 5.10. *For any ν, let C_ν be a non-empty subset of a topological space E_ν, $\theta_\nu : E \to \mathbb{R}$ be a function and $K_\nu : C^{-\nu} \rightrightarrows E_\nu$ be a set-valued map. If the following assumptions hold*

(i) *the set* Fix(S) *is compact, where S is defined as* (9),

(ii) *the set-valued map U defined as* (10) *associated to f^*, is transfer closed valued,*

(iii) *for each finite subset Y of C, there exists $(x, z) \in$* Fix(S) *such that for all $y \in Y$ the following implication holds*

$$y \in K(x) \quad \Longrightarrow \quad \theta_\nu(y) \geq \theta_\nu(z^\nu, y^{-\nu}), \text{ for all } \nu \in J. \tag{17}$$

then there exists a P-equilibrium of the GNEP.

Proof. Since Fix(S) is compact, each set cl($U(y)$) is compact. Relation (17) implies that the family of sets $\{U(y)\}_{y \in C}$ has the finite intersection property. Consequently, by Theorem 4.7, the set PQEP(f^*, K) admits at least one element. Now, since PQEP(f^*, K) $\subset$ PGNEP(θ_ν, K_ν), due to Lemma 5.5(ii), we deduce the result. □

Let us observe that under the additional assumption that bifunction f^* satisfies the upper sign property, one can show, as in Theorem 5.7, that the existence of a P-equilibrium is equivalent to assumption (iii). Nevertheless we didn't include this in Theorem 5.10 because verifying the upper sign-property for f^* could be quite complex.

One can wonder under which hypotheses assumption (ii) of Theorem 17 is automatically satisfied. Thus analogously to Proposition 5.8, we present now some sufficient conditions guaranteeing assumption (ii) of Theorem 17.

Proposition 5.11. *Assume that P has closed graph and for each ν the objective function θ_ν is lower semicontinuous in its player's variable and the constraint map K_ν has open fibres and closed graph. If U is the map defined in* (10) *associated to f^*, then it is transfer closed valued.*

Proof. Clearly, the set $\mathrm{Fix}(S)$ is closed. The lower semicontinuity of each objective function implies the upper semicontinuity of f^* with respect to its first variable. Hence, the result follows from Proposition 4.8. □

Similarly to Corollary 5.9, as a consequence of Theorem 5.10, we can deduce an existence result for discontinuous (lower semi-continuous in fact) classical Nash games. The proof is omitted because it follows the same steps of the one of Corollary 5.9.

Corollary 5.12. *For any ν, let C_ν be a non-empty and compact subset of a topological space E_ν and $\theta_\nu : E \to \mathbb{R}$ be a function. If each objective function is lower semicontinuous with respect to its player's variable and for any $y_1, y_2, \ldots, y_m \in C$ there exists $x \in C$ such that*

$$\theta_\nu(y_i^\nu, y_i^{-\nu}) \geq \theta_\nu(x^\nu, y_i^{-\nu}), \quad \textit{for any } i \textit{ and for any } \nu;$$

then there exists a solution of the NEP.

Acknowledgements. Both authors benefit from the support of Project Math-AmSud SOGGA (23-MATH-13). The second author is supported by Concytec-Prociencia, Peru, through AMSUD 2023-01 code PE501084096-2023.

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