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McShane's Condition for the Modulus of Continuity

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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McShane considered real-valued uniformly continuous functions defined on a subset of a metric space whose modulus of continuity has an affine majorant, and showed that they had uniformly continuous extensions to the entire space. Later it was shown that the same condition on a real-valued uniformly continuous function was both necessary and sufficient for the function to lie in the uniform closure of the real-valued Lipschitz functions. The main purpose of this note is to display conditions that are equivalent to the existence of an affine majorant for the modulus of an arbitrary function between metric spaces, with an emphasis on internal conditions on the function itself. No continuity assumptions on the objective function are made. We also give a simple formula for the uniform distance from a function whose modulus has an affine majorant to the set of Lipschitz functions that is valid for an important class of target spaces.

Keywords: Modulus of continuity, affine function, Lipschitz function, Lipschitz for large distances, Lipschitz in the small, bounded in the small, concave function, subadditive function, coarse map.

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1. Introduction

Let $UC(X,Y)$ (resp. $\mathrm{Lip}(X,Y)$) denote the uniformly continuous (resp. Lipschitz) functions from a metric space $\langle X,d\rangle$ to a metric space $\langle Y,\rho\rangle$. A seminal article of McShane [33] contains two famous extension theorems for real-valued functions. The first states that a real-valued Lipschitz function defined on a subset A of X has a Lipschitz-constant-preserving extension in $\mathrm{Lip}(X,\mathbb{R})$. We must say that his proof has an antecedent in a remark of Whitney [41, p. 63]. The second requires some terminology.

Suppose $f:\langle X,d\rangle\to\langle Y,\rho\rangle$ is an arbitrary function. The *modulus of continuity* of f [10, 33, 40] is the increasing nonnegative extended-real-valued function on $[0,\infty)$ defined by

$$\omega_f(t):=\sup\{\rho(f(x),f(w)):d(x,w)\le t\}.$$

Recall that an *affine function* between real linear spaces is a linear transformation followed by a translation [37]. With this in mind, we say that the modulus of

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continuity of $f : X \to Y$ has an *affine majorant* if for some real numbers α, β we have $\omega_f(t) \leq \alpha t + \beta$ for each $t \geq 0$. If this is so, both α and β must be nonnegative.

McShane's second result says that if A is a metric subspace of the metric space $\langle X, d\rangle$ and $f \in UC(A, \mathbb{R})$ has a modulus of continuity with an affine majorant, then there exists $g \in UC(X, \mathbb{R})$ with $f = g|_A$ such that ω_g has the same affine majorant. One of the present authors has recently displayed a proof of the second result using the first [5, Theorem 3.9]. As a special case, since the modulus of continuity of a bounded function has a constant majorant, each bounded uniformly continuous real-valued function on $\langle A, d\rangle$ has a bounded uniformly continuous extension to the entire space.

This property of the modulus of continuity, along with uniform continuity, are also characteristic of the functions that are in the uniform closure of $\mathrm{Lip}(X, \mathbb{R})$ for an arbitrary metric space $\langle X, d\rangle$ (see most recently [6, Theorem 6.6]). In our opinion, one of the most glaring omissions in an introductory course dealing with analysis on metric spaces is any discussion of the uniform closure of the vector lattice $\mathrm{Lip}(X, \mathbb{R})$, stemming from the reluctance these days to introduce the pivotal concept of modulus of continuity, save to note that the family need not be uniformly closed. For example, the usual square-root function is the uniform limit of a sequence $\langle g_n\rangle$ of Lipschitz functions where for each n,

$$g_n(x) := \begin{cases} \sqrt{n}x & \text{if } \; 0 \leq x \leq \frac{1}{n} \\ \sqrt{x} & \text{if } \; x > \frac{1}{n}. \end{cases}$$

Other descriptions in the literature typically involve the existence of a majorant $h : [0, \infty) \to [0, \infty)$ for the modulus of the potential uniformly continuous limit having a particular property among several possibilities. There is one notable exception to this rule of thumb (see [5, 15, 17, 27]): the potential limit should be *Lipschitz for large distances*, meaning that for each $\delta > 0$, there exists $k_\delta > 0$ such that $d(x, w) \geq \delta \Rightarrow |f(x) - f(w)| \leq k_\delta d(x, w)$.

On the other hand, that each bounded member f of $UC(X, \mathbb{R})$ is a uniform limit of a sequence $\langle g_n\rangle$ of Lipschitz functions is sometimes actually taught students [19, Theorem 6.8]: for each $n \in \mathbb{N}$ define $g_n \in \mathrm{Lip}(X, \mathbb{R})$ by

$$g_n(x) := \inf_{w \in X} \; f(w) + nd(x, w) \quad (x \in X).$$

The function g_n is often called *the lower n-Lipschitz regularization* of f. More generally, for a lower-semicontinuous objective function f having some Lipschitz minorant, the functions g_n so defined will be eventually finite-valued and n-Lipschitz and $\langle g_n\rangle$ will converge pointwise from below to the objective function.

The main purpose of this note is to present in a coherent way conditions on an arbitrary function f between metric spaces equivalent to the existence of an affine majorant for ω_f, in view of the importance of such conditions in extension and approximation problems [27, Theorem 1] and the chaotic way they appear in the literature, often in specialized contexts. Some of our results give internal conditions on the objective function itself. For example, such a function is one whose restriction to each uniformly discrete subset is Lipschitz and that satisfies a modest uniform local growth condition. In our analysis, we do not require uniform continuity or even continuity of our functions. After observing that a function f whose uniform distance to the Lipschitz functions is finite must be one for which ω_f has an affine majorant, we give a simple formula for the uniform distance from f to $\text{Lip}(X, Y)$ valid for an important class of target spaces Y.

2. Preliminaries

To set up further discussion of the modulus of continuity, we first focus on real-valued functions with domain $[0, \infty)$. A function $h : [0, \infty) \to \mathbb{R}$ is called *concave* provided for each $\lambda \in [0, 1]$ and $\{t_1, t_2\} \subseteq [0, \infty)$ we have

$$h(\lambda t_1 + (1 - \lambda)t_2) \geq \lambda h(t_1) + (1 - \lambda)h(t_2).$$

In geometric terms, this means that the *hypograph* of h defined by

$$\text{hypo}(h) := \{(t, \beta) : 0 \leq t \text{ and } \beta \leq h(t)\} \subseteq \mathbb{R}^2$$

is a convex set. Such a function is automatically continuous on $(0, \infty)$. An upper semicontinuous concave function is the infimum of the affine functions that majorize it. All of these statements are analogs of statements for convex functions that can be found in the celebrated monograph of Rockafellar [37].

A nonnegative concave function on $[0, \infty)$ is automatically increasing, that means that $t_1 \leq t_2 \Rightarrow h(t_1) \leq h(t_2)$. An affine function on $[0, \infty)$ is automatically concave; in fact, if $h(t) := \alpha t + \beta$, then $h(\lambda t_1 + (1 - \lambda)t_2) = \lambda h(t_1) + (1 - \lambda)h(t_2)$.

A function $h : [0, \infty) \to \mathbb{R}$ is called *subadditive* provided $h(t_1 + t_2) \leq h(t_1) + h(t_2)$ whenever $\{t_1, t_2\} \subseteq [0, \infty)$. A nonnegative concave function is subadditive as well as increasing (see, e.g., [5, Proposition 3.6]). An example of an increasing nonnegative subadditive function on $[0, \infty)$ that is not concave is given by

$$h(t) = \begin{cases} 1 & \text{if } t \in [0, 2] \\ 2 & \text{otherwise.} \end{cases}$$

The function h is subadditive because $h(t_1) + h(t_2) \geq 2$ whenever $t_1, t_2 \geq 0$.

Next, we turn to metric spaces and the functions between them. All metric spaces will be assumed to contain at least two points. If A is a nonempty subset of $\langle X, d\rangle$, we write $\text{diam}_d(A)$ for its usual diameter. We put $\text{diam}_d(\emptyset) = 0$. We call $A \subseteq X$ *bounded* if its diameter is finite. If $A \neq \emptyset$ and $x \in X$, we write $d(x, A)$ for the usual distance from x to A, i.e., $d(x, A) := \inf\{d(x, a) : a \in A\}$. We write $S_d(x_0, \delta)$ (resp. $B_d(x_0, \delta)$) for the open (resp. closed) ball of radius $\delta > 0$ and center $x_0 \in X$. If $A \subseteq X$ and $\delta > 0$, we call $S_d(A, \delta) := \cup_{a \in A} S_d(a, \delta)$ the *open enlargement of A of radius δ*.

We call $A \subseteq X$ *uniformly discrete* if for some $\delta > 0$, we have $d(a_1, a_2) \geq \delta$ whenever a_1 and a_2 are distinct points of A. Given $\delta > 0$, it is an easy consequence of Zorn's lemma that each nonempty subset of X contains a maximal δ-discrete subset.

Within the family of bounded subsets we have the precompact subsets, also known as the totally bounded subsets. A subset A is *precompact* if there exists a nonempty finite subset F such that some enlargement of F contains A. Between the precompact subsets and bounded subsets are the Bourbaki bounded subsets, also known as the finitely chainable subsets [5, 11, 20]. Their definition requires the notion of a δ-chain.

Definition 2.1. Let a and b be points in a metric space $\langle X, d\rangle$ and let $\delta > 0$. By a *δ-chain of length n* from a to b, we mean a finite sequence $x_0, x_1, x_2, \dots, x_n$ in X such that $x_0 = a, x_n = b$, and $d(x_{j-1}, x_j) < \delta$ for $j = 1, 2, \dots, n$.

In the definition there is no requirement that the nodes of the chain be distinct. In particular, $x_0 = x_n$ is allowed.

Definition 2.2. A subset A of a metric space $\langle X, d\rangle$ is called *Bourbaki bounded* if for each $\delta > 0$ there exists a finite subset F of X and $n \in \mathbb{N}$ such that each point of A can be joined to some point of F by a δ-chain of length n.

Bourbaki bounded subsets are very important in the study of uniformly continuous functions on metric spaces. First, they are precisely those subsets of $\langle X, d\rangle$ on which each uniformly continuous function on X into an arbitrary metric space is bounded [5, 7, 20, 32]. Second, they arise in the most important description of those metric spaces for which $UC(X, R)$ forms a ring [5, 13].

The reader certainly will recall that $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is called *λ-Lipschitz* (where $\lambda \geq 0$) provided for all $x, w \in X,\ \rho(f(x), f(w)) \leq \lambda d(x, w)$. Each Lipschitz function has a smallest Lipschitz constant $L(f)$, and if the target space is a normed linear space, then the assignment $f \mapsto L(f)$ determines a seminorm on $\mathrm{Lip}(X, Y)$. Extending our terminology from Section 1, $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is called *Lipschitz for large distances* provided for each $\delta > 0$, there exists $k_\delta > 0$ such that

$$d(x, w) \geq \delta \Rightarrow \rho(f(x), f(w)) \leq k_\delta d(x, w).$$

Each bounded function is Lipschitz for large distances, and the square-root function, while not Lipschitz, is Lipschitz for large distances. While the Lipschitz functions are not uniformly closed, the functions that are Lipschitz for large distances are.

We make some additional comments relative to the modulus of continuity. We don't much like this terminology, due to de la Vallée Poussin [40], as ω_f actually serves as a growth modulus for f. In fact, some authors now use more suggestive terminology, such as *modulus of expansion* [12]. Whatever name is chosen, the notion of the modulus allegedly goes back to Lebesgue [24, p. 75 and p. 114].

Let $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ be arbitrary. We always have $\omega_f(0) = 0$ and uniform continuity for f amounts to the right continuity of ω_f at $t = 0$. By the monotonicity of ω_f, $r(f) := \lim_{t \to 0^+} \omega_f(t)$ always exists as an extended real number. For the squaring function, we have $\omega_f(t) = \infty$ for all positive t. A function having a modulus of this character will be called a function of *uncontrollable uniform growth*.

The objective function f is λ-Lipschitz if and only if $t \mapsto \lambda t$ is a majorant for ω_f. A function satisfying $\omega_f(t) \leq \lambda t$ for all $t \in [0, t_0]$ for some positive t_0 is called *Lipschitz in the small*. In more concrete terms, this means there exists $t_0 \geq 0$ and $\lambda \geq 0$ such that $\rho(f(x), f(w)) \leq \lambda d(x, w)$ whenever $d(x, w) \leq t_0$. This important notion is due to Luukkainen [31]. Obviously, a function that is Lipschitz in the small and Lipschitz for large distances is already Lipschitz, and conversely. Notably, the Lipschitz in the small real-valued functions form a uniformly dense subfamily of $UC(X, \mathbb{R})$ ([17, Theorem 1] and [8, Theorem 6.1]).

If ω_f has an affine majorant and g has finite uniform distance from f, then so does ω_g. It has been shown that a Lipschitz in the small function is Lipschitz if and only if its modulus of continuity has an affine majorant [6, Proposition 6.4].

A function which has a finite-valued modulus of continuity is called a *coarse map* [1, 12, 23]. The Dirichlet function is an example of a nowhere continuous coarse map. A uniformly continuous coarse map need not be Lipschitz for large distance: let $X = \{1, 2, 4, 8, 16, \ldots\}$ equipped with the usual metric of the line and let f be the squaring function on X. We shall see later that a function whose modulus of continuity has an affine majorant must be Lipschitz for large distances, but not conversely.

3. Functions that are bounded in the small

Following [5, 9], we call a function $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ *bounded in the small* if any of the following equivalent conditions hold:

- $\omega_f(t_0)$ is finite for some positive t_0;
- $r(f)$ is finite;
- for some $\delta > 0$ there exists $\mu > 0$ such that whenever $A \subseteq X$ satisfies $\mathrm{diam}_d(A) \leq \delta$, then $\mathrm{diam}_\rho(f(A)) \leq \mu$.

Clearly, each coarse map as well as each uniformly continuous function is bounded in the small. Boundedness in the small is a uniform local growth condition; the authors of [25] used the alternative phrasing *uniformly locally uniformly bounded* to reflect the fact that centered at each point of X we can find a ball of a common radius such that diameter of the image of the ball has a common bound. Our language was chosen because it parallels the usage Lipschitz in the small, where at each point we can find a ball of common radius on which the objective function is Lipschitz with a Lipschitz constant of common size.

Although it is outside our main line of inquiry, we choose to characterize here the Bourbaki bounded subsets in terms of boundedness in the small, in part because boundedness in the small plays a central role in our forthcoming analysis, in part because Bourbaki boundedness should be better known among the community of analysts.

Theorem 3.1. *Let A be a nonempty subset of a metric space $\langle X, d\rangle$. Then A is Bourbaki bounded if and only if each bounded in the small function from X to an arbitrary metric space $\langle Y, d\rangle$ is bounded on A.*

Proof. For sufficiency, if each bounded in the small function with domain X is bounded when restricted to A, then each uniformly continuous function with domain X restricted to A is bounded. As is well-known, this forces A to be Bourbaki bounded [3, 20]. In fact, it is enough to know that each real-valued Lipschitz in the small function restricted to A is bounded [7, 32]. Conversely, let A be a nonempty

Bourbaki bounded subset of $\langle X, d\rangle$ and let $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ be bounded in the small. Choose $\delta > 0$ such that $\omega_f(\delta) < \infty$. By definition, there exists a finite subset F of X and $n \in \mathbb{N}$ such that each point of A can be joined to some point of F by a δ-chain of length n. It follows that

$$\mathrm{diam}_\rho(f(A)) \leq \mathrm{diam}_\rho(f(F)) + 2n\omega_f(\delta),$$

so that $f|_A$ is bounded. □

The next result is a companion to Theorem 3.1. The equivalence of conditions (1) and (2) in it is a folk fact whose easy proof is left to the reader.

Theorem 3.2. *The following statements are equivalent for a nonempty subset A of a metric space $\langle X, d\rangle$.*

(1) *A is bounded;*
(2) *each coarse mapping f from X to an arbitrary metric space $\langle Y, d\rangle$ is bounded on A;*
(3) *each function f from X to an arbitrary metric space $\langle Y, d\rangle$ that is Lipschitz for large distances is bounded on A.*

Proof. (1) $\Rightarrow$ (3). If A is a singleton, there is nothing to prove. Otherwise, let a_1 and a_2 be distinct points of A and let $\delta = d(a_1, a_2)/2$. It suffices to show that f is bounded on $S_d(a_1, \delta)$ and on $A \cap (X \backslash S_d(a_1, \delta))$. If f fails to be bounded on the ball, then

$$\sup\Big\{\frac{\rho(f(x), f(a_2))}{d(x, a_2)} : x \in X \text{ and } d(x, a_2) \geq \delta\Big\} = \infty,$$

and this violates the Lipschitz for large distances property. In the second case, $\exists \lambda > 0$ such that whenever $d(x, w) \geq \delta$, then $\rho(f(x), f(w)) \leq \lambda d(x, w)$. Then if $a \in A$ with $d(a, a_1) \geq \delta$, we obtain

$$\rho(f(a), f(a_1)) \leq \lambda d(a, a_1) \leq \lambda \mathrm{diam}_d(A).$$

This shows that the values of f restricted to $A \cap (X \backslash S_d(a_1, \delta))$ are contained in a ball with center $f(a_1)$.

(3) $\Rightarrow$ (1). Fixing $a \in A, x \mapsto d(x, a)$ is Lipschitz (for large distances) on X. By condition (3), $f|_A$ is bounded which means A is bounded. □

A much weaker condition than boundedness in the small for a function f is that at each point of the domain, there is a ball of common radius on which f is simply bounded. Such functions were called *uniformly locally bounded* by Hohti [21]. More precisely, $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is uniformly locally bounded if there exists $\delta > 0$ such that for each $x \in X$, $f(S_d(x, \delta))$ is a bounded subset of Y. The usual squaring function is uniformly locally bounded but is not bounded in the small. This weaker property for continuous functions is characteristic of *uniform paracompactness* of the space as introduced by Rice [4, 21, 36].

We finish this section by describing those metric domains on which uniformly locally bounded functions are automatically bounded in the small. To this end, we need another definition.

The *isolation function* for a metric space $\langle X, d\rangle$ is the real-valued (Lipschitz) function on X defined by $I(x) := d(x, X\backslash\{x\})$. Metric spaces in which each sequence $\langle x_n\rangle$ on which $I(x_n) \to 0$ has a convergent subsequence are those on which each continuous function is automatically uniformly continuous and are usually called *UC-spaces* [3, 4, 5]. Metric spaces in which each sequence $\langle x_n\rangle$ on which $I(x_n) \to 0$ has a Cauchy subsequence are precisely those whose completion is a UC-space. A long list of characterizations of such spaces has been compiled by Jain and Kundu [22].

Theorem 3.3. *The following conditions are equivalent for a metric space* $\langle X, d\rangle$.

(1) *each sequence* $\langle x_n\rangle$ *in* X *on which* $I(x_n) \to 0$ *has a Cauchy subsequence;*

(2) *each uniformly locally bounded function on* X *with values in a second metric space* $\langle Y, \rho\rangle$ *is bounded in the small;*

(3) *each continuous uniformly locally bounded real-valued function on* X *is bounded in the small.*

Proof. We need only prove (1) ⇒ (2) and (3) ⇒ (1).

Suppose (1) holds. We show that if $f : X \to Y$ is not bounded in the small, then f is not uniformly locally bounded. Since $r(f) = \infty$, for each n we can find $x_n, w_n \in X$ such that $0 < d(x_n, w_n) < 1/n$ but $\rho(f(x_n), f(w_n)) > n$. By passing to a subsequence, we can assume $\langle x_n\rangle$ is Cauchy. For each $\delta > 0$ there exists k such that $\frac{1}{k} < \frac{\delta}{2}$ and $k \le n \le j$ implies $d(x_n, x_j) < \frac{\delta}{2}$. Thus f fails to bounded on the open ball with center x_k and radius δ.

For (3) ⇒ (1), suppose (1) fails. We prove (3) fails. Let $\langle x_n\rangle$ be a sequence along which the isolation function goes to zero but which has no Cauchy subsequence. We can assume the terms are all distinct. The set of terms is not precompact [38, p. 125], so by passing to a subsequence we may assume that for some $\delta > 0$ and each $n \in \mathbb{N}$ (*i*) $I(x_n) < \frac{\delta}{4}$, and (ii) $d(x_n, x_j) \ge \delta$ whenever $n \ne j$. For each $n \in \mathbb{N}$ we can choose $w_n \ne x_n$ with (iii) $d(w_n, x_n) < I(x_n) + \frac{\delta}{4n}$. Put $\alpha_n := \frac{1}{2} d(x_n, w_n) < \frac{\delta}{4}$, and let S_n be the open ball with center x_n and radius α_n. The key points here are:

- each open ball of radius $\frac{\delta}{4}$ in X can hit at most one S_n;
- each w_n lies in the complement of $\cup_{k=1}^{\infty} S_k$.

Define $f : X \to \mathbb{R}$ by $f(x) := \begin{cases} n - \frac{n}{\alpha_n} d(x, x_n) & \text{if for some } n \in \mathbb{N},\ x \in S_n \\ 0 & \text{otherwise.} \end{cases}$

By the first bullet-point, f is well-defined and Lipschitz restricted to each open ball in X of radius $\frac{\delta}{4}$ (because either the function is identically zero in the ball or is the maximum of the zero function and a second Lipschitz function), so f is globally continuous and uniformly locally bounded. But f fails to be bounded in the small, because by the second bullet point, we have $|f(x_n) - f(w_n)| = n$ while $\lim_{n\to\infty} d(x_n, w_n) = 0$ by (iii). □

4. Functions whose modulus of continuity has an affine majorant

Let $f : \langle X, d\rangle \to \langle Y, \rho\rangle$. Evidently, ω_f having an affine majorant is equivalent to

$$\omega_f(t) = \mathbf{O}(t) \text{ as } t \to \infty.$$

To see this, if for some t_0 and some $\alpha > 0$, we have $\omega_f(t) \leq \alpha t$ for $t \geq t_0$, then f is a coarse map, and for all $t \geq 0$, we have $\omega_f(t) \leq \alpha t + \omega_f(t_0)$ because ω_f is increasing. On the other hand, if $w_f(t) \leq \mu t + \beta$ for all nonnegative t, we must have both $\mu \geq 0$ and $\beta \geq 0$. Taking $\alpha > \mu$, we see that

$$\forall t \geq \frac{\beta}{\alpha - \mu}, \quad \omega_f(t) \leq \alpha t.$$

We note that the condition $\omega_f(t) = \mathbf{O}(t)$ as $t \to \infty$ can also be stated as

$$\mathrm{limsup}_{t\to\infty} \frac{\omega_f(t)}{t} := \mathrm{inf}_{t>0} \, \mathrm{sup}_{s\geq t} \frac{\omega_f(s)}{s} < \infty.$$

We next show that for any function f between the metric spaces that is bounded in the small, then ω_f has an affine majorant if and only if f is Lipschitz for large distances. This auxiliary condition is much weaker than uniform continuity for f.

Theorem 4.1. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces. The following conditions are equivalent for $f : X \to Y$.*

(1) *ω_f has an affine majorant;*

(2) *f is bounded in the small and for each $\delta > 0$, there exists $k_\delta \in \mathbb{N}$ such that $d(x, w) \geq \delta \Rightarrow \rho(f(x), f(w)) \leq k_\delta d(x, w)$;*

(3) *there exists $\delta > 0$ such that $\omega_f(\delta) < \infty$ and there exists $\lambda \geq 0$ such that $d(x, w) \geq \delta \Rightarrow \rho(f(x), f(w)) \leq \lambda d(x, w)$.*

Proof. Assume (1) $t \mapsto \alpha t + \beta$ with $\alpha \geq 0$ and $\beta \geq 0$ is a majorant for ω_f. Then f is a coarse map so that the auxiliary condition in (2) is satisfied. To show that f is Lipschitz for large distances, let $\delta > 0$ be arbitrary. Choose $k_\delta \in \mathbb{N}$ with (#) $k_\delta \delta > \alpha\delta + \beta$. Suppose $d(x, w) \geq \delta$. Since $k_\delta > \alpha$, by (#) we obtain this inequality string:

$$\rho(f(x), f(w)) \leq \omega_f(d(x, w)) \leq \alpha d(x, w) + \beta \leq k_\delta d(x, w),$$

and so f is also Lipschitz for large distances.

Condition (3) is obviously a consequence of condition (2).

Finally, to prove (3) $\Rightarrow$ (1), suppose (3) holds exactly as stated. We claim that $t \mapsto \lambda t + \omega_f(\delta)$ is an affine majorant for ω_f. To see this, let $t \geq 0$ be arbitrary. For $d(x, w) \leq t$, we consider two cases. If $d(x, w) < \delta$, then by the definition of the modulus of continuity, $\rho(f(x), f(w)) \leq \omega_f(\delta)$. Otherwise, $\delta \leq d(x, w) \leq t$ and we compute

$$\rho(f(x), f(w)) \leq \lambda d(x, w) \leq \lambda t.$$

Either way, $\rho(f(x), f(w))) \leq \lambda t + \omega_f(\delta)$. Since $t \geq 0$ was arbitrary, we conclude that for all $t \geq 0$, $\omega_f(t) \leq \lambda t + \omega_f(\delta)$ as required. $\square$

Example 4.2. Equip the following set with the usual metric of the line to produce a metric space $\langle X, d\rangle$:

$$\{2,\ 2\tfrac{1}{2},\ 4,\ 4\tfrac{1}{4},\ 8,\ 8\tfrac{1}{8},\ 16,\ 16\tfrac{1}{16},\ \ldots\}.$$

Define $f : X \to \mathbb{R}$ by $f(2^n) = n$ and $f(2^n + 2^{-n}) = 0$ for $n \in \mathbb{N}$. It is routine to check that f is Lipschitz for large distances, while $\lim_{n\to\infty} f(2^n) - f(2^n + 2^{-n}) = \infty$, so that f is a function of uncontrollable uniform growth, and, in particular, ω_f can have no affine majorant. □

The underlying domain metric space in the last example is rather pathological. On the other hand, if the underlying metric for the domain space is convex (see, e.g. [10, p. 13] or [2]), then for any function f into a second metric space, f is bounded in the small if and only if f is a coarse map if and only if f is Lipschitz for large distances [5, Theorem 25.4].

Keeping condition (3) of Theorem 4.1 in mind, we have this attractive corollary.

Corollary 4.3. *Suppose $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is a coarse map. The following conditions are equivalent:*

(1) *f is Lipschitz for large distances;*

(2) *ω_f has an affine majorant;*

(3) *there exist $\delta, \lambda > 0$ such that $d(x, w) \geq \delta \Rightarrow \rho(f(x), f(w)) \leq \lambda d(x, w)$.*

It is evident that both the real-valued functions that are Lipschitz for large distances and the real-valued functions whose moduli have affine majorants defined on a given metric space form vector spaces. Further, the elementary inequality $||\alpha| - |\beta|| \leq |\alpha - \beta|$ implies that both families are actually vector lattices. As the real-valued Lipschitz functions on a metric space are stable under pointwise product exactly when the metric space is bounded, the following proposition should come as no surprise.

Proposition 4.4. *The following conditions are equivalent for a metric space $\langle X, d\rangle$.*

(1) *$\langle X, d\rangle$ is a bounded metric space;*

(2) *the real-valued functions f on X that are Lipschitz for large distances are stable under pointwise product;*

(3) *the real-valued functions f on X for which ω_f has an affine majorant are stable under pointwise product.*

Proof. Suppose condition (1) holds. We show condition (2) and condition (3) both hold. By Theorem 3.2, each real-valued function on X that is Lipschitz for large distances is bounded on X and the pointwise product of any two will be bounded on X and hence Lipschitz for large distances. If two real-valued functions on X have moduli of continuity with affine majorants, then by Theorem 4.1 both functions are Lipschitz for large distances, and repeating our reasoning, their product is bounded. Thus, the modulus of continuity for the product has an affine majorant.

Conversely, suppose condition (1) fails. Fix $x_0 \in X$ and define $f : X \to \mathbb{R}$ by $f(x) := d(x, x_0)$. Since f is Lipschitz, ω_f has an affine majorant. However, f^2 not only fails to be Lipschitz for large distances but also condition (3) of Corollary 4.3 fails to hold for the square. Thus, conditions (2) and (3) of our current proposition both fail. □

We now come to a second internal characterization of those functions whose moduli have affine majorants involving boundedness in the small.

Theorem 4.5. *Let $f : X \to Y$ be a bounded in the small function between metric spaces $\langle X, d\rangle$ and $\langle Y, \rho\rangle$. Then ω_f has an affine majorant if and only if ($\diamondsuit$) $f|_D$ is Lipschitz for each nonempty uniformly discrete subset D of X.*

Proof. If ω_f has an affine majorant, then by Theorem 4.1, f is Lipschitz for large distances. Hence, ($\diamondsuit$) holds. Conversely, assume that ($\diamondsuit$) holds, and let $\delta > 0$. There exists $\delta_0 \in (0, \delta]$ such that $\omega_f(\delta_0) < \infty$. Let D be a maximal δ_0-discrete subset of X, and let λ be a Lipschitz constant for $f|_D$. Let $x, w \in X$ satisfy $d(x, w) \geq \delta$; by the maximality of D we can find $a, b \in D$ such that $d(x, a) < \delta_0$ and $d(w, b) < \delta_0$. Using the fact that $d(x, w) \geq \delta$, the triangle inequality applied twice yields

$$\begin{aligned}
\rho(f(x), f(w)) &\leq \rho(f(x), f(a)) + \rho(f(b), f(w)) + \rho(f(a), f(b)) \\
&\leq \omega_f(d(x, a)) + \omega_f(d(b, w)) + \lambda d(a, b) \\
&\leq 2\omega_f(\delta_0) + \lambda[d(a, x) + d(x, w) + d(w, b)] \\
&\leq 2\omega_f(\delta_0) \cdot \frac{d(x, w)}{\delta} + \lambda[2\delta_0 \cdot \frac{d(x, w)}{\delta} + d(x, w)] \\
&= [\frac{2\omega_f(\delta_0)\ +\ 2\lambda\delta_0}{\delta} + \lambda]d(x, w).
\end{aligned}$$

Hence, f is Lipschitz for large distances, so by Theorem 4.1, ω_f has an affine majorant. □

There is another notion that is relevant to our discussion. We say that a subset A of a metric space $\langle X, d\rangle$ is *U-embedded* provided each member of $UC(A, \mathbb{R})$ can be extended to a member of $UC(X, \mathbb{R})$. If X is a normed linear space, then a subset is U-embedded exactly when it satisfies a geometric condition that is related to, but is weaker than, convexity. This condition is described in [29, Theorem 3.3].

For our goals here, a different characterization is more relevant [25, Theorem 2]: A is a U-embedded subset of a normed linear space if and only if each real-valued function on A that is bounded in the small is Lipschitz for large distances. We establish a significant generalization of this result below.

But first, as a courtesy to the reader, we give the following simple illustration of a non-U-embedded subset A of the plane equipped with the Euclidean norm [28, Example 7.3]. Consider the parabola $A = \{(x, x^2) : x \in \mathbb{R}\}$ and define the function $f : A \to \mathbb{R}$ by $f(P) :=$ the arc length from $(0, 0)$ to P if P is in the first quadrant, while $f(P) :=$ minus the arc length from $(0, 0)$ to P if P is in the second quadrant. Since the mapping is uniformly continuous, it is bounded in the small. However, f is not Lipschitz for large distances: if $P = (x, x^2)$ where $x > 0$ and $Q = (-x, x^2)$, then

$$|f(P) - f(Q)| = 2f(P) \geq 2x^2 = x||P - Q||_2.$$

Let I be a nonempty set and let $\ell_\infty(I)$ denote the bounded real-valued functions on I, equipped with the supremum norm $||\cdot||_\infty$. When $I = \{1, 2, \ldots, n\}$ we of course get $\mathbb{R}^n$ equipped with the box norm, which is equivalent to the Euclidean norm. We will denote the Euclidean norm by $||\cdot||_2$ as used just above.

The following tool proposition speaks to the ability to uniformly approximate a bounded in the small function f with values in an $\ell_\infty(I)$-space by a uniformly continuous function in the space within $\frac{r(f)}{2}$. We will immediately use this result to show that the modulus of continuity for each bounded in the small function defined on a U-embedded subset of a normed linear space with values in an arbitrary metric target space has an affine majorant. But we find it convenient to isolate part of its proof as a preliminary lemma, as we will call on it again in Section 6.

Lemma 4.6. *Suppose $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is a bounded in the small function between metric spaces. Then for each uniformly continuous function $g : X \to Y$, we have $sup\{\rho(f(x), g(x)) : x \in X\} \geq \frac{r(f)}{2}$.*

Proof. Put $s := \sup\{\rho(f(x), g(x)) : x \in X\}$. Suppose to the contrary that $s < \frac{r(f)}{2}$. Choose $0 < \alpha < r(f) - 2s$. By uniform continuity of g, choose $\delta > 0$ such that if $d(x, w) \leq \delta$, then $\rho(g(x), g(w)) < r(f) - 2s - \alpha$. Suppose $d(x, w) \leq \delta$; the inequality string

$$\begin{aligned}\rho(f(x), f(w)) &\leq \rho(f(x), g(x)) + \rho(f(w), g(w)) + \rho(g(x), g(w))\\ &< 2s + r(f) - 2s - \alpha = r(f) - \alpha\end{aligned}$$

shows that $\omega_f(\delta) < r(f)$, which is a contradiction. Therefore, $s \geq \frac{r(f)}{2}$. □

Proposition 4.7. *Assume that $\langle X, d\rangle$ is a metric space and $f : X \to \ell_\infty(I)$ is bounded in the small. Then there exists $h \in UC(X, \ell_\infty(I))$ such that*

$$sup_{x\in X}\ ||f(x) - h(x)||_\infty = \frac{r(f)}{2}.$$

Proof. Put $Z := X \times I$, and define a metric e on Z by

$$e((x, i), (w, j)) := d(x, w) + \delta(i, j),$$

where δ is the Kronecker delta function on $I \times I$. Define the mapping $g : Z \to \mathbb{R}$ by $g(x, i) := f(x)(i)$. Clearly, if $0 \leq t < 1$, then $\omega_g(t) = \omega_f(t)$, and so $r(g) = r(f)$.

By [35, Theorem 3.1], there is $\hat{h} \in UC(Z, \mathbb{R})$ such that $||g - \hat{h}||_\infty = \alpha := r(g)/2$. Define a function h on X whose values are functions on I by $h(x)(i) := \hat{h}(x, i)$. Since

$$\sup_{i\in I}|h(x)(i)| \leq \sup_{i\in I}|f(x)(i)| + \alpha,$$

$h(x)$ lies in $\ell_\infty(I)$ for each $x \in X$, that is, $h : X \to \ell_\infty(I)$.

Let $\varepsilon > 0$ and choose $\delta_0 > 0$ such that

$$e((x, i), (w, j)) < \delta_0 \Rightarrow |\hat{h}(x, i) - \hat{h}(w, j)| < \varepsilon.$$

As a result, if $d(x, w) < \delta_0$, then $|h(x)(i) - h(w)(i)| < \varepsilon$ for each $i \in I$, and we obtain $||h(x) - h(w)||_\infty \leq \varepsilon$. Therefore, h is uniformly continuous and for each $x \in X$,

$$\begin{aligned}||f(x) - h(x)||_\infty &= \sup_{i\in I}\ |f(x)(i) - h(x)(i)|\\ &= \sup_{i\in I}\ |g(x, i) - \hat{h}(x, i)| \leq \alpha = \frac{r(f)}{2}.\end{aligned}$$

Hence, $s := \sup\{||f(x) - h(x)||_\infty : x \in X\} \leq r(f)/2$. The reverse inequality is a consequence of Lemma 4.6. □

Theorem 4.8. *Assume that $\langle Y, \rho\rangle$ is a metric space and A is a U-embedded subset of a normed linear space $\langle B, ||\cdot||\rangle$. If a mapping $f : A \to Y$ is bounded in the small, then f is Lipschitz for large distances. Hence, ω_f has an affine majorant.*

Proof. There is an isometric embedding e of $\langle Y, \rho\rangle$ into the bounded continuous real-valued functions on Y [38, pp. 84-85], so by extending the target space, we can view e as an isometric embedding of Y into $\ell_\infty(Y)$. Put $g := e \circ f$. Since e is an isometry, we have $r(g) = r(f)$. By Proposition 4.7, there is a uniformly continuous mapping $h : A \to \ell_\infty(Y)$ such that

$$\sup\{||g(a) - h(a)||_\infty : a \in A\} = \frac{r(g)}{2}.$$

By [26, Theorem 3.1], h can be extended to a uniformly continuous mapping

$$\hat{h} : B \to \ell_\infty(Y).$$

Since B is a convex set, by [5, Theorem 25.4], $\hat{h}$ is Lipschitz for large distances, and thus so is h. Since g is a finite uniform distance from h, it follows that g and hence f are also Lipschitz for large distances. Finally, by Theorem 4.1, ω_f has an affine majorant. □

5. Additional majorants for the modulus of continuity

There are a number of other types of majorants that a modulus of continuity can have whose existence is equivalent to the existence of an affine majorant. There is a hierarchy for them in the sense that they can be arranged in descending order of strength. Our focus on affine majorants in part reflects their position in that hierarchy.

Theorem 5.1. *Let f be a function between metric spaces $\langle X, d\rangle$ and $\langle Y, \rho\rangle$. The following conditions are equivalent.*

(1) *ω_f has an affine majorant;*

(2) *ω_f has a concave majorant;*

(3) *ω_f has a subadditive majorant h such that whenever $t \geq 0$ and $\alpha \geq 1$, we have $h(\alpha t) \leq \alpha h(t)$;*

(4) *ω_f has a subadditive majorant.*

Proof. We assert that these conditions are listed in decreasing order of strength. In view of our remarks in the last section, we need only show that a concave nonnegative function h on $[0, \infty)$ satisfies $h(\alpha t) \leq \alpha h(t)$ for $t \geq 0$ and $\alpha \geq 1$. The inequality is clearly valid when $t = 0$, so let t be positive and $\alpha \geq 1$. From concavity we obtain

$$h(t) \geq (1 - \frac{1}{\alpha})\, h(0) + \frac{1}{\alpha}\, h(\alpha t),$$

so by the nonnegativity of $h(0)$, it follows that

$$\alpha h(t) \geq (\alpha - 1)h(0) + h(\alpha t) \geq h(\alpha t).$$

It remains to show that (4) $\Rightarrow$ (1). Suppose g is a subadditive majorant for ω_f. Put $h(t) := g(1)t + g(1)$. For each $t \geq 0$, there is a unique $n \in \{0\} \cup \mathbb{N}$ such that $n \leq t < n+1$. Since ω_f is an increasing function, for each $t \geq 0$, we get

$$\omega_f(t) \leq \omega_f(n+1) \leq g(n+1) \leq (n+1)g(1) \leq (t+1)g(1) = h(t)$$

as required. □

Our proof of (4) $\Rightarrow$ (1) can be rewritten using the floor function by replacing n by $\lfloor t \rfloor$ in this inequality string, where $\lfloor t \rfloor$ is the greatest integer less than or equal to t. Aronszajn and Panitchpakdi [2, p. 406–407] purport to give a proof also using the floor function that condition (4) implies

$$\operatorname{limsup}_{t\to\infty} \frac{\omega_f(t)}{t} < \infty$$

– equivalent to condition (1) – which the present authors find to be flawed. In particular, the first inequality in their proof seems to require that the subadditive majorant be an increasing function. Unfortunately, their result has been cited by subsequent authors (see, e.g., [18]).

The following example shows that several of the conditions in Theorem 5.1 are not automatically equivalent.

Example 5.2. Our intriguing condition (3) of Theorem 5.1 for an increasing, nonnegative real-valued function h on $[0, \infty)$ lies properly between the conditions of concavity and subadditivity for h.

(i) The increasing subadditive function $h(t) := 1$ for $0 \leq t \leq 2$ and $h(t) := 2$ for $t > 2$ fails to satisfy $h(\alpha t) \leq \alpha h(t)$ when $t = \frac{5}{3}$ and $\alpha = \frac{3}{2}$ because

$$h(\frac{3}{2} \cdot \frac{5}{3}) = 2 > \frac{3}{2} = \frac{3}{2} h(\frac{5}{3}).$$

(ii) We next present an increasing, nonnegative, continuous function h satisfying condition (3) of Theorem 5.1 that is not concave. Our function is defined by

$$h(t) := \begin{cases} 1 & \text{if } t \in [0,1] \\ t & \text{otherwise.} \end{cases}$$

To verify subadditivity, we consider three cases for $t_1 \geq 0$ and $t_2 \geq 0$:

(a) $t_1, t_2 \in [0, 1)$; (b) $t_1 \in [0, 1)$ and $t_2 \geq 1$; (c) $t_1 \geq 1$ and $t_2 \geq 1$.

In case (a), as $t_1 + t_2 \leq 2$, $h(t_1 + t_2) \leq 2 = h(t_1) + h(t_2)$. In case (b),

$$h(t_1 + t_2) = t_1 + t_2 < 1 + t_2 = h(t_1) + h(t_2).$$

In case (c), $h(t_1 + t_2) = t_1 + t_2 = h(t_1) + h(t_2)$.

For the scalar multiplication inequality, we consider four cases: (i) $t = 0$ and $\alpha \geq 1$; (ii) $0 < t < 1$ and $1 \leq \alpha \leq \frac{1}{t}$; (iii) $0 < t < 1$ and $\frac{1}{t} < \alpha$; (iv) $t \geq 1$ and $\alpha \geq 1$. As none of the four cases requires any imagination, we only dispense with case (iii), leaving the others to the reader. In this particular case, since $\alpha t > 1$,

$$h(\alpha t) = \alpha t < \alpha = \alpha h(t).$$

□

For completeness, we include this next corollary, as many authors confining their attention to uniformly continuous functions want their concave or subadditive majorant h to also satisfy $h(0) = \lim_{t \to 0^+} h(t) = 0$, perhaps for historical reasons (see, e.g., [2, p. 408], [10, Proposition 2.1] or [18, Satz 4]). The implication (1) ⇒ (3) below was argued somewhat differently by McShane [33].

Corollary 5.3. *Let f be a function between metric spaces $\langle X, d\rangle$ and $\langle Y, \rho\rangle$. The following conditions are equivalent.*

(1) *$f \in UC(X, Y)$ and ω_f has an affine majorant;*

(2) *$f \in UC(X, Y)$ and f is Lipschitz for large distances;*

(3) *ω_f has a concave majorant h with $h(0) = lim_{t \to 0^+} h(t) = 0$;*

(4) *ω_f has a subadditive majorant h with $h(0) = lim_{t \to 0^+} h(t) = 0$.*

Proof. By Theorem 4.1, the first two conditions are equivalent because a uniformly continuous function is bounded in the small. As a concave majorant must already be subadditive, condition (3) implies condition (4). If (4) holds, then $r(f) = 0$ because $h \geq \omega_f \geq 0$ so that f is uniformly continuous. Thus, condition (1) holds by Theorem 5.1. It remains to prove (1) ⇒ (3).

Let $g(t) = \alpha t + \beta$ where $\alpha \geq 0$ and $\beta \geq 0$ be an affine majorant for ω_f. For each $n \in \mathbb{N}$, pick $\mu_n > \alpha$ such that

$$\frac{\mu_n}{n} + \omega_f(\frac{1}{n}) \geq \frac{\alpha}{n} + \beta.$$

Let $h_n : [0, \infty) \to \mathbb{R}$ be the affine function defined by $h_n(t) = \mu_n t + \omega_f(\frac{1}{n})$. By construction, for each $n \in \mathbb{N}$, we have $h_n \geq \omega_f$. Since the intersection of a family of convex sets is convex and $\text{hypo}(\inf_{n\in\mathbb{N}} h_n) = \cap_{n=1}^{\infty}\text{hypo}(h_n)$, $h := \inf_{n\in\mathbb{N}} h_n$ is a concave majorant of ω_f. By the uniform continuity of f,

$$h(0) = \inf_{n\in\mathbb{N}} h_n(0) = \inf_{n\in\mathbb{N}} \omega_f(\frac{1}{n}) = 0.$$

We conclude our proof by observing that for each $n \in \mathbb{N}$,

$$\text{limsup}_{t\to 0+} h(t) \leq \lim_{t\to 0+} h_n(t) = \omega_f(\frac{1}{n}),$$

which implies $\lim_{t\to 0^+} h(t) = 0$. □

Of course, ω_f has an affine majorant h satisfying $\lim_{t\to 0^+} h(t) = 0$ if and only if f is already Lipschitz.

6. Approximating functions whose modulus of continuity has an affine majorant

Our first result not only extends Theorem 1 of [25] but also uses this prior result in its proof. Recall that $L(f)$ denotes the minimal Lipschitz constant of a Lipschitz mapping f.

Lemma 6.1. *Assume that $\langle X, d\rangle$ is a metric space and $f : X \to \ell_\infty(I)$ is bounded. For each $\varepsilon > 0$, there is a Lipschitz function $l : X \to \ell_\infty(I)$ such that*

$$sup_{x\in X}\ ||f(x) - l(x)||_\infty \le \frac{r(f)}{2} + \varepsilon.$$

Proof. Suppose $||f(x)||_\infty \le \mu$ for each $x \in X$. As in the proof of Proposition 4.7, define a metric e on $Z := X \times I$ by $e((x, i), (w, j)) := d(x, w) + \delta(i, j)$ where δ is the Kronecker delta function. Define the mapping $g : Z \to \mathbb{R}$ by $g(x, i) := f(x)(i)$. Then for each $x \in X$ and $i \in I$ we have $|g(x, i)| \le \mu$. As in the earlier proof, $r(g) = r(f)$.

Applying Theorem 1(i) of [25], there is a Lipschitz function $h : Z \to \mathbb{R}$ such that $||g - h||_\infty \le \alpha := \frac{r(g)}{2} + \varepsilon$. Define $l : X \to \ell_\infty(I)$ by $l(x)(i) = h(x, i)$. Since $||g - h||_\infty \le \alpha$, the mapping l is well-defined, and

$$\begin{aligned} ||l(x) - l(w)||_\infty &= \sup_{i\in I}|h(x, i) - h(w, i)| \\ &\le \sup_{i\in I} L(h) \cdot e((x, i), (w, i)) = L(h)d(x, w). \end{aligned}$$

Hence, l is a Lipschitz mapping and we obtain for each $x \in X$

$$||f(x) - l(x)||_\infty = \sup_{i\in I}|f(x)(i) - l(x)(i)| \le ||g - h||_\infty \le \alpha = \frac{r(f)}{2} + \varepsilon,$$

and the result follows. □

For the proof of the main theorem of Section 6, we also need this known fact.

Proposition 6.2. *Let S be a subspace of the metric space $\langle X, d\rangle$. Each Lipschitz mapping $l : S \to \ell_\infty(I)$ can be extended to a member of $Lip(X, \ell_\infty(I))$ with the same Lipschitz constant.*

By [2, Theorem 4 of §2], this proposition is a consequence of the hyperconvexity of the space $\ell_\infty(I)$.

Definition 6.3. A metric space $\langle X, d\rangle$ is called *hyperconvex* [2, 16] provided whenever $\{B_d(x_\alpha, r_\alpha) : \alpha \in \Omega\}$ is a collection of closed balls with $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$ whenever $\{\alpha, \beta\} \subseteq \Omega$, then $\cap_{\alpha\in\Omega} B_d(x_\alpha, r_\alpha)$ is nonempty.

The hyperconvexity of $\mathbb{R}$ equipped with the Euclidean metric is a special case of Helly's theorem for convex sets [37]. With this in mind, the hyperconvexity of $\ell_\infty(I)$ is an immediate consequence of Theorem 3.1 of [16] as applied to $\mathbb{R}^I$ where the distinguished function a in its statement is the zero function. Although the proof indicated by Espínola and Khamsi is natural, for the entertainment of the reader, we thought to offer an alternative order-theoretic proof for our particular case.

Let $\{f_\alpha : \alpha \in \Omega\}$ be a subset of $\ell_\infty(I)$ and let $\{r_\alpha : \alpha \in \Omega\}$ be a set of positive scalars such that whenever $(\alpha, \beta) \in \Omega^2$, we have $||f_\alpha - f_\beta||_\infty \le r_\alpha + r_\beta$. Given $(\alpha, \beta) \in \Omega^2$, let f be the convex combination of f_α and f_β defined by

$$f := \frac{r_\alpha}{r_\alpha + r_\beta} f_\beta + \frac{r_\beta}{r_\alpha + r_\beta} f_\alpha.$$

We leave to the reader as a straightforward computation to show that we have both $||f - f_\alpha||_\infty \leq r_\alpha$ and $||f - f_\beta||_\infty \leq r_\beta$. As a result, we have the inequalities

$$\forall i \in I,\ f_\alpha(i) - r_\alpha \leq f(i) \leq f_\beta(i) + r_\beta.$$

We conclude (#) $\forall(\alpha, \beta) \in \Omega^2\ \forall i \in I,\ f_\alpha(i) - r_\alpha \leq f_\beta(i) + r_\beta$.

Define the mapping $g : I \to \mathbb{R}$ by $g(i) := \sup\ \{f_\alpha(i) - r_\alpha : \alpha \in \Omega\}$. Fix $\beta \in \Omega$. By (#), for each $i \in I,\ f_\beta(i) - r_\beta \leq g(i) \leq f_\beta(i) + r_\beta$. This at once shows that g is bounded and $||f_\beta - g||_\infty \leq r_\beta$. Since $\beta \in \Omega$ was arbitrary, we conclude that $g \in \bigcap_{\alpha\in\Omega} B_{||\cdot||_\infty}(f_\alpha, r_\alpha)$, that is, $\ell_\infty(I)$ is hyperconvex.

We now come to the main result of Section 6.

Theorem 6.4. *Suppose $\langle X, d\rangle$ is a metric space and $f : X \to \ell_\infty(I)$ is an arbitrary function. The following conditions are equivalent.*

(1) *there exists $l \in Lip(X, \ell_\infty(I))$ whose uniform distance from f is finite;*

(2) *the modulus of continuity ω_f has an affine majorant;*

(3) *for each $\varepsilon > 0$, there is a Lipschitz mapping $l : X \to \ell_\infty(I)$ such that*

$$sup_{x\in X}\ ||f(x) - l(x)||_\infty \leq \frac{r(f)}{2} + \varepsilon.$$

Proof. We need only prove (1) ⇒ (2) and (2) ⇒ (3). For (1) ⇒ (2), if the function l is λ-Lipschitz and $\mu := \sup\{||f(x) - l(x)||_\infty : x \in X\} < \infty$, then for each $t \geq 0$, we have $\omega_f(t) \leq \omega_l(t) + 2\mu \leq \lambda t + 2\mu$.

For (2) ⇒ (3), let $t \mapsto \alpha t + \beta$ be an affine majorant for ω_f, where $\alpha, \beta \geq 0$. Noting that $r(f) \leq \beta$, choose $\delta > 0$ such that $\omega_f(\delta) < r(f) + 1$. Let D be a maximal δ-discrete subset of X. We know from Theorem 4.5 that $f|_D$ is Lipschitz. We claim (♡) $L(f|_D) \leq \alpha + \frac{\beta}{\delta}$. To see this, let x and w be distinct points in D. Then we have

$$||f(x) - f(w)||_\infty \leq \omega_f(d(x, w)) \leq \alpha d(x, w) + \beta.$$

Since $d(x, w) \geq \delta$ implies $\beta \leq \frac{\beta}{\delta} d(x, w)$, the number $\alpha + \frac{\beta}{\delta}$ is a Lipschitz constant for the restriction.

By (♡) and Proposition 6.2, $f|_D$ has a Lipschitz extension $l_1 : X \to \ell_\infty(I)$. Let $x \in X$ be arbitrary. By the maximality of D, we can find $p \in D$ such that $d(x, p) < \delta$. Since $l_1(p) = f(p)$, the triangle inequality yields

$$\begin{aligned} ||f(x) - l_1(x)||_\infty &\leq ||f(x) - f(p)||_\infty + ||l_1(p) - l_1(x)||_\infty \\ &\leq \omega_f(\delta) + L(l_1)d(p, x) < r(f) + 1 + L(l_1)\delta. \end{aligned}$$

Since $x \in X$ was arbitrary, $h := f - l_1$ is a bounded mapping, and since l_1 is uniformly continuous, it follows that $r(h) = r(f)$. Hence, by Lemma 6.1, there is $l_2 \in \mathrm{Lip}(X, \ell_\infty(I))$ satisfying

$$\sup_{x\in X}\ ||h(x) - l_2(x)||_\infty \leq \frac{r(h)}{2} + \varepsilon = \frac{r(f)}{2} + \varepsilon.$$

Finally, $l := l_1 + l_2$ is Lipschitz with $\sup\ \{||f(x) - l(x)||_\infty : x \in X\} \leq \frac{r(f)}{2} + \varepsilon$ as required. □

Our first corollary follows from Lemma 4.6 and the fact that each Lipschitz function is uniformly continuous.

Corollary 6.5. *Let f be a function from $\langle X, d\rangle$ to $\ell_\infty(I)$ such that ω_f has an affine majorant. Then the uniform distance from f to $Lip(X, \ell_\infty(I))$ is $\frac{r(f)}{2}$.*

Let $f : X \to \mathbb{R}$ be a non-Lipschitz function defined on a metric space $\langle X, d\rangle$ such that ω_f has an affine majorant. By Corollary 6.5, the uniform distance from f to $\mathrm{Lip}(X, \mathbb{R})$ is exactly $\frac{r(f)}{2}$. But must f have a nearest point in $\mathrm{Lip}(X, \mathbb{R})$? When $X = [0, \infty)$, this is true for the Dirichlet function f (the characteristic function of the rationals); since $\omega_f(t) = 1$ for all $t > 0$, the constant function $l := \frac{1}{2}$ is a best approximation. However, there is no best approximation in $\mathrm{Lip}(X, \mathbb{R})$ for any uniformly continuous non-Lipschitz function f such as the square-root function since $r(f) = 0$. It would be interesting to characterize those f for which the answer is affirmative.

Our next two results are presented simply as applications of our main theorem. We do not claim that the estimates are best possible.

Corollary 6.6. *If $f : \langle X, d\rangle \to \langle \mathbb{R}^n, ||\cdot||_2\rangle$ is a mapping such that ω_f has an affine majorant, then for each $\varepsilon > 0$, there exists $l : \langle X, d\rangle \to \langle \mathbb{R}^n, ||\cdot||_2\rangle$ such that l is Lipschitz and satisfies*

$$sup_{x\in X}\ ||f(x) - l(x)||_2 \le \sqrt{n}\frac{r(f)}{2} + \varepsilon.$$

Proof. We noted earlier that when $I = \{1, 2, \ldots, n\}$, $\ell_\infty(I)$ is $\mathbb{R}^n$ equipped with the box norm which we still of course denote by $||\cdot||_\infty$. Now equivalent norms on a vector space give rise to the same class of Lipschitz functions into it defined on a prescribed metric space, and in this case we have for each $v \in \mathbb{R}^n$,

$$||v||_\infty \le ||v||_2 \le \sqrt{n}||v||_\infty.$$

Define $\hat{f} : \langle X, d\rangle \to \langle \mathbb{R}^n, ||\cdot||_\infty\rangle$ by $\hat{f}(x) = f(x)$. Since $\omega_{\hat{f}} \le \omega_f$, each affine majorant for ω_f will also be an affine majorant for $\omega_{\hat{f}}$. Applying Theorem 6.4 to $\hat{f}$, there is a Lipschitz function l from $\langle X, d\rangle$ to $\langle \mathbb{R}^n, ||\cdot||_\infty\rangle$ such that

$$\mathrm{sup}_{x\in X}\ ||f(x) - l(x)||_\infty \le \frac{r(\hat{f})}{2} + \varepsilon \le \frac{r(f)}{2} + \varepsilon.$$

As remarked, l is also Lipschitz viewed as a function into Euclidean space, and we compute $\mathrm{sup}_{x\in X}\ ||f(x) - l(x)||_2 \le \sqrt{n}r(f)/2 + \varepsilon$. □

For our third corollary, recall that for an index set I, $c_0(I)$ denotes the vector space of all real-valued functions f on I such that $\forall\alpha > 0$, $\{i \in I : |f(i)| > \alpha\}$ is a finite set. Equipped with the supremum norm, this is a subspace of $\ell_\infty(I)$.

Corollary 6.7. *Let $\langle X, d\rangle$ be a metric space and suppose $f : X \to c_0(I)$ is a mapping for which ω_f has an affine majorant. Then for each $\varepsilon > 0$, there exists a Lipschitz mapping $l : X \to c_0(I)$ such that $sup_{x\in X}\ ||f(x) - l(x)||_\infty \le r(f) + \varepsilon$.*

Proof. Let *incl* : $c_0(I) \to \ell_\infty(I)$ be the inclusion mapping and put $g := incl \circ f$. Since $\omega_f = \omega_g$, we can apply Theorem 6.4 to g: for each $\varepsilon > 0$, there is a Lipschitz mapping $l' : X \to \ell_\infty(I)$ such that

$$\sup\{||g(x) - l'(x)||_\infty : x \in X\} \le \frac{r(g)}{2} + \frac{\varepsilon}{2} = \frac{r(f)}{2} + \frac{\varepsilon}{2}.$$

Lindentrauss has shown that there is a Lipschitz retraction $\phi : \ell_\infty(I) \to c_0(I)$ satisfying $L(\phi) \le 2$ [30, Theorem 6a]. Then $l := \phi \circ l'$ is a Lipschitz mapping into $c_0(I)$ and for each $x \in X$,

$$||f(x) - l(x)||_\infty = ||\phi(g(x)) - \phi(l'(x))||_\infty \le 2||g(x) - l'(x)||_\infty \le r(f) + \varepsilon,$$

completing the proof. $\square$

Given metric spaces $\langle X, d\rangle$ and $\langle Y, \rho\rangle$, let $\overline{\mathrm{Lip}(X,Y)}$ denote the *uniform closure* of $\mathrm{Lip}(X,Y)$, i.e., the family of functions $h : X \to Y$ that can be uniformly approximated by members of $\mathrm{Lip}(X,Y)$. From our earlier remarks, each member of $\overline{\mathrm{Lip}(X,Y)}$ is both uniformly continuous and Lipschitz for large distances. The next result shows that in certain cases the converse also holds.

Corollary 6.8. *For each metric space* $\langle X, d\rangle$,

$$\overline{Lip(X,Y)} = \{h \in UC(X,Y) : h \text{ is Lipschitz for large distances}\}$$

provided Y *is any of the spaces* $\ell_\infty(I)$, $c_0(I)$, *or* n*-dimensional Euclidean space.*

Proof. First, let $Y = \ell_\infty(I)$ and suppose $h \in UC(X,Y)$ is Lipschitz for large distances. Since h is bounded in the small, by Theorem 4.1, ω_h has an affine majorant. Hence, since $r(h) = 0$, by the equivalence of conditions (2) and (3) in Theorem 6.4, for each $\varepsilon > 0$, there is a Lipschitz function $l : X \to Y$ such that $\sup\{||h(x) - l(x)||_\infty : x \in X\} \le \varepsilon$. If $Y = n$-dimensional Euclidean space or $c_0(I)$, then by using Corollary 6.6 or Corollary 6.7 in place of Theorem 6.4, the same proof establishes the desired result. $\square$

We note that for $Y = \mathbb{R}$, the assertion of the last corollary is established in [14, Proposition 1].

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