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A Compactness Property of the Minimizing Sequences of a Minimal Surfaces Functional

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We prove the compactness, with respect to the convergence in measure, of the gradients of the minimizing sequences of strictly convex and coercive integral functionals having linear growth.

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1. Introduction

This paper deals with convex integral functionals (see (1)) and is dedicated to Ralph Tyrrell Rockafellar, the scientific father of convex analysis.

In our previous paper ([3]) we dealt with integral functionals J defined on the Sobolev space $\wp$ and we proved the $\wp$ compactness of the minimizing sequences if J is strictly convex and coercive.

In this paper, we consider integral functionals J on the nonreflexive Sobolev space $W_0^{1,1}(\Omega)$ and we prove the compactness, with respect to the convergence in measure, of the gradients of the minimizing sequences if J is strictly convex and coercive.

The framework is classic; we recall the papers [1], [5], [6], [8], [9].

Important tools in our approach are some Real Analysis techniques, introduced in [2] (for the study of nonlinear Dirichlet problems with nonregular data), developed in [3] (for the study of compactness properties of minimizing sequences of integral functionals defined on the Sobolev space $\wp$, $p > 1$) and an argument of Nonlinear Functional Analysis: the Ekeland variational principle.

Thus, given a minimizing sequence, we are able to construct a second minimizing sequence, close to the first one, such that the sequence of the gradients converges, with respect to the convergence in measure, to a vector field $Y(x) \in L^1$; $Y(x)$ can

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be seen as a "generalized" minimum of the functional and a "generalized" solution (loosely related to the definition of "solution généralisée" given in [8]) of the Euler-Lagrange equation.

General cases where Y is a gradient of a function belonging to $W_0^{1,1}(\Omega)$ are an open problem. A slightly particular case where we are able to solve this open problem is studied in the last section.

2. Setting

We study integral problems where the functional is defined by

$$J(v) = \int_\Omega j(x, \nabla v) - \int_\Omega fv, \quad v \in W_0^{1,1}(\Omega), \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, and $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function

$$\begin{cases} \text{measurable with respect to } x \in \Omega, \text{ for every } \xi \in \mathbb{R}^N, \\ \text{convex, continuous with respect to } \xi \text{ in } \mathbb{R}^N \text{ (for almost every } x \text{ in } \Omega). \end{cases} \tag{2}$$

We assume that there exist a function $h(x) \in L^1(\Omega)$ and two real positive constants α, β such that for almost every x in Ω, for every $\xi \in \mathbb{R}^N$, we have

$$\alpha|\xi| \leq j(x, \xi), \tag{3}$$

$$j(x, \xi) \leq \beta|\xi| + h(x), \tag{4}$$

$$f \in L^N(\Omega), \quad \text{with } \|f\|_{L^N(\Omega)} < \alpha\, \mathcal{S}, \tag{5}$$

Here $\mathcal{S}$ is the Sobolev constant for $W_0^{1,1}(\Omega)$, that is, with $1^\star = N/(N-1)$,

$$\mathcal{S} = \inf_{w \in W_0^{1,1}(\Omega) \setminus \{0\}} \frac{\|w\|_{W_0^{1,1}(\Omega)}}{\|w\|_{L^{1^\star}(\Omega)}}.$$

We recall the following definition.

Definition 2.1. A sequence $(y_n)_{n\in\mathbb{N}}$ of functions on Ω, measurable with respect to the Lebesgue measure μ, is said to *converge in measure* to a measurable function y if for each $A > 0$

$$\lim_{n\to\infty} \mu(\{x \in \Omega : |y_n(x) - y(x)| \geq A\}) = 0.$$

3. Compactness with respect to the convergence in measure

Lemma 3.1. *Assume* (3), (4), (5). *Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence of J on $W_0^{1,1}(\Omega)$. Then, up subsequences, there exists $u_* \in L^{\frac{N}{N-1}}(\Omega)$ such that for $N > 1$,*

$$u_n \textit{ converges to } u_* \textit{ weakly in } L^{\frac{N}{N-1}}(\Omega), \textit{ in } L^q(\Omega),\ q < \tfrac{N}{N-1} \textit{ and a.e.} \tag{6}$$

and, for $N = 1$,

$$u_n \textit{ converges to } u_*\ \star\textit{-weakly in } L^\infty(\Omega), \textit{ in } L^q(\Omega),\ q < +\infty \textit{ and a.e.}. \tag{7}$$

Proof. Thanks to (4) and the first part of (5), $J(v)$ is well defined on $W_0^{1,1}(\Omega)$. Moreover the Sobolev inequality and the assumption (3) imply the following inequality

$$J(v) \geq \left(\alpha - \frac{\|f\|_{L^N(\Omega)}}{\mathcal{S}}\right)\|v\|_{W_0^{1,1}(\Omega)}. \tag{8}$$

Thanks to the second part of (5), the above inequality says that the infimum of J on $W_0^{1,1}(\Omega)$ is a real positive number. The sequence $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence of J in $W_0^{1,1}(\Omega)$:

$$\left(\int_\Omega j(x, \nabla u_n) - \int_\Omega f\, u_n\right) \to I = \inf_{v\in W_0^{1,1}(\Omega)} J(v). \tag{9}$$

Note that (9) and the inequality (8) imply that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,1}(\Omega)$; then there exists $R > 0$ such that

$$\|u_n\|_{W_0^{1,1}(\Omega)} \leq R. \tag{10}$$

The above a priori estimate and Rellich's theorem imply that, up subsequences, there exists $u_* \in L^{\frac{N}{N-1}}(\Omega)$ satisfying (6) for $N > 1$ and (7) for $N = 1$.

Moreover, we point out that $u_* \in BV(\Omega)$. □

Now we also assume an assumption slightly stronger than (2) (see (11) below) and we prove the following lemma (the kernel of this paper).

Lemma 3.2. *Assume* (2), (3), (4), (5) *and*

$$j(x, \xi) \text{ is strictly convex with respect to } \xi, \text{ for a.e. } x \in \Omega. \tag{11}$$

Let $(u_n)_{n\in\mathbb{N}}$ and $u_\star$ be given by Lemma 3.1. Then the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the convergence in measure.

Proof. We observe that the convexity of the function $j(x, \xi)$ with respect to ξ and the definition of infimum I give

$$\left[\begin{aligned} I &\leq \int_\Omega j\Big(x, \frac{\nabla u_n + \nabla u_m}{2}\Big) - \int_\Omega f\Big(\frac{u_n + u_m}{2}\Big) \\ &\leq \frac{1}{2}\Big(\int_\Omega j(x, \nabla u_n) - \int_\Omega f\, u_n\Big) + \frac{1}{2}\Big(\int_\Omega j(x, \nabla u_m) - \int_\Omega f\, u_m\Big) \end{aligned}\right. \tag{12}$$

Both members of the right hand side converge to $\frac{1}{2}I$. Furthermore, thanks to the weak (or $\star$-weak for $N = 1$) convergence in $L^{\frac{N}{N-1}}(\Omega)$ of u_p to $u_\star$ as $p \to +\infty$,

$$\lim_{n\to+\infty} \int_\Omega f\, u_n = \lim_{m\to+\infty} \int_\Omega f\, u_m = \int_\Omega f\, u_*.$$

Then, for every $\delta > 0$, there exists $\bar{\nu}(\delta) > 0$ such that $n, m > \bar{\nu}(\delta)$ imply

$$\int_\Omega \Big\{\frac{1}{2} j(x, \nabla u_m) + \frac{1}{2} j(x, \nabla u_n) - j\Big(x, \frac{\nabla u_m + \nabla u_n}{2}\Big)\Big\} < \delta. \tag{13}$$

Let λ and ϵ be positive numbers.

We want to prove that there exists $\nu(\epsilon,\lambda)>0$ such that

$$m,\, n>\nu(\epsilon,\lambda)\Rightarrow \mu(\{|\nabla u_n-\nabla u_m|>\lambda\})<\epsilon. \tag{14}$$

First of all, there exists $k>0$ such that

$$\mu(\{|\nabla u_n|>k\})<\epsilon,\ \mu(\{|\nabla u_m|>k\})<\epsilon,$$

uniformly w.r.t. n, m, thanks to the $W_0^{1,1}(\Omega)$ bound on the sequence $(u_n)_{n\in\mathbb{N}}$. Now we define

$$\begin{cases}\mathcal{K}=\{(\xi,\eta)\in\mathbb{R}^{2N}:|\xi|\le k,\ |\eta|\le k,\ |\xi-\eta|\ge\lambda\},\\ \gamma(x)=\displaystyle\min_{(\xi,\eta)\in\mathcal{K}}\Big[\frac{1}{2}\,j(x,\xi)+\frac{1}{2}\,j(x,\eta)-j\Big(x,\frac{\xi+\eta}{2}\Big)\Big],\end{cases}$$

$$A_{n,m}=\{|\nabla u_n|\le\ k\}\cap\{|\nabla u_m|\le\ k\}\cap\{|\nabla u_n-\nabla u_m|\ge\lambda\}.$$

Thanks to (13), one has, for $n,m>\bar\nu(\delta)$,

$$\int_{A_{n,m}}\gamma(x)\quad\le\quad\int_\Omega\Big\{\frac{1}{2}j(x,\nabla u_m)\ +\ \frac{1}{2}j(x,\nabla u_n)\ -\ j\Big(x,\frac{\nabla u_m+\nabla u_n}{2}\Big)\Big\}\quad<\quad\delta.$$

Now, thanks to Lemma 6.1 (where we use the strict convexity of the function $j(x,\xi)$ with respect to ξ), we can choose δ such that

$$\int_A\gamma(x)\le\delta\ \text{ implies }\ \mu(A)<\epsilon.$$

Thus

$$\mu(A_{n,m})<\epsilon\text{ if }n,m>\bar\nu=\bar\nu(\delta). \tag{15}$$

On the other hand we have

$$\{|\nabla u_n-\nabla u_m|\ge\lambda\}\subset\{|\nabla u_n|>k\}\cup\{|\nabla u_m|>k\}\cup\,A_{n,m}.$$

Then for $n,m>\bar\nu$ one has

$$\mu(\{|\nabla u_n-\nabla u_m|\ge\lambda\})\le 3\,\epsilon.$$

This means that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the convergence in measure. $\square$

Theorem 3.3. *Assume* (2), (3), (4), (5) *and* (11). *Let* $(u_n)_{n\in\mathbb{N}}$ *and* $u_\star$ *be given by Lemma* 3.1. *Then there exists a vector field* $Y(x)$ *such that the sequence* $(\nabla u_n)_{n\in\mathbb{N}}$ *converges in measure to* $Y(x)$. *Moreover* $Y(x)\in(L^1(\Omega))^N$ *and*

$$\int_\Omega j(x,Y(x))-\int_\Omega f\,u_*\le\int_\Omega j(x,\nabla v)-\int_\Omega f\,v,\quad\forall\,v\in W_0^{1,1}(\Omega). \tag{16}$$

Proof. We proved in Lemma 3.2 that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the convergence in measure. Then (see Therem 7.1, Appendix 2) there exists a vector field $Y(x)$ such that

$$\nabla u_n \text{ converges in measure to } Y(x). \tag{17}$$

Fatou's lemma (with respect to the convergence in measure) gives (note that we have $j(x, \nabla u_n) \geq 0$ a.e. for all $n \in \mathbb{N}$).

$$\int_\Omega j(x, Y(x)) \leq \liminf_{n\to+\infty} \int_\Omega j(x, \nabla u_n).$$

Then (9) gives with the weak (or $\star$-weak for $N = 1$) convergence of u_n to $u_\star$ in $L^{1^\star}(\Omega)$

$$\int_\Omega j(x, Y(x)) - \int_\Omega f\, u_* \leq \liminf_{n\to+\infty} \left(\int_\Omega j(x, \nabla u_n) - \int_\Omega f\, u_n \right) = I.$$

Then $$\int_\Omega j(x, Y(x)) - \int_\Omega f\, u_* \leq \int_\Omega j(x, \nabla v) - \int_\Omega f\, v, \quad \forall\, v \in W_0^{1,1}(\Omega).$$

Note that $$\int_\Omega j(x, Y(x)) - \int_\Omega f\, u_* \leq I$$

implies $$\int_\Omega j(x, Y(x)) \leq I + \int_\Omega f\, u_*$$

and $$\alpha \int_\Omega |Y(x)| \leq I + \int_\Omega f\, u_*.$$

Then we deduce that $$Y \in L^1(\Omega)^N. \tag{18}$$

The proof is complete. □

Remark 3.4. In some sense, we can say that, in (16), (u_*, Y) is a generalized minimum.

Remark 3.5. Under the assumptions of Theorem 3.3, we point out that, at this moment, we are not able to say that $Y = \nabla u_*$.

4. Euler-Lagrange equation

Let $(\bar{u}_n)_{n\in\mathbb{N}}$ be any minimizing sequence of J in $W_0^{1,1}(\Omega)$; recall that for the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$ we proved in Theorem 3.3 the existence of $Y \in L^1(\Omega)^N$ such that, up to a subsequence (see (17)),

$$\nabla \bar{u}_n \text{ converges in measure to the vector field } Y.$$

Then the minimizing sequence $(u_n)_{n\in\mathbb{N}}$ built after $(\bar{u}_n)_{n\in\mathbb{N}}$, using Ekeland's variational principle (see [4]), satisfies:

$$J(u_n) \to I,$$

$$\|\bar{u}_n - u_n\|_{W_0^{1,1}(\Omega)} \to 0, \tag{19}$$

$$\|u_n - u_*\|_{L^q(\Omega)} \to 0, \quad \forall\, q < \frac{N}{N-1}. \tag{20}$$

$$\begin{cases} \displaystyle\int_\Omega j(x,\nabla u_n) - \int_\Omega f\, u_n \le \int_\Omega j(x,\nabla v) - \int_\Omega f\, v + \sqrt{\frac{1}{n}} \int_\Omega |\nabla(u_n - v)|, \\ \forall\, v \in W_0^{1,1}(\Omega). \end{cases} \tag{21}$$

Note that (19) and (17) imply

$$\begin{cases} \nabla u_n \ \text{ converges in measure to } \ Y, \\ u_n \text{ converges to } u_* \text{ weakly in } L^{\frac{N}{N-1}}(\Omega) \text{ if } N > 1, \\ u_n \text{ converges to } u_* \ \star\text{-weakly in } L^\infty(\Omega) \text{ if } N = 1. \end{cases} \tag{22}$$

From now on, we consider a slightly less general case of J:

$$\begin{cases} \displaystyle\mathcal{J}(v) = \int_\Omega a(x)\sqrt{1 + |\nabla v|^2} - \int_\Omega f v, \\ a(x) \ \text{ is a measurable function such that } 0 < \alpha \le a(x) \le \gamma, \end{cases} \tag{23}$$

with $\alpha, \gamma \in \mathbb{R}^+$.

Theorem 4.1. *We assume* (2), (3), (4), (5) *and* (23). *Then, the couple* (u_*, Y), $u_* \in L^{\frac{N}{N-1}}(\Omega)$, $Y \in (L^1(\Omega))^N$ *given by Theorem* 3.3 *is a generalized solution of the Euler-Lagrange equation of the functional* J *in the following meaning*

$$\int_\Omega a(x) \frac{Y(x)\,\nabla v(x)}{\sqrt{1 + |Y(x)|^2}} - \int_\Omega f(x)\, v(x) = 0, \quad \forall\, v \in W_0^{1,1}(\Omega). \tag{24}$$

Proof. We first remark here that we can apply Theorem 3.3 since the function $\xi \mapsto \sqrt{1 + |\xi|^2}$ is strictly convex (from $\mathbb{R}^N$ to $\mathbb{R}$).

To prove the fact that (u_*, Y) is a generalized solution of the Euler-Lagrange equation of the functional J, our approach is standard in the use of the Ekeland's variational principle: in (21) we take $v = u_n - t\, w$, with $t \in \mathbb{R}^+$, $w \in W_0^{1,1}(\Omega)$, and we have

$$\int_\Omega j(x, \nabla u_n) \le \int_\Omega j(x, \nabla[u_n - t\, w]) - \int_\Omega f\,[-t\, w] + \sqrt{\frac{1}{n}} \int_\Omega |\nabla(t\, w)|$$

After dividing by $t > 0$, the limit as $t \to 0$ gives

$$\left| \int_\Omega a(x) \frac{\nabla u_n\, \nabla v(x)}{\sqrt{1 + |\nabla u_n|^2}} - \int_\Omega f(x)\, v(x) \right| \le \sqrt{\frac{1}{n}} \int_\Omega |\nabla v|, \quad \forall\, v \in W_0^{1,1}(\Omega). \tag{25}$$

Then the limit as $n \to \infty$, joint with the Lebesgue's theorem and the results of Theorem 3.3, gives (24). □

Remark 4.2. In some sense, we can say that, in (24), Y is a generalized solution of the Euler-Lagrange equation. Moreover this concept is loosely related with the definition of "solution généralisée" given in [8].

5. Functionals with nearly linear growth

In Remark 3.5, we said that we are not able to prove in general that $Y = \nabla u_\star$. Nevertheless, below, we prove that $Y = \nabla u_\star$ if the growth of $j(x,\xi)$ is more than linear with respect to ξ. We consider here the functional

$$\bar{J}(v) = \int_\Omega \left(\sqrt{1+|\nabla v|^2} - 1\right) \log(1+|\nabla v|) - \int_\Omega f(x)\, v(x), \tag{26}$$

which has a "nearly linear growth". The function j defined by

$$j(\xi) = (\sqrt{1+|\xi|^2} - 1\Big) \log(1+|\xi|)$$

does not satisfy (4) (so that we may have $\bar{J}(v) = +\infty$ for some $v \in W_0^{1,1}(\Omega)$) but it is quite easy to prove Theorem 3.3 with $\bar{J}$ instead of J.

Let $(u_n)_{n\in\mathbb{N}}$, Y and $u_\star$ as in Theorem 3.3 (with $\bar{J}$ instead of J). Even if the growth of $\bar{J}$ is nearly linear, it is possible to prove $Y = \nabla u_\star$.

Again we prove (17) and we have (like in (9)), as $n \to +\infty$,

$$\int_\Omega \left(\sqrt{1+|\nabla u_n|^2} - 1\right) \log(1+|\nabla u_n|) - \int_\Omega f(x)\, u_n(x) \to\ I,$$

which implies the existence of C_1 such that, for all $n \in \mathbb{N}$,

$$\int_\Omega \sqrt{1+|\nabla u_n|^2}\ \log(1+|\nabla u_n|) \le C_1.$$

It is classical that these two informations give, thanks to the Vitali theorem, the convergence in L^1 of the sequence $(\nabla u_n)_{n\in\mathbb{N}}$. Indeed the result follows by (17) and by the equi-integrability.

For every measurable subset E of Ω and for every $t \in \mathbb{R}^+$, we have

$$\begin{aligned}\int_E |\nabla u_n| &\le t\,\mu(E) + \frac{1}{\log(1+t)} \int_\Omega \sqrt{1+|\nabla u_n|^2}\ \log(1+|\nabla u_n|)\\ &\le t\,\mu(E) + \frac{C_1}{\log(1+t)},\end{aligned}$$

which implies $$\lim_{\mu(E)\to 0} \int_E |\nabla u_n| \le \frac{C_1}{\log(1+t)}, \quad \forall\, t \in \mathbb{R}^+.$$

That is $$\lim_{\mu(E)\to 0} \int_E |\nabla u_n| = 0, \quad \text{uniformly with respect to } n. \tag{27}$$

Since $\nabla u_n \to Y$ in measure as $n \to +\infty$, we deduce from (27) the convergence in L^1 of the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ to Y. Then, from (6) we deduce that $Y = \nabla u_*$.

6. Appendix 1

Lemma 6.1. *Let (X,μ) be a measurable space, μ a positive measure with $\mu(X)<+\infty$, and*

$$\gamma : X \longrightarrow [0,+\infty]$$

a measurable function such that $\mu(\{x\in X:\gamma(x)=0\})=0$. Then, for every $\epsilon>0$ there exists $\delta>0$ such that the statement

$$A \text{ a measurable subset of } X, \quad \int_A \gamma(x)\,d\mu(x)\le\delta,$$

implies $\mu(A)\le\epsilon$.

Proof. For every $r>0$, we have

$$r\,\mu(\{A\cap\{x:\ r<\gamma(x)\}\}) = \int_{A\cap\{x:\, r<\gamma(x)\}} r\,d\mu \le \int_A \gamma(x)\,d\mu(x).$$

Then
$$\mu\big(\{A\cap\{x:\ r<\gamma(x)\}\}\big)\le\frac{1}{r}\int_A\gamma(x)\,d\mu(x). \tag{28}$$

On the other hand, since the sequence $\{x\in X:\gamma(x)\le\dfrac{1}{m}\}$ is decreasing, we have

$$\mu\Big(\Big\{x:\gamma(x)\le\frac{1}{m}\Big\}\Big)\to\mu\big(\{x:\gamma(x)=0\}\big)=0 \ \text{ as } \ m\to+\infty. \tag{29}$$

Fix $\epsilon>0$. Thus, there exists m_ϵ such that

$$\mu\Big(\Big\{x:\gamma(x)\le\frac{1}{m_\epsilon}\Big\}\Big)\le\frac{\epsilon}{2}.$$

Then, with $r=\dfrac{1}{m_\epsilon}$

$$\begin{aligned}\mu(A)&\le\mu\Big(A\cap\Big\{x:\gamma(x)>r\Big\}\Big)+\mu\Big(\Big\{x:\gamma(x)\le\frac{1}{m_\epsilon}\Big\}\Big)\\&\le\frac{1}{r}\int_A\gamma(x)\,d\mu(x)+\frac{\epsilon}{2}.\end{aligned}$$

We choose $\delta=r\dfrac{\epsilon}{2}$ to conclude. □

7. Appendix 2

We recall that Ω is a bounded open subset of $\mathbb{R}^N$, $N\ge1$. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of measurable functions from Ω to $\mathbb{R}^\nu$ with respect to the convergence in measure (for the Lebesgue measure). In this appendix, we want to prove the existence of a measurable function y such that y_n converges in measure to y as $n\to+\infty$. Reasoning with each component of y_n, there is no restriction to assume that $\nu=1$.

We recall that y_n converges in measure to y (as $n \to +\infty$) if and only (since the Lebesgue measure of Ω is finite)

$$\int_\Omega \frac{|y_n - y|}{1+|y_n - y|} \to 0, \quad \text{as} \quad n \to +\infty,$$

and that
$$d_M(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|}$$

defines a distance on $\widetilde{M}(\Omega)$, the quotient space of measurable functions on Ω with respect to the relation " $= a.e.$".

A less known fact is the completeness of the space $(\widetilde{M}(\Omega), d_M(f,g))$. We will present a proof of this completeness in Theorem 7.1 below; thanks to it we obtain the existence of a measurable function y such that y_n converges in measure to y as $n \to +\infty$.

Theorem 7.1. *The metric space $(\widetilde{M}(\Omega), d_M(f,g))$ is complete.*

Proof. Remember that we work with the Lebesgue measure μ. We will use the inequality

$$\left|\frac{t}{1+|t|} - \frac{s}{1+|s|}\right| \le 2\,\frac{|t-s|}{1+|t-s|}, \quad t,\ s \in \mathbb{R}. \tag{30}$$

Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the convergence in measure, i.e.:

$$\forall\ \epsilon > 0,\ \exists\ n_\epsilon \in \mathbb{N} : \int_\Omega \frac{|y_n(x) - y_m(x)|}{1+|y_n(x) - y_m(x)|} \le \epsilon, \quad \forall\ n, m \ge n_\epsilon. \tag{31}$$

So that, by (30)

$$\forall\ \epsilon > 0\ \exists\ n_\epsilon \in \mathbb{N} : \int_\Omega \left|\frac{y_n(x)}{1+|y_n(x)|} - \frac{y_m(x)}{1+|y_m(x)|}\right| \le 2\,\epsilon,\ \forall\ n, m \ge n_\epsilon, \tag{32}$$

which means that the sequence $(\frac{y_n}{1+|y_n|})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$, which is complete. Then there exists $g \in L^1(\Omega)$ such that $\frac{y_n}{1+|y_n|}$ converges (as $n \to +\infty$) in $L^1(\Omega)$ to g. The convergence in L^1 implies the convergence in measure, so that $\frac{y_n}{1+|y_n|}$ converges in measure to g (and we can choose g such that g is measurable and everywhere defined).

Up to subsequences, we can also assume that $\frac{y_n}{1+|y_n|}$ converges a.e. to g (so that $g(x) \in [-1,1]$ for a.e. $x \in \Omega$).

Since the function $s \mapsto \frac{s}{1+|s|}$ is an increasing bijection between $\mathbb{R}$ and $]-1,1[$, one has for all $x \in \Omega$ such $\frac{y_n(x)}{1+|y_n(x)|}$ converges to $g(x)$ (and then for a.e. $x \in \Omega$)

$$g(x) = 1 \quad \text{if and only if } y_n(x) \to +\infty \text{ as } n \to +\infty,$$

$$g(x) = -1 \quad \text{if and only if } y_n(x) \to -\infty \text{ as } n \to +\infty.$$

We now use Fatou's lemma in (31) with m fixed and $n \to +\infty$; it gives for $m \geq n_\epsilon$

$$\int_{x \in \Omega : g(x)=1} 1 \leq \epsilon,$$

and then as $\epsilon \to 0$, $\qquad \mu\{x \in \Omega : g(x) = 1\} = 0.$

Similarly $\qquad \mu\{x \in \Omega : g(x) = -1\} = 0,$

so that $-1 < g < 1$ a.e.

We can now define the measurable function y from Ω to $\mathbb{R}$ setting

$$y(x) = \frac{g(x)}{1 - |g(x)|}$$

if $\frac{y_n(x)}{1 + |y_n(x)|}$ converges to $g(x)$ and $g(x) \neq \pm 1$, elsewhere we can take $y(x) = 0$.

We then obtain $y_n \to y$ a.e. as $n \to +\infty$.

Now we pass to the limit (with Lebesgue's theorem) as $m \to \infty$ in (31) and we deduce that

$$\forall\, \epsilon > 0,\ \exists\, n_\epsilon \in \mathbb{N} : \int_\Omega \frac{|y_n(x) - y(x)|}{1 + |y_n(x) - y(x)|} \leq \epsilon, \quad \forall\, n \geq n_\epsilon,$$

that is the convergence in measure of y_n to y as $n \to +\infty$.

Finally, a standard argument by contradiction gives the convergence in measure of y_n to y as $n \to +\infty$ without the extraction of a subsequence. $\square$

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