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A Unified View of Polarity for Functions

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We propose a unified view of the polarity of functions, that encompasses all specific definitions, generalizes several well-known properties and provides new results. We show that bipolar sets and bipolar functions are anti-isomorphic complete lattices. Also, we explore three possible notions of polar subdifferential associated with a nonnegative function, and we make the connection with the notion of alignement of vectors.

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1. Introduction

The introduction of [18, Chapter 11, Sect. E] reads as follows: *While most of the major duality correspondences, like convex sets versus sub-linear functions, or polarity of convex cones, fit directly within the framework of conjugate convex functions as in 11.4, others, like polarity of convex sets that aren't necessarily cones but contain the origin, fit obliquely.* We feel that this "obliquely" has to do with the many different ways one finds to define the polar of a function in the literature, especially by restricting the definitions to special classes of functions. In this paper, we propose a unified view of the polarity of functions, that encompasses all specific definitions, generalizes several well-known properties and provides new results, especially an anti-isomorphism between the complete lattices of bipolar sets and of bipolar functions.

Given a pair $(\mathcal{X},\mathcal{Y})$ of (real) vector spaces equipped with a bilinear functional $\langle\,,\rangle\colon \mathcal{X}\times\mathcal{Y}\to\mathbb{R}$, the polar operation is defined without ambiguity over subsets (see [16, Section 14], [18, Chapter 11, Sect. E], [2, §5.16]). The (negative, or one-sided) polar set of a subset $X\subset\mathcal{X}$ is the closed convex set

$$X^{\odot}=\{y\in\mathcal{Y}\mid\langle x,y\rangle\leq 1\,,\ \forall x\in X\}.$$

The situation is not as clear-cut for the polar operation defined over functions. We present the main approaches now.

The polar of a function $f\colon\mathbb{R}^n\to\overline{\mathbb{R}}$ is defined in [16, Section 15], entitled *Polars of convex functions*, as follows: first, for nonnegative positively 1-homogeneous convex functions vanishing at the origin (so-called gauges) in [16, p. 128] by the formula $f^{\circ}(y)=\inf\{\lambda\geq 0\mid\langle x,y\rangle\leq\lambda f(x),\forall x\in\mathbb{R}^n\}$ and, second, extended to

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nonnegative convex functions vanishing at the origin in [16, p. 136] by the formula $f^\circ(y) = \inf\{\lambda \geq 0 \mid \langle x, y\rangle \leq 1 + \lambda f(x), \forall x \in \mathbb{R}^n\}$. It is shown that the two definitions are equivalent for gauges, and that, if $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is nonnegative convex and vanishes at the origin, then the bipolar $f^{\circ\circ}$ is the greatest nonnegative lsc (lower semicontinuous) convex function vanishing at the origin which is majorized by f ($f^{\circ\circ} = \mathrm{cl} f$). Other formulas are related to polarity, like the transform $\mathcal{A}$ for nonnegative convex lsc functions vanishing at zero [4, Equation (2) and after], with expression $(\mathcal{A}f)(y) = \sup_{x\in\mathbb{R}^n}(\langle x, y\rangle - 1)/f(x)$, and the perspective-polar transform, for closed proper convex nonnegative functions [3, Equation (4.1)] (see other examples in [12, Remark 5.1]). In the formulas defining polar functions, and in derived formulas, one is often embarrassed with the treatment of 0 and $+\infty$ values, especially in fraction terms. For instance, the expression $f^\circ(y) = \sup_{x\neq 0}\langle x, y\rangle/f(x)$ in [16, p. 128] is valid when f is finite everywhere and positive except at the origin. By contrast, there is no problem with the treatment of 0 and $+\infty$ values with the way that the polar operation is defined on nonnegative functions in [12, §4] by the formula $f^\circ(y) = \sup_{x\in\mathcal{X}}\big(\langle x, y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f(x))^{-1}\big)$ (we will explain the $\mathbin{\underset{\cdot}{\times}}$ later). To our knowledge, this is the most general formula as it is defined for any nonnegative function (without restriction to positively 1-homogeneous, or convex or vanishing at zero).

In this paper, we propose a unified view of the polarity of functions by proposing a definition of the polar of any function, and then revisit the polar operation on functions as one would do following the tracks of the Fenchel transform. Indeed, the Fenchel transform is defined for any function, and then closed convex lsc functions (we follow the terminology in [17, p. 15]) appear as the class of functions that are equal to their biconjugate (bi-Fenchel transform). We do the same by defining the polar transfom for any function, and then bipolar functions will be defined as those equal to their bipolar transform.

The paper is organized as follows. In Section 2, we present set polarity and the Minkowski functional. We also introduce bipolar sets and show that they form a complete lattice. In Section 3, we develop the polar operation on functions. First, in §3.1, we follow the approach taken in [12, §4]: we recall the definition of the polar of any nonnegative function (the embarrassement with 0 and $+\infty$ values is systematically handled by means of lower $\mathbin{\underset{\cdot}{\times}}$ and upper $\mathbin{\dot{\times}}$ multiplications); we recall that the polar operation is a $\times$-duality as in [12, §4]; we show that the polar of any nonnegative function is a support function (of the polar set of the 0-level set of the Fenchel conjugate). Second, in §3.2, we provide a definition of the polar of any function, we present several properties and we show that the polar of any function is a Minkowski functional (of the 0-level set of the Fenchel conjugate). In Section 4, we present equivalent expressions of the set of bipolar functions, that is, those equal to their bipolar. We show that the complete lattices of bipolar sets and of bipolar functions are anti-isomorphic. In Section 5, we add to [12, §4] by exploring three possible notions of polar subdifferential associated with a nonnegative function. We make the connection with the notion of alignement of vectors. In Appendix A, we study best lsc convex lower approximations of a function. In Appendix B, we provide background on *-dualities, as defined and studied in [12].

2. Set polarity and the Minkowski functional

In §2.1, we recall notions related to functions[1] and to the Fenchel conjugacy. In §2.2, we provide background on set polarity, and we also introduce the notion of bipolar set, and show that bipolar sets form a complete lattice. In §2.3, we provide background on 1-homogeneous functions and on the Minkowski functional.

2.1. Background on functions

We denote $\overline{\mathbb{R}} = [-\infty, +\infty]$, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_{++} =]0, +\infty[$. The *positive part* z_+ of a real number z is $z_+ = \max\{z, 0\}$. For any set $\mathcal{W}$ and any function $h \colon \mathcal{W} \to \overline{\mathbb{R}}$, we introduce different possible notations for

$$
\begin{aligned}
&\text{the } \textit{level sets} && \{h \le r\} = h^{\le r} = \{w \in \mathcal{W} \mid h(w) \le r\}\,,\ \forall r \in \overline{\mathbb{R}}\,, && \text{(1a)}\\
&\text{the } \textit{strict level sets} && \{h < r\} = h^{< r} = \{w \in \mathcal{W} \mid h(w) < r\}\,,\ \forall r \in \overline{\mathbb{R}}\,, && \text{(1b)}\\
&\text{the } \textit{level curves} && \{h = r\} = h^{= r} = \{w \in \mathcal{W} \mid h(w) = r\}\,,\ \forall r \in \overline{\mathbb{R}}\,, && \text{(1c)}\\
&\text{and also} && \{s < h < r\} = \{w \in \mathcal{W} \mid s < h(w) < r\}\,,\ \forall s, r \in \overline{\mathbb{R}}\,. && \text{(1d)}
\end{aligned}
$$

For any function $h \colon \mathcal{W} \to \overline{\mathbb{R}}$, its *epigraph* is $\mathrm{epi} h = \{(w, t) \in \mathcal{W} \times \mathbb{R} \mid h(w) \le t\}$, its *strict epigraph* is $\mathrm{epi}_s h = \{(w, t) \in \mathcal{W} \times \mathbb{R} \mid h(w) < t\}$, its *effective domain* is $\mathrm{dom}\, h = \{w \in \mathcal{W} \mid h(w) < +\infty\}$. A function $h \colon \mathcal{W} \to \overline{\mathbb{R}}$ is said to be *convex* if its epigraph is a convex set, *proper* if it never takes the value $-\infty$ and that $\mathrm{dom}\, h \neq \emptyset$, *lower semi continuous (lsc)* if its epigraph is closed.

For any set $\mathcal{W}$ and subset $W \subset \mathcal{W}$, we denote by $\iota_W \colon \mathcal{W} \to \overline{\mathbb{R}}$ the *indicator function* of the set W, defined by $\iota_W(w) = 0$ if $w \in W$, and $\iota_W(w) = +\infty$ if $w \notin W$. We denote by $\chi_W \colon \mathcal{W} \to \overline{\mathbb{R}}_+$ the *generalized indicator function* of the set W [12, Definition 2.2], defined by $\chi_W(w) = 1$ if $w \in W$, and $\chi_W(w) = +\infty$ if $w \notin W$. Thus, we have that $\chi_W = \iota_W + 1$.

2.2. Set polarity

In §2.2.1, we recall the notions of dual pair and paired vector spaces. In §2.2.2, we recall the definition of the polar of a set, and we introduce the notion of bipolar set. In §2.2.3, we show that bipolar sets form a complete lattice.

2.2.1. Dual pair, paired vector spaces, Fenchel conjugacy

We refer the reader to [2] and [17] for the following backgrounds. We consider a pair $(\mathcal{X}, \mathcal{Y})$ of (real) vector spaces equipped with a bilinear functional $\langle\,,\rangle \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$. Following [17, p. 13], we say that $\mathcal{X}$ and $\mathcal{Y}$ are *paired spaces*, when they have been equipped with topologies that are compatible with respect to the pairing (hence Hausdorff and locally convex topologies). Details on how to generate consistent (or compatible) topologies [2, Definition 5.96] from a pairing is developed in [2, §5.15]. More precisely, $\mathcal{X}$ and $\mathcal{Y}$ is called a *dual pair* in [2, Definition 5.90] when the bilinear functional $\langle\,,\rangle$ separates the points of $\mathcal{X}$ and $\mathcal{Y}$. Then, it is proved in

[1] Adopting usage in mathematics, we follow Serge Lang and use "function" only to refer to mappings in which the codomain is a set of numbers (i.e. a subset of $\mathbb{R}$ or $\mathbb{C}$, or their possible extensions with $\pm\infty$), and reserve the term mapping for more general codomains.

[2, Theorem 5.93] that, from a dual pair $\mathcal{X}$ and $\mathcal{Y}$, one makes paired spaces when $\mathcal{X}$ (resp. $\mathcal{Y}$) is equipped with the weak topology $\sigma(\mathcal{X},\mathcal{Y})$ (resp. with the weak$*$ topology $\sigma(\mathcal{Y},\mathcal{X})$).

Now, we review concepts and notations related to the Fenchel conjugacy (we refer the reader to [17, Sect. 3]). We consider $\mathcal{X}$ and $\mathcal{Y}$ two paired vector spaces. For any functions $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ and $g\colon \mathcal{Y} \to \overline{\mathbb{R}}$, the different conjugates are defined by[2]

$$f^{\star}(y) = \sup_{x\in\mathcal{X}}\big(\langle x,y\rangle - f(x)\big)\,,\ \forall y \in \mathcal{Y}\,, \tag{2a}$$

$$g^{\star'}(x) = \sup_{y\in\mathcal{Y}}\big(\langle x,y\rangle - g(y)\big)\,,\ \forall x \in \mathcal{X}\,, \tag{2b}$$

$$f^{\star\star'}(x) = \sup_{y\in\mathcal{Y}}\big(\langle x,y\rangle - f^{\star}(y)\big)\,,\ \forall x \in \mathcal{X}\,. \tag{2c}$$

We also recall the classic *(Rockafellar-Moreau) subdifferential*

$$\partial_{\star} f(x) = \{y \in \mathcal{Y} \mid f^{\star}(y) = \langle x,y\rangle - f(x)\}\,,\ \forall x \in \operatorname{dom} f\,. \tag{2d}$$

A function $f : \mathcal{X} \to \overline{\mathbb{R}}$, or $g\colon \mathcal{Y} \to \overline{\mathbb{R}}$, is said to be *closed*[3] if it is either lsc and nowhere having the value $-\infty$, or is the constant function $-\infty$. Closed convex functions are the two constant functions $-\infty$ and $+\infty$ united with all proper convex lsc functions[4]. It is proved that the Fenchel conjugacy – indifferently $f \mapsto f^{\star}$ or $g \mapsto g^{\star'}$ – induces a one-to-one correspondence between the closed convex functions and themselves [17, Theorem 5].

2.2.2. Support function, polar of a set, bipolar set

For any subset $X \subset \mathcal{X}$, we denote by $\mathrm{co}X$ (or $\mathrm{co}(X)$) the *convex hull* of X – that is, the smallest convex set in $\mathcal{X}$ containing X – by $\overline{\mathrm{co}}X$ (or $\overline{\mathrm{co}}(X)$) the *closed convex hull* of X – that is, the smallest closed convex set in $\mathcal{X}$ containing X. A subset $K \subset \mathcal{X}$ is said to be a *cone*[5] if $\mathbb{R}_{++}K \subset K$. For any subset[6] $Y \subset \mathcal{Y}$, $\sigma_Y\colon \mathcal{X} \to \overline{\mathbb{R}}$ denotes the *support function of the set* Y – defined by $\sigma_Y(x) = \sup_{y\in Y}\langle x,y\rangle$, for any $x \in \mathcal{X}$ [2, §7.10, p. 288]. In the same way, we define $\sigma_X\colon \mathcal{Y} \to \overline{\mathbb{R}}$ for any subset $X \subset \mathcal{X}$.

The (negative) *polar cone* $X^{\ominus}$ of the subset $X \subset \mathcal{X}$ is the closed convex cone [5, p. 122, Equation (6.28)]

$$X^{\ominus} = \{y \in \mathcal{Y} \mid \langle x,y\rangle \le 0\,,\ \forall x \in X\} = \{\sigma_X \le 0\}\,, \tag{3a}$$

[2] In convex analysis, one does not use $^{\star'}$ and $^{\star\star'}$, but simply * and **. We use $^{\star\star'}$ to be consistent with the notation for general conjugacies.

[3] We follow the terminology in [17, p. 15], although it can be misleading. Indeed, anticipating on the notion of valley function [15], a function taking the value $-\infty$ on a closed subset (neither the empty set nor the whole set) and $+\infty$ outside is lsc but not closed. Some authors [6] use closed in the sense of lsc.

[4] In particular, any closed convex function that takes at least one finite value is necessarily proper convex lower semi continuous. Notice that a function taking the value $-\infty$ on a closed convex subset (neither the empty set nor the whole set) and $+\infty$ outside is convex lsc, but is not closed convex (see Footnote 3).

[5] Hence, a cone does not necessarily contain the origin 0.

[6] We use the letter X for a primal subset, and the letter x for a primal vector. We use the letter Y for a dual subset, and the letter y for a dual vector.

and the same definition holds for $Y \subset \mathcal{Y}$, so that we define

$$X^{\ominus\ominus} = \left(X^{\ominus}\right)^{\ominus} . \tag{3b}$$

The (negative) or (one-sided) *polar set* $X^{\odot}$ of the subset $X \subset \mathcal{X}$ is the closed convex set [2, Definition 5.101, p. 216]

$$X^{\odot} = \{y \in \mathcal{Y} \mid \langle x, y\rangle \leq 1\,,\ \forall x \in X\} = \{\sigma_X \leq 1\}\,, \tag{4a}$$

and the same definition holds for $Y \subset \mathcal{Y}$. For any subset $Y \subset \mathcal{Y}$, the effective domain of the support function σ_Y is $\mathbb{R}_{++}Y^{\odot}$.

We define the (negative) or (one-sided) *bipolar set* $X^{\odot\odot}$ of the subset $X \subset \mathcal{X}$ as the closed convex set

$$X^{\odot\odot} = \left(X^{\odot}\right)^{\odot}\,, \tag{4b}$$

and the same definition holds for $Y \subset \mathcal{Y}$. By the bipolar Theorem [2, Theorem 5.103], we have that

$$X^{\odot\odot} = \overline{\mathrm{co}}(X \cup \{0\})\,. \tag{5}$$

We have not found the following definition of bipolar set in the literature, as most authors simply say that a set is closed convex and contains 0. However, putting a name on this well-known notion will be quite practical for our purposes, especially for the connection with bipolar functions. The three equivalences below are a straightforward consequence of the bipolar Theorem. The notion of polar pair can be found in [16, Theorem 14.6, p. 126].

Definition 2.1. A subset $X \subset \mathcal{X}$ is said to be a *bipolar set* if any of the following three equivalent conditions is satisfied:

1. X is closed convex and contains 0,
2. $X = X^{\odot\odot}$, that is, X is equal to its bipolar,
3. there exists a nonempty set $Z \subset \mathcal{X}$ such that $X = Z^{\odot\odot}$, that is, X is equal to the bipolar of Z.

The same definition holds for a subset $Y \subset \mathcal{Y}$. We denote by $\mathcal{B}p[\mathcal{X}]$ the set of bipolar sets of $\mathcal{X}$.

Let $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ be two subsets. We say that X and Y form a *polar pair* if $Y = X^{\odot}$ and $X = Y^{\odot}$. The two elements of a polar pair are necessarily bipolar sets.

As an example, given a subset $X \subset \mathcal{X}$, we easily check that $X^{\odot}$ is a bipolar set – as $X^{\odot\odot\odot} = X^{\odot}$ follows from (5) – and that $X^{\odot\odot}$ and $X^{\odot}$ form a polar pair.

We will need the following properties.

Proposition 2.2. *Let $\{X_i\}_{i\in I}$ be a family of (primal) subsets of $\mathcal{X}$ (or a family of (dual) subsets of $\mathcal{Y}$). We have that*

$$\Big(\bigcup_{i\in I} X_i\Big)^{\odot} = \bigcap_{i\in I} X_i^{\odot}\,, \tag{6a}$$

$$X_i \text{ is a bipolar set for all } i \in I \implies \Big(\bigcap_{i\in I} X_i\Big)^{\odot} = \overline{\mathrm{co}}\,\Big(\bigcup_{i\in I} X_i^{\odot}\Big)\,. \tag{6b}$$

Proof. Equation (6a) easily follows from the definition (4a) of a (negative) or (one-sided) polar set (see also [2, Item 3, Lemma 5.102]).

Suppose that X_i is a bipolar set for all $i \in I$. Then, Equation (6b) follows from

$$\big(\bigcap_{i\in I} X_i\big)^{\odot} = \big(\bigcap_{i\in I} {X_i}^{\odot\odot}\big)^{\odot}$$

(as, for all $i \in I$, X_i is a bipolar set, hence $X_i = {X_i}^{\odot\odot}$ by Item 2 of Definition 2.1)

$$= \Big(\big(\bigcup_{i\in I} {X_i}^{\odot}\big)^{\odot}\Big)^{\odot} \qquad \text{(by (6a))}$$

$$= \big(\bigcup_{i\in I} {X_i}^{\odot}\big)^{\odot\odot} \qquad \text{(by definition of } Z^{\odot\odot}\text{)}$$

$$= \overline{\mathrm{co}}\big(\bigcup_{i\in I} {X_i}^{\odot} \cup \{0\}\big) \qquad \text{(by the bipolar Theorem in (5))}$$

$$= \overline{\mathrm{co}}\big(\bigcup_{i\in I} {X_i}^{\odot}\big)\ . \qquad \text{(as the polar set } {X_i}^{\odot} \text{ contains 0, for all } i \in I\text{)}$$

This ends the proof. □

2.2.3. The lattice of bipolar sets

The following Proposition 2.3 is easy to show. To our knowledge, it is new (see [2, p. 291–292] that points out that the set of weak$*$ compact convex subsets is a lattice).

Proposition 2.3. *The set of bipolar sets of $\mathcal{X}$, when ordered by inclusion $\subset$, is a complete lattice $(\mathcal{B}p[\mathcal{X}], \wedge, \vee)$ with bottom $\{0\}$ and with top $\mathcal{X}$. The greatest lower bound $\vee$ and the least upper bound $\wedge$ operations are given, for any family $\{X_j\}_{j\in J}$ of bipolar sets of $\mathcal{X}$, by*

$$\bigwedge_{j\in J} X_j = \bigcap_{j\in J} X_j\ , \tag{7a}$$

$$\bigvee_{j\in J} X_j = \overline{\mathrm{co}}\big(\bigcup_{j\in J} X_j\big)\ . \tag{7b}$$

Proof. First, we prove (7a). Let X be a bipolar set such that $X \subset X_j$ for all $j \in J$ (the set $\{0\}$ is always a possibility). Then it is immediate that $X \subset \bigcap_{j\in J} X_j$ and (7a) follows as $\bigcap_{j\in J} X_j$ is a bipolar set, being closed convex and containing 0 as the intersection of closed convex sets containing 0 (Item 1 of Definition 2.1).

Second, we prove (7b). Let X be a bipolar set such that $X_j \subset X$ for all $j \in J$ (the set $\mathcal{X}$ is always a possibility). Then we have $\bigcup_{j\in J} X_j \subset X$ and, as X is closed convex (by Item 1 of Definition 2.1), we obtain that $\bigcup_{j\in J} X_j \subset \overline{\mathrm{co}}(\bigcup_{j\in J} X_j) \subset X$ by the very definition of the closed convex hull. Now, $\overline{\mathrm{co}}(\bigcup_{j\in J} X_j)$ is closed convex and contains 0 as all X_j contain 0. Using Item 1 of Definition 2.1, we immediately obtain that $\overline{\mathrm{co}}(\bigcup_{j\in J} X_j)$ is a bipolar set and (7b) follows.

Obviously, $\{0\}$ is the bottom and $\mathcal{X}$ is the top of the lattice. □

2.3. Background on the Minkowski functional

We define 1-homogeneous functions, present the Minkowski functional and some of their properties.

Definition 2.4. Let $\mathcal{X}$ be a (real) vector space, and $K \subset \mathcal{X}$ be a nonempty cone. We say that a function $f \colon K \to \overline{\mathbb{R}}$ is *(strictly positively)*[7] 1*-homogeneous*, or homogeneous of degree 1, (on the cone K) if

$$f(\lambda x) = \lambda f(x) \, , \ \forall \lambda \in \mathbb{R}_{++} \, , \ \forall x \in K \, . \tag{8}$$

Following [16, Section 15, p. 130], and as recalled in Sect. 1, a function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is said to be a *gauge* if it is nonnegative (strictly positively) 1-homogeneous convex and vanishing at zero ($f(0) = 0$). By [2, Theorem 7.51, p. 288] and [2, Definition 5.45, p. 190], the support function of a nonempty set is convex, lsc, (strictly positively) 1-homogeneous and vanishes at the origin, hence is a lsc gauge.

For a nonnegative function, we will need the following result (although it is well-known that a function is (strictly positively) 1-homogeneous if and only if its epigraph is a cone if and only if its strict epigraph is a cone, we give a proof and an explicit expression of the cone in Item 2).

Proposition 2.5. *Let $K \subset \mathcal{X}$ be a nonempty cone and $f \colon K \to \overline{\mathbb{R}}_+$ be a function. The following statements are equivalent:*

(1) *the function $f \colon K \to \overline{\mathbb{R}}_+$ is (strictly positively)* 1*-homogeneous,*

(2) *the strict epigraph of f has the expression* $\mathrm{epi}_s f = \mathbb{R}_{++}((K \cap \{f < 1\}) \times \{1\})$,

(3) *the strict epigraph of f is a cone included in $K \times \mathbb{R}_{++}$.*

Proof. (1) We prove that Item 1 implies Item 2. We have that

$$(x, \alpha) \in \mathrm{epi}_s f \iff x \in K \text{ and } f(x) < \alpha \text{ and } \alpha > 0$$

(by definition of the strict epigraph of f, and using the assumption that $f \geq 0$, hence that $\alpha > 0$)

$$\iff x \in K \text{ and } \frac{1}{\alpha} f(x) = f(\frac{x}{\alpha}) < 1 \text{ and } \alpha > 0$$

(by the assumption that the function f is (strictly positively) 1-homogeneous in Item 1)

$$\iff \frac{x}{\alpha} \in K \cap \{f < 1\} \text{ and } \alpha > 0 \, , \qquad \text{(as } K \text{ is a cone)}$$

$$\iff (\frac{x}{\alpha}, 1) \in (K \cap \{f < 1\}) \times \{1\} \text{ and } \alpha > 0 \, ,$$

$$\iff (x, \alpha) \in \mathbb{R}_{++}(K \cap \{f < 1\}) \times \{1\} \, .$$

(2) It is straightforward that Item 2 implies Item 3.

[7] The definition of homogeneous function is not stabilized in the literature. For instance, in [2, §5.8, p. 190], a real function defined on a cone is positively homogeneous if Equation (8) holds for all $\lambda \in \mathbb{R}_+$ (thus including $\lambda = 0$); in [16, p. 30], a function on $\mathbb{R}^n$ is positively homogeneous if Equation (8) holds for all $\lambda \in \mathbb{R}_{++}$ (thus excluding $\lambda = 0$). This is why, we prefer to avoid all ambiguity and speak of (strictly positively) when $\lambda \in \mathbb{R}_{++}$. We may sometimes omit the (strictly positively) and speak of a 1-homogeneous function.

(3) We prove that Item 3 implies Item 1. For any $x \in K$ and $\alpha \in \mathbb{R}_{++}$, we have

$$\begin{aligned} f(\alpha x) &= \inf_{(\alpha x,t)\in \mathrm{epi}_s f} t = \inf_{(x,t/\alpha)\in \frac{1}{\alpha}\mathrm{epi}_s f} t \\ &= \alpha \inf_{(x,t/\alpha)\in \mathrm{epi}_s f} \frac{t}{\alpha} \qquad \text{(as } \tfrac{1}{\alpha}\mathrm{epi}_s f = \mathrm{epi}_s f \text{ since this latter is a cone)} \\ &= \alpha \inf_{(x,t')\in \mathrm{epi}_s f} t' = \alpha f(x) \; . \end{aligned}$$

This ends the proof. □

We follow [2, Definition 5.48, p. 191] to introduce Minkowski functionals.[8]

Definition 2.6. Let $X \subset \mathcal{X}$. The *Minkowski functional* associated with the subset $X \subset \mathcal{X}$ is the function $m_X \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ defined by (with the convention that $\inf \emptyset = +\infty$)

$$m_X(x) = \inf\{\lambda > 0 \mid x \in \lambda X\} \; , \; \forall x \in \mathcal{X} \; . \tag{9}$$

We will need the following properties.

Proposition 2.7. *Let $X \subset \mathcal{X}$.*

(1) *The Minkowski functional $m_X \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is a nonnegative (strictly positively) 1-homogeneous function.*

(2) *The strict epigraph $\mathrm{epi}_s m_X$ of the Minkowski functional m_X is the cone*

$$\mathrm{epi}_s m_X = \mathbb{R}_{++}(\{m_X < 1\} \times \{1\}) \; . \tag{10}$$

(3) *For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, we have the implication*

$$\mathrm{epi}_s f = \mathbb{R}_{++}(X \times \{1\}) \implies f = m_X \; . \tag{11}$$

(4) *Conversely, any nonnegative (strictly positively) 1-homogeneous function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is the Minkowski functional m_X of a subset $X \subset \mathcal{X}$, which can be chosen as the subset $X = \{f < 1\}$, that is,*

$$f = m_{\{f<1\}} \; . \tag{12a}$$

When $f(0) = 0$, we have that

$$f = m_{\{f<1\}} = m_{\{f\leq 1\}} \; . \tag{12b}$$

(5) *The Minkowski functional satisfies*

$$m_{(\bigcup_{i\in I} X_i)} = \inf_{i\in I} m_{X_i} \; , \tag{13a}$$

for any family $\{X_i\}_{i\in I}$ of subsets of $\mathcal{X}$,

$$\mathrm{dom}(m_X) = \mathbb{R}_{++}X \; , \tag{13b}$$

$$X \textit{ is a convex set} \implies m_X \textit{ is a convex function} \; . \tag{13c}$$

[8] Also called gauges in [2, Definition 5.48]. The definition of gauge is not stabilized in the literature. For instance, in [16, Section 15, p. 130], gauges are functions of the form $k(x) = \inf\{\lambda \geq 0 \mid x \in \lambda X\}$ for some nonempty convex set X. In [20, p. 4], Minkowski gauges are defined like in (9) but with $\lambda \geq 0$ and for an absorbing set X.

Proof.

(1) Item 1 is well-known and easy to prove.

(2) Item 2 is implied by Item 1 as a consequence of Proposition 2.5 (more precisely, a consequence of the fact that Item 1 implies Item 2 in Proposition 2.5).

(3) We prove Item 3. We consider a subset $X \subset \mathcal{X}$ and a function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ such that $\mathrm{epi}_s f = \mathbb{R}_{++}(X \times \{1\})$. For any $x \in \mathcal{X}$, we have that

$$
\begin{aligned}
f(x) &= \inf_{(x,t)\in \mathrm{epi}_s f} t = \inf_{(x,t)\in \mathbb{R}_{++}(X\times\{1\})} t && \text{(by assumption)}\\
&= \inf_{t>0, x\in tX} t && \text{(as } (x,t) \in \mathbb{R}_{++}(X \times \{1\}) \iff t > 0 \text{ and } x \in tX)\\
&= m_X(x) . && \text{(by definition (9) of the Minkowski functional)}
\end{aligned}
$$

(4) We prove Item 4. Let $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ be a nonnegative (strictly positively) 1-homogeneous function. The implication of Item 2 by Item 1 in Proposition 2.5 gives that (with $K = \mathcal{X}$)

$$
\mathrm{epi}_s f = \mathbb{R}_{++}(\{f < 1\} \times \{1\}) ,
$$

hence (12a) follows from implication (11), proved in Item 3 of this very Proposition 2.7.

Now, the epigraph of the nonnegative (strictly positively) 1-homogeneous function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is given by

$$
\mathrm{epi} f = (\{f = 0\} \times \{0\}) \cup \mathbb{R}_{++}(\{f \leq 1\} \times \{1\}) . \tag{14}
$$

Indeed, we have that

$$
(x, \alpha) \in \mathrm{epi} f \iff f(x) \leq \alpha \text{ and } \alpha \geq 0,
$$

(by definition of the strict epigraph of f, and using the assumption that $f \geq 0$)

$$
\iff \begin{cases} \text{either} & \alpha = 0 \text{ and } f(x) = 0, \text{ as } 0 \leq f(x) \leq 0 , \\ \text{or} & \frac{1}{\alpha} f(x) = f(\frac{x}{\alpha}) \leq 1 \text{ and } \alpha > 0, \text{ as the function } f \\ & \text{is (strictly positively) 1-homogeneous by assumption in Item 1,} \end{cases}
$$

$$
\iff \begin{cases} \text{either} & (x, \alpha) \in (\{f = 0\} \times \{0\}) , \\ \text{or} & (x, \alpha) \in \mathbb{R}_{++}(\{f \leq 1\} \times \{1\}) , \end{cases}
$$

$$
\iff (x, \alpha) \in \{f = 0\} \times \{0\} \cup \mathbb{R}_{++}(\{f \leq 1\} \times \{1\}) .
$$

Thus, we have proved (14). As a consequence, for any $x \in \mathcal{X}$, we get that

$$
\begin{aligned}
f(x) &= \inf_{(x,t)\in \mathrm{epi} f} t\\
&= \inf\{\inf_{(x,t)\in\{f=0\}\times\{0\}} t, \inf_{(x,t)\in\mathbb{R}_{++}(\{f\leq1\}\times\{1\})} t\} && \text{(by (14))}\\
&= \inf\{\inf_{(x,t)\in\{f=0\}\times\{0\}} t, m_{\{f\leq1\}}(x)\}
\end{aligned}
$$

by definition (9) of the Minkowski functional (see the details in the proof of Item 3)

$$= \begin{cases} \inf\{\inf_{t=0} t, m_{\{f\leq 1\}}(x)\} & \text{if } x \in \{f = 0\} \ , \\ \inf\{\inf_{(x,t)\in\emptyset} t, m_{\{f\leq 1\}}(x)\} & \text{if } x \notin \{f = 0\} \ , \end{cases}$$

$$= \begin{cases} \inf\{0, m_{\{f\leq 1\}}(x)\} = 0 & \text{if } x \in \{f = 0\}, \text{ as } m_{\{f\leq 1\}}(x) \geq 0 \ , \\ \inf\{+\infty, m_{\{f\leq 1\}}(x)\} = m_{\{f\leq 1\}}(x) & \text{if } x \notin \{f = 0\} \ . \end{cases}$$

To prove (12b), there remains to show that $f(x) = 0 \implies m_{\{f\leq 1\}}(x) = 0$.

Now, for any $x \in \{f = 0\}$ and any $\alpha > 0$, we have that $x \in \alpha\{f \leq 1\}$ – because $f(\frac{x}{\alpha}) = \frac{1}{\alpha} f(x) = 0 \leq 1$ – and we deduce that $m_{\{f\leq 1\}}(x) = \inf \mathbb{R}_{++} = 0$.

We conclude that $f(x) = 0 = m_{\{f\leq 1\}}(x)$. Thus, we have obtained (12b).

(5) The proofs of Item 5 are left to the reader (see [2, Lemmas 5.49, 5.50]).

This ends the proof. □

3. Polar operation on functions

In this Sect. 3, we consider $\mathcal{X}$ and $\mathcal{Y}$ two (real) vector spaces that are paired (see §2.2.1). In §3.1, we define the polar of any nonnegative function, and study properties of the polar operation. In §3.2, we define the polar of any function, and study properties of the polar operation.

3.1. Polar operation on nonnegative functions

In §3.1.1, we provide background on upper and lower multiplications. In §3.1.2, we follow [12] to define the polar (transform) of any nonnegative function, and we recall that, thus defined, the polarity operation is a $\times$-duality. Then, in §3.1.3, we provide several results about the polar of nonnegative functions, some well-known (and scattered in the literature) and some new. Finally, in §3.1.4, we provide examples of polar transforms as support functions.

3.1.1. Background on upper and lower multiplications

Following Appendix B, we consider $\overline{\mathbb{R}}_+ = [0, +\infty]$ as the canonical enlargement $(\overline{\mathbb{R}}_+, \leq, \dot{\times}, \underset{\cdot}{\times})$ of the complete totally ordered group $(\mathbb{R}_{++}, \leq, \times)$ with the two elements 0 and $+\infty$ by [12, §4], that is, $\overline{\mathbb{R}}_+ = \mathbb{R}_{++} \cup \{0\} \cup \{+\infty\}$ with order extended by $0 \leq \alpha \leq +\infty$, for all $\alpha \in \overline{\mathbb{R}}_+$, and with *upper multiplication* $\dot{\times}$ and *lower multiplication* $\underset{\cdot}{\times}$ given by (see [14, Equations (14.8)–(14.9)], [12, Equations (1.4)–(1.8)])

$$\alpha \dot{\times} \beta = \alpha \underset{\cdot}{\times} \beta = \alpha \times \beta \ , \ \forall \alpha, \beta \in \mathbb{R}_{++} \ , \tag{15a}$$

$$(+\infty) \dot{\times} \alpha = \alpha \dot{\times} (+\infty) = +\infty \ , \ \forall \alpha \in \overline{\mathbb{R}}_+ \ , \tag{15b}$$

$$0 \dot{\times} \alpha = \alpha \dot{\times} 0 = 0 \ , \ \forall \alpha \in \mathbb{R}_{++} \cup \{0\} \ , \tag{15c}$$

$$(+\infty) \underset{\cdot}{\times} \alpha = \alpha \underset{\cdot}{\times} (+\infty) = +\infty \ , \ \forall \alpha \in \mathbb{R}_{++} \cup \{+\infty\} \ , \tag{15d}$$

$$0 \underset{\cdot}{\times} \alpha = \alpha \underset{\cdot}{\times} 0 = 0 \ , \ \forall \alpha \in \overline{\mathbb{R}}_+ \ , \tag{15e}$$

and the inverse operation extended as

$$0^{-1} = +\infty \ , \ (+\infty)^{-1} = 0 \ . \tag{15f}$$

Both upper and lower multiplications are associative and commutative [12, Remark 1.2], and isotone in the following sense

$$\beta \leq \gamma \implies \alpha \dot{\times} \beta \leq \alpha \dot{\times} \gamma \,, \; \alpha \underset{\cdot}{\times} \beta \leq \alpha \underset{\cdot}{\times} \gamma \,. \tag{15g}$$

3.1.2. Definition of the polar of a nonnegative function

We follow [12, §4,§5 B)] (see the background in §B.2) to define the polar (transform) of any nonnegative function.

Definition 3.1. For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, the *polar (transform)* $f^\circ \colon \mathcal{Y} \to \overline{\mathbb{R}}_+$ of the function f is defined by

$$f^\circ(y) = \sup_{x \in \mathcal{X}} \big(\langle x, y \rangle_+ \underset{\cdot}{\times} (f(x))^{-1} \big) \,, \; \forall y \in \mathcal{Y} \,. \tag{16a}$$

For any function $g \colon \mathcal{Y} \to \overline{\mathbb{R}}_+$, the *reverse polar transform* $g^{\circ'} \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ of the function g is defined by

$$g^{\circ'}(x) = \sup_{y \in \mathcal{Y}} \big(\langle x, y \rangle_+ \underset{\cdot}{\times} (g(y))^{-1} \big) \,, \; \forall x \in \mathcal{X} \,. \tag{16b}$$

For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, the *bipolar transform* $f^{\circ\circ'} \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ of the function f is defined by[9]

$$f^{\circ\circ'} = (f^\circ)^{\circ'} \,. \tag{16c}$$

The following Proposition 3.2 is a direct application of [12, §4]. Surprisingly, the Inequality (17c) is not stated in [12].

Proposition 3.2. ([12, Theorem 4.1]) *The polarity mapping* $\overline{\mathbb{R}}_+^{\mathcal{X}} \ni f \mapsto f^\circ \in \overline{\mathbb{R}}_+^{\mathcal{Y}}$ *is a* $\times$*-duality, that is, it satisfies* [12, Definition 2.3]:

$$(\inf_{i \in I} f_i)^\circ = \sup_{i \in I} f_i^\circ \,, \; \forall \{f_i\}_{i \in I} \subset \overline{\mathbb{R}}_+^{\mathcal{X}} \,, \tag{17a}$$

$$(f \dot{\times} \alpha)^\circ = f^\circ \underset{\cdot}{\times} \alpha^{-1} \,, \; \forall \alpha \in \overline{\mathbb{R}}_+ \,, \; \forall f \in \overline{\mathbb{R}}_+^{\mathcal{X}} \,, \tag{17b}$$

$$f^{\circ\circ'} \leq f \,, \; \forall f \in \overline{\mathbb{R}}_+^{\mathcal{X}} \,. \tag{17c}$$

Proof. We follow [12, §4] and the background in §B.2. We define the coupling c between $\mathcal{X}$ and $\mathcal{Y}$ by $c(x,y) = \langle x, y \rangle_+$, for all $x \in \mathcal{X}, y \in \mathcal{Y}$, and also (see [12, Equation (4.11)] and Equation (76a))

$$f^{\mathcal{D}(c)}(y) = \sup_{x \in \mathcal{X}} \big(\langle x, y \rangle_+ \underset{\cdot}{\times} (f(x))^{-1} \big) \,, \; \forall y \in \mathcal{Y} \,. \tag{18}$$

By [12, Theorem 4.1], the mapping $\overline{\mathbb{R}}_+^{\mathcal{X}} \ni f \mapsto f^{\mathcal{D}(c)} \in \overline{\mathbb{R}}_+^{\mathcal{Y}}$ is a $\times$-duality, that is, it satisfies (17) (which corresponds to [12, Equations (2.7)-(2.8)]). The last Inequation (17c) follows from Equation (78) in §B.2. □

[9] We adopt the notation $f^{\circ\circ'}$, and not $f^{\circ\circ}$, to be consistent with the notation for general conjugacies (see also Footnote 2).

3.1.3. Polar transform as a support function

The following Proposition 3.3 gathers properties of the polar transform of a nonnegative function.

1. Item 1 (polar inequality) is not stated in [12] (although it can be easily deduced); it is established in [16, p. 130], but only for functions that are themselves gauges, and for vectors in the respective domains.
2. Item 2 – expressing the polar transform of a nonnegative function $f\colon \mathcal{X} \to \overline{\mathbb{R}}_+$ as the support function $\sigma_{\{f^\star \leq 0\}^{\odot}}$ of the polar set $\{f^\star \leq 0\}^{\odot}$ of the 0-level set $\{f^\star \leq 0\}$ of the Fenchel conjugate[10] $f^\star$ – is stated neither in [12] nor [16] (the set $\{f^\star \leq 0\}$ appears in [16, Theorem 13.5, p. 118], [16, Theorem 14.3, p. 123]).
3. Item 3 is not stated in [12] (although it can be deduced from the proof of [12, Theorem 5.2] which, however, lacks some details). It is established in [16, Theorem 15.1, p. 128], but only for functions that are themselves gauges.
4. Item 4 is stated and proved in [12, Corollary 4.1].
5. As for Item 2, we suspect that Item 5 is new.
6. Finally, Item 6 is related to [16, Theorem 15.4, p. 137], but our assumptions are weaker.

Proposition 3.3. *For any function $f\colon \mathcal{X} \to \overline{\mathbb{R}}_+$, the following statements hold true.*

(1) *Polar inequality* $$\langle x, y \rangle \leq f(x) \dot{\times} f^\circ(y) \, , \ \forall x \in \mathcal{X} \, , \ \forall y \in \mathcal{Y} \, . \tag{19}$$

(2) *The following set*
$$X_f = \{0\} \cup \mathbb{R}_+\{f = 0\} \ \cup \bigcup_{x \in \{0 < f < +\infty\}} \Big\{ \frac{x}{f(x)} \Big\} \, , \tag{20a}$$
is such that
$$\overline{\mathrm{co}} X_f = \{f^\star \leq 0\}^{\odot} \, , \tag{20b}$$
where $\{f^\star \leq 0\}$ is a bipolar set. The polar transform f° is a support function as follows
$$f^\circ = \sigma_{X_f} = \sigma_{\{f^\star \leq 0\}^{\odot}} \, . \tag{21}$$

(3) *The polar transform f° is convex lsc (strictly positively) 1-homogeneous and vanishes at the origin ($f^\circ(0) = 0$) – that is, the function f° is a lsc gauge, with effective domain* $\mathrm{dom} f^\circ = \mathbb{R}_{++}\{f^\star \leq 0\}$.

(4) *The polar transform f° is also given as an infimum by*
$$f^\circ(y) = \inf\{\lambda \in]0, +\infty[\mid \langle x, y \rangle_+ \leq \lambda f(x) \, , \ \forall x \in \mathcal{X}\} \, , \ \forall y \in \mathcal{Y} \, , \tag{22a}$$
$$= \inf\{\lambda \in]0, +\infty[\mid \langle x, y \rangle \leq \lambda f(x) \, , \ \forall x \in \mathcal{X}\} \, , \ \forall y \in \mathcal{Y} \, . \tag{22b}$$

(5) *The bipolar transform $f^{\circ\circ'}\colon \mathcal{X} \to \overline{\mathbb{R}}_+$ satisfies*[11]
$$f^{\circ\circ'} = \sigma_{\{f^\star \leq 0\}} \, . \tag{23}$$
As a consequence, the bipolar transform $f^{\circ\circ'}\colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is convex lsc (strictly positively) 1-homogeneous and vanishes at the origin ($f^{\circ\circ'}(0) = 0$) – that is, the function $f^{\circ\circ'}$ is a lsc gauge.

[10] The classic Fenchel conjugacy $f \mapsto f^\star$ is outlined in §2.2.1.

[11] Equation (23) is valid even if f is not proper, that is, even when $f \equiv +\infty$.

(6) *If* $0 \in \operatorname{dom} f$, *the bipolar transform* $f^{\circ\circ'} : \mathcal{X} \to \overline{\mathbb{R}}_+$ *is the greatest (strictly positively)* 1*-homogeneous proper convex lsc function below* f.

Proof. We consider a function $f : \mathcal{X} \to \overline{\mathbb{R}}_+$.

As a preliminary result, observe that

$$0 \in \{f^\star \leq 0\} \Leftrightarrow \sup_{x \in \mathcal{X}} (\langle x, 0 \rangle - f(x)) \leq 0 \Leftrightarrow \sup_{x \in \mathcal{X}} (-f(x)) \leq 0 \Leftrightarrow f \geq 0 \; . \tag{24}$$

(1) We prove (19). If $\langle x, y \rangle_+ = 0$, the inequality (19) is satisfied. Thus, we suppose that $\langle x, y \rangle_+ \in \mathbb{R}_{++}$. For any $x \in \mathcal{X}$, $y \in \mathcal{Y}$, we have that

$$f(x) \dot{\times} f^\circ(y) \geq f(x) \dot{\times} \big(\langle x, y \rangle_+ \underset{\cdot}{\times} (f(x))^{-1} \big)$$

(by definition (16a) of the polar transform f° and by isotony (15g) of the upper multiplication $\dot{\times}$)

$$\begin{aligned}
&= f(x) \dot{\times} \big(\langle x, y \rangle_+ \dot{\times} (f(x))^{-1} \big) && \text{(because } \langle x, y \rangle_+ \in \mathbb{R}_{++}) \\
&= f(x) \dot{\times} (f(x))^{-1} \dot{\times} \langle x, y \rangle_+ && \text{(by associativity and commutativity of } \dot{\times} \text{)} \\
&\geq 1 \dot{\times} \langle x, y \rangle_+ && \text{(because } f(x) \dot{\times} (f(x))^{-1} \in \{1, +\infty\}) \\
&\geq \langle x, y \rangle \; . && \text{(as } \langle x, y \rangle_+ = \sup(\langle x, y \rangle, 0) \geq \langle x, y \rangle)
\end{aligned}$$

(2) We first prove that $f^\circ = \sigma_{X_f}$ in (21).

By definition (16a) of the polar transform f°, we have that

$$f^\circ = \sup_{x \in \mathcal{X}} \big(\langle x, \cdot \rangle_+ \underset{\cdot}{\times} (f(x))^{-1} \big)$$

where $\langle x, \cdot \rangle$ denotes the continuous linear form $\mathcal{Y} \ni y \mapsto \langle x, y \rangle$,

$$= \sup \Big(\sup_{x \in \{f = +\infty\}} \langle x, \cdot \rangle_+ \underset{\cdot}{\times} 0, \sup_{x \in \{0 < f < +\infty\}} \langle x, \cdot \rangle_+ \times (f(x))^{-1}, \sup_{x \in \{f = 0\}} \langle x, \cdot \rangle_+ \underset{\cdot}{\times} (+\infty) \Big)$$

where we have used (15f), (15b),

$$= \sup \Big(\sup_{x \in \{f = +\infty\}} 0, \sup_{x \in \{0 < f < +\infty\}} \Big\langle \frac{x}{f(x)}, \cdot \Big\rangle_+, \sup_{x \in \{f = 0\}} \langle x, \cdot \rangle_+ \underset{\cdot}{\times} (+\infty) \Big)$$

where we have used (15e). In the above expression with three terms, the middle term is

$$\begin{aligned}
\sup_{x \in \{0 < f < +\infty\}} \Big\langle \frac{x}{f(x)}, \cdot \Big\rangle_+ &= \sup_{x \in \{0 < f < +\infty\}} \sup(0, \Big\langle \frac{x}{f(x)}, \cdot \Big\rangle) \\
&= \sup(\sup_{x \in \{0 < f < +\infty\}} 0, \sup_{x \in \{0 < f < +\infty\}} \Big\langle \frac{x}{f(x)}, \cdot \Big\rangle) \; .
\end{aligned}$$

In the last term, we have that (by (15d), (15e))

$$\langle x, y \rangle_+ \underset{\cdot}{\times} (+\infty) = \begin{cases} 0 & \text{if } \langle x, y \rangle \leq 0 \\ +\infty & \text{if } \langle x, y \rangle > 0 \end{cases} = \iota_{\{\langle x, \cdot \rangle \leq 0\}}(y) = \sigma_{\mathbb{R}_+ x}(y) \; ,$$

so that the last term can be rewritten as

$$\sup_{x\in\{f=0\}} \langle x,\cdot\rangle_+ \dot{\times} (+\infty) = \sup_{x\in\{f=0\}} \sigma_{\mathbb{R}_+ x} = \sigma_{\bigcup_{x\in\{f=0\}} \mathbb{R}_+ x} = \sigma_{\mathbb{R}_+ \bigcup_{x\in\{f=0\}} x} = \sigma_{\mathbb{R}_+\{f=0\}} \ .$$

Thus, finally, we have obtained that

$$\begin{aligned}
f^\circ &= \sup\Big(\sup_{x\in\{f=+\infty\}} 0, \sup\big(\sup_{x\in\{0<f<+\infty\}} 0, \sup_{x\in\{0<f<+\infty\}} \Big\langle \frac{x}{f(x)},\cdot \Big\rangle\big), \sigma_{\mathbb{R}_+\{f=0\}}\Big) \\
&= \sup\Big(\sup_{x\in\{f=+\infty\}} 0, \sup_{x\in\{0<f<+\infty\}} 0, \sup_{x\in\{0<f<+\infty\}} \Big\langle \frac{x}{f(x)},\cdot \Big\rangle, \sigma_{\mathbb{R}_+\{f=0\}}\Big) \\
&= \sup\Big(\sup_{x\in\{0<f\}} 0, \sup_{x\in\{0<f<+\infty\}} \Big\langle \frac{x}{f(x)},\cdot \Big\rangle, \sigma_{\mathbb{R}_+\{f=0\}}\Big) \\
&\qquad\qquad (\text{as } \{0<f\} = \{f=+\infty\} \cup \{0<f<+\infty\}) \\
&= \begin{cases}
\sup\Big(0, \sup_{x\in\{0<f<+\infty\}} \Big\langle \frac{x}{f(x)},\cdot \Big\rangle, \sigma_{\mathbb{R}_+\{f=0\}}\Big) & \text{if } \{0<f\} \neq \emptyset \ , \\
= \sigma_{\underbrace{\{0\} \cup \bigcup_{x\in\{0<f<+\infty\}} \{\frac{x}{f(x)}\} \cup \mathbb{R}_+\{f=0\}}_{X_f}} & \\
\sup\big(-\infty, -\infty, \sigma_{\mathbb{R}_+\{f=0\}}\big) & \text{if } \{0<f\} = \emptyset \ (\text{as } \{0<f<+\infty\} \subset \{0<f\} = \emptyset) \\
= \sigma_{\underbrace{\mathbb{R}_+\{f=0\}}_{X_f}} \ , &
\end{cases} \\
&= \sigma_{X_f} \text{ where } X_f \text{ is given by (20a).}
\end{aligned}$$

Thus, we have shown that $f^\circ = \sigma_{X_f}$, which is the left hand side equality in (21). The right hand side equality in (21) is a consequence of (20b), that we are going to prove now.

We have

$$\begin{aligned}
(\overline{\mathrm{co}} X_f)^\odot &= X_f{}^\odot \qquad \text{(by definition (4a) of a (negative) or (one-sided) polar set)} \\
&= \Big(\{0\} \cup \mathbb{R}_+\{f=0\} \cup \bigcup_{x\in\{0<f<+\infty\}} \{\frac{x}{f(x)}\}\Big)^\odot \\
&\qquad\qquad \text{(by definition (20a) of the set } X_f) \\
&= \{0\}^\odot \cap (\mathbb{R}_+\{f=0\})^\odot \cap \bigcap_{x\in\{0<f<+\infty\}} \{\frac{x}{f(x)}\}^\odot \qquad \text{(by (6a))} \\
&= \mathcal{Y} \cap \{f=0\}^\ominus \cap \bigcap_{x\in\{0<f<+\infty\}} \{\frac{x}{f(x)}\}^\odot \quad \text{(by the polar cone definition (3a))} \\
&= \bigcap_{x\in\{f=0\}} \{y\in\mathcal{Y} \mid \langle x,y\rangle \leq 0\} \cap \bigcap_{x\in\{0<f<+\infty\}} \{y\in\mathcal{Y} \mid \langle \frac{x}{f(x)},y\rangle \leq 1\} \\
&= \bigcap_{x\in\{f=0\}} \{y\in\mathcal{Y} \mid \langle x,y\rangle \leq f(x)\} \cap \bigcap_{x\in\{0<f<+\infty\}} \{y\in\mathcal{Y} \mid \langle x,y\rangle \leq f(x)\} \\
&= \bigcap_{x\in\{f<+\infty\}} \{y\in\mathcal{Y} \mid \langle x,y\rangle \leq f(x)\} = \bigcap_{x\in\mathcal{X}} \{y\in\mathcal{Y} \mid \langle x,y\rangle \leq f(x)\}
\end{aligned}$$

$$= \{y\in\mathcal{Y} \mid \langle x,y\rangle - f(x) \leq 0\,,\ \forall x\in\mathcal{X}\} = \{y\in\mathcal{Y} \mid \sup_{x\in\mathcal{X}}(\langle x,y\rangle - f(x)) \leq 0\}$$
$$= \{y\in\mathcal{Y} \mid f^{\star}(y) \leq 0\} = \{f^{\star} \leq 0\}\,.$$

As the set $\overline{\mathrm{co}}X_f$ is closed convex and contains 0, it is a bipolar set (by Item 1 of Definition 2.1) and we deduce, using the bipolar Theorem expressed in (5), that

$$\overline{\mathrm{co}}X_f = (\overline{\mathrm{co}}X_f)^{\odot\odot} = \{f^{\star} \leq 0\}^{\odot}\,,$$

which exactly is (20b). As we have shown that $f^{\circ} = \sigma_{X_f}$, the right hand side equality in (21) follows from $\sigma_{X_f} = \sigma_{\overline{\mathrm{co}}X_f} = \sigma_{\{f^{\star}\leq 0\}^{\odot}}$.
We also have that $\{f^{\star} \leq 0\}$ is a bipolar set: indeed, it is closed convex as a level set of a closed convex function and $0 \in \{f^{\star} \leq 0\}$ by (24), and we conclude with Item 1 of Definition 2.1.

(3) By the just proven Item 2, the polar transform f° is the support function $\sigma_{\{f^{\star}\leq 0\}^{\odot}}$ of a nonempty set, hence it is convex, lsc, (strictly positively) 1-homogeneous and takes the value 0 at the origin ($f^{\circ}(0) = 0$), as recalled in §2.2.2. The effective domain of the support function $\sigma_{\{f^{\star}\leq 0\}^{\odot}}$ is $\mathbb{R}_{++}\{f^{\star} \leq 0\}^{\odot\odot}$, which is equal to $\mathbb{R}_{++}\{f^{\star} \leq 0\}$, using the fact that $\{f^{\star} \leq 0\}$ is a bipolar set, as proved above in Item 2.

(4) Finally, we prove (22a)–(22b). For any $y \in \mathcal{Y}$, we have that

$$\begin{aligned}
f^{\circ}(y) &= f^{\mathcal{D}(c)}(y) && \text{(by definition (16a) of the polar transform } f^{\circ} \text{ and by (18))}\\
&= \inf\{\alpha \in \overline{\mathbb{R}}_+ \mid \langle x,y\rangle_+ \overset{\times}{\cdot}\, \alpha^{-1} \leq f(x)\,,\ \forall x \in \mathcal{X}\} && \text{(by [12, Equation (4.17)])}\\
&= \inf\{\lambda \in]0,+\infty[\mid \frac{1}{\lambda}\langle x,y\rangle_+ \leq f(x)\,,\ \forall x \in \mathcal{X}\}\,, && \text{(by [12, Equation (4.17)])}\\
&= \inf\{\lambda \in]0,+\infty[\mid \langle x,y\rangle_+ \leq \lambda f(x)\,,\ \forall x \in \mathcal{X}\}\,,\\
&= \inf\{\lambda \in]0,+\infty[\mid \langle x,y\rangle \leq \lambda f(x)\,,\ \forall x \in \mathcal{X}\}\,,
\end{aligned}$$

because $\langle x,y\rangle_+ = \max\{\langle x,y\rangle, 0\}$ and $\lambda f(x) \geq 0$, as $\lambda \in]0,+\infty[$ and $f(x) \in \overline{\mathbb{R}}_+$.

(5) We prove (23) as follows:

$$\begin{aligned}
f^{\circ\circ'} &= (\sigma_{\{f^{\star}\leq 0\}^{\odot}})^{\circ'} && \text{(by definition (16c) of the bipolar transform and by (21))}\\
&= \sigma_{\{f^{\star}\leq 0\}^{\odot\odot}}
\end{aligned}$$

by (26a) proven below (there is no circularity in the reasoning, as (26a) is proven by only using (21) established before)

$$= \sigma_{\{f^{\star}\leq 0\}}\,. \qquad \text{(since } \{f^{\star} \leq 0\} \text{ is a bipolar set as seen in Item 2)}$$

The rest of the assertions in Item 5 are proven in the same way than for Item 3.

(6) Using Equation (24), we obtain that $0 \in \{f^{\star} \leq 0\}$. Thus, $\{f^{\star} \leq 0\} \neq \emptyset$ and, by assumption, we also have $0 \in \mathrm{dom}\, f$. Thus, using Proposition A.2 postponed in Appendix A, we obtain that the greatest lsc convex (strictly positively) 1-homogeneous lower approximation of f is given by $\sigma_{\{f^{\star}\leq 0\}}$. As this function is also proper (as the support function of a nonempty set), we conclude that it is also the greatest lsc proper convex (strictly positively) 1-homogeneous lower approximation of f. Now, using Equation (23), we have that $f^{\circ\circ'} = \sigma_{\{f^{\star}\leq 0\}}$ and the conclusion follows for $f^{\circ\circ'}$.
This ends the proof. □

3.1.4. Examples of polar transforms as support functions

Using Item 2 (Equation (21)) and Item 5 (Equation (23)) in Proposition 3.3, we obtain expressions of the polar transforms of nonnegative support functions, of Minkowski functionals, of indicator functions and of generalized indicator functions as support functions. Equations (26) can be deduced from [16, Corollary 15.1.2, p. 129].

Proposition 3.4.

(1) *Polar transform of a nonnegative support function as a support function.* *For any bipolar sets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$, we have that*

$$\sigma_Y^{\circ} = \sigma_{Y^{\odot}} \ , \tag{26a}$$

$$\sigma_X^{\circ'} = \sigma_{X^{\odot}} \ , \tag{26b}$$

$$\sigma_Y^{\circ\circ'} = \sigma_Y \ . \tag{26c}$$

(2) *Polar transform of a Minkowski functional as a support function.* *For any subsets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$, we have that*

$$m_X^{\circ} = \sigma_{X^{\odot\odot}} \ , \tag{27a}$$

$$m_Y^{\circ'} = \sigma_{Y^{\odot\odot}} \ , \tag{27b}$$

$$m_X^{\circ\circ'} = \sigma_{X^{\odot}} \ . \tag{27c}$$

(3) *Polar transform of an indicator function as a support function.* *For any subsets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$, we have that*

$$\iota_X^{\circ} = \sigma_{X^{\ominus\ominus}} = \iota_{X^{\ominus}} \ , \tag{28a}$$

$$\iota_Y^{\circ'} = \sigma_{Y^{\ominus\ominus}} = \iota_{Y^{\ominus}} \ , \tag{28b}$$

$$\iota_X^{\circ\circ'} = \sigma_{X^{\ominus}} = \iota_{X^{\ominus\ominus}} \ , \tag{28c}$$

(4) *Polar transform of a generalized indicator function as a support function.* *For any subsets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$, we have that*

$$\chi_X^{\circ} = \sigma_{X^{\odot\odot}} \ , \tag{29a}$$

$$\chi_Y^{\circ'} = \sigma_{Y^{\odot\odot}} \ , \tag{29b}$$

$$\chi_X^{\circ\circ'} = \sigma_{X^{\odot}} \ . \tag{29c}$$

Proof. (1) As both X and Y are bipolar sets, they both contain 0 (see Item 1 of Definition 2.1), and thus both $\sigma_X \geq 0$ and $\sigma_Y \geq 0$. As the set Y is nonempty closed convex (see Item 1 of Definition 2.1), the function ι_Y is proper closed convex. As the Fenchel conjugacy $g \mapsto g^{\star}$ induces a one-to-one correspondence between the closed convex functions on $\mathcal{Y}$ and themselves (see [17, Theorem 5] recalled in §2.2.1), we get that $\iota_Y = \iota_Y^{\star'\star} = \sigma_Y^{\star}$ as the equality $\iota_Y^{\star'} = \sigma_Y$ follows from the very definition of the support fonction σ_Y. Thus, we get that $\{\sigma_Y^{\star} \leq 0\} = \{\iota_Y \leq 0\} = Y$ and then, by (21), we obtain (26a).

Because the reverse polar transform (16b) acts like the polar transform (16a) on nonnegative functions, we obtain (26b) in the same fashion.

Finally, by definition (16c) of the bipolar transform, we apply (26a) and then (26b) with $X = Y^{\odot}$, which is a bipolar set, and get $\sigma_Y^{\circ\circ'} = (\sigma_Y^{\circ})^{\circ'} = (\sigma_{Y^{\odot}})^{\circ'} = \sigma_{Y^{\odot\odot}} = \sigma_Y$, since $Y^{\odot\odot} = Y$ as Y is a bipolar set. Thus, we have obtained (26c).

(2) The Minkowski functional in (9) associated with the subset $X \subset \mathcal{X}$ can be written as $m_X = \inf_{\lambda\in\mathbb{R}_{++}}(\lambda + \iota_{\lambda X})$, from which we obtain the Fenchel conjugate

$$
\begin{aligned}
m_X^{\star} &= \big(\inf_{\lambda\in\mathbb{R}_{++}}(\lambda + \iota_{\lambda X})\big)^{\star} \\
&= \sup_{\lambda\in\mathbb{R}_{++}}(-\lambda + \iota_{\lambda X}^{\star}) && \text{(by property of conjugacies)} \\
&= \sup_{\lambda\in\mathbb{R}_{++}}(\lambda(\sigma_X - 1)) && (\text{as } \iota_{\lambda X}^{\star} = \sigma_{\lambda X} = \lambda\sigma_X) \\
&= \begin{cases} 0 & \text{if } \sigma_X \leq 1 \\ +\infty & \text{if } \sigma_X > 1 \end{cases} \\
&= \iota_{\{\sigma_X\leq 1\}} = \iota_{X^{\odot}}\ . && (\text{by definition (4a) of } X^{\odot})
\end{aligned}
$$

Thus, we get that
$$m_X^{\star} = \iota_{X^{\odot}}\ , \quad\text{and}\quad \{m_X^{\star} \leq 0\} = X^{\odot}\ , \tag{30}$$
and then, using (21), we obtain (27a) by $m_X^{\circ} = \sigma_{\{m_X^{\star}\leq 0\}^{\odot}} = \sigma_{X^{\odot\odot}}$.

Because the reverse polar transform (16b) acts like the polar transform (16a) on nonnegative functions, we obtain (27b) in the same fashion.

Finally, by definition (16c) of the bipolar transform, we apply (27a) and then (26b) to the support function of the bipolar set $X^{\odot\odot}$ and get

$$m_X^{\circ\circ'} = (m_X^{\circ})^{\circ'} = (\sigma_{X^{\odot\odot}})^{\circ'} = \sigma_{X^{\odot\odot\odot}} = \sigma_{X^{\odot}}\ ,$$

since $X^{\odot\odot\odot} = X^{\odot}$. Thus, we have obtained (27c).

(3) We have that $\iota_X^{\star} = \sigma_X$, and hence $\{\iota_X^{\star} \leq 0\} = \{\sigma_X \leq 0\} = X^{\ominus}$ by (3a). By (21), we get that $\iota_X^{\circ} = \sigma_{\{\iota_X^{\star}\leq 0\}^{\odot}} = \sigma_{(X^{\ominus})^{\odot}}$, where $(X^{\ominus})^{\odot} = (X^{\ominus})^{\ominus}$ because $X^{\ominus}$ is a cone. By (3b), we get that $\iota_X^{\circ} = \sigma_{X^{\ominus\ominus}}$. Finally, as $X^{\ominus\ominus}$ is cone, we have that $\sigma_{X^{\ominus\ominus}} = \iota_{X^{\ominus\ominus\ominus}} = \iota_{X^{\ominus}}$ since $X^{\ominus\ominus\ominus} = X^{\ominus}$. We have proven (28a).

Because the reverse polar transform (16b) acts like the polar transform (16a) on nonnegative functions, we obtain (28b) in the same fashion.

Finally, from (28a) and (28b), we deduce $\iota_X^{\circ\circ'} = (\iota_X^{\circ})^{\circ'} = \iota_{X^{\ominus}}^{\circ'} = \iota_{X^{\ominus\ominus}}$, where we used $\iota_{X^{\ominus\ominus}} = \sigma_{X^{\ominus}}$ because $X^{\ominus}$ is a cone.

(4) As $\chi_X = \iota_X + 1$, we have that $\chi_X^{\star} = (\iota_X + 1)^{\star} = \iota_X^{\star} - 1 = \sigma_X - 1$, and hence that $\{\chi_X^{\star} \leq 0\} = \{\sigma_X \leq 1\} = X^{\odot}$ by (4a). By (21), we deduce (29a) by $\chi_X^{\circ} = \sigma_{\{\chi_X^{\star}\leq 0\}^{\odot}} = \sigma_{X^{\odot\odot}}$.

Because the reverse polar transform (16b) acts like the polar transform (16a) on nonnegative functions, we obtain (29b) in the same fashion.

Finally, from (29a) and (26b), we deduce $\chi_X^{\circ\circ'} = (\chi_X^{\circ})^{\circ'} = \sigma_{X^{\odot\odot}}^{\circ'} = \sigma_{X^{\odot\odot\odot}} = \sigma_{X^{\odot}}$ since $X^{\odot\odot\odot} = X^{\odot}$. □

When Y is a unit ball, σ_Y is a norm $\|\cdot\|$ and $\sigma_Y^\circ = \sigma_{Y^\odot} = \|\cdot\|_\star$ is the so-called *dual norm.*

3.2. Polar operation on functions

In §3.2.1, we propose an extension of the polar transform from nonnegative to any functions. Then, in §3.2.2, we express the polar transform of any function as a Minkowski functional, and we provide several results about the polar of functions, some well-known (and scattered in the literature) and some new. Finally, in §3.2.3, we provide examples of polar transforms expressed as Minkowski functionals.

3.2.1. Definition of the polar of a function

The equality (22b) is taken as the definition of the polar transform of a gauge in [16, p. 128]. In fact, we can use the formula (22b) to extend Definition 3.1 to all functions, and not necessarily nonnegative ones.

Definition 3.5. For any function $f\colon \mathcal{X} \to \overline{\mathbb{R}}$, we define the *polar transform* $f^\circ\colon \mathcal{Y} \to \overline{\mathbb{R}}_+$ by

$$f^\circ(y) = \inf\{\lambda \in]0,+\infty[\mid \langle x, y\rangle \le \lambda f(x)\,,\ \forall x \in \mathcal{X}\}\,,\ \forall y \in \mathcal{Y}\,. \tag{31a}$$

For any function $g\colon \mathcal{Y} \to \overline{\mathbb{R}}$, the *reverse polar transform* $g^{\circ'}\colon \mathcal{X} \to \overline{\mathbb{R}}_+$ of the function g is defined by

$$g^{\circ'}(x) = \inf\{\lambda \in]0,+\infty[\mid \langle x, y\rangle \le \lambda g(y)\,,\ \forall y \in \mathcal{Y}\}\,,\ \forall x \in \mathcal{X}\,. \tag{31b}$$

For any function $f\colon \mathcal{X} \to \overline{\mathbb{R}}$, we define the *bipolar transform* $f^{\circ\circ'}\colon \mathcal{X} \to \overline{\mathbb{R}}_+$ of the function f by

$$f^{\circ\circ'} = (f^\circ)^{\circ'}\,. \tag{31c}$$

By the formula (22b), which coincides with (31a), the three definitions are consistent with those in Definition 3.1 when $f\colon \mathcal{X} \to \overline{\mathbb{R}}_+$.

3.2.2. Polar transform as a Minkowski functional

We display systematic relationships of polar functions with Minkowski functionals. To our knowledge, the results in Proposition 3.6 are new, if only because they hold for any function, in contrast to [16, Theorem 15.1], [8, Proposition 2.1], [3, Theorem 4.1]) established for functions that are convex, or vanishing at zero, or (strictly positively) 1-homogeneous, or nonnegative.

Proposition 3.6. *For any function $f\colon \mathcal{X} \to \overline{\mathbb{R}}$, we have the following properties.*

(1) *The function $f^\circ\colon \mathcal{Y} \to \overline{\mathbb{R}}_+$ is the Minkowski functional $m_{\{f^\star \le 0\}}$ of the closed convex subset $\{f^\star \le 0\}$:*

$$f^\circ = m_{\{f^\star \le 0\}}\,. \tag{32}$$

As a consequence, the polar transform $f^\circ\colon \mathcal{Y} \to \overline{\mathbb{R}}_+$ is convex (strictly positively) 1-homogeneous[12]*, with effective domain* $\mathrm{dom} f^\circ = \mathbb{R}_{++}\{f^\star \le 0\}$.

[12] Note that here, in contrast to Item 3 in Proposition 3.3, the function f° may not be lsc. As an example, consider the function $f = \sigma_{\{1\}}$ on $\mathcal{X} = \mathbb{R}$. Using Equation (38a), we obtain that $f^\circ = \sigma_{\{1\}}^\circ = m_{\overline{\mathrm{co}}\{1\}} = m_{\{1\}}$. Now, $m_{\{1\}}(x)$ is equal to $+\infty$ for $x \le 0$ and to x for $x > 0$. As a consequence, the function $m_{\{1\}}$ is not lsc at 0.

(2) *If $f(0)=0$, the (Rockafellar-Moreau) subdifferential satisfies*

$$\partial f(0) = \{f^\star = 0\} = \{f^\star \leq 0\} \; . \tag{33a}$$

As a consequence, we have that

$$f^\circ = m_{\{f^\star \leq 0\}} = m_{\{f^\star = 0\}} = m_{\partial f(0)} \; . \tag{33b}$$

(3) *The function $f^{\circ\circ'} \colon \mathcal{Y} \to \overline{\mathbb{R}}_+$ is the Minkowski functional $m_{\{f^\star \leq 0\}^\odot}$ of the bipolar set $\{f^\star \leq 0\}^\odot$:*

$$f^{\circ\circ'} = m_{\{f^\star \leq 0\}^\odot} \; . \tag{34}$$

As a consequence, the bipolar transform $f^{\circ\circ'} \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is convex lsc (strictly positively) 1-homogeneous and vanishes at the origin ($f^{\circ\circ}(0)=0$) – that is, the function $f^{\circ\circ'}$ is a lsc gauge.

Proof. We consider a function $f \colon \mathcal{X} \to \overline{\mathbb{R}}$.

(1) Let $y \in \mathcal{Y}$. We have that

$$\begin{aligned}
f^\circ(y) &= \inf\{\lambda \in]0,+\infty[\mid \langle x,y\rangle \leq \lambda f(x) \, , \; \forall x \in \mathcal{X}\} \, , \\
&\qquad\qquad \text{(by expression (31a) of the $\circ$-polar transform f°)} \\
&= \inf\{\lambda \in]0,+\infty[\mid \langle x,y\rangle + \lambda(-f(x)) \leq 0 \, , \; \forall x \in \mathcal{X}\} \, , \\
&= \inf\{\lambda \in]0,+\infty[\mid \sup_{x\in\mathcal{X}} \big(\langle x, \tfrac{y}{\lambda}\rangle - f(x)\big) \leq 0\} \, , \\
&= \inf\{\lambda \in]0,+\infty[\mid f^\star(\tfrac{y}{\lambda}) \leq 0\} \, , \\
&\qquad\qquad \text{(by definition (2a) of the Fenchel conjugate $f^\star$)} \\
&= m_{\{f^\star \leq 0\}}(y) \; . \qquad \text{(by definition (9) of the Minkowski functional)}
\end{aligned}$$

Thus, we have proven (32). As $\{f^\star \leq 0\}$ is a closed convex subset, the function $f^\circ \colon \mathbb{R}^n \to \overline{\mathbb{R}}_+$ is a nonnegative (strictly positively) 1-homogeneous convex function, by Item 4 in Proposition 2.7 (nonnegative (strictly positively) 1-homogeneous), and by (13c) (convex). The effective domain $\mathrm{dom} f^\circ = \mathbb{R}_{++}\{f^\star \leq 0\}$ by (13b).

(2) If $f(0)=0$, then $0 \in \mathrm{dom}\, f$ and the (Rockafellar-Moreau) subdifferential $\partial f(0)$ in (2d) can be expressed either as

$$\partial f(0) = \{y \in \mathcal{Y} \mid f^\star(y) = \langle 0,y\rangle - f(0)\} = \{y \in \mathcal{Y} \mid f^\star(y) = 0\} = \{f^\star = 0\} \, , \tag{35}$$

or as (using the property that $f^\star(y) \geq \langle 0,y\rangle - f(0) = 0$ by definition (2a))

$$\partial f(0) = \{y \in \mathcal{Y} \mid f^\star(y) \leq \langle 0,y\rangle - f(0)\} = \{y \in \mathcal{Y} \mid f^\star(y) \leq 0\} = \{f^\star \leq 0\} \; . \tag{36}$$

(3) This is a simple application of Item 1. Indeed, we have that

$$\begin{aligned}
f^{\circ\circ'} &= (f^\circ)^{\circ'} && \text{(by definition (31c) of the bipolar transform)} \\
&= (m_{\{f^\star \leq 0\}})^{\circ'} && \text{(by (32) in Item 1)} \\
&= m_{\{f^\star \leq 0\}^\odot} \, ,
\end{aligned}$$

by the expression (37b) of $m_Y^{\circ'} = m_{Y^\odot}$ (there is no circularity in the reasoning, as (37b) is proven by only using (32) established before).

Thus, we have proven (34). We also have that $0 \in \{f^\star \leq 0\}^\odot$, by definition (4a) of $\{f^\star \leq 0\}^\odot$, hence that

$$f^{\circ\circ'}(0) = m_{\{f^\star\leq 0\}^\odot}(0) = \inf\{\lambda > 0 \mid 0 \in \lambda\{f^\star \leq 0\}^\odot\} = \inf \mathbb{R}_{++} = 0. \qquad \square$$

3.2.3. Examples of polar transforms as Minkowski functionals

Using Item 1 (Equation (32)) and Item 3 (Equation (34)) in Proposition 3.6, we obtain expressions of the polar transforms of Minkowski functionals, of support functions, of indicator functions and of generalized indicator functions as Minkowski functionals. Equation (37a) can be found in [16, Theorem 15.1, p. 128].

Proposition 3.7.

(1) *Polar transform of a Minkowski functional as a Minkowski functional.* *For any subsets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$, we have that*

$$m_X^\circ = m_{X^\odot} \ , \tag{37a}$$

$$m_Y^{\circ'} = m_{Y^\odot} \ , \tag{37b}$$

$$m_X^{\circ\circ'} = m_{X^{\odot\odot}} \ . \tag{37c}$$

(2) *Polar transform of a support function*[13] *as a Minkowski functional.* *For any subsets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$, we have that*

$$\sigma_Y^\circ = m_{\overline{\mathrm{co}}Y} \ , \tag{38a}$$

$$\sigma_X^{\circ'} = m_{\overline{\mathrm{co}}X} \ , \tag{38b}$$

$$\sigma_Y^{\circ\circ'} = m_{Y^\odot} \ . \tag{38c}$$

(3) *Polar transform of an indicator function as a Minkowski functional.* *For any subset $X \subset \mathcal{X}$, we have that*

$$\iota_X^\circ = m_{X^\ominus} = \iota_{X^\ominus} \ , \tag{39a}$$

$$\iota_Y^{\circ'} = m_{Y^\ominus} = \iota_{Y^\ominus} \ , \tag{39b}$$

$$\iota_X^{\circ\circ'} = m_{X^{\ominus\ominus}} = \iota_{X^{\ominus\ominus}} \ , \tag{39c}$$

(4) *Polar transform of a generalized indicator function as a Minkowski functional.* *For any subset $X \subset \mathcal{X}$, we have that*

$$\chi_X^\circ = m_{X^\odot} \ , \tag{40a}$$

$$\chi_Y^{\circ'} = m_{Y^\odot} \ , \tag{40b}$$

$$\chi_X^{\circ\circ'} = m_{X^{\odot\odot}} \ . \tag{40c}$$

[13] To the difference of Item 1 in Proposition 3.4, the support functions that we consider here are not supposed to be nonnegative.

Proof.

(1) By (30), we know that $\{m_X^\star \leq 0\} = X^\odot$. Then, using (32), we obtain (37a) by $m_X^\circ = m_{\{m_X^\star \leq 0\}} = m_{X^\odot}$.

Because the reverse polar transform (31b) acts like the polar transform (31a), we obtain (37b) in the same fashion.

Finally, by definition (31c) of the bipolar transform, we apply (37a) and then (37b) and get $m_X^{\circ\circ'} = (m_X^\circ)^{\circ'} = (m_{X^\odot})^{\circ'} = m_{X^{\odot\odot}}$. Thus, we have obtained (37c).

(2) We have that $\sigma_Y^\star = \iota_{\overline{\mathrm{co}}Y}$, from which we get that $\{\sigma_Y^\star \leq 0\} = \overline{\mathrm{co}}Y$ and then, by (32), we obtain (38a) by $\sigma_Y^\circ = m_{\{\sigma_Y^\star \leq 0\}} = m_{\overline{\mathrm{co}}Y}$.

Because the reverse polar transform (31b) acts like the o-polar transform (31a), we obtain (38b) in the same fashion.

Finally, by definition (31c) of the bipolar transform, we apply (38a) and then (37b) and get $\sigma_Y^{\circ\circ'} = (\sigma_Y^\circ)^{\circ'} = m_{\overline{\mathrm{co}}Y}^{\circ'} = m_{(\overline{\mathrm{co}}Y)^\odot} = m_{Y^\odot}$, since $(\overline{\mathrm{co}}Y)^\odot = Y^\odot$. Thus, we have obtained (38c).

function	Fenchel conjugate	0-level set of the Fenchel conjugate	$\circ$-polar transform
f	$f^\star$	$\{f^\star \leq 0\}$	$f^\circ = m_{\{f^\star \leq 0\}}$ by (32)
σ_Y	$\iota_{\overline{\mathrm{co}}Y}$	$\overline{\mathrm{co}}Y$	$\sigma_Y^\circ = m_{\overline{\mathrm{co}}Y}$ by (38a)
$f \geq 0$	$f^\star$	$\{f^\star \leq 0\} \ni 0$	$f^\circ = m_{\{f^\star \leq 0\}} = \sigma_{\{f^\star \leq 0\}^\odot}$ by (32) and (21)
$\sigma_Y \geq 0$ $0 \in \overline{\mathrm{co}}Y$	$\iota_{\overline{\mathrm{co}}Y}$	$\overline{\mathrm{co}}Y$	$\sigma_Y^\circ = m_{\overline{\mathrm{co}}Y} = \sigma_{Y^\odot}$ by (38a) and (26a)
ι_X	σ_X	$X^\ominus$	$\iota_X^\circ = m_{X^\ominus} = \sigma_{X^{\ominus\ominus}} = \iota_{X^\ominus}$ by (39a) and (28a)
$\chi_X (=\iota_X + 1)$	$\sigma_X - 1$	$X^\odot$	$\chi_X^\circ = m_{X^\odot} = \sigma_{X^{\odot\odot}}$ by (40a) and (29a)
m_X	$\iota_{X^\odot}$ by (30)	$X^\odot$	$m_X^\circ = m_{X^\odot} = \sigma_{X^{\odot\odot}}$ by (37a) and by (27a)
$m_{Y^\odot}$	$\iota_{Y^{\odot\odot}}$ by (30)	$Y^{\odot\odot}$	$m_{Y^\odot}^\circ = m_{Y^{\odot\odot}} = \sigma_{Y^\odot}$ by (37a), (27a) and $Y^{\odot\odot\odot} = Y^\odot$

Table 3.1: Fenchel conjugates and polar transforms, for any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}$, any subset $X \subset \mathcal{X}$ (in the primal space) and any subset $Y \subset \mathcal{Y}$ (in the dual space).

(3) We have that $\iota_X^\star = \sigma_X$, and hence $\{\iota_X^\star \leq 0\} = \{\sigma_X \leq 0\} = X^\ominus$ by (3a). By (32), we deduce that $\iota_X^\circ = m_{\{\iota_X^\star \leq 0\}} = m_{X^\ominus}$. Now, as $X^\ominus$ is a cone, we easily see that $m_{X^\ominus} = \iota_{X^\ominus}$ by definition (9) of the Minkowski functional. Thus, we proved (39a).

Because the reverse polar transform (31b) acts like the polar transform (31a), we obtain (39b) in the same fashion.

Finally, using (39a) and (39b), we get (39c).

(4) As $\chi_X = \iota_X + 1$, we have that $\chi_X^\star = (\iota_X + 1)^\star = \iota_X^\star - 1 = \sigma_X - 1$, and hence that $\{\chi_X^\star \leq 0\} = \{\sigma_X \leq 1\} = X^\odot$ by (4a). By (32), we deduce (40a).

Because the reverse polar transform (31b) acts like the polar transform (31a), we obtain (40b) in the same fashion.

Finally, using (40a) and (37b), we get (40c). □

Tables 3.1 and 3.2 are consequences of Propositions 3.3, 3.4, 3.6 and 3.7.

Tables 3.1 and 3.2 lead to the following results.

function	polar of 0-level set of the Fenchel conjugate	$\circ$-polar bitransform
f	$\{f^\star \leq 0\}^\odot$	$f^{\circ\circ'} = m_{\{f^\star\leq 0\}^\odot}$ by (34)
σ_Y	$Y^\odot$	$\sigma_Y^{\circ\circ'} = m_{Y^\odot} = \sigma_{Y^{\odot\odot}}$ by (38c) and (41b)
$f \geq 0$	$\{f^\star \leq 0\}^\odot \ni 0$	$f^{\circ\circ'} = m_{\{f^\star\leq 0\}^\odot} = \sigma_{\{f^\star\leq 0\}}$ by (34) and (23)
ι_X	$X^{\ominus\ominus}$	$\iota_X^{\circ\circ'} = m_{X^{\ominus\ominus}} = \sigma_{X^\ominus} = \iota_{X^{\ominus\ominus}}$ by (39c) and (28c)
$\chi_X(=\iota_X + 1)$	$X^{\odot\odot}$	$\chi_X^{\circ\circ'} = m_{X^{\odot\odot}} = \sigma_{X^\odot}$ by (40c) and by (29c)
m_X	$X^{\odot\odot}$	$m_X^{\circ\circ'} = m_{X^{\odot\odot}} = \sigma_{X^\odot}$ by (37c) and by (27c)
$m_{Y^\odot}$	$Y^\odot$	$m_{Y^\odot}^{\circ\circ'} = m_{Y^\odot} = \sigma_{Y^{\odot\odot}}$ by (37c), $Y^{\odot\odot\odot} = Y^\odot$ and (27c)

Table 3.2: Fenchel biconjugates and bipolar transforms, for any function $f\colon \mathcal{X} \to \overline{\mathbb{R}}$, any subset $X \subset \mathcal{X}$ (in the primal space) and any subset $Y \subset \mathcal{Y}$ (in the dual space)

Proposition 3.8. *For any subset $X \subset \mathcal{X}$ (in the primal space) and any subset $Y \subset \mathcal{Y}$ (in the dual space), we have that*

$$m_{X^\odot} = \sigma_{X^{\odot\odot}} \quad \textit{and} \quad m_{Y^{\odot\odot}} = \sigma_{Y^\odot} \ , \tag{41a}$$

$$m_{Y^\odot} = \sigma_{Y^{\odot\odot}} \quad \textit{and} \quad m_{X^{\odot\odot}} = \sigma_{X^\odot} \ , \tag{41b}$$

which, in the case of a polar pair (see Definition 2.1), gives

$$X, Y \ \textit{polar pair} \implies m_X = \sigma_Y \quad \textit{and} \quad m_Y = \sigma_X \ . \tag{41c}$$

For any nonnegative (strictly positively) 1-homogeneous function $f\colon \mathcal{X} \to \overline{\mathbb{R}}_+$, we have that

$$\{f^\star \leq 0\} = \{f < 1\}^\odot \ , \tag{42a}$$

and, when $f(0) = 0$, we have that

$$\{f^\star \leq 0\} = \{f < 1\}^\odot = \{f \leq 1\}^\odot \ . \tag{42b}$$

Proof. The last two lines of Table 3.1 give (41), which are well-known results (see [16, Theorem 14.5, p. 125], [16, Corollaries 15.1.1-2, p. 129]).

On the one hand, by line 4 of Table 3.2, we have that $f^{\circ\circ'} = \sigma_{\{f^\star \leq 0\}}$. On the other hand, by (12a), we have that $f = m_{\{f<1\}}$. Then, by line 7 of Table 3.2 with $X = \{f < 1\}$, we get that $f^{\circ\circ'} = \sigma_{\{f<1\}^\odot}$. As both sets $\{f^\star \leq 0\}$ and $\{f < 1\}^\odot$ are closed convex, we obtain Equation (42a) from the equality $f^{\circ\circ'} = \sigma_{\{f^\star \leq 0\}} = \sigma_{\{f<1\}^\odot}$. When $f(0) = 0$, by Equation (12b) in Proposition 2.7, we have that $f = m_{\{f\leq 1\}}$. Equation (42b) follows in the same way as above. □

4. Bipolar functions

In this Sect. 4, we consider $\mathcal{X}$ and $\mathcal{Y}$ two (real) vector spaces that are paired (see §2.2.1). We present in a systematic fashion different expressions for the set of bipolar functions defined as follows.

Definition 4.1. We say that a function $f\colon \mathcal{X} \to \overline{\mathbb{R}}$ is a *bipolar function* if

$$f^{\circ\circ'} = f \; . \tag{43}$$

We denote by $\mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}]$ the set of bipolar functions from $\mathcal{X}$ to the extended reals.

The following equivalences – between Item 1 and all but the last Item 7 – have been long established in the class of gauges (see [16, Theorem 15.1, Corollary 15.1, Corollary 15.2]). The equivalence between Item 1 and Item 6 has been established in [12, Theorem 5.2] in the class of nonnegative functions. The equivalence between Item 1 and Item 7 has been established in [12, Corollary 4.3], for a large class of couplings which includes the coupling $\mathcal{X} \times \mathcal{Y} \ni (x, y) \mapsto c(x, y) = \langle x, y\rangle_+$. Once again, the novelty of Proposition 4.2 is to provide equivalence in the class of all functions (and not necessarily convex ones, or lsc, or (strictly positively) 1-homogeneous, or gauges, or even nonnegative), and to display the equivalences in a unified fashion. Item 2 is possibly new.

Proposition 4.2. *For any function $f\colon\mathcal{X}\to\overline{\mathbb{R}}$, the following statements are equivalent.*

(1) *The function f is a bipolar function, that is, $f^{\circ\circ'} = f$.*

(2) *The function f is the Minkowski functional of the bipolar set $\{f^\star \leq 0\}^\odot$ and also the support function of the bipolar set $\{f^\star \leq 0\}$:*

$$f = m_{\{f^\star \leq 0\}^\odot} = \sigma_{\{f^\star \leq 0\}} \; . \tag{44}$$

(3) *There exist a polar pair $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ (that is, $Y = X^\odot$, $X = Y^\odot$) such that $f = m_X = \sigma_Y$.*

(4) *There exists a bipolar (primal) set $X \subset \mathcal{X}$ such that the function f is the Minkowski functional m_X.*

(5) *There exists a bipolar (dual) set $Y \subset \mathcal{Y}$ such that the function f is the support function $f = \sigma_Y$.*

(6) *The function f is a nonnegative (strictly positively) 1-homogeneous convex lsc function satisfying $f(0) = 0$, that is, the function f is a lsc gauge.*

(7) *The function f is the pointwise supremum of a family of functions of the form $\lambda\langle\cdot, y\rangle_+$, with $y \in \mathcal{Y}$ and $\lambda \in \mathbb{R}_{++}$.*

Proof.

(1)⇒(2): Suppose that $f^{\circ\circ'} = f$. By (31c) and (31a), we get that $f^{\circ\circ'} \geq 0$, hence $f \geq 0$. By line 4 in Table 3.2 (or by (34) and (23)), we get (44). By Item 2 in Proposition 3.3, we know that $\{f^\star \leq 0\}$ is a bipolar set. The set $\{f^\star \leq 0\}^\odot$ is bipolar.

(2)⇒(3)⇒(4)⇔(5): Obvious. Note that (4) and (5) are equivalent, using (41c).

(5)⇒(6): Because the support function of a subset containing 0 is a nonnegative (strictly positively) 1-homogeneous convex lsc function satisfying $f(0) = 0$.

(6)⇒(4): Suppose that the function f is a nonnegative (strictly positively) 1-homogeneous convex lsc function satisfying $f(0) = 0$. By (12b), we know that $f = m_{\{f \leq 1\}}$, where $\{f \leq 1\}$ is a closed convex subset containing zero, hence is a bipolar set, by Item 1 of Definition 2.1.

(4)⇒(1): Using (37c).

(1)⇔(7): This is an application of [12, Corollary 4.3] to the case of the coupling $c(x, y) = \langle x, y \rangle_+$.

This ends the proof. □

The one-to-one correspondence between bipolar sets and lsc gauges (or, equivalently, bipolar functions) is outlined at the beginning of [16, Sect. 15]. We show that this correspondence is an anti-isomorphism between complete lattices. To the best of our knowledge, this result is new (and different from [2, p. 292], which points out an isomorphism between the lattice of weak∗ compact convex subsets of $\mathcal{Y}$ and the lattice of continuous gauges on $\mathcal{X}$).

Theorem 4.3. *The set of bipolar functions, ordered by $\leq$, is a complete lattice $(\mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}], \wedge, \vee)$. Consider a family $\{f_i\}_{i \in I}$ of bipolar functions $f_i \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, $i \in I$. The* greatest lower bound $\bigwedge_{i \in I} f_i$ *is given by*

$$\bigwedge_{i \in I} f_i = \sigma_{\bigcap_{i \in I} \{f_i^\star \leq 0\}} = m_{\overline{\mathrm{co}}(\bigcup_{i \in I} \{f_i^\star \leq 0\}^\odot)} , \tag{45a}$$

whereas the least upper bound $\bigvee_{i \in I} f_i$ *is given by the supremum*

$$\bigvee_{i \in I} f_i = \sup_{i \in I} f_i . \tag{45b}$$

The two mappings

$$\begin{aligned} \varphi &: \mathcal{B}p[\mathcal{X}] \to \mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}], \quad X \mapsto m_X = \sigma_{X^\odot}, \\ \theta &: \mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}] \to \mathcal{B}p[\mathcal{X}], \quad f \mapsto \{f^\star \leq 0\}^\odot \end{aligned} \tag{46}$$

define an anti-isomorphism between the complete lattice $(\mathcal{B}p[\mathcal{X}], \wedge, \vee)$ of bipolar sets (see Definition 2.1 *and Proposition* 2.3*) and the complete lattice $(\mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}], \wedge, \vee)$ of bipolar functions.*

Proof.

• First, we prove that the set of bipolar functions, ordered by $\leq$, is a complete lattice $(\mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}], \wedge, \vee)$.

▷ We consider a family $\{f_i\}_{i\in I}$ of bipolar functions $f_i \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, $i \in I$, and we prove (45a). The greatest lower bound $\bigwedge_{i\in I} f_i$ is the greatest bipolar function below $\inf_{i\in I} f_i$, hence is $(\inf_{i\in I} f_i)^{\circ\circ'}$ (using that, for any $h \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, we have that $h^{\circ\circ'} \leq h$ by the Inequality (17c), where $h^{\circ\circ'}$ is a bipolar function). We have that

$$
\begin{aligned}
\bigwedge_{i\in I} f_i &= (\inf_{i\in I} f_i)^{\circ\circ'} && (47)\\
&= (\inf_{i\in I} \sigma_{\{f_i^\star \leq 0\}})^{\circ\circ'} && (\text{as } f_i = \sigma_{\{f_i^\star \leq 0\}}, \text{ for all } i \in I, \text{ by } (44))\\
&= (\sup_{i\in I} (\sigma_{\{f_i^\star \leq 0\}})^{\circ})^{\circ'} && (\text{by } (17\text{a}))\\
&= (\sup_{i\in I} \sigma_{\{f_i^\star \leq 0\}^{\odot}})^{\circ'} && (\text{by } (26\text{a}))\\
&= (\sigma_{\bigcup_{i\in I} \{f_i^\star \leq 0\}^{\odot}})^{\circ'} && (\text{as is well known for support functions})\\
&= \sigma_{(\bigcup_{i\in I} \{f_i^\star \leq 0\}^{\odot})^{\odot}} && (\text{by } (26\text{b}))\\
&= \sigma_{\bigcap_{i\in I} \{f_i^\star \leq 0\}} \; .
\end{aligned}
$$

(by (6a) as $\{f_i^\star \leq 0\}$, $i \in I$, are bipolar sets by Item 2 in Proposition 3.3)

Thus, we have proved the left equality in (45a). The right equality is a consequence of (41c) and (6b).

▷ We consider a family $\{f_i\}_{i\in I}$ of bipolar functions $f_i \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, $i \in I$, and we prove (45b). We are going to show that the supremum $h = \sup_{i\in I} f_i$ is a bipolar function. Indeed, on the one hand, we have that $h^{\circ\circ'} \geq \sup_{i\in I} f_i^{\circ\circ'} = \sup_{i\in I} f_i$, where we have first used that the bipolar operation is isotone, and second that f_i, $i \in I$, are bipolar functions (hence $f_i^{\circ\circ'} = f_i$, $i \in I$, by Definition 4.1). Now, on the other hand, we have that $h^{\circ\circ'} \leq h$ by the Inequality (17c). We conclude that $h^{\circ\circ'} = h$, that is, $h = \sup_{i\in I} f_i$ is a bipolar function. Thus, the least upper bound $\bigvee_{i\in I} f_i = \sup_{i\in I} f_i$, which is (45b).

• Second, we show that the two mappings (46) define a one-to-one correspondence between bipolar sets and bipolar functions.

We show that the mapping φ takes values in $\mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}]$. Indeed, when X is a bipolar set, we obtain that $\varphi(X)^{\circ\circ'} = m_X^{\circ\circ'} = m_{X^{\odot\odot}} = m_X = \varphi(X)$ by line 6 column 3 of Table 3.2, and thus $\varphi(X) \in \mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}]$. It is immediate to check that the mapping θ takes values in $\mathcal{B}p[\mathcal{X}]$.

Now, we have that $\theta \circ \varphi = I_{\mathcal{B}p[\mathcal{X}]}$ as, for $X \in \mathcal{B}p[\mathcal{X}]$, line 6 column 2 of Table 3.2 gives

$$
(\theta \circ \varphi)(X) = \theta(m_X) = \{m_X^\star \leq 0\}^{\odot} = X^{\odot\odot} = X \; . \tag{48a}
$$

We have that $\varphi \circ \theta = I_{\mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}]}$ as, for $f \in \mathcal{B}p[\overline{\mathbb{R}}^{\mathcal{X}}]$, line 1 column 3 of Table 3.2 gives

$$
(\varphi \circ \theta)(f) = \varphi(\{f^\star \leq 0\}^{\odot}) = m_{\{f^\star \leq 0\}^{\odot}} = f^{\circ\circ'} = f \; . \tag{48b}
$$

• Third, we show that the two mappings (46) define an anti-isomorphism between complete lattices. For this purpose, we consider a family $\{X_i\}_{i\in I}$ of bipolar sets, that is, $X_i \in \mathcal{B}p[\mathcal{X}]$, for all $i \in I$. On the one hand, we have that

$$\begin{aligned}
\varphi(\bigwedge_{i\in I} X_i) &= \varphi(\bigcap_{i\in I} X_i) && \text{(by (7a))}\\
&= \sigma_{(\bigcap_{i\in I} X_i)^{\odot}} && \text{(by (46))}\\
&= \sigma_{\overline{\mathrm{co}}(\bigcup_{i\in I} X_i{}^{\odot})} && \text{(by (6b))}\\
&= \sigma_{\bigcup_{i\in I} X_i{}^{\odot}} && \text{(as is well known for support functions)}\\
&= \sup_{i\in I} \sigma_{X_i{}^{\odot}} && \text{(as is well known for support functions)}\\
&= \bigvee_{i\in I} \sigma_{X_i{}^{\odot}} && \text{(by (45b))}\\
&= \bigvee_{i\in I} \varphi(X_i) \; . && \text{(by (46))}
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\varphi(\bigvee_{i\in I} X_i) &= \varphi(\overline{\mathrm{co}}(\bigcup_{i\in I} X_i)) && \text{(by (7b))}\\
&= \sigma_{(\overline{\mathrm{co}}(\bigcup_{i\in I} X_i))^{\odot}} && \text{(by (46))}\\
&= \sigma_{(\bigcup_{i\in I} X_i)^{\odot}} && \text{(using } (\overline{\mathrm{co}}(\textstyle\bigcup_{i\in I} X_i))^{\odot} = (\textstyle\bigcup_{i\in I} X_i)^{\odot})\\
&= \sigma_{\bigcap_{i\in I} X_i{}^{\odot}} && \text{(by (6a))}\\
&= \sigma_{\bigcap_{i\in I}\{\sigma^{\star}_{X_i{}^{\odot}} \leq 0\}} && \text{(as } \{\sigma^{\star}_{Z^{\odot}} \leq 0\} = \{\iota_{Z^{\odot}} \leq 0\} = Z^{\odot})\\
&= \bigwedge_{i\in I} \sigma_{X_i{}^{\odot}} && \text{(by (45a))}\\
&= \bigwedge_{i\in I} \varphi(X_i) \; . && \text{(by (46))}
\end{aligned}$$

This concludes the proof. □

Now recall that the *infimal convolution* (or *inf convolution*) $f \,\square\, g$ of two functions $f \colon \mathcal{X} \to \overline{\mathbb{R}}$ and $g \colon \mathcal{X} \to \overline{\mathbb{R}}$ is the function defined by

$$\mathrm{epi}_s(f \,\square\, g) = \mathrm{epi}_s f + \mathrm{epi}_s g \; . \tag{49}$$

Proposition 4.4. *Consider two bipolar functions $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$ and $g \colon \mathcal{X} \to \overline{\mathbb{R}}_+$. The function $f \wedge g$ is related to the infimal convolution $f \,\square\, g$, as we have*

$$f \wedge g \leq f \,\square\, g \leq \inf\{f, g\} \; . \tag{50}$$

Moreover, when $f \,\square\, g$ is lsc, it coincides with $f \wedge g$.

Proof. We have that

$$
\begin{aligned}
(f \,\square\, g)^{\star\star'} &= (f^{\star} + g^{\star})^{\star'} && \text{(by [20, Theorem 2.3.1 (ix)])} \\
&= (\sigma^{\star}_{\{f^{\star}\leq 0\}} + \sigma^{\star}_{\{g^{\star}\leq 0\}})^{\star'} && \text{(as } f = \sigma_{\{f^{\star}\leq 0\}} \text{ and } g = \sigma_{\{g^{\star}\leq 0\}} \text{ by (44))} \\
&= (\iota_{\{f^{\star}\leq 0\}} + \iota_{\{g^{\star}\leq 0\}})^{\star'}
\end{aligned}
$$

(as $\{f^{\star} \leq 0\}$ and $\{g^{\star} \leq 0\}$ are closed convex sets, using Item 6 in Proposition 4.2)

$$
\begin{aligned}
&= (\iota_{\{f^{\star}\leq 0\}\cap\{g^{\star}\leq 0\}})^{\star'} \\
&= \sigma_{\{f^{\star}\leq 0\}\cap\{g^{\star}\leq 0\}} \\
&= f \wedge g && \text{(by (45a))} \\
&= (\inf\{f,g\})^{\circ\circ'} \; . && \text{(by (47))}
\end{aligned}
$$

We therefore get that

$$
f \wedge g = (\inf\{f,g\})^{\circ\circ'} = (f \,\square\, g)^{\star\star'} \leq f \,\square\, g \leq \inf\{f,g\} \; , \tag{51}
$$

as $f \,\square\, g \leq \inf\{f,g\}$. Indeed, we have that

$$
\mathrm{epi}_s(\inf\{f,g\}) = \mathrm{epi}_s f \cup \mathrm{epi}_s g \subset \mathrm{epi}_s f + \mathrm{epi}_s g \; ,
$$

because $(0,0) \in \mathrm{epi}_s f \cap \mathrm{epi}_s g$, as bipolar functions vanish at the origin.

When the function $f \,\square\, g$ is lsc, we deduce from Equation (51) that $f \wedge g = f \,\square\, g$. □

5. Polar subdifferentials and alignement

In this Sect. 5, we consider $\mathcal{X}$ and $\mathcal{Y}$ two (real) vector spaces that are paired (see §2.2.1). In §5.1, we present three possible definitions for the polar subdifferential of a nonnegative function. The first two definitions are inspired by definitions of subdifferentials of dualities [1, 13]. We propose a third definition which, to our knowledge, is new and will be explored in more detail in §5.2 in relation to the notion of alignement.

5.1. Polar subdifferentials of a nonnegative function

In duality in convex analysis, one uses the (Rockafellar-Moreau) subdifferential (2d) which is defined over the effective domain of a proper function (in general convex lsc, but this is not compulsory). By restricting to proper functions, one avoids the value $-\infty$, which is the bottom of the ordered set $(\overline{\mathbb{R}}, \leq)$. By contrast, with polarity one deals with nonnegative functions than can take the value 0, which is the bottom of the ordered set $(\overline{\mathbb{R}}_+, \leq)$. This explains the three formulas (3), for the polar subdifferential of a nonnegative function, which do not necessarily give the same result, especially when $f(x) = 0$ or $+\infty$.

Definition 5.1. For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, we define

1. the *lower polar subdifferential* of f (inspired by [1, Equation (10a)]) by
$$\partial_\circ f(x) = \{y \in \mathcal{Y} \mid f^\circ(y) = \langle x, y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f(x))^{-1}\} \, , \ \forall x \in \mathcal{X} \, , \tag{52a}$$
2. the *upper polar subdifferential* of f (inspired by [13, Equation (1.7) in Definition 1.2]) by
$$\partial^\circ f(x) = \{y \in \mathcal{Y} \mid f(x) = \langle x, y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f^\circ(y))^{-1}\} \, , \ \forall x \in \mathcal{X} \, , \tag{52b}$$
3. the *middle polar subdifferential* of f (inspired by the equality case in the Fenchel-Young, Cauchy-Schwarz and polar inequalities) by
$$\partial^\circ_\circ f(x) = \{y \in \mathcal{Y} \mid \langle x, y\rangle_+ = f(x) \mathbin{\dot{\times}} f^\circ(y)\} \, , \ \forall x \in \mathcal{X} \, . \tag{52c}$$

The first two expressions (52a) and (52b) are inspired by definitions of subdifferentials of dualities in [1] and [13]. We propose the third expression (52c) which, to the best of our knowledge, is new and will be explored in more detail in §5.2.

Each of the three definitions, for the polar subdifferential of a nonnegative function, in Definition 5.1 have their own advantages as shown in Proposition 5.2.

Proposition 5.2. *For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}_+$, we have the following results.*

(1) *Regarding definition* (52a) *of the lower polar subdifferential $\partial_\circ f$ of f, we have the alternate expressions, for any $x \in \mathcal{X}$,*
$$\begin{aligned}
\partial_\circ f(x) &= \{y \in \mathcal{Y} \mid f^\circ(y) \leq \langle x, y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f(x))^{-1}\} \, , && (53a)\\
&= \{y \in \mathcal{Y} \mid \langle x', y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f(x'))^{-1} \leq \langle x, y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f(x))^{-1} \, , \ \forall x' \in \mathbb{R}^n\} \, , && (53b)\\
&= \{y \in \mathcal{Y} \mid x \in \operatorname*{arg\,max}_{x' \in \mathbb{R}^n} (\langle x', y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f(x'))^{-1})\} \, . && (53c)
\end{aligned}$$

(2) *Regarding definition* (52b) *of the upper polar subdifferential $\partial^\circ f$ of f, we have the property that*
$$\partial^\circ f(x) \neq \emptyset \implies f^{\circ\circ'}(x) = f(x) \, . \tag{54}$$

(3) *Regarding definition* (52c) *of the middle polar subdifferential $\partial^\circ_\circ f$ of f, we have the alternate expression, for any $x \in \mathbb{R}^n$,*
$$\partial^\circ_\circ f(x) = \{y \in \mathcal{Y} \mid \langle x, y\rangle = f(x) \mathbin{\dot{\times}} f^\circ(y)\} \, . \tag{55}$$

Proof.

(1) Equations (53) come from the definition (16a) of $f^\circ(y)$.

(2) Let $y \in \partial^\circ f(x)$. The Implication (54) is a consequence of
$$\begin{aligned}
f^{\circ\circ'}(x) &= \sup_{y' \in \mathcal{Y}} \big(\langle x, y'\rangle_+ \mathbin{\underset{\cdot}{\times}} (f^\circ(y'))^{-1}\big)\\
&\qquad \text{(by definition (16c) of } (f^\circ)^{\circ'} \text{ and by definition (16b) of } g^{\circ'})\\
&\geq \langle x, y\rangle_+ \mathbin{\underset{\cdot}{\times}} (f^\circ(y))^{-1} = f(x) \, . && \text{(by (52b))}
\end{aligned}$$

As $f^{\circ\circ'}(x) \leq f(x)$ by the Inequality (17c), we conclude that $f^{\circ\circ'}(x) = f(x)$.

(3) Equation (55) follows from the fact that $f(x) \mathbin{\dot{\times}} f^\circ(y) \geq 0$. □

5.2. Middle polar subdifferential and alignement

When the function $f\colon \mathcal{X} \to \overline{\mathbb{R}}_+$ is a norm $\|\cdot\|$ on $\mathcal{X}$, then f° is the so-called *dual norm* $\|\cdot\|_\star$ on $\mathcal{Y}$. Couples $(x, y) \in \mathcal{X} \times \mathcal{Y}$ of vectors satisfying $\langle x, y\rangle = \|x\|\|y\|_\star$ are said to be $\|\cdot\|$-*dual* in [11, page 2], to form a *dual vector pair* in [9, Equation (1.11)], to be *dual vectors* in [10, p. 283], or to satisfy *polar alignment* in [7, Definition 2.4].

We propose the following definition that encompasses the above definitions, and goes beyond.

Definition 5.3. Let $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ be a polar pair (that is, $Y = X^{\odot}$, $X = Y^{\odot}$). We call the couple $(x, y) \in \mathcal{X} \times \mathcal{Y}$ *aligned w.r.t.* (X, Y) (or w.r.t.(σ_Y, σ_X)) if [14]

$$0 < \langle x, y\rangle \ \text{ and } \ \langle x, y\rangle = \sigma_Y(x) \dot{\times} \sigma_X(y) \ . \tag{56}$$

In this definition, σ_Y can be replaced by m_X and σ_X by m_Y, because of (41c).

The relationship between alignement and the middle polar subdifferential (52c) is as follows.

Proposition 5.4. *Let $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ be a polar pair, and $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Then, we have that*

$$y \in \partial^\circ_\circ \sigma_Y(x) \iff x \in \partial^\circ_\circ \sigma_X(y) \iff \begin{cases} \textit{either} & x \perp y \quad (\textit{i.e } \langle x, y\rangle = 0) \ , \\ \textit{or} & (x, y) \textit{ is aligned w.r.t. } (X, Y) \ . \end{cases} \tag{57}$$

Proof. The proof follows from the very definition (52c) of the middle polar subdifferentials $\partial^\circ_\circ \sigma_Y$ and $\partial^\circ_\circ \sigma_X$. □

As in [7, §3.3], we relate alignement to well-known geometric objets in convex analysis.

Definition 5.5. For any nonempty closed convex subset $C \subset \mathcal{X}$, the *exposed face* of C by the dual vector $y \in \mathcal{Y}$ is

$$F_\perp(C, y) = \arg\max_{x \in C} \langle x, y\rangle \ , \tag{58}$$

and the *normal cone* $N(C, x)$ at any primal vector $x \in C$ is defined by the conjugacy relation

$$x \in C \ \text{ and } \ y \in N(C, x) \iff x \in F_\perp(C, y) \ , \tag{59}$$

that is, equivalently, by

$$N(C, x) = \{y \in \mathcal{Y} \mid \langle x' - x, y\rangle \le 0 \, , \ \forall x' \in C\} \, , \ \forall x \in C \ . \tag{60}$$

[14] Equation (56) could be replaced by $\sigma_Y(x), \sigma_X(y) \in \mathbb{R}_{++}$ and $\langle x, y\rangle = \sigma_Y(x)\sigma_X(y)$. Indeed, the upper multiplication $\dot{\times}$ can be replaced by the usual multiplication $\times$ since

$$0 < \langle x, y\rangle = \sigma_Y(x) \dot{\times} \sigma_X(y) \iff 0 < \langle x, y\rangle = \sigma_Y(x)\sigma_X(y),$$

because $\sigma_Y(x) \neq +\infty$ and $\sigma_X(y) \neq +\infty$ (else the right hand side would be $+\infty$).

The following Proposition is in the vein of [7, Proposition 3.3], but more detailed. The proof is left to the reader.

Proposition 5.6. *Let $X \subset \mathcal{X}$, $Y \subset \mathcal{Y}$ be a polar pair, and $(x,y) \in \mathcal{X} \times \mathcal{Y}$. Then, we have that*

$$(x,y) \text{ is aligned w.r.t. } (X,Y) \,, \tag{61a}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \langle x,y \rangle = \sigma_Y(x)\sigma_X(y) \,, \tag{61b}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \frac{y}{\sigma_X(y)} \in F_\perp(Y, \frac{x}{\sigma_Y(x)}) \,, \tag{61c}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \frac{y}{\sigma_X(y)} \in F_\perp(Y, x) \,, \tag{61d}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } x \in N(Y, \frac{y}{\sigma_X(y)}) \,, \tag{61e}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \frac{x}{\sigma_Y(x)} \in N(Y, \frac{y}{\sigma_X(y)}) \,, \tag{61f}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \frac{x}{\sigma_Y(x)} \in F_\perp(X, \frac{y}{\sigma_X(y)}) \,, \tag{61g}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \frac{x}{\sigma_Y(x)} \in F_\perp(X, y) \,, \tag{61h}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } y \in N(X, \frac{x}{\sigma_Y(x)}) \,, \tag{61i}$$

$$\iff \sigma_Y(x) \in \mathbb{R}_{++}\,,\ \sigma_X(y) \in \mathbb{R}_{++} \text{ and } \frac{y}{\sigma_X(y)} \in N(X, \frac{x}{\sigma_Y(x)}) \,. \tag{61j}$$

In this Proposition, σ_Y can be replaced by m_X and σ_X by m_Y, because of (41c).

A. Best lsc convex lower approximations of a function

We consider $\mathcal{X}$ and $\mathcal{Y}$ two (real) vector spaces that are paired (see §2.2.1).

Consider $\Lambda \subset \overline{\mathbb{R}}^{\mathcal{X}}$, a nonempty subset of the functions defined on $\mathcal{X}$ and taking values in the extended reals. Then, for any function $f : \mathcal{X} \to \overline{\mathbb{R}}$, we define the subset $\Lambda[f] \subset \Lambda$ by

$$\Lambda[f] = \{h \in \Lambda \mid h(x) \leq f(x)\,,\ \forall x \in \mathcal{X}\}. \tag{62}$$

Then, the *best lower* Λ*-approximation* of f, denoted by $\Lambda[f]^\top$, is defined as the greatest element of $\Lambda[f]$ (which always exists as $+\infty$ is always a possible value).

Now, we are going to consider two cases for the subset $\Lambda \subset \overline{\mathbb{R}}^{\mathcal{X}}$: lsc convex extended functions in §A.1 and lsc convex (strictly positively) 1-homogeneous extended functions in §A.2.

A.1. The case of lsc convex extended functions

Let Γ denote the set of lsc convex extended functions on $\mathcal{X}$. When $\Lambda = \Gamma$, the greatest lsc convex lower approximation of the function f, that is $\Gamma[f]^\top$, is given by [19, Theorem 3.1] that we reproduce in Proposition A.1. Following the terminology of [15], the *valley function* v_A of a subset $A \subset \mathcal{X}$ is defined by $v_A(x) = -\infty$ if $x \in A$ and $v_A(x) = +\infty$ if $x \in \mathcal{X} \setminus A$.

Proposition A.1. ([19, Theorem 3.1]) *For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}$, the greatest lsc convex lower approximation of the function f is given by*

$$\Gamma[f]^{\top} = f^{\star\star'} \dot{+} \iota_{\overline{\mathrm{co}}(\mathrm{dom}\, f)} = \begin{cases} f^{\star\star'} & \text{if } f \text{ is proper}\,, \\ \upsilon_{\overline{\mathrm{co}}(\mathrm{dom}\, f)} & \text{if } f \text{ is not proper}\,. \end{cases} \tag{63}$$

The function $\Lambda[f]^{\top}$ is also classically denoted by $\overline{\mathrm{co}}(f)$.

A.2. The case of lsc convex (strictly positively) 1-homogeneous extended functions

Let $\Gamma_{\mathtt{H}}$ denote the set of lsc convex (strictly positively) 1-homogeneous extended functions on $\mathcal{X}$. For $\Lambda = \Gamma_{\mathtt{H}}$, we give in Equation (64) the expression of $\Gamma_{\mathtt{H}}[f]^{\top}$, the greatest lsc convex (strictly positively) 1-homogeneous lower approximation of the function f which is, to our knowledge, a new result. Recall that $\overline{\mathrm{cone}}X$ is the *closed conical hull* of $X \subset \mathcal{X}$, that is, the smallest closed cone in $\mathcal{X}$ containing X.

Proposition A.2. *For any function $f \colon \mathcal{X} \to \overline{\mathbb{R}}$, such that $0 \in \mathrm{dom}\, f$, the greatest lsc convex (strictly positively) 1-homogeneous lower approximation of f is given by*

$$\Gamma_{\mathtt{H}}[f]^{\top} = \begin{cases} \sigma_{\{f^{\star} \leq 0\}} & \text{if} \quad \{f^{\star} \leq 0\} \neq \emptyset\,, \\ \upsilon_{\overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))} & \text{if} \quad \{f^{\star} \leq 0\} = \emptyset\,. \end{cases} \tag{64}$$

Proof. Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be given. We will successively consider the two disjoint cases: $\{f^{\star} \leq 0\} = \emptyset$ and $\{f^{\star} \leq 0\} \neq \emptyset$.

- We consider the case where $\{f^{\star} \leq 0\} = \emptyset$. We are going to extend [19, Lemma 2.3] in order to obtain that

$$\Gamma_{\mathtt{H}}[f]^{\top} = \upsilon_{\overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))}\;. \tag{65}$$

We denote by $g = \Gamma_{\mathtt{H}}[f]^{\top}$. We observe that the function g is not proper. Indeed, otherwise g would admit a continuous affine minorant and, being also (strictly positively) 1-homogeneous, it is easily deduced that it would admit a continuous linear minorant; thus, as $g \leq f$, the function f would also admit a continuous linear minorant, which would imply that $\{f^{\star} \leq 0\} \neq \emptyset$ (contradiction). Consequently, the function g is not proper and, by [19, Lemma 2.2], we get that $g = \upsilon_{\mathrm{dom}\, g}$.

As the function $g = \upsilon_{\mathrm{dom}\, g}$ is in $\Gamma_{\mathtt{H}}[f]$, it is lsc. Hence, we obtain that $\upsilon_{\mathrm{dom}\, g}$ is lsc and, using the definition of a valley function, that the subset $\mathrm{dom}\, g$ is closed. Combined with the fact that the function g is convex and (strictly positively) 1-homogeneous as an element of $\Gamma_{\mathtt{H}}[f]$, this gives that $\mathrm{dom}\, g$ is a closed convex cone.

As $\upsilon_{\mathrm{dom}\, g} = g \leq f$, we get that $\mathrm{dom}\, f \subset \mathrm{dom}\, g$, from which we obtain that

$$\overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f)) \subset \overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, g)) = \mathrm{dom}\, g.$$

Finally, we observe that $\upsilon_{\overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))}$ is in $\Gamma_{\mathtt{H}}[f]$, and is therefore smaller than g by definition of $g = \Gamma_{\mathtt{H}}[f]^{\top}$. This implies that $\upsilon_{\overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))} \leq \upsilon_{\mathrm{dom}\, g}$ and thus $\mathrm{dom}\, g \subset \overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))$. We conclude that $\mathrm{dom}\, g = \overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))$, from which we derive the equalities $g = \upsilon_{\mathrm{dom}\, g} = \upsilon_{\overline{\mathrm{cone}}(\overline{\mathrm{co}}(\mathrm{dom}\, f))}$.

- We consider the case where $\{f^\star \leq 0\} \neq \emptyset$.

First, the function $\sigma_{\{f^\star \leq 0\}}$ is (strictly positively) 1-homogeneous convex lsc as a support function and smaller than f (by (17c)). Moreover, using using the fact that $\{f^\star \leq 0\} \neq \emptyset$ we obtain that $\sigma_{\{f^\star \leq 0\}}$ is proper. Therefore, we have by definition of $\Gamma_{\mathtt{H}}[f]^\top$ that

$$\sigma_{\{f^\star \leq 0\}} \leq \Gamma_{\mathtt{H}}[f]^\top . \tag{66}$$

Second, as the function $\Gamma_{\mathtt{H}}[f]^\top$ is (strictly positively) 1-homogeneous, by definition, and proper, as it is minorized by a proper function and thus cannot take the value $-\infty$, we have $-\infty < \Gamma_{\mathtt{H}}[f]^\top(0) \leq f(0) < +\infty$ since we have assumed $0 \in \operatorname{dom} f$. We obtain, using Lemma A.3, that the greatest proper lsc convex function majorized by $\Gamma_{\mathtt{H}}[f]^\top$ is $\sigma_{\{\Gamma_{\mathtt{H}}[f]^{\top\star} \leq 0\}}$. Now, as the function $\Gamma_{\mathtt{H}}[f]^\top$ is also lsc convex by definition, it must coincide with its best proper lsc convex lower approximation, that is, we have $\Gamma_{\mathtt{H}}[f]^\top = \sigma_{\{\Gamma_{\mathtt{H}}[f]^{\top\star} \leq 0\}}$.

Third, by definition of $\Gamma_{\mathtt{H}}[f]^\top$, we have $\Gamma_{\mathtt{H}}[f]^\top \leq f$, and thus

$$\{\Gamma_{\mathtt{H}}[f]^{\top\star} \leq 0\} \subset \{f^\star \leq 0\}.$$

We obtain
$$\Gamma_{\mathtt{H}}[f]^\top = \sigma_{\{\Gamma_{\mathtt{H}}[f]^{\top\star} \leq 0\}} \leq \sigma_{\{f^\star \leq 0\}},$$

which combined with (66) gives $\Gamma_{\mathtt{H}}[f]^\top = \sigma_{\{f^\star \leq 0\}}$.

This concludes the proof. □

We give here an instrumental lemma used in the proof of the previous Proposition A.2.

Lemma A.3. *Assume that $h : \mathcal{X} \to \overline{\mathbb{R}}$ is a (strictly positively) 1-homogeneous proper function. Then, the greatest lsc proper convex lower approximation of the function h is given by $\sigma_{\{h^\star \leq 0\}}$.*

Proof. We start by proving some properties of $h^\star$.

(i) If $h^\star(y) > 0$ then $h^\star(y) = +\infty$. Indeed, if $h^\star(y) > 0$ there exists $\alpha > 0$ and $x \in \mathcal{X}$ such that $\langle x, y\rangle - h(x) > \alpha$. Thus, for all $\lambda > 0$, we obtain that

$$h^\star(y) \geq \langle \lambda x, y\rangle - h(\lambda x) = \lambda(\langle x, y\rangle - h(x)) \geq \lambda\alpha , \tag{67}$$

and thus that $h^\star(y) = +\infty$ by letting λ go to $+\infty$.

(ii) If $h^\star(y) \leq 0$, then $h^\star(y) = 0$. Indeed, for all $\lambda > 0$ and $x \in \mathcal{X}$, we have that

$$\lambda(\langle x, y\rangle - h(x)) = \langle \lambda x, y\rangle - h(\lambda x) \leq h^\star(y) \leq 0 ,$$

and the result follows by chosing x such that $h(x) \in \mathbb{R}$, which exists as h is proper, and letting λ go to 0.

Now, we turn to the proof of the Lemma. As h is assumed to be proper, the greatest lsc proper convex lower approximation of the function h is given by the Fenchel biconjugate $h^{\star\star'}$ in (2c) of the function h, which is

$$\begin{aligned}
h^{\star\star'} &= \sup_{y\in\mathcal{Y}}(\langle\cdot,y\rangle - h^\star(y)) \\
&= \sup\big(\sup_{y\in\{h^\star>0\}}(\langle\cdot,y\rangle - h^\star(y)), \sup_{y\in\{h^\star\leq 0\}}(\langle\cdot,y\rangle - h^\star(y))\big) \\
&= \sup\big(\sup_{y\in\{h^\star>0\}}(\langle\cdot,y\rangle - (+\infty)), \sup_{y\in\{h^\star\leq 0\}}(\langle\cdot,y\rangle - 0)\big) \qquad \text{(using (i) and (ii))} \\
&= \sup(-\infty, \sigma_{\{h^\star\leq 0\}}) = \sigma_{\{h^\star\leq 0\}} \; .
\end{aligned}$$

This ends the proof. □

B. Background on *-dualities

We provide here the necessary background on *-dualities, as defined and studied by Martinez-Legaz and Singer in [12].

B.1. Canonical enlargement of a complete totally ordered group

A complete totally ordered (commutative [15]) group is a triplet $(\mathcal{A}, \leq, *)$, where $(\mathcal{A}, \leq)$ is a totally ordered set (either $a \leq b$ or $b \leq a$), $(\mathcal{A}, *)$ is a (commutative) group – such that all translations are isotone ($a \leq b \implies a * c \leq b * c$), and any nonempty (order) bounded subset admits a supremum and an infimum.

As defined and studied in [12, Sect. 1], we describe the *canonical enlargement* $(\overline{\mathcal{A}}, \leq, \dot{*}, \overset{\cdot}{*})$ of a complete totally ordered group $(\mathcal{A}, \leq, *)$ by

$$\overline{\mathcal{A}} = \mathcal{A} \cup \{-\infty\} \cup \{+\infty\} \; , \tag{68}$$

with order extended by

$$-\infty \leq a \leq +\infty \, , \; \forall a \in \overline{\mathcal{A}} \; , \tag{69}$$

and with *upper composition* $\dot{*}$ and *lower composition* $\overset{\cdot}{*}$ given by [12, Equations (1.4)–(1.8)]

$$a \dot{*} b = a \overset{\cdot}{*} b = a * b \, , \; \forall a, b \in \mathcal{A} \; , \tag{70a}$$

$$(+\infty) \dot{*} a = a \dot{*} (+\infty) = +\infty \, , \; \forall a \in \overline{\mathcal{A}} \; , \tag{70b}$$

$$(-\infty) \dot{*} a = a \dot{*} (-\infty) = -\infty \, , \; \forall a \in \mathcal{A} \cup \{-\infty\} \; , \tag{70c}$$

$$(+\infty) \overset{\cdot}{*} a = a \overset{\cdot}{*} (+\infty) = +\infty \, , \; \forall a \in \mathcal{A} \cup \{+\infty\} \; , \tag{70d}$$

$$(-\infty) \overset{\cdot}{*} a = a \overset{\cdot}{*} (-\infty) = -\infty \, , \; \forall a \in \overline{\mathcal{A}} \; . \tag{70e}$$

As $*$, the operations $\dot{*}$ and $\overset{\cdot}{*}$ are associative and commutative on $\overline{\mathcal{A}}$. The unit element e satisfies [12, Equation (1.9)]

$$a \dot{*} e = a \overset{\cdot}{*} e = a \, , \; \forall a \in \overline{\mathcal{A}} \; , \tag{71}$$

[15] See [12, bottom of page 296] for why we restrict to commutative groups.

and we extend the group inverse operation $(\cdot)^{-1}$ by [12, Equation (1.11)]

$$(+\infty)^{-1} = -\infty\,,\ (-\infty)^{-1} = +\infty\,. \tag{72}$$

With the conventions [12, Equation (1.13)]

$$\inf \emptyset = +\infty\,,\ \sup \emptyset = -\infty \qquad \text{as elements of } \overline{\mathcal{A}}\,, \tag{73}$$

we have the following properties, where a, b are any elements of $\overline{\mathcal{A}}$ and $\{a_i\}_{i\in I}$, $\{b_j\}_{j\in J}$, are any collections, indexed by any sets I, J, with values in $\overline{\mathcal{A}}$

$$(\inf_{i\in I} a_i)^{-1} = \sup_{i\in I} a_i^{-1}\,,\ (\sup_{i\in I} a_i)^{-1} = \inf_{i\in I} a_i^{-1}\,, \tag{74a}$$

$$\inf_{i\in I}(a \overset{\cdot}{*} a_i) = a \overset{\cdot}{*} \inf_{i\in I} a_i\,,\ \sup_{i\in I}(a \underset{\cdot}{*} a_i) = a \underset{\cdot}{*} \sup_{i\in I} a_i\,, \tag{74b}$$

$$\inf_{i\in I, j\in J}(a_i \overset{\cdot}{*} b_j) = \inf_{i\in I} a_i \overset{\cdot}{*} \inf_{j\in J} b_j\,,\ \sup_{i\in I, j\in J}(a_i \overset{\cdot}{*} b_j) \leq \sup_{i\in I} a_i \overset{\cdot}{*} \sup_{j\in J} b_j\,, \tag{74c}$$

$$\sup_{i\in I, j\in J}(a_i \underset{\cdot}{*} b_j) = \sup_{i\in I} a_i \underset{\cdot}{*} \inf_{j\in J} b_j\,,\ \inf_{i\in I, j\in J}(a_i \underset{\cdot}{*} b_j) \geq \inf_{i\in I} a_i \underset{\cdot}{*} \inf_{j\in J} b_j\,, \tag{74d}$$

$$b < +\infty \implies \inf_{i\in I}(a_i \underset{\cdot}{*} b) = (\inf_{i\in I} a_i) \underset{\cdot}{*} b\,, \tag{74e}$$

$$-\infty < b \implies \sup_{i\in I}(a_i \overset{\cdot}{*} b) = (\sup_{i\in I} a_i) \overset{\cdot}{*} b\,, \tag{74f}$$

and the following properties, where a, b, c are any elements of $\overline{\mathcal{A}}$,

$$b \leq c \implies a \overset{\cdot}{*} b \leq a \overset{\cdot}{*} c\,,\ a \underset{\cdot}{*} b \leq a \underset{\cdot}{*} c\,, \tag{75a}$$

$$(a \overset{\cdot}{*} b)^{-1} = a^{-1} \underset{\cdot}{*} b^{-1}\,,\ (a \underset{\cdot}{*} b)^{-1} = a^{-1} \overset{\cdot}{*} b^{-1}\,, \tag{75b}$$

$$a \underset{\cdot}{*} b \leq a \overset{\cdot}{*} b\,, \tag{75c}$$

$$a^{-1} \overset{\cdot}{*} b^{-1} \geq (a \overset{\cdot}{*} b)^{-1}\,,\ a^{-1} \underset{\cdot}{*} b^{-1} \leq (a \underset{\cdot}{*} b)^{-1}\,, \tag{75d}$$

$$a \overset{\cdot}{*} a^{-1} \geq e\,,\ a \underset{\cdot}{*} a^{-1} \leq e\,, \tag{75e}$$

$$a \underset{\cdot}{*} b^{-1} \leq e \iff a \leq b \iff e \leq b \overset{\cdot}{*} a^{-1}\,, \tag{75f}$$

$$a \underset{\cdot}{*} b^{-1} \leq c \iff a \leq b \overset{\cdot}{*} c \iff a \underset{\cdot}{*} c^{-1} \leq b\,, \tag{75g}$$

$$c \leq b \overset{\cdot}{*} a^{-1} \iff a \underset{\cdot}{*} c \leq b \iff a \leq b \overset{\cdot}{*} c^{-1}\,, \tag{75h}$$

$$(a \overset{\cdot}{*} b) \underset{\cdot}{*} c \leq a \overset{\cdot}{*} (b \underset{\cdot}{*} c)\,. \tag{75i}$$

B.2. *-duality

Let be given two sets $\mathcal{X}$ (“primal”), $\mathcal{Y}$ (“dual”), together with a *coupling* function $c\colon \mathcal{X} \times \mathcal{Y} \to \overline{\mathcal{A}}$. We define a mapping $\mathcal{D}(c)\colon \overline{\mathcal{A}}^{\mathcal{X}} \to \overline{\mathcal{A}}^{\mathcal{Y}}$ as follows: for any function $f\colon \mathcal{X} \to \overline{\mathcal{A}}$, we define the function $\mathcal{D}(c)f\colon \mathcal{Y} \to \overline{\mathcal{A}}$, denoted $f^{\mathcal{D}(c)}$, by [12, Equation (2.9)]

$$f^{\mathcal{D}(c)}(y) = \sup_{x\in\mathcal{X}} \big(c(x,y) \underset{\cdot}{*} (f(x))^{-1} \big)\,,\ \forall y \in \mathcal{Y}\,. \tag{76a}$$

By [12, Corollary 2.1], and using [12, Lemma 1.5] (which establishes equation (75g)), we have that [16]

$$f^{\mathcal{D}(c)}(y) = \inf\{a \in \mathcal{A} \mid c(x,y) \leq f(x) \dot{*} a \,,\ \forall x \in \mathcal{X}\} \,, \tag{76b}$$

and, by [12, Theory 2.1], $\mathcal{D}(c)$ satisfies the following properties, that define a *-duality [12, Definition 2.3]:

$$(\inf_{i \in I} f_i)^{\mathcal{D}(c)} = \sup_{i \in I} f_i^{\mathcal{D}(c)} \,,\ \forall \{f_i\}_{i \in I} \subset \overline{\mathcal{A}}^{\mathcal{X}} \,, \tag{77a}$$

$$(f \dot{*} a)^{\mathcal{D}(c)} = f^{\mathcal{D}(c)} \,\underset{\cdot}{*}\, a^{-1} \,,\ \forall a \in \overline{\mathcal{A}} \,. \tag{77b}$$

We also have that (deduced from [12, Equation (2.27)]):

$$f^{\mathcal{D}(c)\mathcal{D}(c)'} \leq f \,. \tag{78}$$

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