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Estimating Random Coefficients in Mixed Variational Problems with an Application to the Stochastic Elasticity Imaging Inverse Problem for Tumor Identification

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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This research investigates the estimation of the mean and variance of stochastic coefficients in mixed variational problems, with an application to the elasticity imaging inverse problem for tumor localization. We study an abstract mixed variational problem, where the coefficients and the right-hand side are strongly measurable functions. By establishing pathwise mixed variational problems, we demonstrate unique solvability for each realization and prove the measurability of pathwise solutions. This result facilitates the derivation of an integral formulation, which is crucial for treating the inverse problem as a stochastic optimization problem, as well as developing a discretization framework for both direct and inverse problems. In Sobolev-Bochner spaces, we establish existence results for integral solutions and obtain basic estimates. We analyze the coefficient-to-solution map for the inverse problem, proving its Lipschitz continuity and characterizing its first- and second-order derivatives. Two optimization approaches are introduced: the classical least-squares and energy minimization, with the latter shown to be convex. A pathwise approach addresses the compact embedding constraint, enabling the integral formulation to be discretized and solved using a stochastic Galerkin scheme. Finally, we provide numerical examples that demonstrate the estimation of the unknown coefficient's mean and variance from data using discrete stochastic methods.

Keywords: Stochastic mixed variational problems, stochastic inverse problems, partial differential equations with random data, nearly incompressible elasticity system, stochastic Galerkin method, regularization, finite-dimensional noise.

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1. Introduction

The primary objective of this research is to develop an abstract inversion framework for estimating the mean and variance of random coefficients in stochastic mixed

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variational problems. Many applied modules, such as nearly incompressible elasticity equations, Stokes equations, and fourth-order plate models, can be conveniently expressed as mixed variational problems. Therefore, the abstract inversion framework is applicable to identification problems across these models. A key motivation for this study is to establish robust uncertainty quantification techniques for non-invasive early disease detection, with a particular focus on the elasticity imaging inverse problem of locating cancerous tumors in the human body. To elaborate, we note that various diseases trigger alterations in tissue microstructure, consequently influencing macroscopic tissue properties. These properties, in turn, serve as accessible markers for evaluating both the microstructure of tissues and overall health status. Consider, for example, the progression of breast cancer, where the presence of cancerous cells within the milk duct prompts heightened activity of fibroblasts, thereby increasing collagen recruitment. This augmented collagen density manifests as heightened tissue stiffness on a macroscopic scale, forming discernible tumors. The tumors are typically found via palpation, which assesses the stiffness of the tissue in the vicinity of the cancer. However, while widely practiced, breast palpation is inherently subjective and not a particularly sensitive or discriminating diagnostic technique. In contrast, the elasticity imaging inverse problem revolutionizes this process by enabling the spatial discrimination and quantification of elastic coefficients using a system of linear or nonlinear equations of motion. The technique relies on the measurement of tissue states as captured by medical imaging modalities, such as the motion of tissue in response to a stimulus. This shift toward quantifiable coefficients enhances accuracy and eliminates the subjectivity associated with manual palpation, see [4, 12, 23]. Besides tumor identification, other examples of elasticity imaging inverse problems include the study of cirrhosis in the liver and atherosclerosis in arteries.

From a mathematical standpoint, the existing research on elasticity imaging inverse problem has primarily focused on estimating deterministic coefficients in nearly incompressible linear elasticity equations. However, treating these coefficients and the physical models as deterministic is unsatisfactory because they describe characteristics of a subject's body, where structural variations exist from patient to patient. This strongly implies inherent uncertainty within the model. Additionally, the process of collecting patient data is inherently noisy, and the boundary conditions carry natural uncertainty. This situation is commonly encountered when the displacement field is measured using ultrasound, see Seidl [44]. A sensible approach would be to consider both coefficients and data as stochastic. However, this decision renders the underlying mathematical models stochastic as well, leading to a plethora of technical difficulties and challenges.

Driven by the aforementioned notable gap in existing research, we aim to construct a comprehensive framework for determining both deterministic and stochastic material properties in stochastic mixed variational problems, which represent the weak formulation of a nearly incompressible linear elasticity system, the mathematical model behind the elasticity imaging inverse problem. In this study, we focus on the variational approach, which has emerged as a promising alternative to the Bayesian methodology, addressing some of its limitations. The variational approach involves formulating an optimization problem that provides statistical insights into unknown coefficients. It is particularly effective for estimating distributed and spatially correlated coefficients in partial differential equations (PDEs) with random data. A key advantage of this approach is the wide range of optimization algorithms available, backed by strong theoretical foundations. Additionally, it enables the direct incorporation of coefficient information into the inversion framework, making it a valuable tool for estimating random coefficients and controls in PDEs with random data – an area of growing interest in the field. See [18, 19, 20, 24, 28, 38, 39, 41, 47, 49, 50] and the cited references.

1.1. Literature review

Before proceeding, we will briefly review the contributions of related works. As previously mentioned, numerous authors have explored the use of elastic properties of soft tissue to differentiate between normal and cancerous tissue. Notably, Raghavan and Yagle [43] were among the first to recognize that this problem is best approached within an inverse problem framework, where elasticity can be reconstructed from measured strains and equilibrium equations. Kallel and Bertrand [37] utilized the Newton-Raphson method to adjust a finite-element-based model of the elasticity equations. This model was fitted to a set of axial tissue displacement fields in a least-squares sense, with the displacement fields estimated using a correlation technique applied to ultrasound signals. Barbone et al. [10] analytically demonstrated that the inverse problem of identifying the elasticity modulus lacks a unique solution unless sufficient a priori knowledge of stiffness data is available for a significant portion of the boundary. Their work implies that relative stiffness images, constructed under the assumption of constant boundary stiffness, are valid only if the boundary stiffness is indeed constant. They also explored other factors affecting the uniqueness of solutions. Subsequently, Barbone et al. [11] examined the uniqueness of a generalized N-field inverse problem for elastic modulus identification, where N linearly independent displacement fields in an incompressible elastic material are known. Fehrenbach et al. [25] recovered the spatial distribution of Young's modulus under the assumption that the input data are directional displacements in the plane domain under a small quasistatic compression force. Recently, Arnold et al. [7] developed a computationally efficient numerical procedure for identifying the elasticity modulus by employing clustering techniques. Ammari et al. [5] introduced an innovative optimization approach based on a discrepancy functional. Their work demonstrated that, to recover a complex anomaly within a homogeneous medium, it is essential to decompose the wavefield into a *near-field* component, localized around the anomaly, and a *far-field* component, extending further away. By distinguishing these scales, they achieved a more accurate and localized reconstruction. Ammari et al. [3] explored innovative topological derivative-based detection algorithms aimed at localizing elastic inclusions with vanishingly small characteristic sizes. Gokhale et al. [27] examined an inverse problem in nonlinear elasticity imaging, focusing on the recovery of spatial distributions of hyperelastic material properties from measured displacement fields. Seidl et al. [46] simultaneously inverted for a heterogeneous shear modulus field and traction boundary conditions within an incompressible linear elastic plane stress model. For insights into the dynamic analogue of the elasticity imaging inverse problem, we direct the interested reader to Ji and McLaughlin [35] and the references cited therein. Davies et al. [22] focused on magnetic resonance elastography to study the elastic shear modulus. Bretin et al. [17] investigated the stability of finite element discretization in solving inverse problems from internal data, testing the feasibility of this approach in elastography. Sherina et al. [48] developed an algorithm to estimate the internal displacement field of an object undergoing deformation, utilizing optical coherence tomography images before and after compression. This displacement field is then applied in quantitative elastography to estimate elasticity coefficients. Hubmer et al. [30] examined the Landweber iteration for quantitative static elastography, aiming at estimating the Lamé coefficients from internal displacement field data. Fehrenbach et al. [25] identified Young's modulus by employing the Gauss-Newton method, which notably does not require the calculation of the Jacobian matrix. Awasthi et al. [8] proposed a new physics-based method for the computational design of soft hyperelastic bodies, allowing for controlled deformation. This method takes as input the undeformed shape of a body, a specified external load, and a desired final shape. See also [33, 31, 32].

Though limited in number, interesting studies have explored uncertainty quantification in the elasticity imaging inverse problem. Aguiló et al. [1] conducted an in-depth study on large-scale stochastic coefficient estimation to determine the material properties of a region of interest based on an observed displacement field. They employed a maximum a posteriori (MAP) estimation approach to account for uncertainties in both the data and the model, thereby capturing the associated uncertainty in the coefficient estimates. The study also demonstrated the performance of first-order formulations within MAP Bayesian analysis for stochastic coefficient estimation, see also [2]. Bochud and Rus [14] investigated a probabilistic method to characterize the mechanical properties of tissue-equivalent materials. Patel et al. [42] developed an efficient and accurate Bayesian inversion technique using deep generative models. Franck and Koutsourelakis [26] introduced an efficient Bayesian framework for solving nonlinear, high-dimensional model calibration problems, demonstrating the effectiveness of their approach in the context of nonlinear elastography. Warner et al. [55] presented a stochastic reduced-order model for inverse problems under uncertainty, applying it to the identification of random material coefficients in elasto-dynamical systems, and demonstrating the efficiency of their method. Seidl et al. [45] highlighted the significant challenge in inverse problems with interior data, particularly in addressing uncertainty in the boundary conditions of the forward or state model. They specifically examined a linear plane stress inverse elasticity problem with measured displacement data, where one component of the measured displacement field is known with considerably greater precision than the other.

1.2. Main contributions

The key contributions of this work are as follows:

(i) To estimate coefficients in a stochastic linear elasticity system, we focus on a mixed variational problem, where the coefficients and right-hand side are strongly measurable functions. To this variational problem, we associate a pathwise mixed variational problem. By invoking some known results, we prove the unique solvability of the pathwise problem for each realization and give certain bounds (see Theorem 2.1). A key result in this context is the measurability of the pathwise solutions (see Theorem 2.6), which is established by proving the continuous dependence of the pathwise solution map on all sources of randomness (see Theorem 2.2). This result enables us to consider an integral formulation, which is crucial for treating the inverse problem as a stochastic optimization problem, as well as for developing a comprehensive discretization framework for both direct and inverse problems. For the integral mixed variational problem posed in Sobolev-Bochner spaces, we derive the existence of integral solutions and establish basic estimates for them (see Theorem 2.7).

(ii) For the inverse problem of estimating coefficients in mixed variational problems, we investigate the analytic properties of the coefficient-to-solution map, establishing its Lipschitz continuity (cf. Theorem 3.1). Additionally, we characterize the first- and second-order derivatives of the coefficient-to-solution map for both the pathwise and integral formulations (cf. Theorems 3.3 and 3.4). The inverse problem is formulated through two distinct optimization approaches: the first uses a classical least-squares objective functional, while the second employs energy minimization, which is proven to be convex (see Theorem 4.1). Existence results for the minimization problems are established, with the compact embedding condition being one of the more restrictive assumptions (see Theorem 4.2). A significant contribution is our pathwise approach to addressing this constraint. Specifically, we consider a pathwise formulation of the inverse problem, prove the feasibility of the pathwise solution, and subsequently derive the integral formulation of the inverse problem (see Theorem 5.1). This formulation can then be discretized and solved using a stochastic Galerkin scheme.

(iii) We present a comprehensive discretization framework that employs Stochastic Galerkin techniques. Discrete formulations are provided for the mixed variational problem, the energy objective functionals, and the gradient and Hessian.

(iv) We include two numerical examples to estimate the mean and variance of the unknown coefficient from finite-dimensional noisy data.

1.3. Comparing our key contributions to recent advances in stochastic inverse problems

The variational approach to stochastic inverse problems, which forms the basis of our study, was pioneered by [9], where the authors examined inverse heat conduction problems involving uncertainty in the input data. However, their focus was more on practical considerations than on establishing mathematical rigor. In contrast, Jin and Zou [36] developed a rigorous mathematical framework for the nonlinear stochastic inverse problem of estimating the Robin coefficient from boundary measurements in steady-state heat conduction. A closely related study by Borggaard and Wyk [15] utilized an OLS-based stochastic optimization framework to estimate a random coefficient in the stochastic diffusion equation. This work was further extended by [34], who introduced an inversion framework based on a convex energy least-squares (ELS) approach for estimating stochastic coefficients in the stochastic diffusion equation. The ELS framework was later generalized in [21] to estimate random coefficients in stochastic variational problems using a gradient-based optimization method and providing numerical results for diffusion equations, fourth-order equations, and linear elasticity with random data. Our work builds on [21], particularly in the context of incompressible linear elasticity. However, we emphasize that the incompressible case exhibits the locking effect, which prevents its treatment within the framework of [21], which is only suited for linear stochastic PDEs.

We note that our variational model is not only more general than those in previous studies (see [9, 15, 21, 34, 36]), but it also introduces several improvements, even in the simplest case of the stochastic diffusion equation. Notably, we do not assume that the unknown coefficient is represented as a linear expansion, as was common in prior studies. Instead, we develop a complete discretization framework without such assumptions. Furthermore, we provide an explicit, computable formula for the Hessian, which enables the use of more efficient second-order methods—a significant advancement, as second-order approaches have been largely absent in the context of stochastic inverse problems.

2. Mixed variational problems with random data

This section studies a general mixed variational problem with random data. We begin with a pathwise formulation, addressing the strong measurability of the pathwise solution, which allows us to introduce the integral formulation.

Let B be a real Banach space and let K be a nonempty, closed, and convex subset of B. Let V and Q be separable Hilbert spaces with V^* and Q^* as their dual spaces, respectively. We denote the norm of a normed space N by $\|\cdot\|_N$ and the duality pairing between N and its dual space N^* by $\langle\cdot,\cdot\rangle_N$. We denote the inner product of a Hilbert space N by $(\cdot,\cdot)_N$. The strong convergence and the weak convergence are denote by $\to$ and $\rightharpoonup$, respectively. Let $a : B\times V\times V \to \mathbb{R}$ be a trilinear map which we assume to be symmetric with respect to the second and the third arguments, let $b : V\times Q \to \mathbb{R}$ be a bilinear form, and let $c : Q\times Q \to \mathbb{R}$ be a symmetric bilinear form. We assume that there are positive constants $\kappa_1, \kappa_2, \varsigma_1, \varsigma_2$, and κ_0 such that

$$a(\ell,v,v) \geq \kappa_1\|v\|_V^2, \qquad \text{for all } v\in V,\ \ell\in K, \tag{2.1a}$$

$$|a(\ell,u,v)| \leq \kappa_2\|\ell\|_B\|u\|_V\|v\|_V, \qquad \text{for all } u,v\in V,\ \ell\in B, \tag{2.1b}$$

$$c(q,q) \geq \varsigma_1\|q\|_Q^2, \qquad \text{for all } q\in Q, \tag{2.1c}$$

$$|c(p,q)| \leq \varsigma_2\|p\|_Q\|q\|_Q, \qquad \text{for all } p,q\in Q, \tag{2.1d}$$

$$|b(v,q)| \leq \kappa_0\|v\|_V\|q\|_Q, \qquad \text{for all } v\in V,\ q\in Q. \tag{2.1e}$$

Let $(\Omega,\mathbb{F},\mathbb{P})$ be a complete probability space. We will soon introduce the required measurability conditions for the involved data. For now, assume that $\ell : \Omega\to B$ is such that $\ell(\omega)\in K$ for each $\omega\in\Omega$. Additionally, assume that $f : \Omega\to V^*$ and $g : \Omega\to Q^*$ are such that $f(\omega)\in V^*$ and $g(\omega)\in Q^*$ for each $\omega\in\Omega$. For simplicity, we define $\ell_\omega\equiv\ell(\omega)$, $f_\omega\equiv f(\omega)$, and $g_\omega\equiv g(\omega)$.

For any $\omega\in\Omega$, and fixed $\ell_\omega\in K$, $f_\omega\in V^*$, and $g_\omega\in Q^*$, we consider the pathwise stochastic mixed variational problem of finding

$$(u_\omega,p_\omega) := (u_\omega(\ell_\omega,f_\omega,g_\omega), p_\omega(\ell_\omega,f_\omega,g_\omega)) \in V\times Q$$

such that
$$a(\ell_\omega, u_\omega, v) + b(v, p_\omega) = \langle f_\omega, v\rangle_V, \quad \text{for every } v \in V, \tag{2.2a}$$
$$b(u_\omega, q) - c(p_\omega, q) = \langle g_\omega, q\rangle_Q, \quad \text{for every } q \in Q. \tag{2.2b}$$

We have the following result concerning the above mixed variational problem, where $W := V \times Q$.

Theorem 2.1. *For any $\omega \in \Omega$, and fixed $\ell_\omega \in K$, $f_\omega \in V^*$, and $g_\omega \in Q^*$, the pathwise stochastic mixed variational problem* (2.2) *has a unique solution $w_\omega := (u_\omega, p_\omega) \in W$. Moreover, the following estimates hold:*

$$\|u_\omega\|_V \le c_1 \left(\|f_\omega\|_{V^*} + \|g_\omega\|_{Q^*}\right), \quad c_1 := \max\left\{\frac{1}{\kappa_1}, \frac{1}{\sqrt{\kappa_1 \varsigma_1}}\right\}, \tag{2.3a}$$
$$\|p_\omega\|_Q \le c_2 \left(\|f_\omega\|_{V^*} + \|g_\omega\|_{Q^*}\right), \quad c_2 := \max\left\{\frac{1}{\varsigma_1}, \frac{1}{\sqrt{\kappa_1 \varsigma_1}}\right\}. \tag{2.3b}$$

Proof. For any $\omega \in \Omega$, and fixed $\ell_\omega \in K$, $f_\omega \in V^*$, and $g_\omega \in Q^*$, the existence of a unique solution $(u_\omega, p_\omega) \in W$ of (2.2) is a direct consequence of the Lax-Milgram Lemma. To prove (2.3), we set $(v, q) = (u_\omega, p_\omega)$ in (2.2), subtract the resulting two equations, and use (2.1a) and (2.1c) to obtain

$$\kappa_1 \|u_\omega\|_V^2 + \varsigma_1 \|p_\omega\|_Q^2 \le a(\ell_\omega, u_\omega, u_\omega) + c(p_\omega, p_\omega) \le \|f_\omega\|_{V^*}\|u_\omega\|_V + \|g_\omega\|_{Q*}\|p_\omega\|_Q,$$

which, due to the inequality $2ab \le a^2 + b^2$, which holds for all $a, b \in \mathbb{R}$, can be written as follows:

$$\begin{aligned}\kappa_1 \|u_\omega\|_V^2 + \varsigma_1 \|p_\omega\|_Q^2 &\le \frac{1}{\kappa_1}\|f_\omega\|_{V^*}^2 + \frac{1}{\varsigma_1}\|g_\omega\|_{Q^*}^2 \\ &\le \max\left\{\frac{1}{\kappa_1}, \frac{1}{\varsigma_1}\right\} \left(\|f_\omega\|_{V^*} + \|g_\omega\|_{Q^*}\right)^2. \end{aligned} \tag{2.4}$$

Estimates (2.3a) and (2.3b) follow at once from (2.4). □

In the stochastic mixed variational problem (2.2), three distinct sources of uncertainty are present: the coefficient ℓ and the right-hand sides f and g. In the following result, we first provide basic estimates on the dependence of the unique solution of the mixed variational problem on each source of uncertainty.

Theorem 2.2. *For each $\omega \in \Omega$, for fixed $f_\omega \in V^*$ and $g_\omega \in Q^*$, and for any $\ell_\omega^1, \ell_\omega^2 \in K$, the following estimates hold for the solutions $w_\omega(\ell_\omega^1) = (u_\omega(\ell_\omega^1), p_\omega(\ell_\omega^1))$ and $w_\omega(\ell_\omega^2) = (u_\omega(\ell_\omega^2), p_\omega(\ell_\omega^2))$ of the pathwise mixed variational problem* (2.2)*:*

$$\|u_\omega(\ell_\omega^1) - u_\omega(\ell_\omega^2)\|_V \le \frac{c_1 \kappa_2}{\kappa_1}\left(\|f\|_{V^*} + \|g\|_{Q^*}\right)\|\ell_\omega^1 - \ell_\omega^2\|_B, \tag{2.5a}$$

where $c_1 := \max\left\{\frac{1}{\kappa_1}, \frac{1}{\sqrt{\kappa_1 \varsigma_1}}\right\}$,

$$\|p_\omega(\ell_\omega^1) - p_\omega(\ell_\omega^2)\|_Q \le \frac{c_1 \kappa_2}{\sqrt{\kappa_1 \varsigma_1}}\left(\|f\|_{V^*} + \|g\|_{Q^*}\right)\|\ell_\omega^1 - \ell_\omega^2\|_B. \tag{2.5b}$$

Furthermore, for each $\omega \in \Omega$, for fixed $\ell_\omega \in K$ and $g_\omega \in Q^*$, and for any $f_\omega^1, f_\omega^2 \in V^*$, the following estimates must hold for the solutions $w_\omega(f_\omega^1) = (u_\omega(f_\omega^1), p_\omega(f_\omega^1))$ and $w_\omega(f_\omega^2) = (u_\omega(f_\omega^2), p_\omega(f_\omega^2))$ of the pathwise mixed variational problem (2.2)*:*

$$\|u_\omega(f_\omega^1) - u_\omega(f_\omega^2)\|_V \le \frac{1}{\kappa_1}\|f_\omega^1 - f_\omega^2\|_{V^*}, \tag{2.6a}$$

$$\|p_\omega(f_\omega^1) - p_\omega(f_\omega^2)\|_Q \le \frac{1}{\varsigma_1\sqrt{\kappa_1}}\|f_\omega^1 - f_\omega^2\|_{V^*}. \tag{2.6b}$$

Moreover, for each $\omega \in \Omega$, for fixed $\ell_\omega \in K$ and $f_\omega \in V^*$, and for any $g_\omega^1, g_\omega^2 \in Q^*$, the following estimates must hold for the solutions $w_\omega(g_\omega^1) = (u_\omega(g_\omega^1), p_\omega(g_\omega^1))$ and $w_\omega(g_\omega^2) = (u_\omega(g_\omega^2), p_\omega(g_\omega^2))$ of the pathwise mixed variational problem (2.2)*:*

$$\|u_\omega(g_\omega^1) - u_\omega(g_\omega^2)\|_V \le \frac{1}{\varsigma_1}\|g_\omega^1 - g_\omega^2\|_{Q^*}, \tag{2.7a}$$

$$\|p_\omega(g_\omega^1) - p_\omega(g_\omega^2)\|_Q \le \frac{1}{\kappa_1\sqrt{\varsigma_1}}\|g_\omega^1 - g_\omega^2\|_{Q^*}. \tag{2.7b}$$

Proof. We will provide a proof of the above estimates in the integral formulation given below. □

Before moving forward, we want to revisit certain concepts of measurability that will be used shortly. For details on these topics, see [13].

Definition 2.3. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space, and let $(X, \mathcal{B})$ be a measurable space where X is a Banach space with dual X^* and $\mathcal{B}$ is the σ-algebra of all Borel subsets of X.

(i) A mapping $\chi : \Omega \to X$ is called a *Banach space-valued Borel measurable function* if $\chi^{-1}(B) \in \mathbb{F}$ for every $B \in \mathcal{B}$.

(ii) A mapping $s : \Omega \to X$ is said to be a *finite-valued random variable* if it is constant on each of a finite number of disjoint sets $A_i \in \mathbb{F}$ and equal to 0_X on $\Omega \setminus (\cup_{i=1}^n A_i)$. It is called a simple random variable if it is finitely-valued and $\mathbb{P}(\omega : \|s(\omega)\| > 0) < \infty$.

(iii) A mapping $\chi : \Omega \to X$ is a $\mathbb{P}$-*almost separably-valued random variable* if there exists a set $A_0 \in \mathbb{F}$ such that $\mathbb{P}(A_0) = 0$ and $\chi(\Omega \setminus A_0)$ is separable.

(iv) A mapping $\chi : \Omega \to X$ is a *strong* (or *Bochner*) *random variable* if there exists a sequence $s_n(\omega)$ of simple random variables which converges to $\chi(\omega)$ almost surely.

(v) A mapping $\chi : \Omega \to X$ is called a weak (or Pettis) random variable if the functions $\langle \chi^*, \chi(\omega)\rangle$ are real-valued random variables for each $\chi^* \in X^*$.

Remark 2.4. It is known that a mapping $\chi : \Omega \to X$ is a strong random variable if and only if it is a weak random variable and $\mathbb{P}$-almost separably-valued. This result confirms that if $h : \Omega \to X$ is continuous, then h is strongly measurable. Moreover, if X is separable, then the given notions of random variables coincide. □

We also recall the following result (see [51, Corollary 1.13]):

Theorem 2.5. *Let $(A, \mathcal{A}, \mu)$ be a σ-finite measure space, and let E and F be Banach spaces. If an E-valued function $\mathcal{F}$ is strongly μ-measurable and $\varphi : E \to F$ is continuous, then $\varphi \circ \mathcal{F}$ is strongly μ-measurable.*

The following result gives the measurability of the pathwise solution map associated with (2.2).

Theorem 2.6. *Suppose that $\ell : \omega \in \Omega \mapsto \ell_\omega \in B$, $f : \omega \in \Omega \mapsto f_\omega \in V^*$, and $g : \omega \in \Omega \mapsto g_\omega \in Q^*$ are strongly measurable random variables. Then the solution map $w(\ell, f, g) : \omega \in \Omega \mapsto w_\omega(\ell_\omega, f_\omega, g_\omega) \in W$, associated with the pathwise stochastic mixed variational problem* (2.2) *is strongly measurable.*

Proof. As a direct consequence of Theorem 2.2, it follows that for every fixed $\omega \in \Omega$, we have

$$\lim_{n\to\infty} w(\ell^n_\omega, f^n_\omega, g^n_\omega) = w(\ell_\omega, f_\omega, g_\omega),$$

for every $(\ell^n_\omega, f^n_\omega, g^n_\omega)$ converging to $(\ell_\omega, f_\omega, g_\omega)$ in $B \times V^* \times Q^*$. In particular, the map $w : B \times V^* \times Q^* \to V \times Q$ is continuous. On the other hand, the map $\mathcal{F} : \Omega \to \mathcal{F}(\omega) = (\ell_\omega, f_\omega, g_\omega)$ is strongly measurable by assumption. Thus, the strong measurability of the solution map, which can be viewed as a composition $w \circ \mathcal{F} : \Omega \to V \times Q$, follows at once from Theorem 2.5. The proof is complete. □

As an important implication of the measurability of the solution map $\omega \to w_\omega := (u_\omega, p_\omega)$ of (2.2), proved in Theorem 2.6, we consider the integral stochastic mixed variational problem. For this, we first recall that given a real Banach space X, the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and $p \in [1, \infty)$, the Bochner space $L^p(\Omega; X)$ contains (equivalence classes of) Bochner integrable functions $u : \Omega \to X$ with finite p-th moment, that is,

$$\|u\|_{L^p(\Omega;X)} := \left(\int_\Omega \|u(\omega)\|^p_X d\mathbb{P}(\omega) \right)^{1/p} =: \mathbb{E}\left[\|u\|^p_X\right]^{1/p} < \infty,$$

where $\mathbb{E}$ is the expectation operator. If $p = \infty$, then $L^\infty(\Omega; X)$ is the space of (equivalence classes of) Bochner measurable functions $u : \Omega \to X$ with $\operatorname{ess\,sup}_{\omega\in\Omega} \|u(\omega)\|_X < \infty$. See [40] for details.

Given $\ell \in L^\infty(\Omega; B)$, $f \in L^2(\Omega; V^*)$, and $g \in L^2(\Omega; Q^*)$, we consider the integral stochastic mixed variational problem of finding

$$(u, p) = (u(\ell, f, g), p(\ell, f, g)) \in L^2(\Omega; V) \times L^2(\Omega; Q)$$

such that

$$\mathbb{E}\left[a(\ell, u, v)\right] + \mathbb{E}\left[b(v, p)\right] = \mathbb{E}\left[\langle f, v\rangle_V\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{2.8a}$$

$$\mathbb{E}\left[b(u, q)\right] - \mathbb{E}\left[c(p, q)\right] = \mathbb{E}\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in L^2(\Omega; Q). \tag{2.8b}$$

The following is a unique solvability result for the integral stochastic mixed variational problem (2.8).

Theorem 2.7. *Assume that $\ell \in L^\infty(\Omega; B)$, $f \in L^2(\Omega; V^*)$ and $g \in L^2(\Omega; Q^*)$. Then the integral stochastic mixed variational problem* (2.8) *has a unique solution $(u, p) = (u(\ell, f, g), p(\ell, f, g)) \in L^2(\Omega; V) \times L^2(\Omega; Q)$. Moreover, the following estimates hold:*

$$\|u\|_{L^2(\Omega;V)} \le c_1 \left(\|f\|_{L^2(\Omega;V^*)} + \|g\|_{L^2(\Omega;Q^*)} \right), \quad c_1 := \max\left\{ \frac{1}{\kappa_1}, \frac{1}{\sqrt{\kappa_1 \varsigma_1}} \right\}, \tag{2.9a}$$

$$\|p\|_{L^2(\Omega;Q)} \le c_2 \left(\|f\|_{L^2(\Omega;V^*)} + \|g\|_{L^2(\Omega;Q^*)} \right), \quad c_2 := \max\left\{ \frac{1}{\varsigma_1}, \frac{1}{\sqrt{\kappa_1 \varsigma_1}} \right\}. \tag{2.9b}$$

Proof. The unique solvability of (2.8) follows by using the Lax-Milgram Lemma in the Hilbert space $L^2(\Omega; V) \times L^2(\Omega; Q)$. The bound (2.9) follows by using the arguments used in Theorem 2.1. □

Remark 2.8. The estimate (2.9) can also be proved directly from the estimate (2.3).

3. Analytic properties of the solution map

Let $\mathbb{K} \subset L^\infty(\Omega; B)$ be a nonempty, closed, and convex set of feasible B-valued random variables. For convenience we recall the standing assumption that for any $\omega \in \Omega$, we have $\ell_\omega \in K$.

We have the following result concerning the Lipschitz continuity of the coefficient-to-solution map.

Theorem 3.1. *If $f \in L^2(\Omega; V^*)$ and $g \in L^2(\Omega; Q^*)$ are fixed, then for any $\ell_1, \ell_2 \in \mathbb{K}$, the following estimates hold for the corresponding solutions $w(\ell_1) := (u(\ell_1), p(\ell_1))$ and $w(\ell_2) := (u(\ell_2), p(\ell_2))$ of the integral stochastic mixed variational problem* (2.8):

$$\|u(\ell_1) - u(\ell_2)\|_{L^2(\Omega;V)} \leq \frac{c_1\kappa_2}{\kappa_1} \left(\|f\|_{L^2(\Omega;V^*)} + \|g\|_{L^2(\Omega;Q^*)}\right) \|\ell_1 - \ell_2\|_{L^2(\Omega;B)}, \tag{3.1a}$$

$$\text{where} \quad c_1 := \max\left\{\frac{1}{\kappa_1}, \frac{1}{\sqrt{\kappa_1\varsigma_1}}\right\},$$

$$\|p(\ell_1) - p(\ell_2)\|_{L^2(\Omega;Q)} \leq \frac{c_1\kappa_2}{\sqrt{\kappa_1\varsigma_1}} \left(\|f\|_{L^2(\Omega;V^*)} + \|g\|_{L^2(\Omega;Q^*)}\right) \|\ell_1 - \ell_2\|_{L^2(\Omega;B)}. \tag{3.1b}$$

Furthermore, for fixed $\ell \in \mathbb{K}$ and $g \in L_2(\Omega, Q^*)$, and for any $f_1, f_2 \in L^2(\Omega; V^*)$, the following estimates must hold for the solutions $w(f_1) = (u(f_1), p(f_1))$ and $w(f_2) = (u(f_2), p(f_2))$ of the integral stochastic mixed variational problem (2.8):

$$\|u(f_1) - u(f_2)\|_{L^2(\Omega;V)} \leq \frac{1}{\kappa_1}\|f_1 - f_2\|_{L^2(\Omega;V^*)}, \tag{3.2a}$$

$$\|p(f_1) - p(f_2)\|_{L^2(\Omega;Q)} \leq \frac{1}{\varsigma_1\sqrt{\kappa_1}}\|f_1 - f_2\|_{L^2(\Omega;V^*)}. \tag{3.2b}$$

Moreover, for fixed $\ell \in \mathbb{K}$ and $f \in L^2(\Omega; V^*)$, and for any $g_1, g_2 \in L^2(\Omega; Q^*)$, the following estimates must hold for the solutions $w(g_1) = (u(g_1), p(g_1))$ and $w(g_2) = (u(g_2), p(g_2))$ of the integral stochastic mixed variational problem (2.8):

$$\|u(g_1) - u(g_2)\|_{L^2(\Omega;V)} \leq \frac{1}{\varsigma_1}\|g_1 - g_2\|_{L^2(\Omega;Q^*)}, \tag{3.3a}$$

$$\|p(g_1) - p(g_2)\|_{L^2(\Omega;Q)} \leq \frac{1}{\kappa_1\sqrt{\varsigma_1}}\|g_1 - g_2\|_{L^2(\Omega;Q^*)}. \tag{3.3b}$$

Proof. By the following definitions of the solutions $w(\ell_1) := (u(\ell_1), p(\ell_1))$ and $w(\ell_2) := (u(\ell_2), p(\ell_2))$, we have

$$\mathbb{E}\left[a(\ell_1, u(\ell_1), v)\right] + \mathbb{E}\left[b(v, p(\ell_1))\right] = \mathbb{E}\left[\langle f, v\rangle_V\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{3.4a}$$

$$\mathbb{E}\left[b(u(\ell_1), q)\right] - \mathbb{E}\left[c(p(\ell_1), q)\right] = \mathbb{E}\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in L^2(\Omega; Q), \tag{3.4b}$$

and

$$\mathbb{E}\left[a(\ell_2, u(\ell_2), v)\right] + \mathbb{E}\left[b(v, p(\ell_2))\right] = \mathbb{E}\left[\langle f, v\rangle_V\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{3.5a}$$

$$\mathbb{E}\left[b(u(\ell_2), q)\right] - \mathbb{E}\left[c(p(\ell_2), q)\right] = \mathbb{E}\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in L^2(\Omega; Q). \tag{3.5b}$$

We subtract (3.5) from (3.4), and set $(v,q)=(u(\ell_1)-u(\ell_2),p(\ell_1)-p(\ell_2))$ to obtain

$$\begin{aligned}\mathbb{E}\left[a(\ell_1,u(\ell_1)-u(\ell_2),u(\ell_1)-u(\ell_2))\right]+\mathbb{E}\left[b(u(\ell_1)-u(\ell_2),p(\ell_1)-p(\ell_2))\right]\\ =\mathbb{E}\left[a(\ell_2-\ell_1,u(\ell_2),u(\ell_1)-u(\ell_2))\right]\\ \mathbb{E}\left[b(u(\ell_1)-u(\ell_2),p(\ell_1)-p(\ell_2))\right]=\mathbb{E}\left[c(p(\ell_1)-p(\ell_2),p(\ell_1)-p(\ell_2))\right],\end{aligned}$$

which further leads to

$$\begin{aligned}&\kappa_1\|u(\ell_1)-u(\ell_2)\|^2_{L^2(\Omega;V)}+\varsigma_1\|p(\ell_1)-p(\ell_2)\|^2_{L^2(\Omega;Q)}\\ &\quad\le\mathbb{E}\left[a(\ell_1,u(\ell_1)-u(\ell_2),u(\ell_1)-u(\ell_2))\right]+\mathbb{E}\left[c(p(\ell_1)-p(\ell_2),p(\ell_1)-p(\ell_2))\right]\\ &\quad=\mathbb{E}\left[a(\ell_2-\ell_1,u(\ell_2),u(\ell_1)-u(\ell_2))\right]\\ &\quad\le\mathbb{E}\left[\kappa_2\|\ell_1-\ell_2\|_B\|u(\ell_2)\|_V\|u(\ell_1)-u(\ell_2)\|_V\right]\\ &\quad\le\kappa_2\|\ell_1-\ell_2\|_{L^\infty(\Omega;B)}\|u(\ell_2)\|_{L^2(\Omega;V)}\,\|u(\ell_1)-u(\ell_2))\|_{L^2(\Omega;V)}\\ &\quad\le c_1\kappa_2\left(\|f\|_{L^2(\Omega;V^*)}+\|g\|_{L^2(\Omega;Q^*)}\right)\|\ell_1-\ell_2\|_{L^\infty(\Omega;B)}\|u(\ell_1)-u(\ell_2))\|_{L^2(\Omega;V)},\end{aligned}$$

where we used the bound in (2.9a). This proves (3.1a). The above inequality further implies that

$$\varsigma_1\|p(\ell_1)-p(\ell_2)\|^2_{L^2(\Omega;Q)}\le\frac{c_1^2\kappa_2^2}{\kappa_1}\left(\|f\|_{L^2(\Omega;V^*)}+\|g\|_{L^2(\Omega;Q^*)}\right)^2\|\ell_1-\ell_2\|^2_{L^\infty(\Omega;B)},$$

and hence proving (3.1b). The proofs of (3.2) and (3.3) are based on analogous but simpler arguments. ☐

Remark 3.2. Theorem 3.1 demonstrates that the solution map of (2.8) is Lipschitz continuous with respect to the coefficients and the source terms

We now provide a derivative characterization in the pathwise mixed variational problem (2.2).

Theorem 3.3. *For a fixed $\omega\in\Omega$, and given $\ell_\omega\in K$, let $\delta\ell_\omega$ be such that $\ell_\omega+\delta\ell_\omega\in K$. For the given ℓ_ω, let $w_\omega(\ell_\omega)=(u_\omega(\ell_\omega),p_\omega(\ell_\omega))$ be the solution of* (2.2)*. The first-order derivative of u_ω at ℓ_ω in a direction $\delta\ell_\omega$, denoted by*

$$\delta w_\omega=(\delta u_\omega,\delta p_\omega)=(Du_\omega(\ell_\omega)\delta\ell_\omega,Dp_\omega(\ell_\omega)\delta\ell_\omega),$$

is the unique solution of the pathwise mixed variational problem:

$$a(\ell_\omega,\delta u_\omega,v_\omega)+b(v,\delta p_\omega)=-a(\delta\ell_\omega,u_\omega,v),\quad\text{for all } v\in V,\tag{3.6a}$$

$$b(\delta u_\omega,q)-c(\delta p,q)=0,\quad\text{for all } q\in Q.\tag{3.6b}$$

The second-order derivative of u_ω at ℓ_ω in the direction $(\delta\ell^1_\omega,\delta\ell^2_\omega)$, which satisfies $(\ell_\omega+\delta\ell^1_\omega),(\ell_\omega+\delta\ell^2_\omega)\in K$, denoted by

$$\delta^2w_\omega=D^2w_\omega(\ell_\omega)(\delta\ell^1_\omega,\delta\ell^2_\omega)=(\delta^2u_\omega,\delta^2p_\omega)=(D^2u_\omega(\ell_\omega)(\delta\ell^1_\omega,\delta\ell^2_\omega),D^2p_\omega(\ell_\omega)(\delta\ell^1_\omega,\delta\ell^2_\omega))$$

is the unique solution of the pathwise mixed variational problem:

$$a(\ell_\omega,\delta^2u_\omega,v)+b(v,\delta^2p_\omega)=-a(\delta\ell_2,Du_\omega(\ell_\omega)\delta\ell^1_\omega,v)-a(\delta\ell^1_\omega,Du_\omega(\ell_\omega)\delta\ell^2_\omega,v),\tag{3.7a}$$

for all $v\in V$,

$$b(\delta^2u_\omega,q)-c(\delta^2p_\omega,q)=0,\quad\text{for all } q\in Q.\tag{3.7b}$$

The following is a derivative characterization in the integral stochastic variational problem (2.8).

Theorem 3.4. *For ℓ in $\mathbb{K} \subset L^\infty(\Omega; B)$, let $\delta\ell$ be such that $\ell + \delta\ell \in \mathbb{K}$. For the given ℓ, let $w(\ell) := (u(\ell), p(\ell))$ be the solution of (2.8). The first-order derivative of u at ℓ in a direction $\delta\ell$, denoted by $\delta w := (\delta u, \delta p) := (Du(\ell)\delta\ell, Dp(\ell)\delta\ell)$, is the unique solution of the integral stochastic mixed variational problem:*

$$\mathbb{E}\left[a(\ell, \delta u, v)\right] + \mathbb{E}\left[b(v, \delta p)\right] = -\mathbb{E}\left[a(\delta\ell, u, v)\right], \quad \text{for all } v \in L^2(\Omega; V), \tag{3.8a}$$

$$\mathbb{E}\left[b(\delta u, q)\right] - \mathbb{E}\left[c(\delta p, q)\right] = 0, \quad \text{for all } q \in L^2(\Omega; Q). \tag{3.8b}$$

The second-order derivative of u at ℓ in the direction $(\delta\ell^1, \delta\ell^2)$, which satisfies $(\ell + \delta\ell^1), (\ell + \delta\ell^2) \in \mathbb{K}$, denoted by

$$\delta^2 w = D^2 w(\ell)(\delta\ell^1, \delta\ell^2) = (\delta^2 u, \delta^2 p) = (D^2 u(\ell)(\delta\ell^1, \delta\ell^2), D^2 p(\ell)(\delta\ell^1, \delta\ell^2))$$

is the unique solution of the integral stochastic mixed variational problem:

$$\begin{aligned}\mathbb{E}\left[a(\ell, \delta^2 u, v)\right] + \mathbb{E}\left[b(v, \delta^2 p)\right] = &-\mathbb{E}\left[a(\delta\ell_2, Du(\ell)\delta\ell_1, v)\right] \\ &-\mathbb{E}\left[a(\delta\ell_1, Du(\ell)\delta\ell_2, v)\right],\end{aligned} \tag{3.9a}$$

$$\mathbb{E}\left[b(\delta^2 u, q)\right] - \mathbb{E}\left[c(\delta^2 p, q)\right] = 0, \tag{3.9b}$$

for every $v \in L^2(\Omega; V)$ and for every $q \in L^2(\Omega; Q)$.

Proof. A proof can be obtained by the scheme used in [34]. □

4. Stochastic optimization formulations of the inverse problem

An extensively used approach for studying inverse problems revolves around optimizing the Output Least-Squares (OLS) functional. In the present context, the OLS functional can be formulated as follows:

$$J_{\text{OLS}}(\ell) = \frac{1}{2}\mathbb{E}\left[\|u(\ell) - \tilde{z}\|_V^2\right] + \frac{1}{2}\mathbb{E}\left[\|p(\ell) - \hat{z}\|_Q^2\right]. \tag{4.1}$$

In this equation $\mathbb{E}[\cdot]$ is the expectation with respect to the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $z = (\tilde{z}, \hat{z}) \in L^2(\Omega; \widetilde{Z}) \times L^2(\Omega; \widehat{Z})$ is the data with $\widetilde{Z}$ and $\widehat{Z}$ being real Hilbert spaces and $w(\ell) = (u(\ell), p(\ell))$ is the solution of (2.8) for ℓ.

Minimizing energy functionals has proven to be a fruitful approach in the study of deterministic inverse problems (see [34]). Building on these developments, we propose a new Energy Least-Squares (ELS) functional to address the inverse problem of identifying ℓ in the stochastic mixed variational problem (2.8):

$$\begin{aligned}J(\ell) = &\frac{1}{2}\mathbb{E}\left[a(\ell, u(\ell) - \tilde{z}, u(\ell) - \tilde{z})\right] + \mathbb{E}\left[b(u(\ell) - \tilde{z}, p(\ell) - \hat{z})\right] \\ &- \frac{1}{2}\mathbb{E}\left[c(p(\ell) - \hat{z}, p(\ell) - \hat{z})\right],\end{aligned} \tag{4.2}$$

where $\mathbb{E}[\cdot]$ is again the expectation with respect to the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $z = (\tilde{z}, \hat{z}) \in L^2(\Omega; V) \times L^2(\Omega; Q)$ is the data and $w(\ell) = (u(\ell), p(\ell))$ is the solution of (2.8) that corresponds to ℓ.

The key idea behind defining the above functional is to add the two equations in the mixed formulation (2.8) to obtain a single equation and then minimize the associated energy. Alternatively, these two equations can also be subtracted to form a single equation, which leads to the associated energy being represented by the following Modified Least Squares (MLS) functional:

$$J_{\text{MLS}}(\ell) = \frac{1}{2}\mathbb{E}\left[a(\ell, u(\ell) - \tilde{z}, u(\ell) - \tilde{z})\right] + \frac{1}{2}\mathbb{E}\left[c(p(\ell) - \hat{z}, p(\ell) - \hat{z})\right], \tag{4.3}$$

where all the data specifications are as in (4.2).

Given the non-convex nature of the OLS functional, its utilization in inverse problems poses substantial challenges from both theoretical and computational perspectives. A significant concern arises from the potential entrapment in local minimizers when employing widely used gradient methods to address optimization problems based on OLS. In contrast, the ELS functional presents a noteworthy advantage owing to its convex nature, as established in the following result.

Theorem 4.1. *The ELS functional J, defined in* (4.2) *is convex.*

Proof. We shall compute the derivatives of J at ℓ in a direction $\hat{\ell}$. Utilizing (4.2), we get

$$\begin{aligned} DJ(\ell)(\hat{\ell}) = {} & \frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, u(\ell) - \tilde{z}, u(\ell) - \tilde{z})\right] + \mathbb{E}\left[a(\ell, Du(\ell)\hat{\ell}, u(\ell) - \tilde{z})\right] \\ & - \mathbb{E}\left[c(Dp(\ell)\hat{\ell}, p(\ell) - \hat{z})\right] + \mathbb{E}\left[b(Du(\ell)\hat{\ell}, p(\ell) - \tilde{z})\right] \\ & + \mathbb{E}\left[b(u(\ell) - \tilde{z}, Dp(\ell)\hat{\ell})\right]. \end{aligned} \tag{4.4}$$

We next set $(v, q) = (u(\ell) - \tilde{z}, p(\ell) - \hat{z})$ in the derivative characterization (3.8) to obtain

$$\begin{aligned} \mathbb{E}\left[a(\ell, Du(\ell)\hat{\ell}, u(\ell) - \tilde{z})\right] + \mathbb{E}\left[b(u(\ell) - \tilde{z}, Dp(\ell)\hat{\ell})\right] &= -\mathbb{E}\left[a(\hat{\ell}, u(\ell), u(\ell) - \tilde{z})\right], \\ \mathbb{E}\left[b(Du(\ell)\hat{\ell}, p(\ell) - \hat{z})\right] - \mathbb{E}\left[c(Dp(\ell)\hat{\ell}, p(\ell) - \hat{z})\right] &= 0. \end{aligned}$$

We add the above two equations and substitute the resulting equation in (4.4) to obtain

$$\begin{aligned} DJ(\ell)(\hat{\ell}) &= \frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, u(\ell) - \tilde{z}, u(\ell) - \tilde{z})\right] - \mathbb{E}\left[a(\hat{\ell}, u(\ell), u(\ell) - \tilde{z})\right] \\ &= -\frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, u(\ell) + \tilde{z}, u(\ell) - \tilde{z})\right]. \end{aligned} \tag{4.5}$$

To compute the second-order derivative, we make use of (4.5) and proceed as follows:

$$\begin{aligned} D^2J(\ell)(\hat{\ell}, \hat{\ell}) &= -\frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, Du(\ell)\hat{\ell}, u(\ell) - \tilde{z})\right] - \frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, u(\ell) + \tilde{z}, Du(\ell)\hat{\ell})\right] \\ &= -\frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, u(\ell) - \tilde{z}, Du(\ell)\hat{\ell})\right] - \frac{1}{2}\mathbb{E}\left[a(\hat{\ell}, u(\ell) + \tilde{z}, Du(\ell)\hat{\ell})\right] \\ &= -\mathbb{E}\left[a(\hat{\ell}, u(\ell), Du(\ell)\hat{\ell})\right]. \end{aligned} \tag{4.6}$$

We next set $(v,q) = (Du(\ell)\hat{\ell}, Dp(\ell)\hat{\ell})$ in the derivative characterization (3.8) to obtain

$$\mathbb{E}\big[a(\ell, Du(\ell)\hat{\ell}, Du(\ell)\hat{\ell})\big] + \mathbb{E}\big[b(Du(\ell)\hat{\ell}, Dp(\ell)\hat{\ell})\big] = -\mathbb{E}\big[a(\delta\ell, u(\ell), Du(\ell)\hat{\ell})\big]$$
$$\mathbb{E}\big[b(Du)(\ell)\hat{\ell}, Dp(\ell)\hat{\ell})\big] - \mathbb{E}\big[c(Dp(\ell)\hat{\ell}, Dp(\ell)\hat{\ell})\big] = 0.$$

We utilize the above system in (4.6) to get

$$\begin{aligned} D^2J(\ell)(\hat{\ell},\hat{\ell}) &= \mathbb{E}\big[a(\ell, Du(\ell)\hat{\ell}, Du(\ell)\hat{\ell}\big] + \mathbb{E}\big[b(Du(\ell)\hat{\ell}, Dp(\ell)\hat{\ell})\big] \\ &= \mathbb{E}\big[a(\ell, Du(\ell)\hat{\ell}, Du(\ell)\hat{\ell})\big] + \mathbb{E}\big[c(Dp(\ell)\hat{\ell}, Dp(\ell)\hat{\ell})\big] \\ &\geq \kappa_1 \|Du(\ell)\hat{\ell}\|^2_{L^2(\Omega;V)} + \varsigma_1 \|Dp(\ell)\hat{\ell}\|^2_{L^2(\Omega;Q)}, \end{aligned} \tag{4.7}$$

which proves the desired claim. □

In the following, we examine two regularized optimization problems within the framework of the ELS functional, aiming to address the ill-posed nature of the inverse problem. To achieve regularization, for simplicity, we incorporate a quadratic term. The first regularized optimization problem focuses on identifying deterministic coefficients, while the second problem involves the general optimization framework of estimating a random coefficient. By introducing regularization, we can effectively counteract the ill-posedness of the inverse problem and enhance the robustness of the estimation process. The key objective of considering the two optimization problems is to gain valuable insights into the complexities of coefficient estimation in Hilbert spaces and their associated Bochner spaces.

We consider the regularized optimization problem of finding $\ell \in K \subset H$ by solving

$$\begin{aligned} \min_{\ell\in K} J_\kappa(\ell) := \frac{1}{2}\mathbb{E}\left[a(\ell, u(\ell)-\tilde{z}, u(\ell)-\tilde{z})\right] + \mathbb{E}\left[b(u(\ell)-\tilde{z}, p(\ell)-\hat{z})\right] \\ - \frac{1}{2}\mathbb{E}\left[c(p(\ell)-\hat{z}, p(\ell)-\hat{z})\right] + \frac{\kappa}{2}\|\ell\|^2_H. \end{aligned} \tag{4.8}$$

Here $w(\ell) = (u(\ell), p(\ell)) \in L^2(\Omega;V) \times L^2(\Omega;Q)$ is the unique solution of the integral stochastic mixed variational problem (2.8) that corresponds to the deterministic coefficient ℓ. Additionally, $z = (\tilde{z}, \hat{z}) \in L^2(\Omega;V) \times L^2(\Omega;Q)$ is the data. The term $\|\cdot\|^2_H$ corresponds to quadratic regularizer, where H is Hilbert space that is compactly embedded into B. The set $K \subset H$ represents the set of feasible coefficients.

In the following, we first give an existence result for the regularized optimization problem (4.8).

Theorem 4.2. *The regularized stochastic optimization problem* (4.8) *has a unique solution.*

Proof. Since $J(\ell)$

$$\begin{aligned} &= \frac{1}{2}\mathbb{E}\left[a(\ell, u(\ell)-\tilde{z}, u(\ell)-\tilde{z})\right] + \mathbb{E}\left[b(u(\ell)-\tilde{z}, p(\ell)-\hat{z})\right] - \frac{1}{2}\mathbb{E}\left[c(p(\ell)-\hat{z}, p(\ell)-\hat{z})\right] \\ &\geq \frac{\kappa_1}{2}\|u(\ell)-\tilde{z}\|^2_{L^2(\Omega;V)} - \kappa_0\|u(\ell)-\tilde{z}\|_{L^2(\Omega;V)}\|p(\ell)-\hat{z}\|_{L^2(\Omega;Q)} - \frac{\varsigma_1}{2}\|p(\ell)-\hat{z}\|^2_{L^2(\Omega;Q)} \\ &\geq C, \end{aligned}$$

where C is some constant involving the bounds in (2.9a) and (2.9b) of Theorem 2.7.

Next, by using the just noticed fact that J is bounded below, we infer that the regularized functional J_κ is also bounded below. Consequently, there exists a minimizing sequence $\{\ell_n\}$ in K such that

$$\lim_{n\to\infty} J_\kappa(\ell_n) = \inf\{J_\kappa(\ell)|\ \ell \in K\},$$

which, due to the definition of J_κ, confirms that $\{\ell_n\}$ is bounded in $\|\cdot\|_H$. Since H is compactly imbedded into B, there is a subsequence $\{\ell_n\}$, which converges strongly in $\|\cdot\|_H$ to some $\tilde{\ell} \in K$. For $n \in \mathbb{N}$, let $(u_n, p_n) := (u(\ell_n), p(\ell_n))$ be the solution of the stochastic mixed variational problem (2.8) for ℓ_n. That is,

$$\mathbb{E}\left[a(\ell_n, u_n, v)\right] + \mathbb{E}\left[b(v, p_n)\right] = \mathbb{E}\left[\langle f, v\rangle_V\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{4.9a}$$

$$\mathbb{E}\left[b(u_n, q)\right] - \mathbb{E}\left[c(p_n, q)\right] = \mathbb{E}\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in L^2(\Omega; Q). \tag{4.9b}$$

By proceeding as in the proof of Theorem 2.7., we can show that the sequence $\{(u_n, p_n)\}$ is bounded in $L^2(\Omega; V) \times L^2(\Omega; Q)$, and hence there is a subsequence, still denoted by $\{(u_n, p_n)\}$, that converges weakly to some $(\tilde{u}, \tilde{p})$. We claim that $(\tilde{u}, \tilde{p}) = (u(\tilde{\ell}), p(\tilde{\ell}))$. For this, for $v \in L^2(\Omega; V)$, we rearrange (4.9a) as follows

$$\mathbb{E}\big[a(\tilde{\ell}, \tilde{u}, v)\big] + \mathbb{E}\big[b(v, p_n)\big] = \mathbb{E}\big[a(\ell_n - \tilde{\ell}, u_n, v)\big] - \mathbb{E}\big[a(\tilde{\ell}, u_n - \tilde{u}, v)\big] + \mathbb{E}\big[\langle f, v\rangle_V\big],$$

and then pass the above equation and (4.9b), to the limit $n \to \infty$, to obtain

$$\mathbb{E}[a(\tilde{\ell}, \tilde{u}, v)] + \mathbb{E}\left[b(v, \tilde{p})\right] = \mathbb{E}\left[\langle f, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{4.10a}$$

$$\mathbb{E}\left[b(\tilde{u}, q)\right] - \mathbb{E}\left[c(\tilde{p}, q)\right] = \mathbb{E}\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in L^2(\Omega; Q), \tag{4.10b}$$

and since the above mixed variational problem is uniquely solvable, we deduce that $(\tilde{u}, \tilde{p}) = (u(\tilde{\ell}), p(\tilde{\ell}))$.

We next claim that for $\ell_n \to \tilde{\ell}$ in $\|\cdot\|_B$, we have $J(\ell_n) \to J(\tilde{\ell})$. For this, we set $(v, q) = (u(\ell_n) - \tilde{z}, p(\ell_n) - \hat{z})$ in (4.9), and combine and rearrange the resulting equations to obtain

$$\begin{aligned} 2J(\ell_n) &= \mathbb{E}\left[a(\ell_n, u(\ell_n) - \tilde{z}, u(\ell_n) - \tilde{z})\right] + 2\mathbb{E}\left[b(u(\ell_n) - \tilde{z}, p(\ell_n) - \hat{z})\right] \\ &\quad - \mathbb{E}\left[c(p(\ell_n) - \hat{z}, p(\ell_n) - \hat{z})\right] \\ &= \mathbb{E}\left[\langle f, u(\ell_n) - \tilde{z}\rangle_V\right] + \mathbb{E}\left[\langle g, p(\ell_n) - \hat{z}\rangle_Q\right] - \mathbb{E}\left[a(\ell_n, \tilde{z}, u(\ell_n) - \tilde{z})\right] \\ &\quad - 2\mathbb{E}\left[b(\tilde{z}, p(\ell_n) - \hat{z})\right] - \mathbb{E}\left[c(\hat{z}, p(\ell_n) - \hat{z})\right]. \end{aligned} \tag{4.11}$$

Analogously, we substitute $(v, q) = (u(\tilde{\ell}) - \tilde{z}, p(\tilde{\ell}))$ in (4.10), and rearrange the resulting equations to get

$$\begin{aligned} 2J(\tilde{\ell}) &= \mathbb{E}\big[a(\tilde{\ell}), u(\tilde{\ell}) - \tilde{z}, u(\tilde{\ell}) - \tilde{z})\big] + 2\mathbb{E}\big[b(u(\tilde{\ell}) - \tilde{z}, p(\tilde{\ell}) - \hat{z})\big] \\ &\quad - \mathbb{E}\big[c(p(\tilde{\ell}) - \hat{z}, p(\tilde{\ell}) - \hat{z})\big] \\ &= \mathbb{E}\big[\langle f, u(\tilde{\ell}) - \tilde{z}\rangle_V\big] + \mathbb{E}\big[\langle g, p(\tilde{\ell}) - \hat{z}\rangle_Q\big] - \mathbb{E}\big[a(\tilde{\ell}, \tilde{z}, u(\tilde{\ell}) - \tilde{z})\big] \\ &\quad - 2\mathbb{E}\big[b(\tilde{z}, p(\tilde{\ell}) - \hat{z})\big] - \mathbb{E}\big[c(\hat{z}, p(\tilde{\ell}) - \hat{z})\big]. \end{aligned} \tag{4.12}$$

It follows at once from (4.11) and (4.12) that $J(\ell_n) \to J(\tilde{\ell})$ and $\ell_n \to \tilde{\ell}$ in $\|\cdot\|_B$.

Finally, we return to the regularized ELS functional and note that

$$\begin{aligned}
J_\kappa(\tilde{\ell}) &= \frac{1}{2}\mathbb{E}\big[a(\tilde{\ell}, u(\tilde{\ell}) - \tilde{z}, u(\tilde{\ell}) - \tilde{z})\big] + \mathbb{E}\big[b(u(\tilde{\ell}) - \tilde{z}, p(\ell^*) - \hat{z})\big] \\
&\quad - \frac{1}{2}\mathbb{E}\big[c(p(\tilde{\ell}) - \hat{z}, p(\tilde{\ell}) - \hat{z})\big] + \frac{\kappa}{2}\|\tilde{\ell}\|_H^2 \\
&\le \lim_{n\to\infty}\Big(\frac{1}{2}\mathbb{E}\big[a(\ell_n, u(\ell_n) - \tilde{z}, u(\ell_n) - \tilde{z})\big] + \mathbb{E}\big[b(u(\ell_n) - \tilde{z}, p(\ell_n) - \hat{z})\big] \\
&\quad - \frac{1}{2}\mathbb{E}\big[c(p_n - \hat{z}, p(\ell_n) - \hat{z})\big]\Big) + \liminf_{n\to\infty}\frac{\kappa}{2}\|\ell_n\|^2 \\
&\le \liminf_{n\to\infty}\Big(\frac{1}{2}\mathbb{E}\big[a(\ell_n, u(\ell_n) - \tilde{z}, u(\ell_n) - \tilde{z})\big] + \mathbb{E}\big[b(u(\ell_n) - \tilde{z}, p(\ell_n) - \hat{z})\big] \\
&\quad - \frac{1}{2}\mathbb{E}\big[c(p_n - \hat{z}, p(\ell_n) - \hat{z})\big] + \frac{\kappa}{2}\|\ell_n\|_H^2\Big) \\
&= \inf\left\{J_\kappa(\ell) |\ \ell \in K\right\},
\end{aligned}$$

which proves that $\tilde{\ell}$ is a solution of (4.8). The uniqueness is a consequence of the convexity of the ELS objective (4.2) and the strong convexity of the quadratic regularizer. The proof is complete. □

We now consider the regularized optimization problem to identify a stochastic coefficient $\ell \in \mathbb{K} \subset \mathbb{H}$ by solving

$$\begin{aligned}
\min_{\ell\in\mathbb{K}} \mathbb{J}_\kappa(\ell) &= \frac{1}{2}\mathbb{E}\left[a(\ell, u(\ell) - \tilde{z}, u(\ell) - \tilde{z})\right] + \mathbb{E}\left[b(u(\ell) - \tilde{z}, p(\ell) - \hat{z})\right] \\
&\quad - \frac{1}{2}\mathbb{E}\left[c(p(\ell) - \hat{z}, p(\ell) - \hat{z})\right] + \frac{\kappa}{2}\|\ell\|_{\mathbb{H}}^2. \qquad (4.13)
\end{aligned}$$

Here $u(\ell) = (u(\ell), p(\ell)) \in L^2(\Omega; V) \times L^2(\Omega; Q)$ is the unique solution of the integral stochastic mixed variational problem (2.8) for the coefficient ℓ.

Moreover, $z = (\tilde{z}, \hat{z}) \in L^2(\Omega; V) \times L^2(\Omega; Q)$ is the data. The term $\|\cdot\|_{\mathbb{H}}^2$ is the quadratic regularizer, where $\mathbb{H}$ is Hilbert space that is compactly embedded into $L^\infty(\Omega; B)$, and $\kappa > 0$ is a regularization coefficient. The set $\mathbb{K} \subset \mathbb{H}$ is the set of feasible coefficients.

Prior to moving forward, we state an existence result for (4.13).

Theorem 4.3. *The regularized stochastic optimization problem* (4.13) *has a unique solution.*

Proof. The proof closely parallels the proof of Theorem 4.2, requiring only minor adjustments. □

5. A pathwise approach for stochastic parameter estimation

The solvability of the stochastic optimization problem (4.13) becomes challenging due to the stringent condition that the Hilbert space $\mathbb{H}$ needs to be compactly embedded into the coefficient space $L^\infty(\Omega; B)$. Recent progress in this intricate research area indicates that meeting such embeddings requires stronger regularity prerequisites for the random element. Although these prerequisites are natural for evolutionary problems, they are less suitable within the present context of inverse problems (as seen in works like [6, 51, 52, 54]). As a result, we introduce a novel approach for addressing stochastic inverse problems. This approach allows us to bypass the requirement for the compact embedding of the Bochner spaces.

The core idea is to tackle the inverse problem in a pathwise manner, aiming to attain a unique solution $\bar{\ell}_\omega$ for any arbitrary $\omega \in \Omega$. This leads to the definition of a mapping $\omega \to \bar{\ell}_\omega$, where we subsequently verify its measurability and establish specific bounds on the optimal pathwise solution. This process ultimately enables us to derive the integral formulation, which can then be discretized for a numerical solution.

We recall that the coefficient space B is a Banach space and $\emptyset \neq K \subset B$ is such that for any $\ell \in \mathbb{K} \subset L^\infty(\Omega; B)$, the realizations ℓ_ω belong to the set K. Let H be a separable Hilbert space compactly embedded into B. Let $\mathcal{O} \subset H \cap K$ be an open set. For any fixed $\omega \in \Omega$, let $\emptyset \neq \mathcal{K}_\omega \subset \mathcal{O}$ be closed, convex, and bounded. We assume that the set-valued map $\mathcal{K} : \omega \in \Omega \mapsto \mathcal{K}_\omega$ is measurable, that is, for any open $O \subset \mathcal{O}$, the inverse image $\mathcal{K}^{-1}(O) = \{\omega \in \Omega | \mathcal{K}_\omega \cap O \neq \emptyset\} \in \mathbb{F}$. For the fixed $\omega \in \Omega$, we propose the following pathwise energy least-squares functional

$$\begin{aligned} \mathcal{J}(\ell_\omega) = &\frac{1}{2} a(\ell_\omega, u_\omega(\ell_\omega) - \tilde{z}_\omega, u_\omega(\ell_\omega) - \tilde{z}_\omega) + b(u_\omega(\ell_\omega) - \tilde{z}_\omega, p_\omega(\ell_\omega) - \hat{z}_\omega) \\ &- \frac{1}{2}[c(p_\omega(\ell_\omega) - \hat{z}_\omega, p_\omega(\ell_\omega) - \hat{z}_\omega). \end{aligned} \tag{5.1}$$

Here $w_\omega(\ell_\omega) := (u_\omega(\ell_\omega), p_\omega(\ell_\omega)) \in V \times Q$ is the unique solution of the pathwise mixed variational problem (2.2) that corresponds to the coefficient ℓ_ω. Additionally, $z_\omega := (\tilde{z}_\omega, \hat{z}_\omega) \in V \times Q$ is the data realization.

We next formulate the following pathwise regularized ELS-based optimization problem

$$\begin{aligned} \min_{\ell_\omega \in \mathcal{K}_\omega} \mathcal{J}_\kappa(\ell_\omega) = &\frac{1}{2} a(\ell_\omega, u_\omega(\ell_\omega) - \tilde{z}_\omega, u_\omega(\ell_\omega) - \tilde{z}_\omega) + b(u_\omega(\ell_\omega) - \tilde{z}_\omega, p_\omega(\ell_\omega) - \hat{z}_\omega) \\ &- \frac{1}{2}[c(p_\omega(\ell_\omega) - \hat{z}_\omega, p_\omega(\ell_\omega) - \hat{z}_\omega) + \frac{\kappa}{2}\|\ell_\omega\|_H^2, \end{aligned} \tag{5.2}$$

where $\kappa > 0$ is a regularization parameter, the square norm $\|\cdot\|_H^2$ is the regularizer, and $w_\omega(\ell_\omega) := (u_\omega(\ell_\omega), p_\omega(\ell_\omega)) \in V \times Q$ is the unique solution of the pathwise mixed variational problem (2.2) that corresponds to the coefficient ℓ_ω.

We have the following result concerning the above optimization problem.

Theorem 5.1. *For any fixed $\omega \in \Omega$, (5.2) has a unique minimizer $\bar{\ell}_\omega \in \mathcal{K}_\omega$ satisfying the following variational inequality as a necessary and sufficient optimality condition:*

$$\langle \nabla \mathcal{J}(\bar{\ell}_\omega), \ell - \bar{\ell}_\omega \rangle_H + \kappa \langle \bar{\ell}_\omega, \ell - \bar{\ell}_\omega \rangle_H \geq 0, \quad \text{for all } \ell \in \mathcal{K}_\omega, \tag{5.3}$$

where the derivative $\nabla \mathcal{J}$ of $\mathcal{J}$ at $\bar{\ell}_\omega$ in a direction ℓ_ω is given by

$$(\nabla \mathcal{J}(\bar{\ell}_\omega), \ell_\omega)_H = -\frac{1}{2} a(\ell_\omega, u_\omega(\bar{\ell}_\omega) + \tilde{z}_\omega, u_\omega(\bar{\ell}_\omega) - \tilde{z}_\omega). \tag{5.4}$$

Assume that $f \in L^2(\Omega; V^*)$, $g \in L^2(\Omega; Q^*)$, $\tilde{z} \in L^2(\Omega; V)$, $\hat{z} \in L^2(\Omega; Q)$, the map $\mathcal{K} : \omega \mapsto \mathcal{K}_\omega$ is measurable, and there exists $\ell^0 \in L^2(\Omega; H)$ such that any realization $\ell^0_\omega \in \mathcal{K}_\omega$. Then the map $\omega \to \bar{\ell}_\omega$ is measurable. Moreover, $\bar{\ell} \in L^2(\Omega; H)$.

Proof. The existence of a unique solution to (5.2) can be established by employing the reasoning used in the proof of Theorem 4.2. Furthermore, the variational inequality (5.3) is a necessary and sufficient optimality condition by classical arguments, where as the derivative form (5.4) follows by a calculation similar to that led to (4.5), but using the pathwise derivative characterizations. To show the map $\omega \to \bar{\ell}_\omega$ is measurable, we recall that the map $w(\ell) = (u(\ell), p(\ell))$ is measurable. Moreover the maps $z = (\tilde{z}, \hat{z})$, and $\mathcal{K}$ are measurable by assumption. Thus [29, Theorem 6.1.1]) applies to deduce the desired measurability of the solution map $\bar{\ell}$ into the separable Hilbert space H.

Let us now prove a bound on the unique solution $\bar{\ell}_\omega$. For the fixed $\ell^0 \in L^2(\Omega; H)$, by the strong monotonicity of $\nabla \mathcal{J}_\kappa$, we have

$$\langle (\nabla \mathcal{J} + \kappa I)(\bar{\ell}_\omega) - (\nabla \mathcal{J} + \kappa I)(\ell^0_\omega), \bar{\ell}_\omega - \ell^0_\omega \rangle_H \geq \kappa \|\bar{\ell}_\omega - \ell^0_\omega\|^2_H,$$

which, due to (5.3), can be written as

$$\langle (\nabla \mathcal{J} + \kappa I)(\ell^0_\omega), \ell^0_\omega - \bar{\ell}_\omega \rangle_H \geq \kappa \|\bar{\ell}_\omega - \ell^0_\omega\|^2_H + \langle (\nabla \mathcal{J} + \kappa I)(\bar{\ell}_\omega), \ell^0_\omega - \bar{\ell}_\omega \rangle_H \geq \kappa \|\bar{\ell}_\omega - \ell^0_\omega\|^2.$$

We rearrange the above inequality as follows

$$\langle \nabla \mathcal{J} \ell^0_\omega, \ell^0_\omega - \bar{\ell}_\omega \rangle_H + \kappa \langle \ell^0_\omega, \ell^0_\omega \rangle_H - \kappa \langle \ell^0_\omega, \bar{\ell}_\omega \rangle_H \geq \kappa \Big(\|\bar{\ell}_\omega\|^2_H + \|\ell^0_\omega\|^2_H - 2\langle \bar{\ell}_\omega, \ell^0_\omega \rangle_H \Big),$$

which further yields

$$\begin{aligned} \kappa \|\bar{\ell}_\omega\|^2_H &\leq \langle \nabla \mathcal{J} \ell^0_\omega, \ell^0_\omega - \bar{\ell}_\omega \rangle_H + \kappa \langle \ell^0_\omega, \bar{\ell}_\omega \rangle \\ &\leq \frac{\kappa}{2} \|\ell^0_\omega\|^2_H + \frac{\kappa}{2} \|\bar{\ell}_\omega\|^2_H - \frac{1}{2} a(\ell^0_\omega - \bar{\ell}_\omega, u_\omega(\ell^0_\omega) + \tilde{z}_\omega, u_\omega(\ell^0_\omega) - \tilde{z}_\omega), \end{aligned}$$

and consequently

$$\frac{\kappa}{2} \|\bar{\ell}_\omega\|^2_H \leq \frac{\kappa}{2} \|\ell^0_\omega\|^2_H + \frac{\kappa_2}{2} \|\ell^0_\omega - \bar{\ell}_\omega\|_B \|u_\omega(\ell^0_\omega) + \tilde{z}_\omega\|_V \|u_\omega(\ell^0_\omega) - \tilde{z}_\omega\|_V.$$

Now using the elementary inequalities $ab \leq \frac{a^2}{4} + b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$, the fact that H is continuously imbedded in B, and the bounds, we rearrange the above inequality as follows:

$$\begin{aligned} \kappa \|\bar{\ell}_\omega\|^2_H &\leq \kappa \|\ell^0_\omega\|^2_H + 2\kappa_2 \left[\|\ell^0_\omega\| + \|\bar{\ell}_\omega\|_B \right] \left[\|u_\omega(\ell^0_\omega)\|^2_V + \|\tilde{z}_\omega\|^2_V \right] \\ &\leq \kappa \|\ell^0_\omega\|^2_H + \frac{\kappa}{2} \|\bar{\ell}_\omega\|^2_H + \frac{\kappa}{2} \|\ell^0_\omega\|^2_H + c \left[\|f_\omega\|^4_{V^*} + \|g_\omega\|^4_{Q^*} + \|\tilde{z}_\omega\|^4_V \right], \end{aligned}$$

or $$\|\bar{\ell}_\omega\|^2_H \leq \tilde{c} \left[\|\ell^0_\omega\|^2_H + \|f_\omega\|^4_{V^*} + \|g_\omega\|^4_{Q^*} + \|\tilde{z}_\omega\|^4_V \right]$$

for positive constants c and $\tilde{c}$. This inequality at once yields that $\mathbb{E}\left[\|\bar{\ell}_\omega\|^2_H \right] < \infty$ and hence confirming that $\bar{\ell} \in L^2(\Omega; H)$. The proof is thus complete. □

6. Stochastic Galerkin based discretization

Mixed variational problems and objective functions need to be discretized for the numerical resolution of stochastic inverse problems. In this section, we establish the foundation of a comprehensive discretization framework using a stochastic Galerkin finite element method.

6.1. Finite-dimensional noise

The central aspect of the numerical handling of PDEs with random data and the related inverse problem is the presumption of finite-dimensional noise, which allows for the transformation of the abstract sample space Ω into a more manageable domain.

Definition 6.1. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space and let $(X, \mathcal{A}_X)$ be a measurable space. A function $\tilde{v} : \Omega \to X$ is called *finite-dimensional noise* if there exist measurable mappings $\xi_k : \Omega \to \mathbb{R}$ for $k = 1, \dots, M$, with $M < \infty$, and a measurable map $v : \mathbb{R}^M \to X$ such that $\tilde{v}(\omega) = v(\xi_1(\omega), \dots, \xi_M(\omega))$. In other words, the dependence on ω is solely through $\xi_1, \dots, \xi_M$. $\square$

Let $\Gamma \subset \mathbb{R}^M$ with $\Gamma \in \mathcal{B}(\mathbb{R}^M)$ be such that $\mathbb{P}((\xi_1, \dots, \xi_M) \in \Gamma) = 1$ and let $\mathbb{P}_\xi = \mathbb{P}_{\xi_1, \dots, \xi_M}$ be the distribution of random vector $\xi = (\xi_1, \dots, \xi_M)$. Then instead of working with the abstract probability space $(\Omega, \mathbb{F}, \mathbb{P})$, we can use the concrete probability space $(\Gamma, \mathcal{B}(\Gamma), \mathbb{P}_\xi)$. In this work, we additionally assume that $\xi_1, \dots, \xi_M$ are independent random variable, and $\Gamma = \Gamma_1 \times \cdots \times \Gamma_M$ with $\mathbb{P}(\xi_i \in \Gamma_i) = 1$, for $i = 1, \dots, M$. Moreover, if $\xi = (\xi_1, \dots, \xi_M)$ is a random vector with distribution absolutely continuous with respect to Lebesgue measure, then there exists a Lebesgue measurable density $\rho = \rho_\xi$ with which we can calculate expectations. For example, if X is a separable Banach space with norm $\|\cdot\|_X$, then

$$\int_\Gamma \|v(\xi)\|_X^p \, d\mathbb{P}_\xi = \int_\Gamma \rho(y) \|v(y)\|_X^p \, dy =: \mathbb{E}_\rho \left[\|\tilde{v}\|_X^p \right].$$

In this context, instead of working with the Bochner Hilbert space $L^2(\Omega; V)$, where V is a given Hilbert space, we can use the corresponding weighted Hilbert space $L^2_\rho(\Gamma; V)$. In this space, the norm and inner product are defined as follows:

$$\|\tilde{v}\|^2_{L^2(\Omega;V)} = \mathbb{E}_\rho \left[\|v\|_V^2 \right] = \int_\Gamma \rho(y) \|v(y)\|_V^2 \, dy, \tag{6.1}$$

$$\langle \tilde{v}, \tilde{w} \rangle_{L^2(\Omega;V)} = \mathbb{E}_\rho \left[\langle v(y), w(y) \rangle_V \right] = \int_\Gamma \rho(y) \langle v(y), w(y) \rangle_V \, dy. \tag{6.2}$$

6.2. Finite dimensional noise integral formulations and discretization

In this section, we work with the parametric form of the mixed variational problem. For this we take the feasible coefficients $\ell : \Gamma \to K \subset B$, with $\Gamma \subset \mathbb{R}^M$ and the source terms $f : \Gamma \to V^*$ and $g : \Gamma \to Q^*$, which are all measurable functions. We also assume that $f \in L^2(\Gamma; V^*)$, $g \in L^2(\Gamma; Q^*)$, and $\ell \in L^\infty(\Gamma; B)$, that is, $\operatorname{esssup}_{y \in \Gamma} \|\ell(y)\|_B < \infty$.

We consider the following parametric mixed variational problem associated with (2.8): Given $\ell \in L^\infty_\rho(\Gamma; B)$, find $(u(\ell), p(\ell)) \in L^2_\rho(\Gamma; V) \times L^2_\rho(\Gamma; Q)$ such that

$$\mathbb{E}_\rho \left[a(\ell, u(\ell), v) \right] + \mathbb{E}_\rho \left[b(v, p(\ell)) \right] = \mathbb{E}_\rho \left[\langle f, v \rangle_V \right], \quad \text{for every } v \in L^2_\rho(\Gamma; V), \tag{6.3a}$$

$$\mathbb{E}_\rho \left[b(u(\ell), q) \right] - \mathbb{E}_\rho \left[c(p(\ell), q) \right] = \mathbb{E}_\rho \left[\langle g, q \rangle_Q \right], \quad \text{for every } q \in L^2_\rho(\Gamma; Q). \tag{6.3b}$$

Let $\mathcal{V}_{hk}$ be a finite-dimensional subspace of $V_\rho := L^2_\rho(\Gamma; V)$ and let $\mathcal{Q}_{hk}$ be a finite dimensional subspace of $Q_\rho := L^2_\rho(\Gamma; Q)$. An element $w_{hk} = (u_{hk}, p_{hk}) \in \mathcal{V}_{hk} \times \mathcal{Q}_{hk}$ is a stochastic Galerkin solution of (6.3) if

$$\mathbb{E}_\rho\left[a(\ell, u_{hk}, v)\right] + \mathbb{E}_\rho\left[b(v, p_{hk})\right] = \mathbb{E}_\rho\left[\langle f, v\rangle_V\right], \quad \text{for every } v \in \mathcal{V}_{hk}, \tag{6.4a}$$

$$\mathbb{E}_\rho\left[b(u_{hk}, q)\right] - \mathbb{E}_\rho\left[c(p_{hk}, q)\right] = \mathbb{E}_\rho\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in \mathcal{Q}_{hk}. \tag{6.4b}$$

To form finite-dimensional subspaces of V_ρ and Q_ρ, we will perform a tensor product of the underlying basis functions. Let $\mathcal{V}_h$ be an n-dimensional subspace of V and let $\mathcal{Q}_h$ be a k-dimensional subspaces of Q. The sets $\{\psi_1, \psi_2, \dots, \psi_n\}$, and $\{\chi_1, \chi_2, \dots, \chi_k\}$ serve as the bases of $\mathcal{V}_h$ and $\mathcal{Q}_h$, respectively. Additionally, we assume that $\mathcal{S}_k$ be an s-dimensional subspace of $L^2_\rho(\Gamma)$ with basis $\{\phi_1, \phi_2, \dots, \phi_s\}$.

We shall assume that the basis $\{\phi_1, \phi_2, \dots, \phi_s\}$ is orthonormal with respect to the density ρ, that is, $\int_\Gamma \rho\phi_n\phi_m dy=\delta_{nm}$, where δ_{nm} is the Kronecker delta: $\delta_{nm} = 1$ for $n = m$, $\delta_{nm} = 0$ for $n \neq m$. We now define the following ns-dimensional and ks-dimensional subspaces of V_ρ and Q_ρ for solving (6.3):

$$\mathcal{V}_{hk} := \mathcal{V}_h \otimes \mathcal{S}_k := \operatorname{span}\{\psi_i\phi_j |\ i = 1, \dots, n,\ j = 1, \dots, s\}.$$

$$\mathcal{Q}_{hk} := \mathcal{Q}_h \otimes \mathcal{S}_k := \operatorname{span}\{\chi_i\phi_j |\ i = 1, \dots, k,\ j = 1, \dots, s\}.$$

Then, any $v \in \mathcal{V}_{hk}$ has the representation

$$v = \sum_{i=1}^{n}\sum_{j=1}^{s} V_{ij}\psi_i\phi_j = \sum_{j=1}^{s}\left[\sum_{i=1}^{n} V_{ij}\psi_i\right]\phi_j = \sum_{j=1}^{s}\widehat{V}_j\phi_j, \text{ where } \widehat{V}_j \equiv \sum_{i=1}^{n} V_{ij}\psi_i.$$

Analogously, any $q \in \mathcal{Q}_{hk}$ has the representation

$$q = \sum_{i=1}^{k}\sum_{j=1}^{s} Q_{ij}\chi_i\phi_j = \sum_{j=1}^{s}\left[\sum_{i=1}^{k} Q_{ij}\chi_i\right]\phi_j = \sum_{j=1}^{s}\widehat{Q}_j\phi_j, \text{ with } \widehat{Q}_j \equiv \sum_{i=1}^{k} Q_{ij}\chi_i.$$

For convenience, we introduce the following vectorized notation:

$$\mathbb{V} = \operatorname{vec}(V_{ij}) = \begin{bmatrix}\widehat{V}_1\\ \widehat{V}_2\\ \vdots\\ \widehat{V}_s\end{bmatrix} \in \mathbb{R}^{ns}, \text{ with } \widehat{V}_j := \begin{bmatrix}V_{1j}\\ V_{2j}\\ \vdots\\ V_{nj}\end{bmatrix} \in \mathbb{R}^{n}. \tag{6.5}$$

$$\mathbb{Q} = \operatorname{vec}(Q_{ij}) = \begin{bmatrix}\widehat{Q}_1\\ \widehat{Q}_2\\ \vdots\\ \widehat{Q}_s\end{bmatrix} \in \mathbb{R}^{ks}, \text{ with } \widehat{Q}_j := \begin{bmatrix}Q_{1j}\\ Q_{2j}\\ \vdots\\ Q_{kj}\end{bmatrix} \in \mathbb{R}^{k}. \tag{6.6}$$

We substitute $u_{hk} = \sum_{i_1=1}^{n}\sum_{j_1=1}^{s} U_{i_1j_1}\psi_{i_1}\phi_{j_1}$, and $p_{hk} = \sum_{h_1=1}^{k}\sum_{j_1=1}^{s} P_{h_1j_1}\chi_{h_1}\phi_{j_1}$, into (6.4) and choose $v = \psi_{i_2}\phi_{j_2}$ and $q = \chi_{h_2}\phi_{j_2}$, where $i_2 = 1, \dots, n$, $j_2 = 1, \dots, s$, and $h_2 = 1, \dots, k$, to obtain

$$\sum_{i_1=1}^{n}\sum_{j_1=1}^{s} U_{i_1 j_1}\mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}a(\ell,\psi_{i_1},\psi_{i_2})\right] + \sum_{h_1=1}^{k}\sum_{j_1=1}^{s} P_{h_1 j_1}\mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}b(\psi_{i_2},\chi_{h_1})\right]$$
$$= \mathbb{E}_\rho\left[\phi_{j_2}\langle f,\psi_{i_2}\rangle\right], \tag{6.7a}$$
$$\sum_{i_1=1}^{n}\sum_{j_1=1}^{s} U_{i_1 j_1}\mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}b(\psi_{i_1},\chi_{h_2})\right] - \sum_{h_1=1}^{k}\sum_{j_1=1}^{s} P_{h_1 j_1}\mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}c(\chi_{h_1},\chi_{h_2})\right]$$
$$= \mathbb{E}_\rho\left[\phi_{j_2}\langle g,\chi_{h_2}\rangle\right], \tag{6.7b}$$

We next define three block matrices $\mathbb{K}$, $\mathbb{B}$, and $\mathbb{C}$, and two block vectors $\mathbb{F}$ and $\mathbb{G}$. The matrix $\mathbb{K}$ is of size $ns \times ns$ consisting of $s \times s$ blocks with each block of size $n \times n$, the matrix $\mathbb{B}$ is of size $ks \times ns$ consisting of $s \times s$ blocks with each block of size $k \times n$, and the matrix $\mathbb{C}$ is of size $ks \times ks$ consisting of $s \times s$ blocks with each block of size $k \times k$. The block load vector $\mathbb{F}$ is of size $n \times s$ consisting of $s \times 1$ components, with each component of size $n \times 1$ and the block load vector $\mathbb{G}$ of size $k \times s$ consisting of $s \times 1$ blocks, each of size $k \times 1$. The blocks are defined as follows:

$$\left(K_\ell^{j_1 j_2}\right)_{i_1 i_2} = \mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}a(\ell,\psi_{i_1},\psi_{i_2})\right], \tag{6.8}$$
$$\left(B^{j_1 j_2}\right)_{h_1 i_2} = \mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}b(\psi_{i_2},\chi_{h_1})\right], \tag{6.9}$$
$$\left(C^{j_1 j_2}\right)_{h_1 h_2} = \mathbb{E}_\rho\left[\phi_{j_1}\phi_{j_2}c(\chi_{h_1},\chi_{h_2})\right], \tag{6.10}$$
$$\left(F^{j_2}\right)_{i_2} = \mathbb{E}_\rho\left[\phi_{j_2}\langle f,\psi_{i_2}\rangle\right], \tag{6.11}$$
$$\left(G^{j_2}\right)_{h_2} = \mathbb{E}_\rho\left[\phi_{j_2}\langle g,\chi_{h_2}\rangle\right]. \tag{6.12}$$

where $j_1, j_2 = 1,\dots,s$, $i_1, i_2 = 1,\dots,n$, $h_1, h_2 = 1,\dots,k$. Then the discrete mixed problem (6.7) can be written as the system

$$\begin{bmatrix} \mathbb{K}(\ell) & \mathbb{B}^\top \\ \mathbb{B} & -\mathbb{C} \end{bmatrix}_{(ns+ks)\times(ns+ks)} \begin{bmatrix} \mathbb{U} \\ \mathbb{P} \end{bmatrix}_{(ns+ks)\times 1} = \begin{bmatrix} \mathbb{F} \\ \mathbb{G} \end{bmatrix}_{(ns+ks)\times 1},$$

where

$$\mathbb{K}(\ell) := \begin{bmatrix} K_\ell^{11} & \dots & K_\ell^{1s} \\ \vdots & \ddots & \vdots \\ K_\ell^{s1} & \dots & K_\ell^{ss} \end{bmatrix}, \quad \mathbb{B} := \begin{bmatrix} B^{11} & \dots & B^{1s} \\ \vdots & \ddots & \vdots \\ B^{s1} & \dots & B^{ss} \end{bmatrix}, \quad \mathbb{C} := \begin{bmatrix} C^{11} & \dots & C^{1s} \\ \vdots & \ddots & \vdots \\ C^{s1} & \dots & C^{ss} \end{bmatrix},$$

$$\mathbb{F} := \begin{bmatrix} F^1 \\ \vdots \\ F^s \end{bmatrix}, \quad \mathbb{G} := \begin{bmatrix} G^1 \\ \vdots \\ G^s \end{bmatrix}.$$

To derive the discrete form of the regularized ELS and OLS functionals, we will first derive a discrete form of the inner product in V_σ. For $v_{hk}, w_{hk} \in \mathcal{V}_h \otimes \mathcal{S}_k$, with the representations

$$v_{hk} = \sum_{i_1=1}^{n}\sum_{j_1=1}^{s} V_{i_1 j_1}\psi_{i_1}\phi_{j_1}, \quad \text{and} \quad w_{hk} = \sum_{i_2=1}^{n}\sum_{j_2=1}^{s} W_{i_2 j_2}\psi_{i_2}\phi_{j_2}, \tag{6.13}$$

we have

$$\begin{aligned}\langle v_{hk}, w_{hk}\rangle_{V_\rho} &:= \mathbb{E}_\rho\left[\langle v_{hk}, w_{hk}\rangle_V\right]\\ &= \mathbb{E}_\rho\left[\left\langle \sum_{i_1=1}^{n}\sum_{j_1=1}^{s} V_{i_1 j_1}\psi_{i_1}\phi_{j_1}, \sum_{i_2=1}^{n}\sum_{j_2=1}^{s} W_{i_2 j_2}\psi_{i_2}\phi_{j_2}\right\rangle_V\right]\\ &= \sum_{i_1,i_2=1}^{n}\sum_{j_1,j_2=1}^{s} V_{i_1 j_1} W_{i_2 j_2}\langle\psi_{i_1},\psi_{i_2}\rangle_V \langle\phi_{j_1},\phi_{j_2}\rangle_\rho\\ &= \sum_{j_1,j_2=1}^{s} \widehat{V}_{j_1}^\top K_V \widehat{W}_{j_2}\langle\phi_{j_1},\phi_{j_2}\rangle_\rho = \mathbb{V}^\top (I_s\otimes K_V)\,\mathbb{W}, \end{aligned}\tag{6.14}$$

where

$$\widehat{V}_{j_1} = \begin{bmatrix} V_{1,j_1}\\ \vdots\\ V_{n,j_1}\end{bmatrix} \in \mathbb{R}^n, \quad \mathbb{V} = \begin{bmatrix}\widehat{V}_1\\ \vdots\\ \widehat{V}_s\end{bmatrix}\in\mathbb{R}^{ns}, \quad \widehat{W}_{j_2} = \begin{bmatrix} W_{1,j_2}\\ \vdots\\ W_{n,j_2}\end{bmatrix}\in\mathbb{R}^n, \quad \mathbb{W} = \begin{bmatrix}\widehat{W}_1\\ \vdots\\ \widehat{K}_s\end{bmatrix}\in\mathbb{R}^{ns},$$

$(K_V)_{i_1 i_2} = \langle\psi_{i_1},\psi_{i_2}\rangle_V$ for $i_1, i_2 = 1,\dots,n$, and I_s is the s by s identity matrix. In the same fashion, we can derive a discrete form of the inner product in Q_σ. For $p_{hk}, q_{hk} \in \mathcal{Q}_h\otimes\mathcal{S}_k$, with the representations

$$p_{hk} = \sum_{h_1=1}^{k}\sum_{j_1=1}^{s} P_{h_1 j_1}\chi_{h_1}\phi_{j_1}, \quad\text{and}\quad q_{hk} = \sum_{h_2=1}^{k}\sum_{j_2=1}^{s} Q_{h_2 j_2}\chi_{h_2}\phi_{j_2}, \tag{6.15}$$

we have

$$\begin{aligned}\langle p_{hk}, q_{hk}\rangle_{Q_\rho} &:= \mathbb{E}_\rho\left[\langle p_{hk}, q_{hk}\rangle_Q\right]\\ &= \mathbb{E}_\rho\left[\left\langle \sum_{h_1=1}^{k}\sum_{j_1=1}^{s} P_{h_1 j_1}\chi_{h_1}\phi_{j_1}, \sum_{h_2=1}^{k}\sum_{j_2=1}^{s} Q_{h_2 j_2}\chi_{h_2}\phi_{j_2}\right\rangle_Q\right]\\ &= \mathbb{P}^\top (I_s\otimes K_Q)\,\mathbb{Q}, \end{aligned}\tag{6.16}$$

where

$$\widehat{P}_{j_1} = \begin{bmatrix} P_{1,j_1}\\ \vdots\\ P_{k,j_1}\end{bmatrix} \in \mathbb{R}^k, \quad \mathbb{P} = \begin{bmatrix}\widehat{P}_1\\ \vdots\\ \widehat{P}_s\end{bmatrix}\in\mathbb{R}^{ks}, \quad \widehat{Q}_{j_2} = \begin{bmatrix} Q_{1,j_2}\\ \vdots\\ Q_{k,j_2}\end{bmatrix}\in\mathbb{R}^k, \quad \mathbb{Q} = \begin{bmatrix}\widehat{Q}_1\\ \vdots\\ \widehat{Q}_s\end{bmatrix}\in\mathbb{R}^{ks}$$

and $(K_Q)_{h_1 h_2} = \langle\chi_{h_1},\chi_{h_2}\rangle_Q$ for $h_1, h_2 = 1,\dots,k$. For a discrete representation of the regularization space $H_\rho := L^2(\Gamma; H)$, we assume that $\mathcal{L}_h$ is an m-dimensional subspace of the space H with basis denoted by $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. Let $\mathcal{H}_{hk}$ be finite-dimensional subspace of H_ρ. As before, we take the tensor product of the basis φ_i and ϕ_j and employ the following mp-dimensional subspace:

$$\mathcal{H}_{hk} := \mathcal{L}_h\otimes\widetilde{\mathcal{S}}_k := \operatorname{span}\{\varphi_i\eta_j \mid i = 1,\dots,m,\ j = 1,\dots,p\}.$$

Then, any $\ell\in\mathcal{L}_h\otimes\widetilde{\mathcal{S}}_k$ has the representation

$$\ell = \sum_{i=1}^{m}\sum_{j=1}^{p} L_{ij}\varphi_i\eta_j = \sum_{j=1}^{p}\left[\sum_{i=1}^{m} L_{ij}\varphi_i\right]\eta_j = \sum_{j=1}^{p}\widehat{L}_j\eta_j, \quad\text{with } \widehat{L}_j := \sum_{i=1}^{m} L_{ij}\varphi_i. \tag{6.17}$$

We will use the following vectorized notation for the discrete coefficient:

$$\mathbb{L} = \mathrm{vec}(L_{ij}) = \begin{bmatrix} \widehat{L}_1 \\ \widehat{L}_2 \\ \vdots \\ \widehat{L}_p \end{bmatrix} \in \mathbb{R}^{mp}, \text{ with } \widehat{L}_j := \begin{bmatrix} L_{1j} \\ L_{2j} \\ \vdots \\ L_{mj} \end{bmatrix} \in \mathbb{R}^m. \tag{6.18}$$

Now, using (6.13) and (6.17), we have

$$\mathbb{E}_\rho \left[a(\ell, v, w)\right] = \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{s} \mathbb{E}_\rho \left[a\left(\ell, V_{i_1j_1}\psi_{i_1}\phi_{j_1}, W_{i_2j_2}\psi_{i_2}\phi_{j_2}\right)\right] = \langle \mathbb{W}, \mathbb{K}(\mathbb{L})\mathbb{V}\rangle, \tag{6.19}$$

where, for $i_1, i_2 = 1, ..., n$ and $j_1, j_2 = 1, ..., s$, the blocks of matrix $\mathbb{K}(\mathbb{L})$ are given by

$$\begin{aligned}
\left(K(\mathbb{L})^{j_1j_2}\right)_{i_1i_2} &= \mathbb{E}_\rho \left[\phi_{j_1}\phi_{j_2} a\left(\ell, \psi_{i_1}, \psi_{i_2}\right)\right] \\
&= \mathbb{E}_\rho \left[\phi_{j_1}\phi_{j_2} a\left(\sum_{i=1}^{m}\sum_{j=1}^{p} L_{ij}\eta_j\varphi_i, \psi_{i_1}, \psi_{i_2}\right)\right] \\
&= \sum_{j=1}^{p} \mathbb{E}_\rho \left[\phi_{j_1}\phi_{j_2}\eta_j\right] a\left(\sum_{i=1}^{m} L_{ij}\varphi_i, \psi_{i_1}, \psi_{i_2}\right) && (6.20) \\
&= \sum_{j=1}^{p} \mathbb{E}_\rho \left[\eta_j\phi_{j_1}\phi_{j_2}\right] a\left(\widehat{L}_j, \psi_{i_1}, \psi_{i_2}\right) && (6.21) \\
&= \sum_{j=1}^{p} G^j_{j_1j_2} D^j_{i_1i_2}
\end{aligned}$$

where, for $j_1, j_2 = 1, \dots, s$ and $i_1, i_2 = 1, \dots, n$,

$$G^j_{j_1j_2} := \mathbb{E}_\rho \left[\eta_j\phi_{j_1}\phi_{j_2}\right], \quad \text{and} \quad D^j_{i_1i_2} := a\big(\widehat{L}_j, \psi_{i_1}, \psi_{i_2}\big).$$

The matrix $\mathbb{K}(\mathbb{L})$ can be expressed as a sum of p Kronecker products of matrices G^j and D^j,

$$\mathbb{K}(\mathbb{L}) = \sum_{j=1}^{p} G^j \otimes D^j. \tag{6.22}$$

From an alternative point of view, we also have

$$\begin{aligned}
\mathbb{E}_\rho \left[a(\ell, v, w)\right] &= \mathbb{E}_\rho \left[a\left(\sum_{i=1}^{m}\sum_{j=1}^{p} L_{ij}\varphi_i\eta_j, v, \sum_{i_1=1}^{n}\sum_{j_1=1}^{s} W_{i_1j_1}\psi_{i_1}\phi_{j_1}\right)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{i_1=1}^{n}\sum_{j_1=1}^{s} L_{ij}W_{i_1j_1}\mathbb{E}_\rho \left[\eta_j\phi_{j_1} a\left(\varphi_i, v, \psi_{i_1}\right)\right], \\
&= \langle \mathbb{W}, \mathbb{S}(\mathbb{V})\mathbb{L}\rangle, && (6.23)
\end{aligned}$$

where $\mathbb{S}(\mathbb{V})$ is a block matrix size $ns \times mp$ consisting of $s \times p$ blocks, each of size $n \times m$, that is,

$$\mathbb{S}(\mathbb{V}) := \begin{bmatrix} S_v^{11} & \dots & S_v^{1p} \\ \vdots & \vdots & \vdots \\ S_v^{s1} & \dots & S_v^{sp} \end{bmatrix}, \tag{6.24}$$

with a general block, with $j_2 = 1, \dots, s$, $j = 1, \dots, p$, $i = 1, \dots, m$, and $i_2 = 1, \dots, n$, is given by

$$\begin{aligned} \left(S_v^{j_2 j}\right)_{i_2 i} &= \mathbb{E}_\rho\left[\eta_j \phi_{j_2} a\left(\varphi_i, v, \psi_{i_2}\right)\right] \\ &= \mathbb{E}_\rho\left[\eta_j \phi_{j_2} a\left(\varphi_i, \sum_{i_1=1}^{n} \sum_{j_1=1}^{s} V_{i_1 j_1} \psi_{i_1} \phi_{j_1}, \psi_{i_2}\right)\right] \\ &= \sum_{i_1=1}^{n} \sum_{j_1=1}^{s} V_{i_1 j_1} \mathbb{E}_\rho\left[\eta_j \phi_{j_1} \phi_{j_2}\right] a\left(\varphi_i, \psi_{i_1}, \psi_{i_2}\right). \end{aligned}$$

As in (6.21), the above blocks can be expressed as follows:

$$\left(S_v^{j_2 j}\right)_{i_2 i} = \sum_{j_1=1}^{s} \mathbb{E}_\rho\left[\eta_j \phi_{j_1} \phi_{j_2}\right] a\left(\varphi_i, \widehat{V}_{j_1}, \psi_{i_2}\right). \tag{6.25}$$

We note that due to (6.19) and (6.23), we have

$$\mathbb{S}(\mathbb{V})\mathbb{L} = \mathbb{K}(\mathbb{L})\mathbb{V}, \quad \text{for every } \mathbb{V} \in \mathbb{R}^{ns},\ \mathbb{L} \in \mathbb{R}^{mp}. \tag{6.26}$$

Remark 6.2. The matrix $\mathbb{S}$ introduced in (6.26) plays a crucial role in deriving explicit formulas for the objective functionals ELS/OLS, as well as their derivatives and Hessians. The key idea is that the coefficient, which appears implicitly within $\mathbb{K}$, becomes explicit through the intervention of $\mathbb{S}$. This transition simplifies the computation of derivatives. Analogous to deterministic inverse problems, the matrix $\mathbb{S}$ is referred to as the block adjoint stiffness matrix. □

6.3. Discrete ELS

We will now present the explicit formulas for the regularized ELS objective. We note that the parameterized form of the ELS functional is given by

$$J(\ell) = \frac{1}{2}\mathbb{E}_\rho\left[a(\ell, u_\ell - \tilde{z}, u_\ell - \tilde{z})\right] + \mathbb{E}_\rho\left[b(u_\ell - \tilde{z}, p_\ell - \hat{z})\right] - \frac{1}{2}\mathbb{E}_\rho\left[c(p_\ell - \hat{z}, p_\ell - \hat{z})\right], \tag{6.27}$$

where $(\tilde{z}, \hat{z}) \in L^2(\Gamma; V) \times L^2(\Gamma; Q)$ is the data and $w_\ell = (u_\ell, p_\ell)$ is a solution of (6.3), that is,

$$\begin{aligned} \mathbb{E}_\rho\left[a(\ell, u(\ell), v)\right] + \mathbb{E}_\rho\left[b(v, p(\ell))\right] &= \mathbb{E}_\rho\left[\langle f, v\rangle_V\right], \quad \text{for every } v \in L^2_\rho(\Gamma; V), \\ \mathbb{E}_\rho\left[b(u(\ell), q)\right] - \mathbb{E}_\rho\left[c(p(\ell), q)\right] &= \mathbb{E}_\rho\left[\langle g, q\rangle_Q\right], \quad \text{for every } q \in L^2_\rho(\Gamma; Q). \end{aligned}$$

For the data $\tilde{z} \in \mathcal{V}_{hk}$, we take the following expression:

$$\tilde{z} = \sum_{i=1}^{n} \sum_{j=1}^{s} \widetilde{Z}_{ij} \psi_i \phi_j = \sum_{j=1}^{s} \left[\sum_{i=1}^{n} \widetilde{Z}_{ij} \psi_i\right] \phi_j = \sum_{j=1}^{s} \widetilde{Z}_j \phi_j, \quad \text{where } \widetilde{Z}_j := \sum_{i=1}^{n} \widetilde{Z}_{ij} \psi_i.$$

We again employ the vectorized notation as follows:

$$\widetilde{\mathbb{Z}} = \mathrm{vec}(\widetilde{Z}_{ij}) = \begin{bmatrix} \widetilde{Z}_1 \\ \widetilde{Z}_2 \\ \vdots \\ \widetilde{Z}_s \end{bmatrix} \in \mathbb{R}^{ns}, \quad \text{with } \widetilde{Z}_j := \begin{bmatrix} \widetilde{Z}_{1j} \\ \widetilde{Z}_{2j} \\ \vdots \\ \widetilde{Z}_{nj} \end{bmatrix} \in \mathbb{R}^n.$$

In a complete analogous fashion, for the data $\hat{z} \in \mathcal{Q}_{hk}$, we take the expression

$$\hat{z} = \sum_{i=1}^{k}\sum_{j=1}^{s} \widehat{Z}_{ij}\chi_i\phi_j = \sum_{j=1}^{s}\Big[\sum_{i=1}^{k}\widehat{Z}_{ij}\chi_i\Big]\phi_j = \sum_{j=1}^{s}\widehat{Z}_j\phi_j, \quad \text{where } \widehat{Z}_j \equiv \sum_{i=1}^{k}\widehat{Z}_{ij}\chi_i.$$

Once again, we again employ the vectorized notation:

$$\widehat{\mathbb{Z}} = \mathrm{vec}(\widehat{Z}_{ij}) = \begin{bmatrix} \widehat{Z}_1 \\ \widehat{Z}_2 \\ \vdots \\ \widehat{Z}_s \end{bmatrix} \in \mathbb{R}^{ks}, \quad \text{with } \widehat{Z}_j := \begin{bmatrix} \widehat{Z}_{1j} \\ \widehat{Z}_{2j} \\ \vdots \\ \widehat{Z}_{kj} \end{bmatrix} \in \mathbb{R}^k.$$

By the discretization scheme described above, we obtain the discrete form of the ELS function (6.27):

$$\begin{aligned}
\mathbb{J}(\mathbb{L}) &= \frac{1}{2}\begin{bmatrix} \mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}} \\ \mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}} \end{bmatrix}^\top \begin{bmatrix} \mathbb{K}(\mathbb{L}) & \mathbb{B}^\top \\ \mathbb{B} & -\mathbb{C} \end{bmatrix} \begin{bmatrix} \mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}} \\ \mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}} \end{bmatrix} \\
&= \frac{1}{2}(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{K}(\mathbb{L})(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}}) + (\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{B}^\top(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}}) \\
&\quad - \frac{1}{2}(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}})^\top\mathbb{C}(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}}).
\end{aligned} \tag{6.28}$$

To compute the gradient and the Hessian of the ELS objective, we will discretize the parametric derivative characterization: For a parameter $\ell \in L^2(\Gamma; B)$, the derivative of the solution map $\ell \to (u_\ell, p_\ell) \in L^2(\Gamma; V) \times L^2(\Gamma; Q)$ in a direction $\delta\ell$, denoted by $(\delta u, \delta p) \in L^2(\Gamma; V) \times L^2(\Gamma; Q)$, is the unique solution of the following parametric mixed variational problem:

$$\mathbb{E}_\rho\left[a(\ell, \delta u, v)\right] + \mathbb{E}_\rho\left[b(v, \delta p)\right] = -\mathbb{E}_\rho\left[a(\delta\ell, u, v)\right], \quad \text{for all } v \in L^2(\Gamma; V), \tag{6.29a}$$

$$\mathbb{E}_\rho\left[b(\delta u, q)\right] - \mathbb{E}_\rho\left[c(\delta p, q)\right] = 0, \quad \text{for all } q \in L^2(\Gamma; Q). \tag{6.29b}$$

Using the finite-dimensional subspaces $\mathcal{V}_{hk}$ and $\mathcal{Q}_{hk}$ of V_ρ and Q_ρ, we consider the discrete form of the above derivative characterization that seeks $\delta w_{hk} = (\delta u_{hk}, \delta p_{hk}) \in \mathcal{V}_{hk} \times \mathcal{Q}_{hk}$ by solving the following finite-dimensional mixed variational problem:

$$\mathbb{E}_\rho\left[a(\ell, \delta u_{hk}, v)\right] + \mathbb{E}_\rho\left[b(v, \delta p_{hk})\right] = -\mathbb{E}_\rho\left[a(\delta\ell, u_{hk}, v)\right], \quad \text{for all } v \in \mathcal{V}_{hk}, \tag{6.30a}$$

$$\mathbb{E}_\rho\left[b(\delta u_{hk}, q)\right] - \mathbb{E}_\rho\left[c(\delta p_{hk}, q)\right] = 0, \quad \text{for all } q \in \mathcal{Q}_{hk}. \tag{6.30b}$$

To discretize the above system, we denote by $\delta\mathbb{W} = (\delta\mathbb{U}, \delta\mathbb{P})$, and as before, obtain

$$\mathbb{K}(\mathbb{L})\delta\mathbb{U} + \mathbb{B}^\top\delta\mathbb{P} = -\mathbb{K}(\delta\mathbb{L})\mathbb{U}(\mathbb{L}), \tag{6.31a}$$

$$\mathbb{B}(\delta\mathbb{U}) - \mathbb{C}\delta\mathbb{P} = 0. \tag{6.31b}$$

We rewrite the above system into the following equivalent form:

$$(\mathbb{U}(\mathbb{L})-\widetilde{\mathbb{Z}})^\top\mathbb{K}(\mathbb{L})\delta\mathbb{U}+(\mathbb{U}(\mathbb{L})-\widetilde{\mathbb{Z}})^\top\mathbb{B}^\top\delta\mathbb{P}-(\mathbb{U}(\mathbb{L})-\widetilde{\mathbb{Z}})^\top\mathbb{K}(\delta\mathbb{L})\mathbb{U}(\mathbb{L}) = 0, \quad (6.32a)$$

$$(\delta\mathbb{U})^\top\mathbb{B}^\top(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}}) - \delta\mathbb{P}^\top\mathbb{C}(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}})^\top = 0. \quad (6.32b)$$

It follows from equation (6.31) by a direct computation that

$$\begin{aligned}\delta\mathbb{P} = \mathbb{C}^{-1}\mathbb{B}\delta\mathbb{U} \quad \text{and} \quad & \mathbb{K}(\mathbb{L})\delta\mathbb{U} + \mathbb{B}^\top\delta\mathbb{P} = -\mathbb{K}(\delta\mathbb{L})\mathbb{U}(\mathbb{L})\\ \Longrightarrow\ & \mathbb{K}(\mathbb{L})\delta\mathbb{U} + \mathbb{B}^\top\mathbb{C}^{-1}\mathbb{B}\delta\mathbb{U} = -\mathbb{S}(\mathbb{U}(\mathbb{L}))\delta\mathbb{L}\\ \Longrightarrow\ & \delta\mathbb{U} = -\left(\mathbb{K}(\mathbb{L}) + \mathbb{B}^\top\mathbb{C}^{-1}\mathbb{B}\right)^{-1}\mathbb{S}(\mathbb{U}(\mathbb{L}))\delta\mathbb{L} \qquad (6.33)\end{aligned}$$

We now return to (6.28) and compute the derivative of $\mathbb{J}(\mathbb{L})$ in the direction $\delta\mathbb{L}$ and by using the above system obtain

$$\begin{aligned}D\mathbb{J}(\mathbb{L})(\delta\mathbb{L}) &= \delta\mathbb{U}^\top\mathbb{K}(\mathbb{L})(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}}) + \frac{1}{2}(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{K}(\delta\mathbb{L})(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})\\ &\quad - \delta\mathbb{P}^\top\mathbb{C}(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}}) + \delta\mathbb{U}^\top\mathbb{B}^\top(\mathbb{P}(\mathbb{L}) - \widehat{\mathbb{Z}}) + (\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{B}^\top\delta\mathbb{P}\\ &= -(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{K}(\delta\mathbb{L})\mathbb{U}(\mathbb{L}) + \frac{1}{2}(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{K}(\delta\mathbb{L})(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})\\ &= -\frac{1}{2}(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{K}(\delta\mathbb{L})(\mathbb{U}(\mathbb{L}) + \widetilde{\mathbb{Z}})\\ &= -\frac{1}{2}(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}})^\top\mathbb{S}(\mathbb{U}(\mathbb{L}) + \widetilde{\mathbb{Z}})\delta\mathbb{L} = -\frac{1}{2}\mathbb{U}(\mathbb{L})^\top\mathbb{S}(\mathbb{U}(\mathbb{L}))\delta\mathbb{L} + \frac{1}{2}\mathbb{Z}^\top\mathbb{S}(\mathbb{Z})\delta\mathbb{L},\end{aligned}$$

where the matrix $\mathbb{S}$ is defined in (6.24). Therefore,

$$\nabla\mathbb{J}(\mathbb{L}) = -\frac{1}{2}\mathbb{S}(\mathbb{U}(\mathbb{L}) + \widetilde{\mathbb{Z}})^\top(\mathbb{U}(\mathbb{L}) - \widetilde{\mathbb{Z}}) = -\frac{1}{2}\mathbb{S}(\mathbb{U}(\mathbb{L}))^\top\mathbb{U}(\mathbb{L}) + \frac{1}{2}\mathbb{S}(\widetilde{\mathbb{Z}})^\top\widetilde{\mathbb{Z}}.$$

In an analogous way, we can derive the following formula for the second variation of the objective functional using equation (6.33)

$$\begin{aligned}D^2\mathbb{J}(\mathbb{L})(\delta\mathbb{L}, \delta\mathbb{L}) &= -\frac{1}{2}\mathbb{U}^\top\mathbb{S}(\delta\mathbb{U})\delta\mathbb{L} - \frac{1}{2}\delta\mathbb{U}^\top\mathbb{S}(\mathbb{U})\delta\mathbb{L} = -\frac{1}{2}\mathbb{U}^\top\mathbb{K}(\delta\mathbb{L})\delta\mathbb{U} - \frac{1}{2}\delta\mathbb{U}^\top\mathbb{S}(\mathbb{U})\delta\mathbb{L}\\ &= -\frac{1}{2}\left(\mathbb{U}^\top\mathbb{K}(\delta\mathbb{L}) + \delta\mathbb{L}^\top\mathbb{S}(\mathbb{U})^\top\right)\delta\mathbb{U}\\ &= \frac{1}{2}\left(\mathbb{U}^\top\mathbb{K}(\delta\mathbb{L}) + \delta\mathbb{L}^\top\mathbb{S}(\mathbb{U})^\top\right)\left(\mathbb{K}(\mathbb{L}) + \mathbb{B}^\top\mathbb{C}^{-1}\mathbb{B}\right)^{-1}\mathbb{S}(\mathbb{U})\delta\mathbb{L}\\ &= \delta\mathbb{L}^\top\mathbb{S}(\mathbb{U})^\top\left(\mathbb{K}(\mathbb{L}) + \mathbb{B}^\top\mathbb{C}^{-1}\mathbb{B}\right)^{-1}\mathbb{S}(\mathbb{U})\delta\mathbb{L},\end{aligned}$$

Therefore, the Hessian matrix of the objective functional is

$$\nabla^2\mathbb{J}(\mathbb{L}) = \mathbb{S}(\mathbb{U})^\top\left(\mathbb{K}(\mathbb{L}) + \mathbb{B}^\top\mathbb{C}^{-1}\mathbb{B}\right)^{-1}\mathbb{S}(\mathbb{U}).$$

7. Stochastic elasticity imaging inverse problem

The elasticity imaging inverse problem consists of identifying stochastic Lamé coefficients describing the elastic properties of the isotropic heterogeneous elastic body (a membrane in $\mathbb{R}^3$) under a stochastic load. To be specific, given a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and a bounded domain $D \subset \mathbb{R}^3$ with its sufficiently smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2$, the following stochastic linear elasticity system models the displacement in the elastic body induced by the stochastic load f, stochastic boundary condition g, and stochastic boundary traction h:

$$-\nabla \cdot [2\mu(\omega,x)\epsilon_u + \lambda(\omega,x)\mathrm{tr}\,(\epsilon_u)\, I] = f(\omega,x), \text{ in } D, \tag{7.1a}$$

$$u(\omega,x) = g(\omega,x), \text{ on } \Gamma_1, \tag{7.1b}$$

$$[2\mu(\omega,x)\epsilon_u + \lambda(\omega,x)\mathrm{tr}\,(\epsilon_u)\, I]\, n = h(\omega,x), \text{ on } \Gamma_2. \tag{7.1c}$$

Here the random field $u(\omega,x)$ is the displacement vector, n is the outward-pointing unit normal to ∂D, and the random fields $\mu : \Omega \times D \to \mathbb{R}$ and $\lambda : \Omega \times D \to \mathbb{R}$ are the Lamé coefficients. Denoting by ∇u, the gradient of the vector-valued function u, the linearized strain tensor $\varepsilon_u := \frac{1}{2}\left(\nabla u(\omega,x) + \nabla u(\omega,x)^\top\right)$ is the local deformation of the elastic body, where $\mathrm{tr}(\varepsilon_u)$ is the trace of ε_u. The stress tensor σ is a measure of the elastic response of the elastic object described by the strain. In the context of (7.1), our focus will be on the inverse problem of estimating the stochastic coefficients $\mu(\omega,x)$ and $\lambda(\omega,x)$ from a measurement of u.

In the context of (7.1), the direct problem involves finding the stochastic displacement vector u when the stochastic functions g and h, the variable coefficients μ and λ, and the body force f are all given. In the elasticity imaging inverse problem, the primary goal is to extract statistical information about the stochastic Lamé coefficient μ based on the statistical data from measurements z. The identification of μ is prioritized over λ due to the composition of soft tissue, which consists of cells, collagen, and other components immersed in a water-based fluid. The tissue's structural properties, which can vary significantly in stiffness depending on the pathological state, are characterized by the shear modulus μ. In contrast, the fluid's properties correspond to the compressibility modulus $M = \lambda + 2\mu \approx \lambda$, where $\lambda \gg \mu$. Materials with this characteristic are referred to as nearly incompressible, and traditional finite element methods become ineffective due to the locking effect.

For simplicity, in (7.1) we take $g = 0$ and define

$$V = \{v \in H^1(D) \times H^1(D) : \ v = 0 \text{ on } \Gamma_1\}.$$

Then, for a fixed $\omega \in \Omega$, the pathwise weak form of the elasticity system (7.1) seeks $u = u(\omega,x) \in V$ such that, for every $v \in V$,

$$\int_D 2\mu(\omega,x)\epsilon(u)\cdot\epsilon(v) + \int_D \lambda(\omega,x)(\mathrm{div}\, u)(\mathrm{div}\, v) = \int_D f(\omega,x)v + \int_{\Gamma_2} vh(\omega,x). \tag{7.2}$$

In this work, we focus on the scenario where the system (7.1) models the response of an isotropic, nearly incompressible elastic object subjected to specific body forces and boundary tractions, characterized by the condition $\lambda \gg \mu$ uniformly. Given the inefficiency of classical finite element discretization in this context, particularly due to the locking effect, we adopt commonly used techniques to introduce a pathwise pressure term for a fixed $\omega \in \Omega$. We define the pressure as $p = p(\omega,x) \in Q = L^2(D)$ through the following identity:

$$p(\omega,x) = \lambda(\omega,x)\mathrm{div}(u(\omega,x)), \tag{7.3}$$

which can be reformulated in variational form as follows:

$$\int_D \mathrm{div}(u(\omega,x))q - \int_D \frac{1}{\lambda(\omega,x)}p(\omega,x)q = 0, \quad \text{for all } q \in Q. \tag{7.4}$$

Using (7.3), we convert the weak form (7.2) into the following variational problem: for the fixed $\omega \in \Omega$, find $u = u(\omega, x) \in V$ such that

$$\int_D 2\mu(\omega, x)\epsilon(u) \cdot \epsilon(v) + \int_D p(\operatorname{div} v) = \int_D fv + \int_{\Gamma_2} vh, \quad \text{for every } v \in V, \tag{7.5}$$

where the pressure p remains an unknown variable.

We assume that there are positive constants $\underline{\mu}$, $\overline{\mu}$, $\underline{\lambda}$, and $\overline{\lambda}$ such that

$$0 < \underline{\mu} \le \mu(\omega, x) \le \overline{\mu}, \quad \forall(\omega, x) \in \Omega \times D,$$
$$0 < \underline{\lambda} \le \lambda(\omega, x) \le \overline{\lambda}, \quad \forall(\omega, x) \in \Omega \times D.$$

It follows that the problem of finding $u \in V$ satisfying (7.2) has now been reformulated as the problem of finding $(u, p) \in V \times Q$ satisfying (7.4) and (7.5). Evidently, for a fixed $\omega \in \Omega$, by defining $a : L^\infty(D) \times V \times V \to \mathbb{R}$, $b : V \times Q \to \mathbb{R}$, and $c : Q \times Q \to \mathbb{R}$ by

$$\begin{aligned} a(\mu, u, v) &= \int_D 2\mu(\omega, x)\epsilon(u) \cdot \epsilon(v) \\ b(u, p) &= \int_D p(\operatorname{div} v), \\ c(q, q) &= \int_D \frac{1}{\lambda(\omega, x)} p(\omega, x) q, \end{aligned}$$

equations (7.4) and (7.5) can be subsumed by the abstract mixed variation problem (2.2).

8. Computational experiments

We now present the result of a computational experiment of a two-dimensional nearly incompressible stochastic elasticity problem (7.1). In both examples, the unknown coefficient $\ell(y, x)$ to be identified is the shear modulus $\mu(y, x)$, expressed as

$$\ell(y, x) = \sum_{i=1}^{m} \sum_{j=1}^{p} L_{ij} \varphi_i(x) \eta_j(y)$$

from the known data

$$\widetilde{z}(y, x) = \sum_{i=1}^{n} \sum_{j=1}^{s} \widetilde{Z}_{ij} \psi_i(x) \phi_j(y) \quad \text{and} \quad \widehat{z}(y, x) = \sum_{i=1}^{k} \sum_{j=1}^{s} \widehat{Z}_{ij} \chi_i(x) \phi_j(y),$$

where $\widetilde{z}$ and $\widehat{z}$ are the observation of displacement u and pressure p, respectively. Since the examples are synthetic, the data

$$\mathbb{Z} = \begin{bmatrix} \widetilde{\mathbb{Z}} \\ \widehat{\mathbb{Z}} \end{bmatrix} = \begin{bmatrix} \operatorname{vec}(\widetilde{Z}_{ij}) \\ \operatorname{vec}(\widehat{Z}_{ij}) \end{bmatrix}$$

is derived from the stochastic-Galerkin solution of the direct problem. We choose $\lambda = 4999$ in both examples.

For coefficient identification, we used the ELS objective function. The optimization problems were solved using the trust-region-reflective algorithm, provided by MATLAB via the `fmincon` function. Originally proposed in [16], this algorithm is also referred to by the authors as the Subspace Trust Region Interior Reflective (STIR) method. STIR extends the traditional trust region approach, which was limited to handling unconstrained problems, to also solve box-constrained problems. Additionally, STIR reduces the dimension of the trust region subproblem to two, enhancing the algorithm's efficiency for large-scale problems. This algorithm requires both the gradient and the Hessian of the objective function to operate effectively.

We denote the degree of stochasticity M, which is the dimension of the random vector y. In all examples, we assume that each component of $y(\omega) = (y_1(\omega), \cdots, y_M(\omega))^\top$ is identically and independently distributed, and the distribution of each component is known a priori.

Relative errors in the mean and variance between two scalar random fields are computed by the following formulas. For example, the relative errors between coefficient ℓ and the exact coefficient $\bar{\ell}$ are computed as:

$$\varepsilon_{\text{mean}}(\ell, \bar{\ell}) = \frac{\sqrt{\int_D (\mathbb{E}[\bar{\ell}(\omega, x)] - \mathbb{E}[\ell(\omega, x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[\bar{\ell}(\omega, x)\right]^2 dx}}, \tag{8.1a}$$

$$\varepsilon_{\text{var}}(\ell, \bar{\ell}) = \frac{\sqrt{\int_D (\text{Var}[\bar{\ell}(\omega, x)] - \text{Var}[\ell(\omega, x)])^2 dx}}{\sqrt{\int_D \text{Var}\left[\bar{\ell}(\omega, x)\right]^2 dx}}. \tag{8.1b}$$

Also, we compute relative errors in the mean and variance between two random vector fields using the following formulas. For example, we solve the direct problem using the estimated coefficient ℓ to obtain the simulated data u_ℓ. We then measure the error by comparing u_ℓ with the exact data z,

$$\varepsilon_{\text{mean}}(u_\ell, z) = \frac{\sqrt{\int_D \|\mathbb{E}[z(\omega, x)] - \mathbb{E}[u_\ell(\omega, x)]\|_2^2 \, dx}}{\sqrt{\int_D \|\mathbb{E}\left[z(\omega, x)\right]\|_2^2 \, dx}}, \tag{8.2a}$$

$$\varepsilon_{\text{var}}(u_\ell, z) = \frac{\sqrt{\int_D \|\text{Var}[z(\omega, x)] - \text{Var}[u_\ell(\omega, x)]\|_2^2 \, dx}}{\sqrt{\int_D \|\text{Var}\left[z(\omega, x)\right]\|_2^2 \, dx}}. \tag{8.2b}$$

Example 8.1. The exact coefficient $\bar{\ell}$ and the exact solution $\bar{u}$ of the PDE are defined by

$$\begin{aligned}
\bar{\ell}(y, x_1, x_2) &= 2 + 1.5 \exp\left(-15(x_1 - 0.3)^2 - 15(x_2 - 0.3)^2\right) y_1 \\
&\quad + 1.5 \exp\left(-15(x_1 - 0.7)^2 - 15(x_2 - 0.7)^2\right) y_2 \\
\bar{u}_1(y, x_1, x_2) &= \frac{1}{2} \sin(2\pi x_2)(-1 + \cos(2\pi x_1)) y_1 + \frac{1}{2} \sin(\pi x) \sin(\pi y) y_2 \\
\bar{u}_2(y, x_1, x_2) &= \frac{1}{2} \sin(2\pi x_1)(1 - \cos(2\pi x_2)) y_1 + \frac{1}{2} \sin(\pi x) \sin(\pi y) y_2
\end{aligned} \tag{8.3}$$

where the domain $D = [0,1]^2$ and $y_i \sim U(0,1)$ are independent and uniformly distributed over the closed interval $[0,1]$, for $i = 1,\dots,2$. Therefore, the density function $\rho = \rho(y_1, y_2) = 1$. Note that this example satisfies the homogeneous Dirichlet condition, $u|_{\partial D} = 0$. And the load function $f(\omega, x_1, x_2)$ can be computed accordingly.

We use the same rectangular mesh $\mathcal{T}_h$ for both the coefficient ℓ and the each component of the solution u within the spatial domain D, employing order-1 rectangular polynomials. The mesh density, denoted by N, is defined as the number of triangle edges along the interval $[0,1]$. It is important to note that the coefficient ℓ has additional degrees of freedom at the nodes on the boundary of D because it does not conform to the homogeneous boundary condition.

We choose the basis for the subspace $\mathcal{S}_k$ to be the complete polynomials of total degree 1 (see [40]):

$$\mathcal{S}_k = \text{span}\left\{ \prod_{i=1}^{M} P_{\alpha_i}(y_i) \;\middle|\; \alpha_i = 0,1, \sum_{i=1}^{M} \alpha_i \le 1 \right\},$$

where $P_{\alpha_i}(y_i)$ is the orthonormal (on the interval $[0,1]$ with uniform weight) Legendre polynomial of degree α_i in y_i , that is, $P_0 = 1$ and $P_1(y_i) = 2\sqrt{3}y_i - \sqrt{3}$. Therefore, the basis for $\mathcal{S}_k$ is

$$\phi_1(y) = P_0^2 = 1, \quad \phi_i(y) = P_0 P_1(y_{4-i}) = 2\sqrt{3}y_{4-i} - \sqrt{3}, \text{ for } i = 2,3.$$

For the subspace $\widetilde{\mathcal{S}}_k$, we choose a slightly different basis,

$$\widetilde{\mathcal{S}}_k = \text{span}\left\{ \prod_{i=1}^{M} \widetilde{P}_{\alpha_i}(y_i) \;\middle|\; \alpha_i = 0,1, \sum_{i=1}^{M} \alpha_i \le 1 \right\}, \quad \text{where } \widetilde{P}_0 = 1,\; \widetilde{P}_1(y_i) = y_i.$$

Therefore, the basis is $\eta_1(y) = 1, \eta_i(y) = y_{4-i}$ for $i = 2,3$.

We use the (squares of) $L^2(\Omega, H^1(D))$ norm as the regularization term:

$$\begin{aligned}
R(\ell) &= \frac{\kappa}{2}\mathbb{E}\left[\|\ell(\omega, x_1, x_2)\|^2_{H^1(D)} \right] = \frac{\kappa}{2}\int_{[0,1]^2} \rho(y)\, \|\ell(y, x_1, x_2)\|^2_{H^1(D)}\, dy \\
&= \frac{\kappa}{2}\int_{[0,1]^2} \rho(y) \Big\| \sum_{j=1}^{p} \widehat{L}_j(x_1,x_2)\eta_j(y) \Big\|^2_{H^1(D)} dy \\
&= \frac{\kappa}{2}\int_{[0,1]^2} \rho(y) \Big\{ \int_D \Big(\sum_{j=1}^{p} \widehat{L}_j(x_1,x_2)\eta_j(y) \Big)^2 + \\
&\quad + \Big(\sum_{j=1}^{p} \nabla\widehat{L}_j(x_1,x_2)\eta_j(y) \Big) \cdot \Big(\sum_{j=1}^{p} \nabla\widehat{L}_j(x_1,x_2)\eta_j(y) \Big) dx \Big\} dy. \\
&= \frac{\kappa}{2}\sum_{j_1,j_2=1}^{k} \Big(\int_{[0,1]^M} \rho(y)\eta_{j_1}(y)\eta_{j_2}(y) dy \Big) \Big(\int_D \widehat{L}_{j_1}\widehat{L}_{j_2} dx + \int_D \nabla\widehat{L}_{j_1}\cdot\nabla\widehat{L}_{j_2} dx \Big) \\
&= \frac{\kappa}{2}\mathbb{L}^\top \left(\Psi \otimes (Q_L + K_L) \right) \mathbb{L},
\end{aligned}$$

where
$$\Psi_{i,j} = \int_{[0,1]^2} \rho(y)\eta_i\eta_j dy \qquad \text{for every } i,j = 1,\dots,p,$$
$$(K_L)_{i,j} = \int_D \varphi_j(x)\varphi_i(x)dx \qquad \text{for every } i,j = 1,\dots,m,$$
and
$$(Q_L)_{i,j} = \int_D \nabla\varphi_j(x)\nabla\varphi_i(x)dx \quad \text{for every } i,j = 1,\dots,m.$$

Furthermore, we choose $\kappa = $ 1e-04.

N	20	30	40	50	60
DoF of u	2166	5046	9126	14406	20886
DoF of p	1200	2700	4800	7500	10800
DoF of ℓ	1323	2883	5043	7803	11163
$\epsilon_{\text{mean}}(u,\bar{u})$	8.883E-03	3.956E-03	2.227E-03	1.426E-03	9.901E-04
$\epsilon_{\text{var}}(u,\bar{u})$	1.718E-02	7.685E-03	4.333E-03	2.776E-03	1.929E-03
$\epsilon_{\text{mean}}(p/\lambda,\bar{p}/\lambda)$	4.106E-03	1.827E-03	1.028E-03	6.580E-04	4.570E-04
$\epsilon_{\text{var}}(p/\lambda,\bar{p}/\lambda)$	8.196E-03	3.650E-03	2.055E-03	1.316E-03	9.138E-04
$\epsilon_{\text{mean}}(l,\bar{l})$	1.210E-02	9.339E-03	9.947E-03	1.177E-02	2.158E-02
$\epsilon_{\text{var}}(l,\bar{l})$	1.165E-01	8.586E-02	9.628E-02	1.279E-01	4.032E-01
$\epsilon_{\text{mean}}(u_\ell,\widetilde{z})$	5.066E-04	3.462E-04	3.895E-04	5.424E-04	1.493E-03
$\epsilon_{\text{var}}(u_\ell,\widetilde{z})$	2.420E-03	1.393E-03	1.707E-03	2.663E-03	8.960E-03
$\epsilon_{\text{mean}}(p_\ell/\lambda,\widehat{z}/\lambda)$	5.693E-06	4.464E-06	4.750E-06	5.610E-06	1.036E-05
$\epsilon_{\text{var}}(p_\ell/\lambda,\widehat{z}/\lambda)$	1.519E-05	1.035E-05	1.159E-05	1.524E-05	3.806E-05
Running Time (s)	4.105641	31.961711	169.086764	445.16195	741.556365

Table 1: Results of Example 8.1 under different mesh sizes.

We solve the direct problem using the exact $\bar{\ell}$ and f with the stochastic Galerkin method. From 4th to 7th rows of Table 1, we assess the accuracy by evaluating the relative differences in mean and variance between the numerical solution (u,p) and the exact solution $(\bar{u},\bar{p})$, using the error functionals defined in (8.1) and (8.2).

Next, we present the numerical results of the inverse problem using the ELS objective and a regularization parameter of 1e-04 in the 9th to 10th rows of Table 1.

We then solve the direct problem using the estimated coefficient ℓ to obtain the simulated data (u_ℓ,p_ℓ). We then measure the error by comparing (u_ℓ,p_ℓ) with the exact data $(\widetilde{z},\widehat{z})$. The results are shown in the 11th to 14th line of Table 1.

The total CPU time for the optimization process is reported in the last row of Table 1. Reconstructions of the the expectation and variance of the estimated coefficient ℓ versus the exact coefficient $\bar{\ell}$ (with $N = 30$) in Figure 1.

In the following example, we consider a model with more degree of stochasticity $M = 5$.

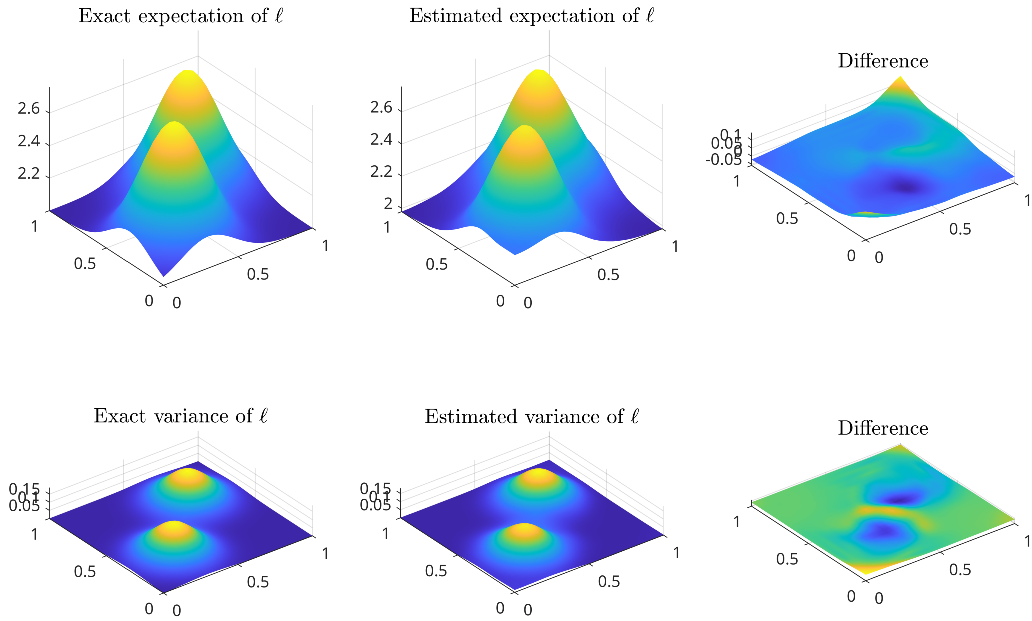


Figure 1: Mean and variance of exact and estimated coefficients with $N = 30$ in Example 8.1.

Example 8.2. The exact coefficient $\bar{\ell}$ and the exact solution $\bar{u}$ of the PDE are defined by

$$\bar{\ell}(\omega, x_1, x_2) = 1 + x_1^2 + x_2^2 + \sum_{n=1}^{M} \cos\left(\frac{n\pi x_1}{2M+2}\right) \cos\left(\frac{n\pi x_2}{2M+2}\right) y_n,$$

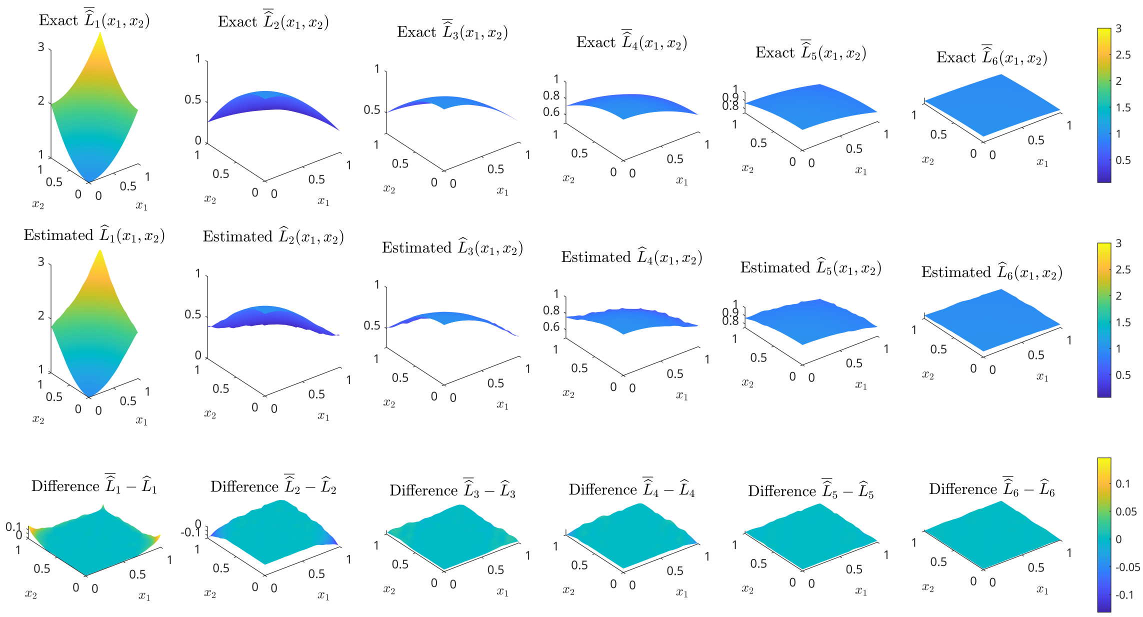
$$\bar{u}_1(\omega, x_1, x_2) = \frac{1}{2} \sum_{n=1}^{M} \sin(n\pi x_1) \sin(n\pi x_2) y_n,$$

$$\bar{u}_2(\omega, x_1, x_2) = -\frac{1}{2} \sum_{n=1}^{M} \sin(n\pi x_1) \sin(n\pi x_2) y_n,$$

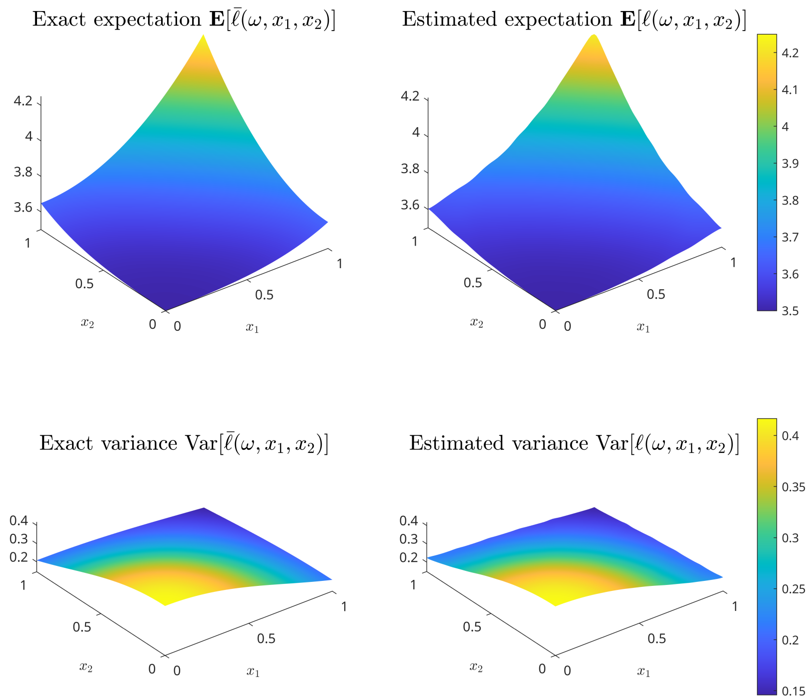
N	20	30	40	50	60
DoF of u	4332	10092	18252	28812	41772
DoF of p	2400	5400	9600	15000	21600
DoF of ℓ	2646	5766	10086	15606	22326
$\epsilon_{\text{mean}}(u, \bar{u})$	1.566E-02	7.260E-03	4.144E-03	2.670E-03	1.861E-03
$\epsilon_{\text{var}}(u, \bar{u})$	2.394E-02	1.136E-02	6.543E-03	4.234E-03	2.959E-03
$\epsilon_{\text{mean}}(p/\lambda, \bar{p}/\lambda)$	7.142E-02	3.244E-02	1.839E-02	1.181E-02	8.216E-03
$\epsilon_{\text{var}}(p/\lambda, \bar{p}/\lambda)$	1.395E-01	6.487E-02	3.708E-02	2.391E-02	1.667E-02
$\epsilon_{\text{mean}}(l, \bar{l})$	1.347E-03	1.251E-03	1.226E-03	1.218E-03	1.215E-03
$\epsilon_{\text{var}}(l, \bar{l})$	4.515E-03	4.101E-03	4.024E-03	3.924E-03	3.910E-03
$\epsilon_{\text{mean}}(u_\ell, \widetilde{z})$	6.777E-06	6.212E-06	6.273E-06	6.182E-06	6.204E-06
$\epsilon_{\text{var}}(u_\ell, \widetilde{z})$	2.446E-05	2.202E-05	2.200E-05	2.175E-05	2.173E-05
$\epsilon_{\text{mean}}(p_\ell/\lambda, \widehat{z}/\lambda)$	3.696E-07	2.996E-07	3.026E-07	2.916E-07	2.930E-07
$\epsilon_{\text{var}}(p_\ell/\lambda, \widehat{z}/\lambda)$	2.934E-06	2.850E-06	2.809E-06	2.762E-06	2.748E-06
Running Time (s)	23.338934	221.513128	945.509571	2277.17927	7663.46633

Table 2: Results of Example 8.2 under different mesh sizes.

where $M = 5$, the domain $D = [0,1]^2$ and $y_i \sim U(0,1)$ are independent and uniformly distributed over the closed interval $[0,1]$, for $i = 1,\ldots,5$. The load term f is computed symbolically.



Figur 2: Exact and estimated coefficients with $N = 60$ in Example 8.2.



Figur 3: Mean and variance of exact and estimated coefficients with $N = 60$ in Example 8.2.

In this example, we apply the same mesh rectangular mesh and the same regularization as in Example 8.1.

In the same way as in Example 8.1, we choose the basis for the subspace $\mathcal{S}_k$ to be the complete polynomials of total degree 1, and $\widetilde{\mathcal{S}}_k$ to be the set of monomials. However in this example, the number of basis for $\mathcal{S}_k$ and $\widetilde{\mathcal{S}}_k$ is larger due to larger M. The basis for $\mathcal{S}_k$ is

$$\phi_1(y) = P_0^2 = 1, \quad \phi_i(y) = P_0 P_1(y_{8-i}) = 2\sqrt{3} y_{8-i} - \sqrt{3}, \text{ for } i = 2, \dots, 6$$

and the basis for $\widetilde{\mathcal{S}}_k$ is $\eta_1(y) = 1$, $\eta_i(y) = y_{8-i}$ for $i = 2, \dots, 6$.

We repeat the same procedure described in Example 8.1 and the result is shown in Table 2. Reconstructions of the components of the coefficient ℓ and the corresponding exact components (with $N = 60$) are shown in Figure 2. Additionally, we visualize the expectation and variance of the estimated coefficient ℓ versus the exact coefficient $\bar{\ell}$ in Figure 3.

9. Concluding remarks

We have developed a convex inversion framework for estimating the mean and variance of random coefficients in stochastic mixed variational problems.

The numerical examples used to assess the effectiveness of our energy least-squares approach are analytical, where an explicit solution representation was available and served as the data for the inverse problem. In these tests, we had access to both the spatial and stochastic components of the data. However, in practical applications, measured data must be preprocessed to reduce noise or be transformed into a truncated Karhunen-Loève representation to fit the required format. Real-world data typically consists of multiple samples, which can be aggregated and processed using a truncated Karhunen-Loève approach. A key challenge is obtaining a reliable estimate of the joint density function that governs the random vector in this representation (see [53]). Estimating this joint density is a complex task, often tackled through parametric or nonparametric methods. A future goal is to thoroughly evaluate existing joint density estimation techniques and incorporate them into the preprocessing of measured data for use in the identification process.

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