Journal of Convex Analysis
Volume 32 (2025), No. 2, 359–374

# Slopes and the Moreau-Rockafellar Theorem

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*Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.*

Received: November 23, 2023
Accepted: February 9, 2024

Properties of local and global slopes of a function and its approximate critical points sets are studied in relation to determination of the function.

*Keywords:* Slope, determination of a function.

*2020 Mathematics Subject Classification:* 49J52, 47J22.

## 1. Introduction

Slopes have been studied in relation to gradient flows, e.g. [1, 11], but they are well defined where the differential inclusions are problematic, for example in the absence of any linear-like structure like in general metric spaces. The evolution of the concept is excellently explained in the monograph [15], see also [12]. Many striking and surprising results have been obtained to show that in many cases slope alone determines the function, see e.g. [23, 29]. One drawback of this very active and recent development is the use of transfinite induction. Here we remove any such reference and in the process develop the topic a bit further.

Why it is important to avoid the use of transfinite induction is explained briefly in the Appendix.

In this article we consider results like: let $f$ and $g$ be continuous on a complete metric space and let the local slope of $f$ be everywhere bigger than that of $g$. The case $f$ and $g$ differentiable on a Banach space with $\|f'(x)\| > \|g'(x)\|$ whenever $f'(x) \neq 0$, is interesting enough. Then under a certain condition the infimum of $f - g$ is attained on the set of approximately critical points of $f$.

Our approach, based on an idea from [8], is to consider for an arbitrary fixed $x_0$ the minimisation problem

$$f(x) \to \min \quad \text{subject to} \quad f(x) - g(x) \leq f(x_0) - g(x_0),$$

and observe that – because of the slopes condition – any (approximately) critical point cannot be on the boundary of the feasible set, so it is actually a critical point of $f$.

The study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project SUMMIT BG-RRP-2.004-0008-C01.

ISSN 0944-6532 / $ 2.50 

Even more can be said in the case of global slopes, where we can estimate the infimum of $f-g$ through the infima of $f$ and $g$. This fully primal result almost immediately yields the celebrated Moreau-Rockafellar Theorem [21, 25].

This work has been completed before the appearance of the recent preprint [9]. There a new general result is obtained via transfinite induction which we do not cover here. It would be interesting to check if the tools we develop here would work for it as well.

The paper is organised as follows. In Section 2 we give the necessary definitions. Section 3 is devoted to the properties of the slopes. Section 4 contains the results.

## 2. Definitions

Here we fix concepts and notations we will use in the paper.

We consider functions defined on a metric space $(M,\rho)$ with values in $\mathbb{R}\cup\{+\infty\}$. Although it is somewhat standard in such setting to adopt the convention $\infty-\infty=\infty$, we find this approach prone to errors, thus we avoid it. So, "$\infty-\infty$" is undefined. Therefore, if nothing more is specified, formula like for example $\{x:\ f(x)-g(x)\le\lambda\}$ means those $x$ for which at least one of $f(x)$ and $g(x)$ is finite, and $f(x)-g(x)\le\lambda$. This said, we usually take care to write more explicitly, so for $\lambda\in\mathbb{R}$ we would rather write the equivalent formulae $\{x\in\operatorname{dom} f:\ f(x)-g(x)\le\lambda\}$ where, as usual for $f:M\to\mathbb{R}\cup\{+\infty\}$, the *domain* of $f$ is

$$\operatorname{dom} f := \{x:\ f(x)<\infty\}.$$

We denote the *sub-level set* of $f$ below the level $\lambda\in\mathbb{R}$ by

$$L_\lambda(f) := \{x:\ f(x)\le\lambda\}.$$

Following the convention above,

$$L_\lambda(f-g) := \{x\in\operatorname{dom} f:\ f(x)-g(x)\le\lambda\}.$$

For $f$ which is proper, i.e. such that $\operatorname{dom} f\neq\varnothing$ and bounded below, and $\varepsilon\ge 0$, we denote

$$\varepsilon\text{-}\operatorname{argmin} f := \{x: f(x)\le\inf f+\varepsilon\},$$

hence, $\varepsilon\text{-}\operatorname{argmin} f = L_{(\inf f+\varepsilon)}(f)$.

If $x\in\operatorname{dom} f$ then the *local slope* of $f$ at $x$ is defined as

$$|\nabla f|(x) := \limsup_{\substack{y\to x\\ y\neq x}} \frac{[f(x)-f(y)]^+}{\rho(x,y)}.$$

Here $[t]^+ := \max\{0,t\} = (t+|t|)/2$, so $|\nabla f|(x)=0$ if $x$ is a local minimum to $f$, and if not

$$|\nabla f|(x) = \limsup_{\substack{y\to x\\ y\neq x}} \frac{f(x)-f(y)}{\rho(x,y)}.$$

In the latter statement we explicit that $|\nabla f|(x)=0$ at the isolated points of $\operatorname{dom} f$, where the limit above is undefined. This notion was introduced by De Giorgi, Marino and Tosques [11] and used thereafter for study of concepts as descent curves (e.g. [12]), error bounds (e.g. [3]), subdifferential calculus and metric regularity (e.g. the monograph [15] and references therein).

We may consider $|\nabla f|$ as a function from $\operatorname{dom} f$ to $[0,\infty]$. Moreover,

$$|\nabla(rf)|(x) = r|\nabla f|(x), \quad \forall r \geq 0,\ \forall x \in \operatorname{dom} f. \tag{1}$$

Because $[a+b]^+ \leq [a]^+ + [b]^+$ we have

$$|\nabla(f+g)| \leq |\nabla f| + |\nabla g|, \tag{2}$$

which for $g : M \to \mathbb{R}$ implies

$$|\nabla(f-g)| \geq |\nabla f| - |\nabla g|, \quad \forall x \in \operatorname{dom}|\nabla g|. \tag{3}$$

Let us denote $\qquad \varepsilon\text{-crit} f := \{x : \ |\nabla f|(x) \leq \varepsilon\}.$

In other words, $\varepsilon\text{-crit} f = L_\varepsilon(|\nabla f|)$. If the space $M$ is complete and the function $f$ is lower semicontinuous and bounded below, then Ekeland Variational Principle gives that $\inf |\nabla f| = 0$, thus $\varepsilon\text{-crit} f = \varepsilon\text{-argmin}\,|\nabla f|$.

The *global slope* of $f$ at $x \in \operatorname{dom} f$ is defined as

$$|\widetilde{\nabla} f|(x) := \sup_{y \neq x} \frac{[f(x)-f(y)]^+}{\rho(x,y)},$$

and we can consider $|\widetilde{\nabla} f|$ as a function from $\operatorname{dom} f$ to $[0,\infty]$. The notion is introduced by Thibault and Zagrodny in [29].

Obviously, $|\widetilde{\nabla} f| \geq |\nabla f|$ on $\operatorname{dom} f$. It is also clear that the subadditivity (1), (2) and (3) hold for the global slope $|\widetilde{\nabla}\cdot|$ as well.

By analogy we define $\qquad \varepsilon\text{-Crit} f := L_\varepsilon(|\widetilde{\nabla} f|).$

It is immediate that

$$\varepsilon\text{-Crit} f = \{x \in \operatorname{dom} f : \ f(y) \geq f(x) - \varepsilon\rho(y,x),\ \forall y\}.$$

## 3. Properties of the slopes

In this section we consider some properties of the local and the global slope of a given function.

**Lemma 3.1.** *Let $(M,\rho)$ be a metric space and let $a \in M$ be fixed.*

*Set* $\qquad \varphi(x) := -\log \rho(x,a).$

*Then*
$$|\widetilde{\nabla}\varphi|(x) \leq \frac{1}{\rho(x,a)}, \quad \forall x \neq a. \tag{4}$$

**Proof.** Fix $x \neq a$ and let $y \neq a$ be such that $\varphi(y) < \varphi(x)$, that is, $\rho(y,a) > \rho(x,a)$.

Using the inequality $\qquad \log(t+s) \leq \log t + \dfrac{s}{t}, \quad \forall t,s > 0,$

due to the concavity of the logarithm, we get

$$\log \rho(y,a) \leq \log(\rho(x,a) + \rho(x,y)) \leq \rho(x,a) + \rho(x,y)/\rho(x,a).$$

Therefore, $\varphi(y) \geq \varphi(x) - \rho(x,y)/\rho(x,a)$, that is,

$$\frac{\varphi(x)-\varphi(y)}{\rho(x,y)} \leq \frac{1}{\rho(x,a)}.$$

As $y$ with $\varphi(y) < \varphi(x)$ was arbitrary, we get (4). $\square$

**Lemma 3.2.** *Let $(M,\rho)$ be a metric space. Let $g : M \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Let $\lambda \in \mathbb{R}$. Consider the function*

$$g_1(x) := \min\{g(x), \lambda\}, \quad \forall x \in M.$$

*Then* $\quad |\nabla g_1|(x) \leq |\nabla g|(x)$ *and* $|\widetilde{\nabla} g_1|(x) \leq |\widetilde{\nabla} g|(x), \quad \forall x \in \operatorname{dom} g.$

**Proof.** Fix $x \in \operatorname{dom} g$. It is enough to show that

$$[g_1(x) - g_1(y)]^+ \leq [g(x) - g(y)]^+, \quad \forall y \in M, \tag{5}$$

because then we can just take lim sup or sup.

**Case 1.** $g(x) \leq \lambda$.
Then $g_1(x) = g(x)$ and so $[g_1(x) - g_1(y)]^+ = [g(x) - g_1(y)]^+$ for any $y \in M$.
If $y$ is such that $g_1(y) \geq g(x)$ then also

$$g(y) \geq g(x) \text{ and } [g(x) - g_1(y)]^+ = [g(x) - g(y)]^+ = 0.$$

If, on the other hand, $y$ is such that $g_1(y) < g(x)$ then, of course, $g_1(y) = g(y)$. So we obtain $[g(x) - g_1(y)]^+ = [g(x) - g(y)]^+$.
We see that in this case (5) holds even with equality.

**Case 2.** $g(x) > \lambda$.
Then $g_1(x) = \lambda$ and so $[g_1(x) - g_1(y)]^+ = [\lambda - g_1(y)]^+$.
If $y$ is such that $g(y) \geq \lambda$ then by definition

$$g_1(y) = \lambda \text{ and } [g_1(x) - g_1(y)]^+ = 0 \leq [g(x) - g(y)]^+.$$

If $y$ is such that $g(y) < \lambda$ then $g_1(y) = g(y)$ and $[g_1(x) - g_1(y)]^+ = [\lambda - g(y)]^+$. Since $t \to [t]^+$ is an increasing function, $[\lambda - g(y)]^+ \leq [g(x) - g(y)]^+$.
So, (5) holds. $\square$

Note that $\varepsilon$-Crit $f$ is the set of those points where the Tikhonov regularisation of $f$ coincides with $f$:

$$f(x) = f_{1/\varepsilon}(x) := \inf\{f(y) + \varepsilon\rho(y,x)\}.$$

As the Tikhonov regularisation is Lipschitz, clearly a lower semicontinuous $f$ is Lipschitz on the closed set $\varepsilon$-Crit $f$, but this can also be proved directly:

**Lemma 3.3.** *Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function on a metric space $(M,\rho)$. Then $\varepsilon$-Crit $f$ is a closed set on which $f$ is $\varepsilon$-Lipschitz.*

**Proof.** Fix $\varepsilon > 0$ and a convergent sequence $(x_n)_1^\infty \subset \varepsilon$-Crit $f$. Let

$$\lim_{n\to\infty} x_n = \bar{x}.$$

By definition we have $f(\bar{x}) \geq f(x_n) - \varepsilon\rho(\bar{x}, x_n)$, which, after taking a limit, gives

$$f(\bar{x}) \geq \limsup_{n\to\infty} f(x_n).$$

But $f$ is given to be lower semicontinuous at $\bar{x}$, so $\lim_{n\to\infty} f(x_n) = f(\bar{x})$.

Let $y \in \operatorname{dom} f$ be arbitrary. By definition,

$$f(y) \geq f(x_n) - \varepsilon\rho(y, x_n), \text{ so } f(y) \geq f(\bar{x}) - \varepsilon\rho(y, \bar{x}).$$

In particular $\bar{x} \in \operatorname{dom} f$ and, therefore, $\bar{x} \in \varepsilon\text{-Crit}\, f$.

We have proved that $\varepsilon\text{-Crit}\, f$ is closed.

Let $x, y \in \varepsilon\text{-Crit}\, f$. By definition,

$$f(y) - f(x) \geq -\varepsilon\rho(y, x) \text{ and } f(x) - f(y) \geq -\varepsilon\rho(x, y),$$

so we obtain $|f(y) - f(x)| \leq \varepsilon\rho(y, x)$. □

**Lemma 3.4.** *Let $f, g : M \to \mathbb{R} \cup \{+\infty\}$ and let for some $\lambda > 0$,*

$$L_\lambda(f - g) \cap \operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing. \tag{6}$$

*Let $M_1 := L_\lambda(f - g)$, and $f_1 := f \restriction_{M_1}$. If $x \in M_1$ and*

$$|\nabla f|(x) > |\nabla g|(x), \tag{7}$$

*then we have* $|\nabla f_1|(x) = |\nabla f|(x)$.

**Proof.** Take $x \in M_1$ satisfying (7). Since $|\nabla f|(x) > 0$, by definition there is a sequence $y_n \to x$ such that

$$\lim_{n\to\infty} \frac{f(x) - f(y_n)}{\rho(x, y_n)} = |\nabla f|(x). \tag{8}$$

Let $r \in \mathbb{R}$ be such that $|\nabla f|(x) > r > |\nabla g|(x)$. By definition,

$$\frac{g(x) - g(y_n)}{\rho(x, y_n)} < r$$

for all $n$ large enough. By (8) $\dfrac{f(x) - f(y_n)}{\rho(x, y_n)} > r$ for all $n$ large enough. So,

$$\frac{g(x) - g(y_n)}{\rho(x, y_n)} < r < \frac{f(x) - f(y_n)}{\rho(x, y_n)}, \quad \forall n > K.$$

This implies $f(y_n) - g(y_n) < f(x) - g(x) \leq \lambda$ for all $n > K$.

That is, $y_n \in M_1$ for all $n > K$. Then (8) becomes

$$\lim_{n\to\infty} \frac{f_1(x) - f_1(y_n)}{\rho(x, y_n)} = |\nabla f|(x),$$

meaning that $|\nabla f_1|(x) \geq |\nabla f|(x)$, while the inverse inequality is trivial. □

Of course, Lemma 3.4 has a variant for global slopes with a shorter proof. We give the details for completeness.

**Lemma 3.5.** *Let $f, g : M \to \mathbb{R} \cup \{+\infty\}$ and let (6) be satisfied for some $\lambda > 0$. Let $M_1 := L_\lambda(f-g)$, and let $f_1 := f\restriction_{M_1}$. If $x \in M_1$ and*

$$|\widetilde{\nabla} f|(x) > |\widetilde{\nabla} g|(x), \tag{9}$$

*then* $|\widetilde{\nabla} f_1|(x) = |\widetilde{\nabla} f(x)|$.

**Proof.** Fix $x \in M_1$ such that (9) holds. Take any $r > 0$ such that

$$|\widetilde{\nabla} g|(x) < r < |\widetilde{\nabla} f|(x).$$

By definition, there is $y \in M$ such that $f(x) - f(y) > r\rho(x, y)$. On the other hand, $g(x) - g(\cdot) \le r\rho(x, \cdot)$, so, in particular, $g(x) - g(y) \le r\rho(x, y)$. Thus,

$$f(x) - f(y) > g(x) - g(y) \quad \text{and} \quad f(y) - g(y) < f(x) - g(x) \le \lambda$$

meaning that $y \in M_1$ and so $f_1(x) - f_1(y) > r\rho(x, y) \Rightarrow |\widetilde{\nabla} f_1|(x) > r$.
Therefore, $|\widetilde{\nabla} f_1|(x) \ge |\widetilde{\nabla} f|(x)$. □

Let us recall the famous Ekeland Variational Principle, see e.g. [24, p. 45]. Let $(M, \rho)$ be a complete metric space and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function bounded below. Let $\varepsilon > 0$ and $x_0 \in \varepsilon$-argmin $f$. Then for every $\lambda > 0$ there exists $x_\lambda$ such that

(i) $\lambda\rho(x_0, x_\lambda) \le f(x_0) - f(x_\lambda)$;

(ii) $\lambda\rho(x, x_\lambda) + f(x) > f(x_\lambda), \quad \forall x \ne x_\lambda$;

(iii) $\rho(x_0, x_\lambda) \le \varepsilon/\lambda$.

Here we will use a variant of the Ekeland Variational Principle, c.f. [2], namely

**Proposition 3.6.** *Let $(M, \rho)$ be a complete metric space. Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function bounded below and let $x_0 \in \operatorname{dom} f$. Then for every $\lambda > 0$ there exists $x_\lambda \in \lambda$-Crit $f$ such that*

$$f(x_\lambda) \le f(x_0) - \lambda\rho(x_0, x_\lambda).$$

**Proof.** If $f$ attains its minimum at $x_0$, then take $x_\lambda = x_0$. Otherwise, define $\varepsilon := f(x_0) - \inf f$ and apply the Ekeland Variational Principle at $x_0$ for $\lambda > 0$ to get $x_\lambda$. Then (ii) means that $x_\lambda \in \lambda$-Crit $f$, while (i) is the claimed inequality. □

The graphical density of $\operatorname{dom}|\widetilde{\nabla}|$ in $\operatorname{dom} f$ is pretty standard. We sketch a proof for the reader's convenience.

**Lemma 3.7.** *Let $(M, \rho)$ be a complete metric space and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and bounded below. Then for each $x \in \operatorname{dom} f$ there is a sequence $(x_n)_1^\infty \subset \operatorname{dom}|\widetilde{\nabla} f|$ such that*

$$(x, f(x)) := \lim_{n\to\infty} (x_n, f(x_n)). \tag{10}$$

**Proof.** By adding $-\inf f$ to $f$ we may assume that $\inf f = 0$. Let $\bar{x} \in \operatorname{dom} f$. Let $\lambda_n \to \infty$. By Proposition 3.6 there are $x_n \in \lambda_n$-Crit $f \subset \operatorname{dom}|\widetilde{\nabla} f|$ such that $f(x_n) \le f(\bar{x}) - \lambda_n\rho(x_n, \bar{x})$. Since $f(x_n) \ge 0$, we have $\rho(x_n, \bar{x}) \le f(\bar{x})/\lambda_n \to 0$, as $n \to \infty$.

Also, $f(x_n) \le f(\bar{x})$, so $\limsup_{n\to\infty} f(x_n) \le f(\bar{x})$, but $\liminf_{n\to\infty} f(x_n) \ge f(\bar{x})$ by the lower semicontinuity of $\bar{x}$. $\square$

In fact, we will be rather using the following consequence of the graphical density.

**Lemma 3.8.** *Let $(M,\rho)$ be a complete metric space. Let $f,g : M \to \mathbb{R}\cup\{+\infty\}$ be lower semicontinuous and let $f$ be also proper and bounded below. Then*

$$\inf_{\operatorname{dom} f}(f-g) = \inf_{\operatorname{dom}|\widetilde{\nabla} f|}(f-g). \tag{11}$$

**Proof.** Let $r\in\mathbb{R}$ be arbitrary such that $r > \inf_{\operatorname{dom} f}(f-g)$.
Let $\bar{x}\in\operatorname{dom} f$ be such that $r > f(\bar{x}) - g(\bar{x})$. By Lemma 3.7 there is a sequence $(x_n)_1^\infty \subset \operatorname{dom}|\widetilde{\nabla} f|$ such that (10) is fulfilled for $x=\bar{x}$.

By the lower semicontinuity of $g$ we obtain

$$g(\bar{x}) \le \liminf_{n\to\infty} g(x_n) \Rightarrow -g(\bar{x}) \ge \limsup_{n\to\infty}(-g(x_n)).$$

Therefore, $f(\bar{x}) - g(\bar{x}) \ge \limsup_{n\to\infty}(f(x_n)-g(x_n))$, that is, $r > f(x_n) - g(x_n)$ for all large enough $n$. $\square$

**Lemma 3.9.** *Let $(M,\rho)$ be a complete metric space. Let $f,g : M \to \mathbb{R}\cup\{+\infty\}$ and let for some $\lambda > 0$ and $s > 0$*

$$L_\lambda(f-g)\cap s\text{-}\operatorname{Crit} f \cap \operatorname{dom} g \ne \varnothing. \tag{12}$$

*Let $M_1 := L_\lambda(f-g)\cap s\text{-}\operatorname{Crit} f$, and $f_1 := f\restriction_{M_1}$. If $x\in M_1$ and (9) holds at $x$ then $|\widetilde{\nabla} f_1|(x) = |\widetilde{\nabla} f|(x)$.*

**Proof.** Fix $x\in M_1$ such that (9) holds. Take any $r>0$ such that

$$|\widetilde{\nabla} g|(x) < r < |\widetilde{\nabla} f|(x).$$

In particular, because $x\in s\text{-}\operatorname{Crit} f$, we have

$$r < s. \tag{13}$$

By definition, there is $y\in M$ such that $f(x)-f(y) > r\rho(x,y)$. By Proposition 3.6 applied at $y$ there is $z\in r\text{-}\operatorname{Crit} f$ such that $f(z)\le f(y) - r\rho(y,z)$. The triangle inequality gives

$$f(x) - f(z) > r\rho(x,z). \tag{14}$$

Since by definition $g(x)-g(z) > r\rho(x,z)$, as in the proof of Lemma 3.5 we derive that $z\in L_\lambda(f-g)$.

But also $z\in r\text{-}\operatorname{Crit} f$ and (13) give $z\in s\text{-}\operatorname{Crit} f$ and so, (14) becomes

$$f_1(x) - f_1(z) > r\rho(x,z),$$

meaning that $|\widetilde{\nabla} f_1|(x) \ge |\widetilde{\nabla} f|(x)$. $\square$

## 4. Results

We start with a result of Daniilidis and Salas [10]. Recently it has been independently proved without transfinite induction in [9].

**Proposition 4.1.** *Let $(M,\rho)$ be a metric space and let $\tau$ be some topology on $M$. Assume that $f,g : M \to \mathbb{R}$ are $\tau$-continuous and such that*

$$|\nabla f|(x) > |\nabla g|(x), \quad \forall x \notin 0\text{-crit}\, f. \tag{15}$$

*If $f$ has $\tau$-compact sub-level sets then*

$$\inf(f-g) = \inf_{0\text{-crit}\, f}(f-g).$$

**Proof.** Let $x_0$ be arbitrary and let $\lambda := f(x_0) - g(x_0)$. Let

$$M_1 := L_\lambda(f-g), \quad f_1 := f\restriction_{M_1}.$$

It is clear that $M_1$ is $\tau$-closed and $f_1$ has $\tau$-compact sub-level sets. Let $\bar{x} \in \operatorname{argmin} f_1$, the latter being non-empty by Weierstrass Theorem.

If $\bar{x} \notin 0\text{-crit}\, f$, then (15) and Lemma 3.4 give $|\nabla f_1|(\bar{x}) > 0$ and, therefore, $\bar{x}$ cannot be a minimum, a contradiction.

Thus $\bar{x} \in 0\text{-crit}\, f$ and $(f-g)(\bar{x}) \le \lambda = (f-g)(x_0)$. □

We will develop the above result in cases where completeness replaces compactness. The easiest case to work with is that of Lipschitz functions because then the slopes are properly defined everywhere and one does not have to meddle with domains.

**Proposition 4.2.** *Let $(M,\rho)$ be a complete metric space and let $f,g : M \to \mathbb{R}$ be Lipschitz and such that $f$ is bounded below and*

$$|\widetilde{\nabla} f|(x) \ge |\widetilde{\nabla} g|(x), \quad \forall x \in M. \tag{16}$$

*Then*
$$\inf(f-g) = \inf_{\varepsilon\text{-Crit}\, f}(f-g), \quad \forall \varepsilon > 0. \tag{17}$$

**Proof.** By adding a constant to both $f$ and $g$ we may assume that $f \ge 0$.

Let $x_0 \in M$ and $\varepsilon > 0$ be arbitrary. Set $g_1(x) := \min\{g(x_0), g(x)\}$. Consider

$$h(x) := (1+\varepsilon)f(x) - g_1(x).$$

From (1) and (2) it follows that

$$|\widetilde{\nabla} h|(x) \ge (1+\varepsilon)|\widetilde{\nabla} f|(x) - |\widetilde{\nabla} g_1|(x) = \varepsilon|\widetilde{\nabla} f|(x) + (|\widetilde{\nabla} f|(x) - |\widetilde{\nabla} g_1|(x)).$$

From Lemma 3.2 it follows that $|\widetilde{\nabla} f|(x) - |\widetilde{\nabla} g_1|(x) \ge 0$ and so

$$|\widetilde{\nabla} h|(x) \ge \varepsilon|\widetilde{\nabla} f|(x), \quad \forall x \in M.$$

Now $h \ge -g(x_0)$, so from Prop. 3.6 there is $\bar{x} \in \varepsilon^2\text{-Crit}\, h$ such that $h(x_0) \ge h(\bar{x})$.

So,
$$\varepsilon^2 \ge |\widetilde{\nabla} h|(\bar{x}) \ge \varepsilon|\widetilde{\nabla} f|(\bar{x}) \Rightarrow \bar{x} \in \varepsilon\text{-Crit}\, f,$$

and $(1+\varepsilon)f(x_0) - g(x_0) = h(x_0) \ge h(\bar{x}) = (1+\varepsilon)f(\bar{x}) - g_1(\bar{x}) \ge f(\bar{x}) - g(\bar{x})$, because $f \ge 0$ and $g_1 \le g$.

Since $\varepsilon > 0$ was arbitrary, we obtain

$$f(x_0) - g(x_0) \geq \lim_{\varepsilon \to 0} \inf_{\varepsilon\text{-Crit}\, f} (f - g).$$

Since $x_0$ was arbitrary, we get

$$\inf(f - g) \geq \lim_{\varepsilon \to 0} \inf_{\varepsilon\text{-Crit}\, f} (f - g),$$

which is clearly equivalent to (17), because $\lim_{\varepsilon\to 0} \inf_{\varepsilon\text{-Crit}\, f}(f-g) \geq \inf_{\varepsilon\text{-Crit}\, f}(f-g)$ for any $\varepsilon > 0$ and $\inf(f-g) \leq \inf_{\varepsilon\text{-Crit}\, f}(f-g)$, as the infimum over the whole space is less or equal to the infimum over any non-empty subset. $\square$

Of course, Proposition 4.2 has a version – and that with the same proof – for the local slope $|\nabla|$ and $\varepsilon$-crit set. The following result in the style of Thibault and Zagrodny [29], however, is $|\widetilde{\nabla}|$-specific.

**Theorem 4.3.** *Let $(M, \rho)$ be a complete metric space and let $f, g : M \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semicontinuous. Let $f$ be bounded below and such that*

$$|\widetilde{\nabla} g|(x) \leq |\widetilde{\nabla} f|(x), \quad \forall x \in \operatorname{dom} |\widetilde{\nabla} f|. \tag{18}$$

*Then*

$$\inf_{\operatorname{dom} f} (f - g) \geq \inf f - \inf g. \tag{19}$$

Note that, since $\inf_{\operatorname{dom} f} (f - g) < \infty$, the inequality (19) implies that $\inf g \neq -\infty$. Therefore, (19) can be rewritten as in [29]:

$$f(x) - \inf f \geq g(x) - \inf g, \quad \forall x \in M. \tag{20}$$

Indeed, if $f(x) = \infty$ then (20) is trivial; otherwise it follows from (19).

**Proof.** By subtracting $\inf f$ from both functions we may assume that

$$\inf f = 0. \tag{21}$$

Because of (11) we have to show only that $\inf_{\operatorname{dom} |\widetilde{\nabla} f|} (f - g) \geq -\inf g$.

Assume the contrary, i.e., there is $x_0 \in \operatorname{dom} |\widetilde{\nabla} f|$ such that $f(x_0) - g(x_0) < -\inf g$. Note that $x_0 \in \operatorname{dom} g$ because of (18). Let $c > 1$ be such that

$$\lambda := cf(x_0) - g(x_0) < -\inf g.$$

Let $s := |\widetilde{\nabla} f|(x_0)$. Set

$$M_1 := L_\lambda(cf - g) \cap s\text{-Crit}\, f, \quad f_1 := cf \restriction_{M_1}, \quad g_1 := g \restriction_{M_1}.$$

Obviously, $x_0 \in M_1$. From Lemma 3.3 we know that $s$-Crit $f$ is closed and $f$ is Lipschitz on $s$-Crit $f$. Also, (18) implies that $s\text{-Crit}\, f \subset s\text{-Crit}\, g$ and thus $g$ is also Lipschitz on $s$-Crit $f$. So, $(cf - g) \restriction_{s\text{-Crit}\, f}$ is Lipschitz and thus $L_\lambda(cf - g) \cap s\text{-Crit}\, f$ is closed. That is, $M_1$ is closed and $f_1$ and $g_1$ are Lipschitz.

Since $g(x) \geq cf(x) - \lambda$ for $x \in M_1$, we have $\inf g_1 \geq c \inf f(x) - \lambda = -\lambda > \inf g$. That is, there are $a \in M \setminus M_1$ and $r > 0$ such that

$$g_1(x) = g(x) > g(a) + r, \quad \forall x \in M_1.$$

By definition, $$|\widetilde{\nabla} g|(x) \geq \frac{r}{\rho(x,a)}, \quad \forall x \in M_1.$$

By (16) we have $$|\widetilde{\nabla} f|(x) \geq \frac{r}{\rho(x,a)}, \quad \forall x \in M_1.$$

Now (18) implies $|\widetilde{\nabla}(cf)| > |\widetilde{\nabla} g|$ on $M_1$ and Lemma 3.9 applied to $cf$ and $g$ gives $|\widetilde{\nabla} f_1| = c|\widetilde{\nabla} f| \restriction_{M_1}$. In particular,

$$|\widetilde{\nabla} f_1|(x) > \frac{r}{\rho(x,a)}, \quad \forall x \in M_1. \tag{22}$$

Let $\varphi : M_1 \to \mathbb{R}$ be defined as $\varphi(x) := -r \log \rho(x,a)$. Lemma 3.1 and (22) show that

$$|\widetilde{\nabla} f_1|(x) > |\widetilde{\nabla} \varphi|(x), \quad \forall x \in M_1.$$

Consider on $M_1$ the function $h(x) := 2f(x) - \varphi(x)$.
Since $h(x) \geq r \log \rho(x,a) \geq r \log \operatorname{dist}(a, M_1) > -\infty$, the function $h$ is proper lower semicontinuous and bounded below.
Let $\varepsilon_n \searrow 0$. By Prop. 3.6 applied to $h$ at $x_0$, there are $x_n \in \varepsilon_n\text{-Crit}\, h$ such that

$$h(x_n) \leq h(x_0), \quad \forall n \in \mathbb{N}.$$

Since $h(x_n) \geq r \log \rho(x_n, a)$ this means that the sequence $(\rho(x_n,a))_1^\infty$ is bounded.

On the other hand, however, (3) and $|\widetilde{\nabla} f| > |\widetilde{\nabla} \varphi|$ show that $|\widetilde{\nabla} h| > |\widetilde{\nabla} f|$, so (22) gives $\rho(x,a) > r/\varepsilon_n \to \infty$, as $n \to \infty$, contradiction. □

Our next aim is to give more precise localisation of the infimum set.

**Proposition 4.4.** *Let $(M, \rho)$ be a complete metric space and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and bounded below function. Let $g : M \to \mathbb{R}$ be a continuous function such that*

$$|\nabla f|(x) > |\nabla g|(x), \quad \forall x \in \operatorname{dom} f \setminus 0\text{-crit}\, f. \tag{23}$$

*Then for each $\varepsilon > 0$ we have $f - g \geq \inf_{\varepsilon\text{-crit}\, f}(f-g)$, or, more precisely, for each $x_0 \in M$ there exists $x \in \varepsilon\text{-crit}\, f$ such that*

$$f(x) \leq f(x_0) - \varepsilon\rho(x, x_0), \tag{24}$$

*and* $$f(x_0) - g(x_0) \geq f(x) - g(x). \tag{25}$$

**Proof.** The claim is clear if $f(x_0) = \infty$. Let $x_0 \in \operatorname{dom} f$. Let $\lambda := f(x_0) - g(x_0)$ and

$$M_1 := L_\lambda(f-g), \quad f_1 := f \restriction_{M_1}.$$

By Proposition 3.6 applied to $f_1$ on $M_1$ there exists $x \in M_1$, that is, satisfying (25); such that (24) is satisfied and $x \in \varepsilon\text{-Crit}\, f_1$.

If $|\nabla f|(x) = 0$ we are done, because $x \in \varepsilon\text{-crit}\, f$.

If $|\nabla f|(x) > 0$ we apply Lemma 3.4 to get $|\nabla f|(x) = |\nabla f_1|(x)$. Since $x \in \varepsilon\text{-Crit}\, f_1$ it follows that $|\nabla f_1|(x) \leq \varepsilon$, so $|\nabla f|(x) \leq \varepsilon$ and we are done. □

**Theorem 4.5.** *Let $(M,\rho)$ be a complete metric space and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and bounded below function. Let $g : M \to \mathbb{R}$ be a continuous function such that (23) holds.*

*For any $x_0 \in \operatorname{dom} f$ at least one of the following two assertions holds:*

(i) *There exists $\overline{x} \in 0\text{-crit}\, f$ such that $f(x_0) - g(x_0) \ge f(\overline{x}) - g(\overline{x})$;*

(ii) *For every sequence $\varepsilon_n \searrow 0$ there exists a sequence $x_n \in \varepsilon_n\text{-crit}\, f$ which has no convergent subsequences, and satisfies*

$$\sum_{n=1}^{\infty} \varepsilon_n \rho(x_n, x_{n-1}) < \infty, \tag{26}$$

*and*
$$f(x_n) - g(x_n) \ge f(x_{n+1}) - g(x_{n+1}), \quad \forall n \in \{0, \mathbb{N}\}. \tag{27}$$

**Proof.** Let us assume first that (i) is false. Fix a sequence $\varepsilon_n \searrow 0$ and proceed by induction.

**Basic step.** Set $\lambda_0 := f(x_0) - g(x_0)$, $M_0 := L_{\lambda_0}(f - g)$, $f_0 := f\restriction_{M_0}$ and take $x_1 \in \varepsilon_1\text{-Crit}\, f_0$ such that

$$f(x_1) \le f(x_0) - \varepsilon_1 \rho(x_1, x_0) \ \text{ and } \ f(x_0) - g(x_0) \ge f(x_1) - g(x_1).$$

The existence of such $x_1$ follows from Proposition 4.4.

**Induction step:** Given $x_n \in \varepsilon_n\text{-Crit}\, f_{n-1}$, set

$$\lambda_n := f(x_n) - g(x_n),\ M_n := L_{\lambda_n}(f - g),\ f_n := f\restriction_{M_n}$$

and take $x_n \in \varepsilon_{n+1}\text{-Crit}\, f_n$ such that

$$f(x_{n+1}) \le f(x_n) - \varepsilon_{n+1}\rho(x_{n+1}, x_n), \ \text{ and } \ f(x_n) - g(x_n) \ge f(x_{n+1}) - g(x_{n+1}). \tag{28}$$

The existence of $x_{n+1}$ follows again from Proposition 4.4.

As $M_{n+1} \subset M_n$, and since any subsequence of the sequence $\{x_n\}_n$ has the same structure, we may assume without loss of generality that $\{x_n\}_n$ itself converges.
Let $x_n \to \overline{x}$. Since

$$f(x_0) - g(x_0) \ge f(x_n) - g(x_n), \quad \forall n \in \mathbb{N},$$

by continuity of $g$ at $\overline{x}$ and by lower semicontinuity of $f$ at $\overline{x}$, after passing to limit, we obtain that
$$f(x_0) - g(x_0) \ge f(\overline{x}) - g(\overline{x}).$$

Since (i) is assumed false, $\overline{x} \notin 0\text{-crit}\, f$, and therefore $|\nabla f|(\overline{x}) > |\nabla g|(\overline{x})$.
Now set $\overline{\lambda} := f(\overline{x}) - g(\overline{x})$, $\overline{M} := L_{\overline{\lambda}}(f - g)$, $\overline{f} := f\restriction_{\overline{M}}$.
On the one hand, from Lemma 3.4 it follows that $|\nabla\overline{f}|(\overline{x}) = |\nabla f|(\overline{x})$ and the above inequality yields $|\nabla\overline{f}|(\overline{x}) > 0$.

On the other hand, $\overline{M} \subset M_n$ and $\overline{f} = f_n\restriction_{\overline{M}}$ for all $n$. Fix $y \in \overline{M}$.
Since $x_n \in \varepsilon_n\text{-Crit}\, f_{n-1}$ we have that

$$f(x_n) - f(y) \le \varepsilon_n \rho(x_n, y),$$

and passing to limit yields $f(\overline{x}) - f(y) \le 0$. Hence, $\overline{x} \in 0\text{-Crit}\,\overline{f}$, and in particular $|\nabla\overline{f}|(\overline{x}) = 0$, which is a contradiction.

Therefore, $\{x_n\}_n$ has no convergent subsequences, (27) is fulfilled by construction, while for (26) note that by (28) we have

$$\varepsilon_{n+1}\rho(x_{n+1},x_n) \le f(x_n) - f(x_{n+1}), \quad \forall n \in \{0,\mathbb{N}\},$$

hence

$$\sum_{n=0}^{\infty} \varepsilon_{n+1}\rho(x_{n+1},x_n) \le f(x_0) - \inf f.$$

The proof is complete. □

Not surprisingly, Proposition 4.4 has a compete analogue for the global slope.

**Proposition 4.6.** *Let $(M,\rho)$ be a complete metric space and let $f : M \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function bounded below. Let $g : M \to \mathbb{R}$ be a continuous function such that*

$$|\widetilde{\nabla} f|(x) > |\widetilde{\nabla} g|(x), \quad \forall x \in \operatorname{dom} f \setminus 0\text{-Crit}\, f. \tag{29}$$

*Then for each $\varepsilon > 0$ we have $f - g \ge \inf_{\varepsilon\text{-Crit}\, f}(f-g)$, or, more precisely, for each $x_0 \in M$ there exists $x \in \varepsilon\text{-Crit}\, f$ such that (24) and (25) hold.*

**Proof.** One can repeat the proof of Proposition 4.4 using Lemma 3.5. □

**Corollary 4.7.** *Let $(M,\rho)$ be a complete metric space and let $f : M \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and bounded below function. Let $g : M \to \mathbb{R}$ be a continuous function such that (18) is fulfilled. Then for each $\varepsilon > 0$, $x_0 \in M$ and $r \in (0,1)$ there exists $x \in \varepsilon\text{-Crit}\, f$ such that (24) holds and*

$$f(x_0) - rg(x_0) \ge f(x) - rg(x). \tag{30}$$

**Proof.** We apply Proposition 4.6 to $f$ and $rg$. It is clear that (18) implies (29) for $rg$, because $|\widetilde{\nabla}(rg)| = r|\widetilde{\nabla} g|$ and $0 < r < 1$. □

Now we can remove the continuity assumption, but once again, only for the global slope $|\widetilde{\nabla}|$.

**Theorem 4.8.** *Let $(M,\rho)$ be a complete metric space and let $f : M \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function bounded below. Let $g : M \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function such that (18) holds.*

*Then for each $\varepsilon > 0$, $x_0 \in \operatorname{dom}|\widetilde{\nabla} f|$ and $r \in (0,1)$ there exists $x \in \varepsilon\text{-Crit}\, f$ such that (24) and (30) hold. More precisely,*

$$\inf_{\operatorname{dom} f}(f - rg) = \inf_{\varepsilon\text{-Crit}\, f}(f - rg). \tag{31}$$

**Proof.** Let $x_0 \in \operatorname{dom}|\widetilde{\nabla} f|$ and let $s := |\widetilde{\nabla} f|(x_0) < \infty$. We can also assume that $s > 0$, because otherwise there is nothing to prove.

Let $\lambda := f(x_0) - rg(x_0)$ and

$$M_1 := s\text{-Crit}\, f \cap L_\lambda(f - rg), \quad f_1 := f\restriction_{M_1}, \quad g_1 := rg\restriction_{M_1}.$$

Obviously, $x_0 \in M_1$. From Lemma 3.3 we know that $s\text{-Crit}\, f$ is closed and $f$ is Lipschitz on $s\text{-Crit}\, f$. Also, (18) implies that $s\text{-Crit}\, f \subset s\text{-Crit}\, g$ and thus $g$ is also Lipschitz on $s\text{-Crit}\, f$. So, $(f - rg)\restriction_{s\text{-Crit}\, f}$ is Lipschitz and thus $L_\lambda(f-rg) \cap s\text{-Crit}\, f$ is closed. That is, $M_1$ is closed and $g_1$ is Lipschitz.

It is clear that $|\widetilde{\nabla} g_1| \leq r|\widetilde{\nabla} g|$ on $M_1$. On the other hand, Lemma 3.9 gives that $|\widetilde{\nabla} f_1| = |\widetilde{\nabla} f|$ on $M_1$. Therefore, (18) translates to

$$|\widetilde{\nabla} f_1|(x) > |\widetilde{\nabla} g_1|(x), \quad \forall x \in M_1 : \ |\widetilde{\nabla} f_1|(x) > 0,$$

which is (29) for $f_1$ and $g_1$. We can apply Proposition 4.6 to $f_1$ and $g_1$ and use once again that $|\widetilde{\nabla} f_1| = |\widetilde{\nabla} f| \restriction_{M_1}$.

Now (31) follows from Lemma 3.8. □

## 5. The Moreau-Rockafellar Theorem

We finally show that slopes provide quite different and modern approach to Moreau-Rockafellar Theorem. For a discussion on the known proofs see [17]. In relation to this, note that the recent proof [19] is also quite classical in flavour.

If $(X, \|\cdot\|)$ is a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a convex function, its *subdifferential* at $x_0 \in \operatorname{dom} f$ is the subset of the topological dual $X^*$ given by

$$\partial f(x_0) := \{p \in X^* : p(x - x_0) \leq f(x) - f(x_0), \ \forall x \in X\},$$

and $\partial f(x_0) = \varnothing$ whenever $x_0 \notin \operatorname{dom} f$. As usual, $\operatorname{dom} \partial f := \{x : \partial f(x) \neq \varnothing\}$, see e.g. [27].

For any convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ it is easy to see that the local slope and global slope coincide at any point.

**Theorem 5.1 (Moreau-Rockafellar).** *Let $(X, \|\cdot\|)$ be a Banach space and let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions. If*

$$\partial g(x) \subset \partial f(x), \quad \forall x \in X, \tag{32}$$

*then we have for some constant $c \in \mathrm{R}$ that $f(x) = g(x) + c$, for all $x \in X$.*

**Proof.** From the maximal monotonicity of $\partial f$, see [26] or [24, Theorem 3.24], and (32) it follows that

$$\partial g(x) \equiv \partial f(x), \quad \forall x \in X. \tag{33}$$

Take $x_0 \in \operatorname{dom} \partial g$ and take $p \in \partial g(x_0)$. So, $p$ satisfies

$$g(x) \geq g(x_0) + p(x - x_0), \quad \forall x \in X.$$

From (32) it follows that $p \in \partial f(x_0)$, so

$$f(x) \geq f(x_0) + p(x - x_0), \quad \forall x \in X.$$

Set $f_1(x) := f(x) - f(x_0) - p(x - x_0)$, $g_1(x) := g(x) - g(x_0) - p(x - x_0)$, for all $x \in X$. It is clear that $\min f_1 = \min g_1 = 0$. From the Sum Theorem, see e.g. [24, Theorem 3.16], and (33) we have that

$$\partial g_1(x) \equiv \partial f_1(x), \quad \forall x \in X. \tag{34}$$

Because for the proper convex lower semicontinuous function $f_1$ it holds that

$$\operatorname{dom}|\widetilde{\nabla} f_1| \equiv \operatorname{dom}|\nabla f_1| \equiv \operatorname{dom}\partial f_1$$

and $$|\widetilde{\nabla} f_1|(x) = |\nabla f_1|(x) = \min\{\|p\| : p \in \partial f_1(x)\}, \quad \forall x \in \operatorname{dom}\partial f_1,$$

see [29, Lemma 3.2], and the same is true for $g_1$, from (34) it holds that

$$|\widetilde{\nabla} f_1|(x) = |\widetilde{\nabla} g_1|(x) \quad \forall x \in \operatorname{dom}|\widetilde{\nabla} f_1| \equiv \operatorname{dom}|\widetilde{\nabla} g_1|.$$

Since we have $\min f_1 = \min g_1 = 0$, Theorem 4.3 implies both $f_1(x) \geq g_1(x)$ and $g_1(x) \geq f_1(x)$ for all $x \in X$. That is, $f_1(x) = g_1(x)$ for all $x \in X$, which means that

$$f(x) = g(x) + (f(x_0) - g(x_0)). \qquad \square$$

## Appendix

Even in trying to solve simple problems one is tempted to use transfinite induction. Let us take for example the problem of finding the middle point of a segment using only ruler and compass.

Let the segment $[a, b]$ is given. Denote the distance between the points $a$ and $b$ by $d(a, b)$. For finding the middle point $m$ consider the following transfinite procedure, given in [16]:

$l \leftarrow a$; $r \leftarrow b$

till $l \neq r$:

  take an arbitrary point $x$ in $[l, r]$ different from $l$ and $r$

    if $d(l, x) < d(x, r)$ then:

      $l \leftarrow x$;

      $r \leftarrow$ the point on $[l, r]$ at distance $d(l, x)$ to $r$;

    if $d(l, x) > d(x, r)$ then:

      $l \leftarrow$ the point on $[l, r]$ at distance $d(x, r)$ to $l$;

      $r \leftarrow x$;

    if $d(l, x) = d(x, r)$ then stop ($x \equiv m$).

The segments $[l, r]$ are nested and their lengths monotonly decrease. Transfering the construction on the real line, the latter implies that we are bypassing at least one new rational number each time, meaning that at some countable ordinal we will get the middle point.

Although the above transfinite procedures works, in practice one uses the well known finite procedure of construction of the bisector. Admittedly, the latter uses the plane, but it is anyway good to know.

**Acknowledgements.** The authors express their sincere gratitude for the fruitful discussions and exchange on the subject to Aris Daniilidis, Robert Deville, Alexander D. Ioffe, Lionel Thibault, and Dariusz Zagrodny.

## References


[1] L. Ambrosio, N. Gigli, G. Savaré: *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel (2008).

[2] D. Azé: *A unified theory for metric regularity of multifunctions*, J. Convex Analysis 13/2 (2006) 225–252.

[3] D. Azé, J.-N. Corvellec: *On some variational properties of metric spaces*, J. Fixed Point Theory 5 (2009) 185–200.

[4] J. M. Borwein, J. D. Vanderwerff: *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, Cambridge (2010).

[5] T. Z. Boulmezaoud, P. Cieutat, A. Daniilidis: *Gradient flows, second-order gradient systems and convexity*, SIAM J. Optim. 28/3 (2018) 2049–2066.

[6] F. H. Clarke: *Optimization and Nonsmooth Analysis*, Wiley, New York (1983).

[7] R. Correa, P. Gajardo, L. Thibault, D. Zagrodny: *Existence of minimizers on drops*, SIAM J. Optim. 23 (2013) 1154–1166.

[8] A. Daniilidis: *Personal communication*, Borovets, July 2023.

[9] A. Daniilidis, T. M. Le, D. Salas: *Metric compatibility and determination in complete metric spaces*, Math. Zeitschrift 308/4 (2024), art.no. 62, 31p.

[10] A. Daniilidis, D. Salas: *A determination theorem in terms of the metric slope*, Proc. Amer. Math. Soc. 150 (2022) 4325–4333.

[11] E. De Giorgi, A. Marino, M. Tosques: *Problems of evolution in metric spaces and maximal decreasing curve*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. 8/3 (1980) 180–187.

[12] D. Drusvyatskiy, A. D. Ioffe, A. S. Lewis: *Curves of descent*, SIAM J. Control Optim. 53/1 (2015) 114–138.

[13] Y. Giga, N. Hamamuki, A. Nakayasu: *Eikonal equations in metric spaces*, Trans. Amer. Math. Soc. 367/1 (2015) 49–66.

[14] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu: *Variational Methods in Partially Ordered Spaces*, Springer, New York (2003).

[15] A. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer Monographs in Mathematics, Springer, Cham (2017).

[16] N. Ivanov: *Personal communication*, Chernomoretz (2023).

[17] M. Ivanov, N. Zlateva: *A new proof of the integrability of the subdifferential of a convex function on a Banach space*, Proc. Amer. Math. Soc. 136/5 (2008) 1787–1793.

[18] A. A. Khan, C. Tammer, C. Zalinescu: *Set-Valued Optimization: An Introduction with Applications*, Springer, Berlin (2015).

[19] M. Konstantinov, N. Zlateva: *Epsilon subdifferential method and integrability*, J. Convex Analysis 29/2 (2022) 571-582.

[20] Q. Liu, N. Shanmugalingam, X. Zhou: *Equivalence of solutions of eikonal equation in metric spaces*, J. Diff. Equations 272 (2021) 979–1014.

[21] J. J. Moreau: *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. France 93 (1965) 273–299.

[22] M. Muratori, G. Savaré: *Gradient flows and evolution variational inequalities in metric spaces. I: Structural properties*, J. Funct. Anal. 278/4 (2020), art. no. 108347, 67p.

[23] P. Pérez-Aros, D. Salas, E. Vilches: *Determination of convex functions via subgradients of minimal norm*, Math. Programming A 190/1-2 (2021) 561–583.

[24] R. R. Phelps: *Convex Functions, Monotone Operators and Differentiability*, 2nd ed., Lecture Notes in Mathematics 1364, Springer, Berlin (1993).

[25] R. T. Rockafellar: *Characterization of the subdifferentials of convex functions*, Pacific J. Math. 17 (1966) 497–510.

[26] R. T. Rockafellar: *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. 33 (1970) 209–216.

[27] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).

[28] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften Vol. 317, Springer, Berlin (1998).

[29] L. Thibault, D. Zagrodny: *Determining functions by slopes*, Comm. Contemp. Math. 25/7 (2023), art. no. 225014, 30p.