Journal of Convex Analysis
Volume 32 (2025), No. 2, 375–398

Strong Metric (Sub)Regularity in Optimal Control

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: September 29, 2024
Accepted: October 13, 2024

This is mainly a survey on the properties of Strong Metric Regularity (SMR) and Strong Metric subRegularity (SMsR) of mappings representing first order optimality conditions (so-called *optimality mappings*) of optimization problems in infinite dimensional spaces. The focus is on the optimality mappings associated with optimal control problems for ODE systems or PDEs. We especially emphasize an extension of the concepts of SMR and SMsR which involves two metrics either in the domain or in the image spaces. The paper shows the relevance of this extension in optimal control.

Keywords: Variational analysis, metric regularity, optimization, optimal control.

2020 Mathematics Subject Classification: 49K40, 90C31, 49M05.

1. Introduction

In this paper, we present a survey about recent results on two properties of strong metric regularity of mappings, focusing on the so-called optimality mappings in calculus of variations and optimal control. These regularity properties imply useful stability properties of the solutions of such problems and estimates for the convergence rate of various kinds of numerical approximations.

We begin with two definitions. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. Let $d^\circ_{\mathcal{X}}$ and $d^\circ_{\mathcal{Y}}$ be additional metrics in the sets $\mathcal{X}$ and $\mathcal{Y}$, respectively. The symbols $\mathcal{O}_{\mathcal{X}}(x)$ and $\mathcal{O}^\circ_{\mathcal{X}}(x)$ denote open neighborhoods of $x \in \mathcal{X}$ with respect to $d_{\mathcal{X}}$ and $d^\circ_{\mathcal{X}}$, correspondingly. The meaning of $\mathcal{O}_{\mathcal{Y}}(y)$ and $\mathcal{O}^\circ_{\mathcal{Y}}(y)$ is similar for $y \in \mathcal{Y}$. Given

This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No. BG-RRP-2.004-0008-C01. The last author is also supported by the Austrian Science Foundation (FWF) under grant No. I-4571-N.

ISSN 0944-6532 / $ 2.50

a set-valued mapping $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$, the inverse mapping, $\mathcal{F}^{-1} : \mathcal{Y} \rightrightarrows \mathcal{X}$, is the set-valued mapping defined as $\mathcal{F}^{-1}(y) := \{x \in \mathcal{X} : y \in \mathcal{F}(x)\}$.

Definition 1.1. The set-valued mapping $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is *strongly metrically Hölder sub-regular* (SMHsR) at $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ (with metrics $d_{\mathcal{X}}, d^\circ_{\mathcal{X}}$ in $\mathcal{X}$ and $d_{\mathcal{Y}}, d^\circ_{\mathcal{Y}}$ in $\mathcal{Y}$) if $\hat{y} \in \mathcal{F}(\hat{x})$ and there exist numbers κ, $\beta > 0$ and neighborhoods $\mathcal{O}_{\mathcal{X}}(\hat{x})$ and $\mathcal{O}_{\mathcal{Y}}(\hat{y})$ such that

$$\forall (x, y) \in \mathcal{O}_{\mathcal{X}}(\hat{x}) \times \mathcal{O}_{\mathcal{Y}}(\hat{y}) \ \text{ with } \ y \in \mathcal{F}(x) \quad \Longrightarrow \quad d^\circ_{\mathcal{X}}(x, \hat{x}) \le \kappa\, d^\circ_{\mathcal{Y}}(y, \hat{y})^\beta. \qquad (1)$$

Definition 1.2. The set-valued mapping $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is *strongly metrically Hölder regular* (SMHR) around $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ (with metrics $d_{\mathcal{X}}, d^\circ_{\mathcal{X}}$ in $\mathcal{X}$ and $d_{\mathcal{Y}}, d^\circ_{\mathcal{Y}}$ in $\mathcal{Y}$) if $\hat{y} \in \mathcal{F}(\hat{x})$ and there exist numbers κ, $\beta > 0$ and neighborhoods $\mathcal{O}_{\mathcal{X}}(\hat{x})$ and $\mathcal{O}_{\mathcal{Y}}(\hat{y})$ such that the mapping $\mathcal{F}^{-1}(\cdot) \cap \mathcal{O}_{\mathcal{X}}(\hat{x})$ is non-empty valued on $\mathcal{O}_{\mathcal{Y}}(\hat{y})$ and

$$\begin{aligned} &\forall (x, y), (x', y') \in \mathcal{O}_{\mathcal{X}}(\hat{x}) \times \mathcal{O}_{\mathcal{Y}}(\hat{y}) \ \text{ with } \ y \in \mathcal{F}(x), \ \ y' \in \mathcal{F}(x') \\ &\qquad \Longrightarrow \quad d^\circ_{\mathcal{X}}(x, x') \le \kappa\, d^\circ_{\mathcal{Y}}(y, y')^\beta. \qquad (2) \end{aligned}$$

Obviously (by taking $y = \hat{y}$ in the definition), SMHsR implies that $\mathcal{F}^{-1}(\hat{y})$ is a singleton, while $\mathcal{F}^{-1}(y)$ may be empty for $y \neq \hat{y}$. Similarly, if F is SMHR then $\mathcal{F}^{-1}(\cdot) \cap \mathcal{O}_{\mathcal{X}}(\hat{x})$ is a single-valued Hölder continuous function in $\mathcal{O}_{\mathcal{Y}}(\hat{y})$ (Lipschitz continuous if $\beta = 1$).

If some of the regularity properties defined above are fulfilled with $\beta = 1$ we skip the name "Hölder" and shorten the acronyms as SMsR and SMR.

The elements $y \in \mathcal{Y}$ can be considered as "disturbances" of the reference point $\hat{y}$, so that it is all about stability of the solution $\hat{x}$ of $\hat{y} \in \mathcal{F}(x)$ with respect to ("small") disturbances y. In case of linear spaces $\mathcal{X}$ and $\mathcal{Y}$, consideration of two metrics in the domain space $\mathcal{X}$ is often aimed to ensure differentiability of the functions involved in $\mathcal{F}$ in the stronger norm, presumably $d_{\mathcal{X}} \ge d^\circ_{\mathcal{X}}$, while the stronger norm in the image space $\mathcal{Y}$, presumably d_Y, ensures appropriate properties of the disturbances $y \in \mathcal{Y}$ for which (1), resp. (2), is fulfilled.

Below we make some remarks concerning the history of the strong metric regularity properties defined above (we do not discuss in this paper the non-strong versions of these properties[1]), starting with the Lipschitz case $\beta = 1$, and with a single metric in each space: $d_{\mathcal{X}} = d^\circ_{\mathcal{X}}$ and $d_{\mathcal{Y}} = d^\circ_{\mathcal{Y}}$. The name *strong regularity* was introduced by Robinson [53], while the name *metric regularity* was coined by Borwein [9]. The property of SMsR was introduced by Dontchev [22] and named in this way in [29]; see also [30, Chapter 3.9] and the recent paper [14]. However, versions of these properties have been used under several other names by Bonnans, Shapiro, Klatte, Kummer and others, [7, 8, 42].

Hölder regularity notions ($\beta < 1$), as defined above, were studied by Frankowska and Quincampoix [35], although the idea to study nonlinear estimations such as (1) or (2) go back to work by Borwein and Zhuang (1988), Penot (1989), Frankowska (1987, 1990). The general topic of *nonlinear regularity* is reviewed in detail by Ioffe [39].

[1] See e.g. [30, Chapter 3] for the notions of Metric Regularity and Metric Subregularity. For important classes of mappings between finite dimensional spaces this notions actually coincide: KKT mappings [30, Theorem 4I.2], maximal monotone mappings [54], variational inequalities over convex polyhedral sets [27].

Two norms in the image space $\mathcal{Y}$ were first used by Quincampoix and Veliov in [51], where the property of *strong bi-metric regularity* was introduced. Utilization of two norms also in the domain space $\mathcal{X}$ proved to be useful; further quotations will be given in the corresponding sections. We mention that the two regularity properties with four norms defined above can be considered as a special case of the *regularity on a fixed set*, which is a non-local property; see Ioffe [38]. However, the analysis in the present paper is local and we deal with sets that are not necessarily fixed – these are the neighborhoods $\mathcal{O}_{\mathcal{X}}(\hat{x})$ and $\mathcal{O}_{\mathcal{Y}}(\hat{y})$ in Definitions 1.1 and 1.2, which may be taken sufficiently small.

The present survey is far from being comprehensive: a lot is missing concerning the general studies of regularity of mappings or, specifically, of mappings associated with variational inequalities. The paper does not include the numerous results about stability of solutions of optimization problems with respect to perturbations, for which the regularity theory of mappings is an important tool. Only a little is mentioned about applications of ideas and results in the regularity theory to numerical approximations. We refer to the books by Bonnans and Shapiro [8], Dontchev and Rockafellar [30], Mordukhovich [44], Klatte and Kummer [43], Ioffe [39] for further information. Instead, in this paper we focus on regularity properties of the optimality mapping associated with optimal control problems, especially in cases when two norms in either the domain or in the image space naturally appear in metric regularity issues.

In the next section we present some basic abstract results about stability of the two regularity defined above. In Section 3 we present sufficient conditions for the SMsR property of a mapping defined by the Karush-Kuhn-Tucker system for mathematical programming problems in Banach spaces. Although this result concerns infinite-dimensional problems and is applicable to problems of calculus of variations, it does not apply to problems of optimal control, where the control constraint cannot be represented by a finite number of inequalities. Results about SMsR and SMR of the optimality mapping associated with various optimal control problems are surveyed in the subsequent three sections: coercive Mayer-type optimal control problems for ODE systems, affine optimal control problems for ODE systems, an optimal control problem for a semi-linear parabolic equations. In all cases, we focus on results where consideration of two norms either in $\mathcal{X}$ or in $\mathcal{Y}$ is essential. In addition, we give a broader literature review of related results.

2. Preliminaries

Below we present two theorems of Robinson's kind, claiming invariance of the metric regularity properties under appropriately "small" perturbations, in particular under "linearization". The notions and notations introduced in the introduction will be used. In addition we denote by $\mathbb{B}_{\mathcal{Y}}(y;\gamma)$ the ball in the space $(\mathcal{Y}, d_{\mathcal{Y}})$ centered at y and with radius γ.

Theorem 2.1. *Assume that $\mathcal{Y}$ is a linear space and the metrics $d_{\mathcal{Y}}$ and $d^{\circ}_{\mathcal{Y}}$ are shift-invariant. Assume also that $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is SMsR (with metrics $d_{\mathcal{X}}, d^{\circ}_{\mathcal{X}}$ in $\mathcal{X}$ and $d_{\mathcal{Y}}, d^{\circ}_{\mathcal{Y}}$ in $\mathcal{Y}$) at $(\hat{x}, \hat{y})$ with neighborhoods $\mathcal{O}_{\mathcal{X}}(\hat{x})$, $\mathcal{O}_{\mathcal{Y}}(\hat{y})$ and a constant $\kappa > 0$ (see Definition 1.1 with $\beta = 1$). Let $\mathcal{O}'_{\mathcal{X}}(\hat{x})$ be a neighborhood of $\hat{x}$ and $\varphi : \mathcal{O}'_{\mathcal{X}}(\hat{x}) \to \mathcal{Y}$. Let the numbers μ, γ, and the neighborhood $\mathcal{O}'_{\mathcal{Y}}(\hat{y})$ of $\hat{y}$ satisfy the relations*

$$\begin{array}{lll} \mathcal{O}'_{\mathcal{X}}(\hat{x}) \subset \mathcal{O}_{\mathcal{X}}(\hat{x}), & \mathcal{O}'_{\mathcal{Y}}(\hat{y}) + \mathbb{B}_{\mathcal{Y}}(0;\gamma) \subset \mathcal{O}_{\mathcal{Y}}(\hat{y}), & \mu\kappa < 1, \\ d_{\mathcal{Y}}(\varphi(x),\varphi(\hat{x})) \le \gamma, & d^\circ_{\mathcal{Y}}(\varphi(x),\varphi(\hat{x})) \le \mu\, d^\circ_{\mathcal{X}}(x,\hat{x}) & \forall\, x \in \mathcal{O}_{\mathcal{X}}(\hat{x}). \end{array}$$

Then the mapping $\mathcal{F}+\varphi$ is SMsR at $(\hat{x},\hat{y}+\varphi(\hat{x}))$ with neighborhoods $\mathcal{O}'_{\mathcal{X}}(\hat{x})$ and $\mathcal{O}'_{\mathcal{Y}}(\hat{y})+\varphi(\hat{x})$ and a constant $\kappa' := \kappa/(1-\mu\kappa)$.

Proof. The proof adapts that of Theorem 2.1 in [14]. We take some $x \in \mathcal{O}'_{\mathcal{X}}(\hat{x})$, $y \in \mathcal{O}'_{\mathcal{Y}}(\hat{y})+\varphi(\hat{x})$ and $y \in \mathcal{F}(x)+\varphi(x)$. Then $x \in \mathcal{O}_{\mathcal{X}}(\hat{x})$ and

$$\mathcal{F}(x) \ni y-\varphi(x) \in \mathcal{O}'_{\mathcal{Y}}(\hat{y})+\varphi(\hat{x})-\varphi(x) \subset \mathcal{O}'_{\mathcal{Y}}(\hat{y})+\mathbb{B}_{\mathcal{Y}}(0;\gamma) \subset \mathcal{O}_{\mathcal{Y}}(\hat{y}).$$

Hence,

$$\begin{aligned} d^\circ_{\mathcal{X}}(x,\hat{x}) &\le \kappa d^\circ_{\mathcal{Y}}(y-\varphi(x),\hat{y}) \le \kappa[d^\circ_{\mathcal{Y}}(y-\varphi(\hat{x}),\hat{y}) + d^\circ_{\mathcal{Y}}(\varphi(\hat{x}),\varphi(x))] \\ &\le \kappa d^\circ_{\mathcal{Y}}(y-\varphi(\hat{x}),\hat{y}) + \mu\kappa d^\circ_{\mathcal{X}}(x,\hat{x}), \end{aligned}$$

which implies the SMsR property of $\mathcal{F}+\varphi$ at $(\hat{x},\hat{y}+\varphi(\hat{x}))$ with the specified neighborhoods and constant κ'. □

Notice that in the case $\varphi(\hat{x}) = 0$ the mapping $\mathcal{F}+\varphi$ is SMsR at the same point $(\hat{x},\hat{y})$ as $\mathcal{F}$ under the assumptions that the values of φ are sufficiently small in $d_{\mathcal{Y}}$ and φ is calm at $\hat{x}$ with respect to $d^\circ_{\mathcal{X}}$ and $d^\circ_{\mathcal{Y}}$ with a sufficiently small constant μ. This is exactly the case in the Banach space setting if an additive single-valued and Fréchet differentiable component of $\mathcal{F}$ is linearized and φ is the difference between this component and its linearization.

Theorem 2.2. *Assume that the metric space $(\mathcal{X}, d_{\mathcal{X}})$ is complete, and $d_X = d^\circ_{\mathcal{X}}$. Also assume that $\mathcal{Y}$ is a linear space, the metrics $d_{\mathcal{Y}}$ and $d^\circ_{\mathcal{Y}}$ are shift-invariant, and $d^\circ_{\mathcal{Y}} \le d_{\mathcal{Y}}$. Let $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ be SMR (with metrics $d_{\mathcal{X}}, d^\circ_{\mathcal{X}}$ in $\mathcal{X}$ and $d_{\mathcal{Y}}, d^\circ_{\mathcal{Y}}$ in $\mathcal{Y}$) around $(\hat{x},\hat{y})$ with neighborhoods $\mathcal{O}_{\mathcal{X}}(\hat{x}) = \mathbb{B}_{\mathcal{X}}(\hat{x};a)$, $\mathcal{O}_{\mathcal{Y}}(\hat{y}) = \mathbb{B}_{\mathcal{Y}}(\hat{y};b)$ and a constant $\kappa > 0$ (see Definition 1.1 with $\beta = 1$). In addition, let $\varphi : B_{\mathcal{X}}(\hat{x};a') \to \mathcal{Y}$, where $a' > 0$. Finally, let the numbers $b' > 0$, μ and γ satisfy the relations*

$$\begin{array}{llll} 0 < a' \le a, & b'+\gamma \le b, & \mu\kappa < 1, & \kappa(b'+\gamma) \le (1-\kappa\mu)a', \\ d_{\mathcal{Y}}(\varphi(x),0) \le \gamma, & d^\circ_{\mathcal{Y}}(\varphi(x),\varphi(x')) \le \mu\, d_{\mathcal{X}}(x,x') & & \forall\, x,x' \in \mathbb{B}_{\mathcal{X}}(\hat{x};a'). \end{array}$$

Then the mapping $\mathcal{F}+\varphi$ is SMR at $(\hat{x},\hat{y})$ with neighborhoods $\mathbb{B}_{\mathcal{X}}(\hat{x};a')$ and $\mathbb{B}_{\mathcal{Y}}(\hat{y};b')$, and a constant $\kappa' := \kappa/(1-\mu\kappa)$.

Proof. The proof is a modification of that of Theorem 4.2 in [50]. By assumption, the mapping $y \mapsto s(y) := \mathcal{F}^{-1}(y) \cap \mathbb{B}_{\mathcal{X}}(\hat{x};a)$ is a Lipschitz continuous function on $\mathbb{B}_{\mathcal{Y}}(\hat{y};b)$ with respect to the metrics $d^\circ_{\mathcal{X}}$ and $d^\circ_{\mathcal{Y}}$. For every $x \in \mathbb{B}_{\mathcal{X}}(\hat{x};a') \subset \mathbb{B}_{\mathcal{X}}(\hat{x};a)$ and $y \in \mathbb{B}_{\mathcal{Y}}(\hat{y};b')$ we have

$$d_{\mathcal{Y}}(y-\varphi(x),\hat{y}) \le d_{\mathcal{Y}}(y-\hat{y},0) + d_{\mathcal{Y}}(\varphi(x),0) \le b'+\gamma \le b,$$

thus $s(y-\varphi(x))$ is defined for every such x and y.

For an arbitrary fixed $y \in \mathbb{B}_{\mathcal{Y}}(\hat{y};b')$ we consider the mapping

$$B_{\mathcal{X}}(\hat{x};a') \ni x \mapsto Z_y(x) := s(y-\varphi(x)).$$

We shall prove that the mapping Z_y has a unique fixed point by using the contraction mapping theorem in the form of [30, Theorem 1A.2].

For this purpose we denote $\lambda = \kappa\mu < 1$ and estimate

$$\begin{aligned} d_{\mathcal{X}}(\hat{x}, Z_y(\hat{x})) &= d_{\mathcal{X}}(s(\hat{y}), s(y - \varphi(\hat{x}))) \le \kappa\, d^\circ_{\mathcal{Y}}(\hat{y}, y - \varphi(\hat{x})) \\ &\le \kappa\, d_{\mathcal{Y}}(\hat{y}, y - \varphi(\hat{x})) \le \kappa(b' + \gamma) \le (1 - \kappa\mu)a' = (1 - \lambda)a'. \end{aligned}$$

Moreover, for x, $x' \in B_{\mathcal{X}}(\hat{x}; a')$ we have

$$\begin{aligned} d_{\mathcal{X}}(Z_y(x), Z_y(x')) &= \kappa\, d^\circ_{\mathcal{Y}}(y - \varphi(x), y - \varphi(x')) = \kappa\, d^\circ_{\mathcal{Y}}(\varphi(x), \varphi(x')) \\ &\le \kappa\mu\, d_{\mathcal{X}}(x, x') = \lambda\, d_{\mathcal{X}}(x, x'). \end{aligned}$$

Then, according to [30, Theorem 1A.2], there exists a unique $x = x(y) \in B_{\mathcal{X}}(\hat{x}; a')$ such that $x = s(y - \varphi(x))$. The latter implies that $y - \varphi(x) \in \mathcal{F}(x)$, which implies $x \in (\mathcal{F} + \varphi)^{-1}(y) \cap B_{\mathcal{X}}(\hat{x}; a')$. Moreover, $x(y)$ is the unique element of the set $(\mathcal{F} + \varphi)^{-1}(y) \cap B_{\mathcal{X}}(\hat{x}; a')$. Indeed, if $x' \in (\mathcal{F} + \varphi)^{-1}(y) \cap B_{\mathcal{X}}(\hat{x}; a')$ we obtain $y \in \mathcal{F}(x') + \varphi(x')$, hence $y - \varphi(x') \in \mathcal{F}(x')$. Since we have $y - \varphi(x') \in B_{\mathcal{Y}}(\hat{y}; b)$ and $x' \in B_{\mathcal{X}}(\hat{x}; a')$, it also holds that $x' = s(y - \varphi(x'))$. Thus $x' = x(y)$. Thus the mapping $B_{\mathcal{Y}}(\hat{y} + \varphi(\hat{x}); b') \ni y \mapsto (\mathcal{F} + \varphi)^{-1}(y) \cap B_{\mathcal{X}}(\hat{x}, a')$ is single-valued.

Now, take two arbitrary elements y, $y' \in B_{\mathcal{Y}}(\hat{y} + \varphi(\hat{x}); b')$ and let $x = s(y - \varphi(x))$ and $x' = s(y' - \varphi(x'))$ be the unique solutions of $y \in \mathcal{F}(x) + \varphi(x)$ in $B_{\mathcal{X}}(\hat{x}; a')$ corresponding to y and y', respectively. Then

$$\begin{aligned} d_{\mathcal{X}}(x, x') &= d_{\mathcal{X}}(s(y - \varphi(x)), s(y' - \varphi(x'))) \le \kappa d^\circ_{\mathcal{Y}}(y - \varphi(x), y' - \varphi(x')) \\ &\le \kappa d^\circ_{\mathcal{Y}}(y, y') + \kappa d^\circ_{\mathcal{Y}}(\varphi(x), \varphi(x')) \le \kappa d^\circ_{\mathcal{Y}}(y, y') + \kappa\mu d_{\mathcal{X}}(x, x'). \end{aligned}$$

Hence,
$$d_{\mathcal{X}}(x, x') \le \frac{\kappa}{1 - \kappa\mu}\, d^\circ_{\mathcal{Y}}(y, y') \le \kappa' d^\circ_{\mathcal{Y}}(y, y'),$$

which completes the proof. □

In the proofs of the results mentioned in the next sections, the case $d_{\mathcal{X}} = d^\circ_{\mathcal{X}}$ is actually used, while the relation $d^\circ_{\mathcal{Y}} \le d_{\mathcal{Y}}$ with nonequivalent metrics is substantial.

Remark 2.3. The above theorems are used to establish that a mapping of the type $\mathcal{F} + \varphi$ is SMR or SMsR at (around, respectively) $(\hat{x}, \hat{y})$ if and only if $x \rightrightarrows \mathcal{F}(x) + \varphi'(\hat{x})(x - \hat{x})$ is such, provided that φ is Fréchet differentiable. Since Theorems 2.1 and 2.2 have no counterpart in the Hölder case $\beta < 1$, one cannot use this linearization approach for nonlinearities in $\mathcal{F} + \varphi$ even with a Fréchet differentiable φ. If the inclusion $0 \in \varphi + \mathcal{F}$ represents first order optimality conditions for an optimization problem, it should be linear-quadratic in order to obtain Hölder regularity. Such a situation will be shown in Sections 4 and 5. □

Characterizations of the SMsR and SMR properties of mappings between finite dimensional spaces in terms of graphical derivatives of $\mathcal{F}$ are known, see Chapters 4.4 and 4.5 in [30]. Being established for finite dimensional spaces, these results concern only the case of a single metric in each of the spaces $\mathcal{X}$ and $\mathcal{Y}$. In the case of Banach spaces, but also with single metrics, it is possible to adapt characterizations of the Metric Subregularity property by coderivatives of the mapping $\mathcal{F}$ (see the survey paper [57]) to the SMsR property. There are no such characterization of neither the SMsR nor of the SMR in the 4-metrics case.

3. Mathematical programming problems in Banach spaces

Let X and Y be Banach spaces. Consider the problem

$$\min \varphi(x), \quad f(x) \le 0, \qquad g(x) = 0, \tag{MP}$$

where $\varphi : X \to \mathbb{R}$, $f = (f_1, \dots, f_m)^* : X \to \mathbb{R}^m$, $g : X \to Y$ are C^1 mappings. The symbol $*$ denotes transposition; when applied to a space it denotes the dual space.

Let $\hat{x} \in X$ satisfy the constraints $f(x) \le 0$, $g(x) = 0$. Denote the set of active indices by

$$I = \{i \,:\, f_i(\hat{x}) = 0\}.$$

The following assumption is called Mangasarian-Fromovitz constraint qualification (MFCQ).

Assumption 3.1. *$g'(\hat{x})X = Y$ and the gradients $f_i'(\hat{x})$, $i \in I$, are positively linearly independent on the subspace* $\ker g'(\hat{x})$.

Recall that the functionals $l_1, \dots, l_k \in X^*$ are *positively linearly independent on a subspace $L \subset X$* if

$$\sum_{i=1}^{k} \lambda_i l_i(x) = 0 \quad \forall\, x \in L, \quad \lambda_i \ge 0 \quad \forall\, i \quad \Longrightarrow \quad \lambda_i = 0 \quad \forall\, i.$$

Let $\mathbb{R}^{m*}$ denote the space of row vectors of dimension m. The Karush-Kuhn-Tucker (KKT) theorem reads as follows.

Theorem 3.2. *If $\hat{x}$ is a local minimizer in problem* (MP) *and Assumption* 3.1 *holds, then there exist $\hat{\lambda} \in \mathbb{R}^{m*}$ and $\hat{y}^* \in Y^*$ such that*

$$\varphi'(\hat{x}) + \hat{\lambda} f'(\hat{x}) + \hat{y}^* g'(\hat{x}) = 0, \quad g(\hat{x}) = 0, \quad f(\hat{x}) \le 0, \quad \hat{\lambda} f(\hat{x}) = 0, \quad \hat{\lambda} \ge 0.$$

Let us fix a point $(\hat{x}, \hat{\lambda}, \hat{y}^*)$ that satisfies the system of relations in the above theorem (called KKT system). We shall equivalently reformulate the KKT system as a variational inequality:

$$L'_x(x, \lambda, y^*) = 0, \quad g(x) = 0, \quad f(x) \in N_{\mathbb{R}^m_+}(\lambda),$$

where $L(x, \lambda, y^*) = \varphi(x) + \lambda f(x) + y^* g(x)$ is the Lagrangian, and

$$N_{\mathbb{R}^m_+}(\lambda) := \begin{cases} \left\{\alpha \in \mathbb{R}^m \,:\, \langle \alpha, \beta - \lambda \rangle \le 0 \text{ for all } \beta \in \mathbb{R}^m_+ \right\} & \text{if } \lambda \in \mathbb{R}^m_+, \\ \emptyset & \text{if } \lambda \notin \mathbb{R}^m_+ \end{cases}$$

is the normal cone to $\mathbb{R}^m_+$ at $\lambda \in \mathbb{R}^m$. Obviously, the condition $f(\hat{x}) \in N_{\mathbb{R}^m_+}(\hat{\lambda})$ incorporates the conditions $f(\hat{x}) \in \mathbb{R}^m_+$, $\hat{\lambda} \ge 0$, and the *complementary slackness* condition $\hat{\lambda} f(\hat{x}) = 0$. Thus the KKT optimality condition can be recast as the inclusion $0 \in \mathcal{F}(x, \lambda, y^*)$, where the set-valued mapping $\mathcal{F}$ is defined as

$$\mathcal{F}(x, \lambda, y^*) := \begin{pmatrix} f(x) - N_{R^m_+}(\lambda) \\ g(x) \\ L'_x(x, \lambda, y^*) \end{pmatrix}.$$

Our aim in this subsection is to obtain sufficient conditions for the SMsR property of the mapping $\mathcal{F}$. For this purpose we first have to define the domain and the image spaces $\mathcal{X}$ and $\mathcal{Y}$ so that $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$. Assume that in the Banach space $(X, \|\cdot\|)$ there is another (weaker) norm $\|\cdot\|^\circ$, that is $\|x\|^\circ \le \|x\|$ for every $x \in X$. We set

$$X' = \{\zeta \in X^* \,:\, \zeta \text{ is continuous with respect to } \|\cdot\|^\circ\}$$

and define for any $\zeta \in X'$ the norm $\|\zeta\|' := \sup_{\|x\|^\circ \le 1} |\zeta(x)|$.

Then we define the spaces $\mathcal{X} := X \times \mathbb{R}^{m*} \times Y^*$ and $\mathcal{Y} := \mathbb{R}^m \times Y \times X'$.

In the space $\mathcal{X}$ we define following two norms

$$\|s\|_{\mathcal{X}} := \|x\| + |\lambda| + \|y^*\| \quad \text{and} \quad \|s\|^\circ_{\mathcal{X}} := \|x\|^\circ + |\lambda| + \|y^*\|, \qquad s = (x, \lambda, y^*) \in \mathcal{X}.$$

In the space $\mathcal{Y}$ we define

$$\|z\|_{\mathcal{Y}} = \|z\|^\circ_{\mathcal{Y}} = |\xi| + \|\eta\| + \|\zeta\|' \qquad z = (\xi, \eta, \zeta) \in \mathcal{Y}.$$

Observe that due to Assumption 2 we have $f_i'(\hat{x}) \in X'$ and $y^* g'(\hat{x}) \in X'$. Thus $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$.

We make the following "two-norm differentiability" assumptions for the mappings φ, f and g.

Assumption 3.3. *There exists a neighborhood $\hat{O}$ of $\hat{x}$ (in the norm $\|\cdot\|$) such that the following conditions are fulfilled for all $\Delta x \in X$ such that $\hat{x} + \Delta x \in \hat{O}$:*

(i) *The operator g is continuously Fréchet differentiable in $\hat{O}$ in the norm $\|\cdot\|$, and the derivative $g'(\hat{x})$ is a continuous operator with respect to the norm $\|\cdot\|^\circ$; moreover, the following representation holds true*

$$g(\hat{x} + \Delta x) = g(\hat{x}) + g'(\hat{x})\Delta x + r(\Delta x) \quad \text{with} \quad \|r(\Delta x)\|_Y \le \theta(\|\Delta x\|)\|\Delta x\|^\circ,$$

where $\theta(t) \to 0$ as $t \to 0+$.

(ii) *There exists a bilinear mapping $Q : X \times X \to Y$ such that*

$$g'(\hat{x} + \Delta x) = g'(\hat{x}) + Q(\Delta x, \cdot) + \bar{r}(\Delta x), \quad \|Q(x_1, x_2)\|_Y \le C\|x_1\|^\circ \|x_2\|^\circ,$$

where C is a constant and $\bar{r}$ satisfies

$$\sup_{\|x\|^\circ \le 1} \|\bar{r}(\Delta x)(x)\|_Y \le \theta(\|\Delta x\|)\|\Delta x\|^\circ.$$

(iii) *φ and f satisfy similar conditions with bilinear mappings Q_0 and Q_f.*

Define the quadratic functional $\Omega : X \to \mathbb{R}$ as

$$\Omega(x) := Q_0(x, x) + \hat{\lambda} Q_f(x, x) + \hat{y}^* Q(x, x),$$

and the so-called *critical cone* at the point $\hat{x}$ as

$$K := \{x \in X \,:\, \varphi'(\hat{x})x \le 0, \quad f_i'(\hat{x})x \le 0 \ \text{ for } \ i \in I, \quad g'(\hat{x})x = 0\}.$$

Assumption 3.4. *There exists a constant $c_0 > 0$ such that*

$$\Omega(x) \ge c_0 \, (\|x\|^\circ)^2 \quad \forall \, x \in K.$$

Assumption 3.5. *(strict MFCQ) The image $g'(\hat{x})X$ is closed and*

$$\textstyle\sum_{i\in I} \lambda_i f_i'(\hat{x}) + y^* g'(\hat{x}) = 0 \quad \text{with } \lambda_i \geq 0 \text{ when } \hat{\lambda}_i = 0$$
$$\implies \lambda_i = 0 \quad \forall\, i \in I, \quad y^* = 0.$$

The strict MFCQ (Assumptions 3.5) implies MFCQ (Assumptions 3.1). In particular, strict MFCQ implies the condition $g'(\hat{x})X = Y$. Moreover, strict MFCQ implies that $(\hat{x}, \hat{\lambda}, \hat{y}^*)$ is the unique KKT point for the fixed $\hat{x}$.

Theorem 3.6. *Let $\hat{s} = (\hat{x}, \hat{\lambda}, \hat{y}^*) \in \mathcal{X}$ be a KKT point for problem* (MP)*, and let assumptions* 3.3–3.5 *be fulfilled at this point. Then the KKT optimality mapping $\mathcal{F}$ is Strongly Metrically sub-Regular at the point $(\hat{s}, 0)$.*

More precisely, there exist constants $a > 0$, $b > 0$ and $\kappa \geq 0$ such that for every $y = (\xi, \eta, \zeta) \in \mathcal{Y}$ such that $\|y\|_{\mathcal{Y}} \leq b$ and for every solution $s = (x, \lambda, y^*) \in \mathcal{X}$ of the inclusion $z = (\xi, \eta, \zeta) \in \mathcal{F}(s)$ with $\|x - \hat{x}\| \leq a$ it holds that

$$\|x - \hat{x}\|^\circ + |\lambda - \hat{\lambda}| + \|y^* - \hat{y}^*\| \leq \kappa(|\xi| + \|\eta\| + \|\zeta\|').$$

A proof of the above theorem is given in [46] and [47].

Additional references:

The above result about SMsR of the optimality mapping is formulated under *strict Mangasarian-Fromovitz conditions* together with *second-order sufficient conditions.* In the case of finite-dimensional spaces X and Y the result is known from Dontchev and Rockafellar (1998) [28, Theorem 2.6] and Cibulka, Dontchev and Kruger (2018) [14, Section 7.1]. We mention that in the first of the quoted papers also local non-emptiness of $\mathcal{F}^{-1}$ is proved, as well as a number of related results that substantially use the finite dimensionality. More about regularity properties of problem (MP) in the finite-dimensional case can be found in [42, Chapter 8] and [8, Chapter 5.2].

The study of SMR of the KKT mapping goes back to Robinson [53]; see [43] for more recent results.

Results in Hilbert spaces can be found in [24] and the references therein, however, a single norm is used in the domain space $\mathcal{X}$ and the SMR property is considered, therefore the assumptions are stronger.

Various Lipschitz stability results related to problem (MP) (in Banach spaces) and the associated Lagrange multipliers are obtained in [8, Chapter 4], which, as far as we can see, do not imply the result in the present subsection.

4. Mayer type optimal control problem

In this section we consider the following Mayer's type optimal control problem:

$$\min \varphi(x(0), x(1)), \tag{3}$$

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. in} \quad [0, 1], \tag{4}$$

$$G(u(t)) \leq 0 \quad \text{a.e. in} \quad [0, 1], \tag{5}$$

where $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, and $G : \mathbb{R}^m \to \mathbb{R}^k$ are of class C^2, $x \in W^{1,1}$, $u \in L^\infty$. Again, we investigate the property of *Strong Metric sub-Regularity* (SMsR) of the *optimality mapping*, associated with the system of first order necessary optimality conditions (Pontryagin's conditions in local form) for problem (3)–(5).

In this and the next sections we use the following standard notations. The euclidean norm and the scalar product in $\mathbb{R}^n$ (the elements of which are regarded as column-vectors) are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. The transpose of a matrix (or vector) E is denoted by $E^\top$. For a function $\psi : \mathbb{R}^p \to \mathbb{R}^r$ of the variable z we denote by $\psi_z(z)$ its derivative (Jacobian), represented by an $(r \times p)$-matrix. If $r = 1$, $\nabla_z \psi(z) = \psi_z(z)^\top$ denotes its gradient (a vector-column of dimension p).
Also for $r = 1$, $\psi_{zz}(z)$ denotes the second derivative (Hessian), represented by a $(p \times p)$-matrix. For a function $\psi : \mathbb{R}^{p+q} \to \mathbb{R}$ of the variables (z, v), $\psi_{zv}(z, v)$ denotes its mixed second derivative, represented by a $(p \times q)$-matrix. The space $L^k([0, T], \mathbb{R}^r)$, with $k = 1, 2$ or $k = \infty$, consists of all (classes of equivalent) Lebesgue measurable r-dimensional vector-functions defined on the interval $[0, T]$, for which the standard norm $\|\cdot\|_k$ is finite. Often the specification $([0, T], \mathbb{R}^r)$ will be omitted in the notations. As usual, $W^{1,k} = W^{1,k}([0, T], \mathbb{R}^r)$ denotes the space of absolutely continuous functions $x : [0, T] \to \mathbb{R}^r$ for which the first derivative belongs to L^k. The norm in $W^{1,k}$ is defined as $\|x\|_{1,k} := \|x\|_k + \|\dot{x}\|_k$. As before, $\mathbb{B}_X(x; r)$ denotes the ball of radius r centered at x in a metric space X.

According to (5), the set of admissible control values is

$$U := \{v \in \mathbb{R}^m \,:\, G(v) \le 0\}.$$

Let G_i denote the i-th component of the vector G. For any $v \in U$ define the set of active indices

$$I(v) = \{i \in \{1, \dots, k\} \,:\, G_i(v) = 0\}.$$

Assumption 4.1. (regularity of the control constraints) *The set U is nonempty and at each point $v \in U$ the gradients $G_i'(v)$, $i \in I(v)$ are linearly independent.*

Define the *augmented Hamiltonian*

$$\bar{H}(x, u, p, \lambda) = p\, f(x, u) + \lambda\, G(u), \quad p \in \mathbb{R}^{n*}, \quad \lambda \in \mathbb{R}^{k*}.$$

If the pair $(\hat{x}, \hat{u}) \in W^{1,1} \times L^\infty$ is a weak local minimizer in the problem (3)–(5) then there exists a unique pair $(\hat{p}, \hat{\lambda}) \in W^{1,1} \times L^\infty$ such that the following *optimality system* of equations and inequalities is satisfied:

$$-\dot{\hat{x}}(t) + \bar{H}_p(\hat{x}(t), \hat{u}(t), \hat{p}(t), \hat{\lambda}) = 0 \quad \text{a.e. in} \quad [0, 1]\,, \tag{6}$$

$$\dot{\hat{p}}(t) + \bar{H}_x(\hat{x}(t), \hat{u}(t), \hat{p}(t), \hat{\lambda}) = 0 \quad \text{a.e. in} \quad [0, 1]\,, \tag{7}$$

$$(-\hat{p}(0), \hat{p}(1)) = \varphi'(\hat{x}(0), \hat{x}(1))\,, \tag{8}$$

$$\bar{H}_u(\hat{x}(t), \hat{u}(t), \hat{p}(t), \hat{\lambda}) = 0, \quad \text{a.e. in} \quad [0, 1]\,, \tag{9}$$

$$G(\hat{u}(t)) \le 0 \quad \text{a.e. in} \quad [0, 1]\,, \tag{10}$$

$$\hat{\lambda}(t) \ge 0, \quad \hat{\lambda}(t) G(\hat{u}(t)) = 0 \quad \text{a.e. in} \quad [0, 1]\,. \tag{11}$$

Here, the $\bar{H}_p$, $\bar{H}_x$, $\bar{H}_u$ denote derivatives of $\bar{H}$ with respect to the argument indicated as subscript. Notice that here and below, the dual variables p and λ are treated as row vectors, while x, u, f, and G are column vectors.

Below, the quadruple $\hat{s} := (\hat{x}, \hat{u}, \hat{p}, \hat{\lambda})$ will be any *reference solution* of the optimality system (6)–(11) (not necessarily resulting from an optimal pair) in the space

$$\mathcal{X} = W^{1,1} \times L^\infty \times W^{1,1} \times L^\infty.$$

We consider this space with the following two norms:

$$\|s\|_{\mathcal{X}} = \|x\|_{1,1} + \|u\|_\infty + \|p\|_{1,1} + \|\lambda\|_\infty, \quad \|s\|_{\mathcal{X}}^\circ = \|x\|_{1,1} + \|u\|_2 + \|p\|_{1,1} + \|\lambda\|_2. \tag{12}$$

Moreover, we define the space

$$\mathcal{Y} = L^1 \times L^\infty \times \mathbb{R}^{2n*} \times L^\infty \times L^\infty,$$

also endowed with two norms of its elements $y = (\xi, \pi, \nu, \rho, \eta)$:

$$\|y\|_{\mathcal{Y}} = \|\xi\|_1 + \|\pi\|_1 + |\nu| + \|\rho\|_\infty + \|\eta\|_\infty,\ \|y\|_{\mathcal{Y}}^\circ = \|\xi\|_1 + \|\pi\|_1 + |\nu| + \|\rho\|_2 + \|\eta\|_2. \tag{13}$$

Observe that conditions (10)–(11) can be equivalently reformulated as an inclusion

$$G(\hat{u}(t)) \in N_{\mathbb{R}^k_+}(\hat{\lambda}(t)),$$

with the standard definition of the normal cone $N_{\mathbb{R}^k_+}$ (see Subsection 3). Then one can reformulate the optimality system as a variational inequality for $(x, u, p, \lambda) \in \mathcal{X}$:

$$\mathcal{F}(x, u, p, \lambda) \ni 0,$$

where the *optimality mapping* $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is defined as

$$\mathcal{F}(x, u, p, \lambda) := \left(\begin{array}{c} -\dot{x} + \bar{H}_p(x, \hat{u}, \hat{p}, \lambda) \\ \dot{p} + \bar{H}_x(x, u, p, \lambda) \\ (-\hat{p}(0), \hat{p}(1)) - \varphi'(\hat{x}(0), \hat{x}(1)) \\ \bar{H}_u(x, u, p, \lambda) \\ G(u) - \mathbb{N}_+(\lambda), \end{array} \right).$$

and $\mathbb{N}_+(\lambda) := \{\eta \in L^\infty \ :\ \eta(t) \in N_{\mathbb{R}^k_+}(\lambda(t)) \ \text{ a.e. in } [0, 1]\}$.

Notice that $\mathbb{N}_+(\lambda)$ is a proper subset of the normal cone at λ to the set of all non-negative L^∞-functions.

In order to obtain fine sufficient conditions for strong metric sub-regularity of the mapping $\mathcal{F}$ at the reference point $(\hat{x}, \hat{u}, \hat{p}, \hat{\lambda})$ we make one more assumption. For its formulation we use the notations $w := (x, u)$, $\hat{w} := (\hat{x}, \hat{u})$, $q := (x(0), x(1))$, $\hat{q} := (\hat{x}(0), \hat{x}(1))$. Define the usual *critical cone*:

$$K = \left\{ w \in W^{1,1} \times L^\infty : \begin{array}{c} \dot{x}(t) = f'(\hat{w}(t))w(t) \quad \text{for a.e. } t \in [0, 1], \\ G_j'(\hat{u}(t))u(t) \le 0 \ \text{ for a.e. } t \text{ with } G_j(\hat{u}(t)) = 0, \\ G_j'(\hat{u}(t))u(t) = 0 \ \text{ for a.e. } t \text{ with } \hat{\lambda}_j(t) > 0,\ j = 1, ..., k \end{array} \right\}.$$

Next, for any $\Delta > 0$ we define the extended critical cone

$$K_\Delta = \left\{ w \in W^{1,1} \times L^\infty : \begin{array}{ll} \dot{x}(t) = f'(\hat{w}(t))w(t) & \text{for a.e. } t \in [0,1], \\ G'_j(\hat{u}(t))u(t) \le 0 & \text{for a.e. } t \text{ for which } G_j(\hat{u}(t)) = 0, \\ G'_j(\hat{u}(t))u(t) = 0 & \text{for a.e. } t \text{ with } \hat{\lambda}_j(t) > \Delta,\ j = 1, ..., k \end{array} \right\}.$$

Notice that the cones K_Δ form a non-increasing family as $\Delta \to 0+$ and $K \subset K_\Delta$ for any $\Delta > 0$.

Define the following *quadratic form* of $w \in W^{1,1} \times L^\infty$:

$$\Omega(w) := \langle \varphi''(\hat{q})q, q \rangle + \int_0^1 \langle \bar{H}_{ww}(\hat{w}(t), \hat{p}(t), \hat{\lambda}(t))w(t), w(t) \rangle \,\mathrm{d}t. \tag{14}$$

Assumption 4.2. *There exist constants $\Delta > 0$ and $c_\Delta > 0$ such that*

$$\Omega(w) \ge c_\Delta \big(|x(0)|^2 + \|u\|_2^2\big) \quad \forall\, w \in K_\Delta. \tag{15}$$

Theorem 4.3. *Let Assumptions* 4.1 *and* 4.2 *be fulfilled. Then the optimality mapping $\mathcal{F}$ is strongly metrically sub-regular (in the four metrics defined in* (12), (13)*) at the reference point $((\hat{x}, \hat{u}, \hat{p}, \hat{\lambda}), 0_\mathcal{Y})$, where $0_\mathcal{Y}$ is the origin in $\mathcal{Y}$.*

In detail, the theorem can be formulated as follows. There exist reals $\delta > 0$ and $\kappa > 0$ such that if $z = (\xi, \pi, \nu, \rho, \eta)$ satisfies the inequality

$$\|\xi\|_1 + \|\pi\|_1 + |\nu| + \|\rho\|_\infty + \|\eta\|_\infty \le \delta, \tag{16}$$

then for any solution $s = (x, u, p, \lambda)$ of the perturbed inclusion $z \in \mathcal{F}(s)$ such that $\|x - \bar{x}\|_\infty + \|u - \bar{u}\|_\infty \le \delta$ the following estimation holds:

$$\|x-\hat{x}\|_{1,1}+\|u-\hat{u}\|_2+\|p-\hat{p}\|_{1,1}+\|\lambda-\hat{\lambda}\|_2 \le \kappa\,(\|\xi\|_1+\|\pi\|_1+|\nu|+\|\rho\|_2+\|\eta\|_2). \tag{17}$$

Observe that the set of points s for which $\|x - \bar{x}\|_\infty + \|u - \bar{u}\|_\infty \le \delta$ is a "cylindric" neighborhood of $\hat{s}$ in the metric $\|\cdot\|_\mathcal{X}$. Also, the inequality in (17) is satisfied in the norms $\|\cdot\|_\mathcal{X}^\circ$ and $\|\cdot\|_\mathcal{Y}^\circ$, while the neighborhoods of $\hat{s}$ and zero are with respect to $\|\cdot\|_\mathcal{X}$ and $\|\cdot\|_\mathcal{Y}$.

The proof of the above theorem is given in [48].

Additional remarks and references

A similar result as Theorem 4.3 is obtained in [24], where however, the property of SMR is considered for which the conditions are stronger.

An important feature of Assumption 4.2 is that the quadratic functional Ω has quadratic growth with respect to the norm $\|u\|_2$ of the control component of the elements of the extended critical cone K_Δ. Such a condition is generally called *coercivity* (see e.g. [23]). In particular, this condition is satisfied if (15) holds for every $u \in \mathcal{U} - \hat{u}$ ($\mathcal{U}$ is the set of admissible controls) with the x-part of w being the corresponding solution of the linearised equation $\dot{x}(t) = f'(\hat{w}(t))w(t)$ as in the definition of the critical cone. Assumption 4.2 is generally weaker than the latter one. In [23, Theorem 5], Dontchev and Hager prove a property similar to SMR (the last term was not in use in that time) under coercivity condition in which (15) is required for all $u \in \mathcal{U} - \mathcal{U}$. This coercivity condition proved to be sufficient for SMR and finds numerous applications in studying convergence of numerical methods for ODE optimal control, see e.g. [6, 13, 25, 26, 31, 36]. Most of these results can, in fact, be obtained based on the SMsR property.

5. An affine optimal control problem

In this section, we investigate the properties of SMsR and SMR of the optimality mapping associated with problems that are affine with respect to the control:

$$\min\Big\{J(u) := \int_0^T [w(t,x(t)) + \langle d(t,x(t)), u(t)\rangle]\,\mathrm{d}t\Big\}, \tag{18}$$

subject to
$$\dot{x}(t) = a(t,x(t)) + B(t,x(t))u(t), \quad x(0) = x^0, \tag{19}$$

$$u(t) \in U, \quad t \in [0,T]. \tag{20}$$

The state vector $x(t)$ belongs to $\mathbb{R}^n$, the control function u has values $u(t)$ that belong to a given set U in $\mathbb{R}^m$ for a.e. $t \in [0,T]$. Correspondingly, w is a scalar function on $[0,T] \times \mathbb{R}^n$, d is an m-dimensional vector function, a and B are vector-/matrix-valued functions with appropriate dimensions. The initial state x^0 and the final time $T > 0$ are fixed. The set of admissible control functions u, denoted in the sequel by $\mathcal{U}$, consists of all Lebesgue measurable and bounded functions $u : [0,T] \to U$. We use the abbreviations

$$f(t,x,u) = a(t,x) + B(t,x)u, \qquad g(t,x,u) = w(t,x) + \langle d(t,x), u\rangle. \tag{21}$$

Obviously the coercivity assumption in Section 4 is not fulfilled in general. For problem (18)–(20) we make the following assumption.

Assumption 5.1. *The set U is convex and compact; the functions*

$$f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \text{ and } g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

have the form as in (21) *and are two times differentiable in* (t,x), *and the derivatives are Lipschitz continuous*[2].

Define the Hamiltonian associated with problem (18)–(20) as usual:

$$H(t,x,p,u) := g(t,x,u) + \langle p, f(t,x,u)\rangle, \quad p \in \mathbb{R}^n.$$

Similarly as to the previous subsection, the Pontryagin (local) minimum principle can be written in the form of a generalized equation (differential variational inequality)

$$0 \in \mathcal{F}(x,p,u) := \begin{pmatrix} -\dot{x} + f(\cdot,x,u) \\ \dot{p} + \nabla_x H(\cdot,y) \\ \nabla_u H(\cdot,y) + N^{1,\infty}_{\mathcal{U}}(u), \end{pmatrix}, \quad y := (x,p,u), \tag{22}$$

[2] The assumption of *global* Lipschitz continuity is made for convenience. Since the analysis in this paper is local (in a neighborhood of a reference trajectory $\hat{x}(\cdot)$ in the uniform metric), it can be replaced with *local* Lipschitz continuity. The differentiability assumption in t can also be relaxed, especially in the consideration of the SMsR property.

where $$N^{1,\infty}_{\mathcal{U}}(u) := \{\rho \in W^{1,\infty} \,:\, \langle \rho, u' - u\rangle \le 0 \quad \forall\, u' \in \mathcal{U}\}$$
is a subset of the normal cone to $\mathcal{U}$ at u, considered as a subset of L^1.

Assumption 5.1 implies that there exists a number $M > 0$ such that for any $u \in \mathcal{U}$ the corresponding solution x of (19) and also the solution p of the adjoint equation $\dot p + \nabla_x H(\cdot, y) = 0$, $p(T) = 0$, exist on $[0, T]$ and

$$\max\{|x(t)|,\, |\dot x(t)|,\, |p(t)|,\, |\dot p(t)|\} \le M \quad \text{for a.e. } t \in [0, T]. \tag{23}$$

In what follows, $\bar M$ will be a fixed number larger that M.

The set-valued mapping $\mathcal{F}$ is considered as acting from the space

$$\mathcal{X} := \{(x, p, u) \in W^{1,1} \times W^{1,1} \times L^1 \,:\, x(0) = x^0,\, p(T) = 0\} \cap \left(\mathbb{B}^2_{W^{1,\infty}}(0; \bar M) \times \mathcal{U}\right)$$

(where $\mathbb{B}^2_{W^{1,\infty}}(0; \bar M) := \mathbb{B}_{W^{1,\infty}}(0; \bar M) \times \mathbb{B}_{W^{1,\infty}}(0; \bar M)$) with the norm

$$\|(x, p, u)\|_{\mathcal{X}} := \|x\|_{1,1} + \|p\|_{1,1} + \|u\|_1,$$

to the space $$\mathcal{Y} := \{(\xi, \pi, \rho)\} = L^\infty \times L^\infty \times W^{1,\infty}$$

endowed with the following two norms

$$\begin{aligned} \|(\xi, \pi, \rho)\|_{\mathcal{Y}} &:= \|\xi\|_\infty + \|\pi\|_\infty + \|\rho\|_{1,\infty}, \\ \|(\xi, \pi, \rho)\|^\circ_{\mathcal{Y}} &:= \|\xi\|_1 + \|\pi\|_1 + \|\rho\|_\infty. \end{aligned}$$

Notice that $F(\mathcal{X}) \subset \mathcal{Y}$, thanks to the independence of $\nabla_u H(\cdot, y)$ of u.

Let us fix a reference solution $\hat s = (\hat x, \hat p, \hat u)$ of the inclusion $0 \in \mathcal{F}(s)$. We mention that such always exists since on Assumption 5.1 problem (18)–(20) has a solution $(\hat x, \hat u)$. To shorten the notations we skip arguments with "hat" in functions, shifting the "hat" on the top of the notation of the function, so that $\hat f(t) := f(t, \hat x(t), \hat u(t))$, $\hat d(t) = d(t, \hat x(t))$, $\hat H(t) := H(t, \hat x(t), \hat u(t), \hat p(t))$, $\hat H(t, u) := H(t, \hat x(t), u, \hat p(t))$, etc. Moreover, denote

$$\hat A(t) := f_x(t,\hat x(t), \hat u(t)), \quad \hat B(t) := f_u(t, \hat x(t), \hat u(t)) = B(t, \hat x(t))$$
$$\hat \sigma(t) := \nabla_u \hat H(t) = \hat B(t)^\top \hat p(t) + \hat d(t).$$

Let us introduce the following functional of $L^1 \ni v \mapsto \Gamma(v) \in \mathbb{R}$:

$$\Gamma(v) := \int_0^T \Big[\langle \hat H_{xx}(t) x(t), x(t)\rangle + 2\langle \hat H_{ux}(t) x(t), v(t)\rangle\Big]\, \mathrm{d}t,$$

where x is the solution of the equation $\dot x = \hat A x + \hat B u$ with initial condition $x(0) = 0$.

Assumption 5.2. *There exist numbers c_0, $\alpha_0 > 0$ and $\gamma_0 > 0$ such that*

$$\int_0^T \langle \sigma(t), v(t)\rangle\, \mathrm{d}t + \Gamma(v) \ge c_0 \|v\|_1^2, \tag{24}$$

holds for every $v = u - u'$ with $u, u' \in \mathcal{U} \cap B_{L^1}(\hat u; \alpha_0)$, and for every function $\sigma \in \mathbb{B}_{W^{1,\infty}}(\hat \sigma; \gamma_0) \cap (-N_{\mathcal{U}}(u'))$.

Remark 5.3. Notice that, in contrast to the coercivity Assumption 4.2, the above growth condition: (i) is in the space L^1 for the control variation; (ii) contains a linear term (not only quadratic). □

The following theorem is proved in [20].

Theorem 5.4. *Let Assumption* 5.1 *be fulfilled for problem* (18)–(20) *and let* $\hat{s} = (\hat{x}, \hat{p}, \hat{u})$ *be a solution of the optimality system* (22) *for which Assumption* 5.2 *is fulfilled. Let, in addition, the matrix* $\hat{H}_{ux}(t)\hat{B}(t)$ *be symmetric for a.e.* $t \in [0, T]$. *Then the optimality mapping* $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ *is strongly metrically regular around* $(\hat{s}, 0)$ *with respect to the above defined norm in* $\mathcal{X}$ *and norms in* $\mathcal{Y}$.

The following assumption is stronger but more compact than Assumption 5.2.

Assumption 5.5. *There exist numbers* c_0, $\alpha_0 > 0$ *and* $\gamma_0 > 0$ *such that*

$$\int_0^T |\langle \sigma(t), v(t) \rangle| \,\mathrm{d}t + \Gamma(v) \geq c_0 \|v\|_1^2, \tag{25}$$

for every function $\sigma \in \mathbb{B}_{W^{1,\infty}}(\hat{\sigma}; \gamma_0)$ *and for every* $v \in \mathcal{U} - \mathcal{U}$ *with* $\|v\|_1 \leq \alpha_0$.

Obviously Assumption 5.5 implies 5.2, since for $\sigma \in -N_{\mathcal{U}}(u')$ and $u \in \mathcal{U}$ it holds that $\langle \sigma(t), u(t) - u'(t) \rangle \geq 0$.

Now, we focus on the first-order term in (24) under an additional condition.

Assumption 5.6. *The set* U *is a convex and compact polyhedron. Moreover, there exist numbers* $\kappa > 0$ *and* $\tau > 0$ *such that for every unit vector* e *parallel to some edge of* U *and for every* $s \in [0, T]$ *for which* $\langle \hat{\sigma}(s), e \rangle = 0$ *it holds that*

$$|\langle \hat{\sigma}(t), e \rangle| \geq \kappa |t - s| \qquad t \in [s - \tau, s + \tau] \cap [0, T].$$

Proposition 5.7. *Let the Assumptions* 5.1 *and* 5.6 *be fulfilled. Then there exist numbers* c_0, $\alpha_0 > 0$ *and* $\gamma_0 > 0$ *such that*

$$\int_0^T |\langle \sigma(t), v(t) \rangle| \,\mathrm{d}t \geq c_0 \|v\|_1^2, \tag{26}$$

for every function $\sigma \in \mathbb{B}_{W^{1,\infty}}(\hat{\sigma}; \gamma_0)$ *and for every* $v \in \mathcal{U} - \mathcal{U}$ *with* $\|v\|_1 \leq \alpha_0$.

This proposition implies that the linear term in (24) alone can ensure satisfaction of Assumption 5.2. The term $\Gamma(v)/\|v\|_1^2$ may take even negative values, provided that they are smaller than c_0 in (26).

Strong metric subregularity

Now we briefly discuss the SMsR property by slightly modifying results in [45].

Assumption 5.8. *There exist numbers* c_0 *and* $\alpha_0 > 0$ *such that* (24) *is fulfilled with* $\sigma = \hat{\sigma}$ *for every* $v \in \mathcal{U} - \hat{u}$ *with* $\|v\|_1 \leq \alpha_0$.

Notice that the last assumption is weaker than Assumption 5.2 in two respects: first, the function σ in the latter assumption is fixed to $\hat{\sigma}$, and second, the variation v in Assumption 5.8 has the more specific form $v = \mathcal{U} - \hat{u}$ instead on $v \in \mathcal{U} - \mathcal{U}$. Of course, Remark 5.3 also applies to Assumption 5.2.

Moreover, the domain space $\mathcal{X}$ and the range space $\mathcal{Y}$ of the optimality mapping $\mathcal{F}$ in (22) change, so that $\mathcal{Y}$ becomes substantially larger (that is, a larger class of disturbances is allowed):

$$\begin{aligned}
\text{now} \qquad & \mathcal{X} := \{(x,p,u) \in W^{1,1} \times W^{1,1} \times L^1 : \, x(0) = x^0, \, p(T) = 0\} \\
& \text{with } \|(x,p,u)\|_{\mathcal{X}} := \|x\|_{1,1} + \|p\|_{1,1} + \|u\|_1 \\
\text{and} \qquad & \mathcal{Y} := \{(\xi,\pi,\rho)\} = L^1 \times L^1 \times L^\infty \\
& \text{with } \|(\xi,\pi,\rho)\|_{\mathcal{Y}} = \|(\xi,\pi,\rho)\|^\circ_{\mathcal{Y}} = \|\xi\|_1 + \|\pi\|_1 + \|\rho\|_\infty.
\end{aligned}$$

It was proved in [45] that under Assumptions 5.1 and 5.8 the optimality mapping $\mathcal{F}$ is strongly metrically subregular at the reference pair $(\hat{x},\hat{p},\hat{u})$ and 0 in the latter spaces $\mathcal{X}$ and $\mathcal{Y}$.

An analog of Proposition 5.7 is also valid for the SMsR. Namely, under Assumptions 5.1 and 5.6, the inequality

$$\int_0^T \langle \hat{\sigma}(t), u - \hat{u} \rangle \,\mathrm{d}t \geq c_0 \|u - \hat{u}\|_1^2$$

holds for every $u \in \mathcal{U}$ – sufficiently close to $\hat{u}$ in L^1.

We mention that Assumption 5.8 is also a sufficient condition for strict week local optimality in L^1. Even more, it was proved in [45] that the condition similar to Assumption 5.2 with (24) replaced with

$$\int_0^T \langle \sigma(t), v(t) \rangle \,\mathrm{d}t + \frac{1}{2}\Gamma(v) \geq c_0 \|v\|_1^2$$

is also sufficient for strict optimality. One can prove that the latter condition is weaker than Assumption 5.2 in the case where $\Gamma(v)$ may take negative values.

Finally, Assumption 5.8 can be weakened by replacing (24) with

$$\int_0^T \langle \sigma(t), u(t) - \hat{u}(t) \rangle \,\mathrm{d}t + \Gamma(u - \hat{u}) \geq c_0 \|u - \hat{u}\|_1^\kappa,$$

for every $u \in \mathcal{U} \cap B_{L^1}(\hat{u};\alpha_0)$, where $\kappa \in (1,2]$. This condition implies strong Hölder subregularity of $\mathcal{F}$, however if $\kappa < 2$ this is true provided that the considered problem is linear-quadratic, that is,

$$f(t,x,u) = A(t)x + B(t)u, \quad g(t,x,u) = w(t)x + W(t)x + C(t)u.$$

Additional references

The SMR property is investigated for affine problems only in a few papers, starting with linear systems in [51], where two norms in the image space were used. The need of these two norms for a relevant representation of the concept of SMR for affine problems is explained in detail in [52]. The non-linear affine case is considered in [20], where Assumption 5.2 is introduced and Theorem 5.4 is proved. A condition similar to Assumption 5.6 is introduced [2] in connection with a method for solving affine optimal control problems by optimal localization of the switching points. A similar condition is used for box-like sets U in [32] to obtain results that are related to (but stronger then) SMsR. The weaker Assumption 5.8 is introduced in [45], under which SMsR of the optimality mapping is proved. There are numerous related results that investigate stability with respect to perturbations of the solutions of affine problems under various assumptions on the structure of the optimal control (bang-singular, bang-singular-bang, etc.), see e.g. [34, 49].

There is an amount of literature on direct discretization methods for affine problems, where the challenge is the usual discontinuity of the optimal controls. The property of SMsR is either explicitly used or is hidden behind the technicalities of the proofs. Here, we mention [3, 4, 5, 45] for error estimates of the Euler scheme and [55] for higher order estimation of a discretization technique based on Volterra-Fliess expansion of the right-hand side of the ODE.

We also mention a series of works by Hager and co-authors that use the switching points as free variables and allow combinations of bang and singular arcs, see e.g. [1, 37]. As noticed in [14] (and earlier in [7] under different conditions), SMsR of the optimality system in principle implies convergence of the Newton method. For affine optimal control problems this is made explicit in [33, 50].

6. Optimal control of parabolic equations

This section is devoted to the study of strong metric Hölder sub-regularity of the optimality mapping of an optimal control problem for a semilinear parabolic PDE. Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Given a finite time $T > 0$, we denote by $Q := \Omega \times (0,T)$ the space-time cylinder, and its lateral boundary by $\Sigma := \partial\Omega \times (0,T)$. We consider the optimal control problem

$$\min_{u \in \mathcal{U}} \left\{ J(u) := \int_Q L_0(x,t,y(x,t)) + g(x,t)u(x,t) \,\mathrm{d}x\,\mathrm{d}t \right\}, \tag{27}$$

subject to
$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathcal{A}\right) y + f(\cdot, y) = u & \text{in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 & \text{on } \Omega. \end{cases} \tag{28}$$

The set of feasible controls is

$$\mathcal{U} := \{ u \in L^\infty(Q) |\ u_a \leq u \leq u_b \ \text{ for a.a. } (x,t) \in Q \},$$

where $u_a, u_b \in L^\infty(Q)$ with $u_a < u_b$ a.e in Q.

Assumption 6.1. (i) *The operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$, is given by*

$$\mathcal{A}y := -\sum_{i,j=1}^{n} \partial_{x_j}(a_{i,j}(x)\partial_{x_i} y),$$

where $a_{i,j} \in L^\infty(\Omega)$. Further, the $a_{i,j}$ satisfy the uniform ellipticity condition

$$\exists \lambda_{\mathcal{A}} > 0 : \ \lambda_{\mathcal{A}} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i \xi_j \quad \textit{for all } \xi \in \mathbb{R}^n \ \textit{ and a.a. } x \in \Omega.$$

(ii) *The functions* $f : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ *and*

$$L(x,t,y,u) := L_0(x,t,y) + g(x,t)u : Q \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

are Carathéodory function of class C^2 *with respect to the second variable. In addition,*

$$\begin{cases} f(\cdot,\cdot,0)\in L^\infty(Q) \text{ and } \dfrac{\partial f}{\partial y}(s,x,y) \geq 0 \ \forall y\in\mathbb{R}, \ \ L_0(\cdot,\cdot,0)\in L^1(Q),\ g\in L^\infty(Q), \\ \forall M>0\ \exists C_M, L_M, \varepsilon>0 \text{ such that } \forall\, |u|,|y|,|y_1|,|y_2| \leq M \text{ with } |y_2-y_1| \leq \varepsilon, \\ \Big|\dfrac{\partial f}{\partial y}(s,x,y)\Big| + \Big|\dfrac{\partial^2 f}{\partial y^2}(s,x,y)\Big| \leq C_M, \\ \Big|\dfrac{\partial L}{\partial y}(s,x,y,u)\Big| + \Big|\dfrac{\partial^2 L}{\partial y^2}(x,y,u)\Big| \leq C_M, \\ \Big|\dfrac{\partial^2 f}{\partial y^2}(s,x,y_2) - \dfrac{\partial^2 f}{\partial y^2}(s,x,y_1)\Big| \leq L_M|y_2-y_1|, \\ \Big|\dfrac{\partial^2 L}{\partial y^2}(s,x,y_2,u) - \dfrac{\partial^2 L}{\partial y^2}(s,x,y_1,u)\Big| \leq L_M|y_2-y_1| \end{cases}$$

for almost every $(s,x) \in Q$.

As usual, the Hilbert space $W(0,T)$ consists of all of functions in $L^2(0,T;\, H_0^1(\Omega))$ that have a distributional derivative in $L^2(0,T;\, H^{-1}(\Omega))$, which is endowed with the norm

$$\|y\|_{W(0,T)} := \|y\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial y/\partial t\|_{L^2(0,T;H^{-1}(\Omega))}.$$

By definition, $y \in L^\infty(Q) \cap W(0,T)$ is a solution of (28) if

$$\int_Q \Big(\frac{\mathrm{d}}{\mathrm{d}t}\, y + \mathcal{A}y\Big)\psi \,\mathrm{d}x\mathrm{d}t = \int_Q (-f(\cdot,y)+u)\psi \,\mathrm{d}x\mathrm{d}t, \text{ for all } \psi \in L^2(0,T,H_0^1(\Omega)).$$

Below we formulate as propositions some known facts that will be used in the formulation of the main result in this sub-section. Further on, r will be a fixed number satisfying $r > \max\{2, 1+n/2\}$.

Proposition 6.2. ([11, 12]) *For every* $u \in L^r(Q)$ *there exists a unique solution* y_u *of* (28). *Moreover, if* $u_k \rightharpoonup u$ *weakly in* $L^r(Q)$, *then*

$$\|y_{u_k} - y_u\|_{L^\infty(Q)} + \|y_{u_k} - y_u\|_{L^2(0,T;H_0^1(\Omega))} \to 0.$$

This propositions implies, in particular, that a globally optimal solution of the considered problem does exist in $L^r(Q)$, see e.g. [56, Theorem 5.7], hence in $L^\infty(Q)$, due to the control constraints.

We denote the control-to-state operator $G_r : L^r(Q) \to W(0,T) \cap L^\infty(Q)$, assigning to each u, the unique state y_u, i.e., $G_r(v) := y_u$.

Proposition 6.3. *The control-to-state operator is of class* C^2 *and for all functions* $u, v, w \in L^r(Q)$, *it holds that* $z_{u,v} := G_r'(u)v$ *is the solution of*

$$\begin{cases} \Big(\dfrac{\partial}{\partial t} + \mathcal{A}\Big) z + f_y(x,t,y_u) z = v \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \ \ z(\cdot,0) = 0 \text{ on } \Omega, \end{cases} \tag{29}$$

and $\omega_{u,(v,w)} := G_r''(u)(v,w)$ *is the solution of*

$$\begin{cases} \Big(\frac{\partial}{\partial t} + \mathcal{A}\Big)z + f_y(x,t,y_u)z = -f_{yy}(x,t,y_u)z_{u,v}z_{u,w} \text{ in } Q, \\ z = 0 \text{ on } \Sigma,\ z(\cdot,0) = 0 \text{ on } \Omega. \end{cases} \tag{30}$$

Proposition 6.4. *The functional* $J : L^r(Q) \longrightarrow \mathbb{R}$ *is of class* C^2*. Moreover, given* $u, v, v_1, v_2 \in L^r(Q)$ *we have*

$$J'(u)v = \int_Q \Big(\frac{dL_0}{dy}(x,t,y_u)z_{u,v} + gv\Big)\,\mathrm{d}x\mathrm{d}t = \int_Q (p_u + g)v\,\mathrm{d}x\mathrm{d}t,$$

$$J''(u)(v_1,v_2) = \int_Q \Big(\frac{\partial^2 L_0}{\partial y^2}(x,t,y_u,u) - p_u\frac{\partial^2 f}{\partial y^2}(x,t,y_u)\Big)z_{u,v_1}z_{u,v_2}\,\mathrm{d}x\mathrm{d}t.$$

Here, $p_u \in W(0,T) \cap C(\bar{Q})$ *is the unique solution of the adjoint equation*

$$\begin{cases} \Big(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}^*\Big)p + \frac{\partial f}{\partial y}(x,t,y_u)p = \frac{\partial L}{\partial y}(x,t,y_u,u) \textit{ in } Q, \\ p = 0 \textit{ on } \Sigma,\ p(\cdot,T) = 0 \textit{ on } \Omega. \end{cases}$$

We introduce the Hamiltonian $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (x,t,y,p,u) \mapsto H(x,t,y,p,u) \in \mathbb{R}$, associated with the problem in the usual way:

$$H(x,t,y,p,u) := L(x,t,y,u) + p(u - f(x,t,y)).$$

Proposition 6.5. *If* $\bar{u}$ *is a weak local minimizer for problem* (27)–(28)*, then there exist unique elements* $\bar{y}, \bar{p} \in W(0,T) \cap L^\infty(Q)$ *such that*

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{A})\bar{y} + f(x,t,\bar{y}) = \bar{u} \text{ in } Q, \\ \bar{y} = 0 \text{ on } \Sigma,\ \bar{y}(\cdot,0) = y_0 \text{ on } \Omega. \end{cases} \tag{31}$$

$$\begin{cases} \Big(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}^*\Big)p = \frac{\partial H}{\partial y}(x,t,\bar{y},\bar{p},\bar{u}) \text{ in } Q, \\ \bar{p} = 0 \textit{ on } \Sigma,\ \bar{p}(\cdot,T) = 0 \textit{ on } \Omega. \end{cases} \tag{32}$$

$$\int_Q \frac{\partial H}{\partial u}(x,t,\bar{y},\bar{p},\bar{u})(u - \bar{u})\,\mathrm{d}x\mathrm{d}t \geq 0 \quad \forall u \in \mathcal{U}. \tag{33}$$

Having the optimality system (31)–(33), we define in the next paragraphs the optimality mapping corresponding to problem (27)–(28). Given the initial data y_0 in (28), we introduce the set

$$D(\mathcal{L}) := \Big\{y \in W(0,T) \cap L^\infty(Q) \Big| \Big(\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}\Big)y \in L^r(Q), y(\cdot,0) = y_0\Big\}. \tag{34}$$

Let us define the operator $\mathcal{L} := \frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A} : D(\mathcal{L}) \to L^r(Q)$. Here, the expression $\Big(\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}\Big)y \in L^r(Q)$ means that there exists a function $\phi \in L^r(Q)$ such that

$$\int_Q \Big(\frac{\mathrm{d}}{\mathrm{d}t}y + \mathcal{A}y\Big)\psi \;\mathrm{d}x \;\mathrm{d}t = \int_Q \phi\psi \;\mathrm{d}x \;\mathrm{d}t, \text{ for all } \psi \in L^2(0,T,H_0^1(\Omega)).$$

Of course, this is satisfied with $\phi = u - f$ if y solves the PDE (28). To address the adjoint equation, we define the mapping $\mathcal{L}^* : D(\mathcal{L}^*) \to L^r(Q)$, $\mathcal{L}^* := (-\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}^*)$,

where $\quad D(\mathcal{L}^*) := \Big\{ p \in W(0,T) \cap L^\infty(Q) \Big| \Big(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathcal{A}^* \Big) p \in L^r(Q), p(\cdot,T) = 0 \Big\}.$

Using the mappings $\mathcal{L}$ and $\mathcal{L}^*$, the state equation (28) and the adjoint equation (32) can be written in a short form:

$$\mathcal{L}y = u - f(\cdot, y), \qquad \mathcal{L}^* p = \frac{\partial H}{\partial y}(\cdot, y_u, p, u).$$

We remind the definition of normal cone to the set $\mathcal{U}$ at $u \in L^1(Q)$:

$$N_{\mathcal{U}}(u) := \begin{cases} \left\{ \nu \in L^\infty(Q) \,\middle|\, \int_Q \nu(v-u)\,\mathrm{d}x\mathrm{d}t \le 0 \quad \forall v \in \mathcal{U} \right\} & \text{if } u \in \mathcal{U}, \\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases}$$

Let C be a fixed positive constant. Denote by $\mathbb{B}^r_C$ the closed ball of radius C in the space $L^r(Q)$, centered at the origin. Define the metric spaces

$$\mathcal{X} := D(\mathcal{L}) \times D(\mathcal{L}^*) \times \mathcal{U} \quad \text{and} \quad \mathcal{Y} := \mathbb{B}^r_C \times \mathbb{B}^r_C \times L^\infty(Q), \tag{35}$$

where the elements of the space $\mathcal{X}$ are triples $\psi := (y, p, u)$, while the elements of $\mathcal{Y}$ are denoted by $\zeta := (\xi, \eta, \rho)$. The metrics in these spaces are defined as follows:

$$\begin{aligned} d_{\mathcal{X}}(\psi_1, \psi_2) &:= \|y_1 - y_2\|_{L^2(Q)} + \|p_1 - p_2\|_{L^2(Q)} + \|u_1 - u_2\|_{L^1(Q)}, \\ d_{\mathcal{Y}}(\zeta_1, \zeta_2) &:= \|\xi_1 - \xi_2\|_{L^2(Q)} + \|\eta_1 - \eta_2\|_{L^2(Q)} + \|\rho_1 - \rho_2\|_{L^\infty(Q)}. \end{aligned} \tag{36}$$

Notice that the norms used in the space $\mathcal{Y}$ are weaker than the norm in which the ball $\mathbb{B}^r_C$ is taken, because $r > 2$ for $n \in \{2, 3\}$.

The first order necessary optimality condition for problem (27)–(28) in Proposition 6.5 can be rewritten as the inclusion

$$0 \in \mathcal{F}(y, p, u) \quad \text{with} \quad \mathcal{F}(y, p, u) := \begin{pmatrix} \mathcal{L}y + f(\cdot, y) - u \\ \mathcal{L}^* p - \frac{\partial H}{\partial y}(\cdot, y, p, u) \\ \frac{\partial H}{\partial u}(\cdot, y, p, u) + N_{\mathcal{U}}(u) \end{pmatrix}. \tag{37}$$

To study the SMsR property of the optimality mapping we consider the elements $(\xi, \eta, \rho) \in \mathcal{Y}$ as disturbances in the optimality system: $(\xi, \eta, \rho) \in \mathcal{F}(y, p, u)$.

Let $(\bar{y}, \bar{p}, \bar{u}) \in \mathcal{X}$ be a reference solution of the inclusion (37). The next assumption is crucial for obtaining the SMsR property of the optimality mapping $\mathcal{F}$. It was proved in [17] that it is also a sufficient condition for weak local optimality in L^1 of the reference control $\bar{u}$.

Assumption 6.6. ([17]) *There exist positive constants c, α and $\gamma \in (4/6, 1]$ for $n \in \{1, 2\}$ and $\gamma \in (5/6, 1]$ for $n = 3$ such that*

$$J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2 \ge c\|u - \bar{u}\|^{1+1/\gamma}_{L^1(Q)} \tag{38}$$

for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(Q)} < \alpha$.

Our main result is the following theorem in which the statement for the case $\gamma = 1$ was proven in [17]. The statement for $\gamma \in (4/6, 1]$ for $n \in \{1, 2\}$ and $\gamma \in (5/6, 1]$ for $n = 3$ follows by straight forward adaptations.

Theorem 6.7. ([17]) *Let Assumption* 6.1 *be satisfied and let Assumption* 6.6 *be fulfilled for the reference solution* $\bar\psi = (y_{\bar u}, p_{\bar u}, \bar u)$ *of* $0 \in \mathcal{F}(\psi)$. *Then the mapping* $\mathcal{F}$ *is strongly metrically Hölder sub-regular at* $(\bar\psi, 0)$. *More precisely, for every* $\varepsilon \in (0, 1/2]$ *there exist positive constants* α_n *and* κ_n, $n = 1, 2, 3$, *(with* α_1 *and* κ_1 *independent of* ε*) such that for all* $\psi = (y, p, u) \in \mathcal{X}$ *with* $\|u - \bar u\|_{L^1(Q)} \le \alpha_n$ *and* $\zeta = (\xi, \eta, \rho) \in \mathcal{Y}$ *satisfying* $\zeta \in \mathcal{F}(\psi)$, *the following inequalities are satisfied.*

$$\|u - \bar u\|_{L^1(Q)} \le \kappa_n \Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\gamma\theta_0},$$

$$\|y_u^\zeta - y_{\bar u}\|_{L^2(Q)} + \|p_u^\zeta - p_{\bar u}\|_{L^2(Q)} \le \kappa_n \Big(\|\rho\|_{L^\infty(Q)} + \|\xi\|_{L^2(Q)} + \|\eta\|_{L^2(Q)} \Big)^{\gamma\theta},$$

where

$$\begin{aligned} \theta_0 = \theta &= 1 && \textit{if } n = 1, \\ \theta_0 = \theta &= 1 - \varepsilon && \textit{if } n = 2, \\ \theta_0 = \frac{10}{11} - \varepsilon,\ \ \theta &= \frac{9}{11} - \varepsilon && \textit{if } n = 3. \end{aligned}$$

Additional results in the spirit of Theorem 6.7 can be found in [17].

Additional remarks and references

Results in the spirit of Theorem 6.7 were established in [16] for affine optimal control problems subject to elliptic PDEs with Robin-type boundary conditions. Similar results were shown under the designation of solution stability in [10] considering several distinct assumptions on the joint growth of the first and second variation of the objective functional which are, in the elliptic case, weaker than Assumption 6.6.

In [18] the Strong Metric Hölder Subregularity of the optimality mapping was investigated for an optimal control problem governed by a semilinear parabolic PDE as in the discussion above, but with the additional constraint that the controls have a fixed state-distribution. This allowed the authors, due to the stronger a-priori estimates for the involved PDEs, to improve the estimations of Theorem 6.7 substantially.

The approach introduced in [17] relies on $L^s - L^1$ type estimates for the linearized PDEs allowing the estimations that are key in the proof of [17, Lemma 11]. Motivated by this result, the notion of Strong Metric Hölder Subregularity was further investigated for optimal control problems constrained by the 2-dimensional Navier-Stokes equation in [15] and for the Boussinesq system in [41]. Regarding Assumption 6.6, it is important to notice, that it can be satisfied only by bang-bang optimal controls. Still in [15], it was shown that for stability under perturbations in the normal cone, and thus SMHsR, it is necessary to have a growth which is, depending on the sign of the second variation, either close to Assumption 6.6 or agrees with it. This highlights the generality of Assumption 6.6 in the bang-bang situation, which is also supported by the results in [21] which state that the occurrence of bang-bang optimal controls is generic in affine optimal control problems. Subsequently, in the case of a linear quadratic control problem that possesses a nonnegative second variation, it was shown in [19], that for those problems, the notion of SMsR is not only equivalent to a growth of the objective functional as in Assumption 6.6 and discussed in [15], but also to a Polyak-Lojasiewicz-type inequality. The implication of Assumption 6.6 for the investigation of error estimates for the numerical solution was evidenced in [40] where it was shown that for an optimal control problem subject to a semilinear elliptic PDE, Assumption 6.6 is sufficient for error estimates for the numerical approximation generated by a Finite Element Method (FEM) with piecewise constant controls, and that the notion of Strong Metric Hölder Subregularity can be applied for the investigation of error estimates for the numerical approximation generated by a FEM with a variational discretization.

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