Journal of Convex Analysis
Volume 32 (2025), No. 2, 447–466

Functions on a Convex Set which are Both ω-Semiconvex and ω-Semiconcave II

Václav Kryštof
Charles University, Faculty of Mathematics and Physics, Praha, Karlín, Czech Republic
vaaclav.krystof@gmail.com

Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

Received: March 14, 2024
Accepted: May 7, 2024

In a recent article [*Functions on a convex set which are both ω-semiconvex and ω-semiconcave I*, J. Convex Analysis 29 (2022) 837–856] we proved with L. Zajíček that if $G \subset \mathbb{R}^n$ is an unbounded open convex set that does not contain a translation of a convex cone with non-empty interior, then there exist $f : G \to \mathbb{R}$ and a concave modulus ω such that $\lim_{t\to\infty} \omega(t) = \infty$, f is both semiconvex and semiconcave with modulus ω and $f \notin C^{1,\omega}(G)$. Here we improve the previous result as follows: If G is as above and $\omega(t) = t^\alpha$ for some $\alpha \in (0, 1)$, then there exists $f : G \to \mathbb{R}$ that is both semiconvex and semiconcave with modulus ω and $f \notin C^{1,\alpha}(G)$. This result has immediate consequences concerning a first-order quantitative converse Taylor theorem and the problem whether $f \in C^{1,\alpha}(G)$ whenever f is smooth in a corresponding sense on all lines.

Keywords: Semiconvex function with general modulus, semiconcave function with general modulus, $C^{1,\alpha}$ function, $C^{1,\omega}$ function, unbounded open convex set.

2010 Mathematics Subject Classification: 26B25.

1. Introduction

If $G \subset \mathbb{R}^n$ is an open convex set, then $f : G \to \mathbb{R}$ is said to be (classically) semiconvex if there exists $C \geq 0$ such that the function

$$x \mapsto f(x) + C|x|^2, \quad x \in G,$$

is convex. We say that $f : G \to \mathbb{R}$ is (classically) semiconcave if $-f$ is semiconvex. Then it is well-known (see [5, Section 3] for references) that

$$f : G \to \mathbb{R} \text{ is both semiconvex and semiconcave } \iff f \in C^{1,1}(G). \tag{1}$$

The notion of semiconvex, resp. semiconcave, functions is generalized to the notion of funtions that are semiconvex, resp. semiconcave, with modulus ω (see [1, Definition 2.1.1]). We will call these functions ω-semiconvex, resp. ω-semiconcave, for short (see Definition 2.1 and Definition 2.3 below). Note that a function is semiconvex, resp. semiconcave, if and only if it is ω-semiconvex, resp. ω-semiconcave, with $\omega(t) = Ct$ for some $C \geq 0$ (see [1, Proposition 1.1.3]). For the theory and application of ω-semiconvex, resp. ω-semiconcave, functions, see the monograph [1].

This article deals with functions that are both ω-semiconvex and ω-semiconcave. Let $G \subset \mathbb{R}^n$ be an open convex set and ω a modulus. It is easy to show that if

ISSN 0944-6532 / \$ 2.50

$f \in C^{1,\omega}(G)$, then there exists $C > 0$ such that f is both $(C\omega)$-semiconvex and $(C\omega)$-semiconcave (see [1, Proposition 2.1.2]). So we will consider the natural problem whether also the opposite implication holds. This implication can be equivalently formulated as follows:

$$f : G \to \mathbb{R} \text{ is both } \omega\text{-semiconvex and } \omega\text{-semiconcave} \Longrightarrow f \in C^{1,\omega}(G). \tag{2}$$

Recall (see (1)) that implication (2) holds if $\omega(t) = Ct$ for some $C \geq 0$. Further, by Remark 2.4 (i) below and [5, p. 839], implication (2) holds if either G is bounded or contains a translation of a convex cone with non-empty interior. However, (2) doesn't hold in general. Namely, we proved (see [5, Theorem 1.9]) that if $G \subset \mathbb{R}^n$ is an unbounded open convex set which does not contain a translation of a convex cone with non-empty interior, then there exists a concave modulus ω such that $\lim_{t\to\infty} \omega(t) = \infty$ and (2) doesn't hold. The main disadvantage of [5, Theorem 1.9] is that ω depends on G and may be irregular. In this article we improve [5, Theorem 1.9] in the following way:

Theorem 1.1. *Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an unbounded open convex set that doesn't contain a translation of a cone with non-empty interior. Let $\alpha \in (0, 1)$ and set $\omega(t) := t^\alpha$ for all $t \geq 0$. Then there exists $f : G \to \mathbb{R}$ that is both ω-semiconvex and ω-semiconcave and $f \notin C^{1,\alpha}(G)$.*

So Theorem 1.1 gives a positive answer to [5, Question 6.10 (3), (4)]. By (31), Theorem 1.1 is a special case of Theorem 5.5 below which works with more general moduli (namely those with property (*) below).

The strategy of the proof of Theorem 5.5 is similar to that used in [5] where we reduced (using the notion of recessive cones) the case of an open convex subset of $\mathbb{R}^n$ to the case of an open convex subset of $\mathbb{R}^2$ of the special form $G_\eta := \{(x, y) \in \mathbb{R}^2 : x > 0,\ |y| < \eta(x)\}$ where $\eta : (0, \infty) \to (0, \infty)$ is concave and satisfies $\lim_{t\to\infty} \eta(t)/t = 0$. However, the construction of $f : G_\eta \to \mathbb{R}$ in the proof of Proposition 4.4 is much more sophisticated than the corresponding construction used in [5]. The present construction is based on the notion of modulus ω_η (see Definition 3.1 below and for the motivation Remark 3.2 below).

It's worth mentioning that if ω is a modulus, then functions that are both $(C\omega)$-semiconvex and $(C\omega)$-semiconcave for some $C \geq 0$ can be characterized via "Taylor approximation of the first order" or via "$C^{1,\omega}$-smoothness on all lines". See Lemma 2.5 below for a precise formulation. Thus Theorem 1.1 can be reformulated as follows:

Corollary 1.2. *Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an unbounded open convex set that doesn't contain a translation of a cone with non-empty interior. Let $\alpha \in (0, 1)$ and set $\omega(t) := t^\alpha$ for all $t \geq 0$. Then there exists $f : G \to \mathbb{R}$ that is Fréchet differentiable,*

(i) *for every $a \in G$ and $v \in \mathbb{R}^n$, $|v| = 1$, the derivative of the function $t \mapsto f(a+tv)$ is uniformly continuous on $\{t \in \mathbb{R} : a + tv \in G\}$ with modulus ω,*

(ii) *$|f(x + h) - f(x) - f'(x)[h]| \leq |h|^{1+\alpha}$ for all $x, x + h \in G$ and*

(iii) *$f \notin C^{1,\alpha}(G)$.*

Let us remark that Corollary 1.2 improves [5, Theorem 6.8]. Note also that Theorem 5.5 can be reformulated (using Lemma 2.5) in the same way as Theorem 1.1 and so we can get a more general statement than Corollary 1.2.

2. Preliminaries

Throughout this article, the norm on $\mathbb{R}^n$ is always Euclidean. For $A \subset \mathbb{R}^n$ the symbol $\operatorname{span}(A)$ denotes the linear span of A. By a cone in $\mathbb{R}^n$ we mean a set $C \subset \mathbb{R}^n$ such that $\lambda x \in C$ for each $x \in C$ and $\lambda > 0$. If $G \subset \mathbb{R}^n$ is an open set, $f : G \to \mathbb{R}$, $x \in G$ and $h \in \mathbb{R}^n$, then $f'(x)$ denotes the Fréchet derivative of f at x and $f'(x)[h]$ denotes its evaluation at h.

Definition 2.1. We denote by $\mathcal{M}$ the set of all $\omega : [0, \infty) \to [0, \infty)$ which are non-decreasing and satisfy $\lim_{h \to 0+} \omega(h) = 0$. The members of $\mathcal{M}$ are called *moduli*.

Definition 2.2. Let $G \subset \mathbb{R}^n$ be an open set, $\omega \in \mathcal{M}$ and $\alpha \in (0, 1]$. Then we denote by $C^{1,\omega}(G)$ the set of all Fréchet differentiable $f : G \to \mathbb{R}$ such that f' is uniformly continuous with modulus $C\omega$ for some $C > 0$. If $\omega(t) = t^\alpha$ for every $t \geq 0$, then we also write $C^{1,\alpha}(G)$ instead of $C^{1,\omega}(G)$.

Definition 2.3. Let $G \subset \mathbb{R}^n$ be an open convex set and $\omega \in \mathcal{M}$.

- We say that $f : G \to \mathbb{R}$ is *semiconvex with modulus* ω (or ω-semiconvex for short) if, for every $x, y \in G$ and $\lambda \in [0, 1]$,
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + \lambda(1 - \lambda)|x - y|\omega(|x - y|).$$
- We say that $f : G \to \mathbb{R}$ is *semiconcave with modulus* ω (or ω-semiconcave) if $-f$ is semiconvex with modulus ω.

Remark 2.4. Let $G \subset \mathbb{R}^n$ be an open convex set, $f : G \to \mathbb{R}$ and $\omega \in \mathcal{M}$. Then the following hold (see [1, Theorem 2.1.7] and [1, Theorem 3.3.7]):

(i) If f is ω-semiconvex or ω-semiconcave, then f is continuous.

(ii) If f is both ω-semiconvex and ω-semiconcave, then $f \in C^1(G)$.

As already mentioned in the introduction, we have the following:

Lemma 2.5. *Let $G \subset \mathbb{R}^n$ be an open convex set, $f : G \to \mathbb{R}$ and $\omega \in \mathcal{M}$. Then the following are equivalent:*

(i) *There exists $C_1 > 0$ such that f is both $(C_1\omega)$-semiconvex and $(C_1\omega)$-semiconcave.*

(ii) *There exists $C_2 > 0$ such that for every $a \in G$ and $v \in \mathbb{R}^n$, $|v| = 1$, the function $t \mapsto f(a + tv)$ is differentiable on $\{t \in \mathbb{R} : a + tv \in G\}$ and its derivative is uniformly continuous with modulus $C_2\omega$.*

(iii) *f is Fréchet differentiable and there exists $C_3 > 0$ such that*
$$|f(x + h) - f(x) - f'(x)[h]| \leq C_3|h|\omega(|h|), \quad x, x + h \in G.$$

Proof. It follows from Remark 2.4 (i) and [5, Proposition 6.4 (i), (iv)] that (i) implies (ii) and (iii). Further, it follows easily from [4, Lemma 2.3 (i)] and [4, Proposition 2.4 (ii)] that (ii) implies (i). Finally, (iii) implies (i) by [5, Proposition 6.4 (ii)]. □

Recall that $\omega \in \mathcal{M}$ is called *sub-additive* if $\omega(x + h) \leq \omega(x) + \omega(h)$ for all $x, h \geq 0$.

Remark 2.6. Let $\omega \in \mathcal{M}$. Then the following hold:

(i) If ω is concave, then it is sub-additive.

(ii) If ω is sub-additive, then there exists a concave $\varphi \in \mathcal{M}$ such that $\omega \le \varphi \le 2\omega$.

Both of these statements are well-known. The proof of (i) is easy. Statement (ii) is due to Stechkin (see [2, Lemma 2.3] for references).

Remark 2.7. Let $f : (0, \infty) \to [0, \infty)$ be concave. Then the following hold:

(i) f is non-decreasing.

(ii) The function $t \mapsto f(t)/t$, $t > 0$, is non-increasing.

(iii) For every $t > 0$ and $c \ge 1$ we have $f(ct) \le cf(t)$.

All these facts are easy and well-known. Note that (iii) is just a reformulation of (ii). However, a useful one.

As in [5] we will use the following auxiliary notation:

Definition 2.8. Let $n \in \mathbb{N}$ and $\omega \in \mathcal{M}$. Then we denote by $\mathcal{G}_n(\omega)$ the set of all open convex sets $G \subset \mathbb{R}^n$ for which there exist $a \in G$, $x \in \mathbb{R}^n \setminus \{0\}$ and $f \in C^1(G)$ such that f is both ω-semiconvex and ω-semiconcave, $M := \{a + \lambda x : \lambda \ge 0\} \subset G$ and f' is uniformly continuous on M with modulus $C\omega$ for no $C > 0$.

Remark 2.9. Let $n \in \mathbb{N}$ and $\omega \in \mathcal{M}$. Then:

(i) By Remark 2.4 (ii) we can equivalently write $f : G \to \mathbb{R}$ instead of $f \in C^1(G)$ in the definition of $\mathcal{G}_n(\omega)$.

(ii) If $G \in \mathcal{G}_n(\omega)$, then (2) clearly doesn't hold.

(iii) If $b \in \mathbb{R}^n$ and $G + b \in \mathcal{G}_n(\omega)$, then it is easy to check that $G \in \mathcal{G}_n(\omega)$.

For the proof of the following lemma see [5, Example 5.1].

Lemma 2.10. *Let $\omega \in \mathcal{M}$ satisfy* $\liminf_{h \to 0+} \frac{\omega(h)}{h} > 0$ *and* $\lim_{h \to \infty} \frac{\omega(h)}{h} = 0$. *Then* $\mathbb{R} \times (0, 1) \in \mathcal{G}_2(\omega)$.

As in [5], we will work with recession cones of unbounded convex sets $A \subset \mathbb{R}^n$ and use the notation $\operatorname{rec}(A)$ (instead of 0^+A used in [6]).

Definition 2.11. Let $\emptyset \ne A \subset \mathbb{R}^n$ be a convex set. Then we set

$$\operatorname{rec}(A) := \{x \in \mathbb{R}^n : (\forall a \in A)(\forall \lambda \ge 0)\ a + \lambda x \in A\}.$$

Now we recall some basic properties of recession cones. For the proof see [5, Lemma 2.10].

Lemma 2.12. *Let $n \in \mathbb{N}$ and let $\emptyset \ne G \subset \mathbb{R}^n$ be an open convex set. Then the following hold:*

(i) $\operatorname{rec}(G) = \operatorname{rec}(\overline{G})$.

(ii) $\operatorname{rec}(G)$ *is a closed convex cone and* $0 \in \operatorname{rec}(G)$.

(iii) $\operatorname{rec}(G) = \{x \in \mathbb{R}^n : (\exists a \in A)(\forall \lambda \ge 0)\ a + \lambda x \in A\}$.

(iv) *If $G \subset \mathbb{R}^n$ is unbounded, then* $1 \le \dim(\operatorname{span}(\operatorname{rec}(G)))$.

(v) $\dim(\operatorname{span}(\operatorname{rec}(G))) = n$ *if and only if G contains a translation of a cone with non-empty interior.*

We will also need the following special case of [5, Lemma 5.11]:

Lemma 2.13. *Let $m \in \mathbb{N}$, $m \geq 2$, and $\omega \in \mathcal{M}$. Let $G \subset \mathbb{R}^m$ be an open convex set and $L : \mathbb{R}^m \to \mathbb{R}^2$ a linear surjection. Suppose that ω is concave, $L(G) \in \mathcal{G}_2(\omega)$ and* $\operatorname{rec}(L(G)) \subset L(\operatorname{rec}(G))$*. Then $G \in \mathcal{G}_m(\omega)$.*

The following lemma is an easy consequence of [6, Theorem 9.1].

Lemma 2.14. *Let $m, n \in \mathbb{N}$, let $G \subset \mathbb{R}^n$ be an unbounded open convex set and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear surjection such that*

$$\operatorname{rec}(G) \cap L^{-1}(\{0\}) \subset \operatorname{rec}(G) \cap (-\operatorname{rec}(G)).$$

Then $\operatorname{rec}(L(G)) = L(\operatorname{rec}(G))$.

Proof. By Lemma 2.12 (i) we have $\operatorname{rec}(G) = \operatorname{rec}(\overline{G})$. So $\operatorname{rec}(\overline{G}) \cap L^{-1}(\{0\}) \subset \operatorname{rec}(\overline{G}) \cap (-\operatorname{rec}(\overline{G}))$ and thus [6, Theorem 9.1] (applied to $C := \overline{G}$) implies that $L(\overline{G})$ is closed and $\operatorname{rec}(L(\overline{G})) = L(\operatorname{rec}(\overline{G}))$. Hence we easily obtain that $L(\overline{G}) = \overline{L(G)}$. Therefore, $\operatorname{rec}(L(\overline{G})) = \operatorname{rec}(L(G))$ by Lemma 2.12 (i) and the equality $\operatorname{rec}(L(G)) = L(\operatorname{rec}(G))$ follows. $\square$

3. Modulus ω_η

One of the main ingredients of this article is the introduction of modulus ω_η:

Definition 3.1. Let $\eta : (0, \infty) \to (0, \infty)$ and $\omega \in \mathcal{M}$. Then for every $h > 0$ we set

$$\omega_\eta(h) := \inf \Big\{ \sum_{j=1}^{n} \max \Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\} \omega(x_j - x_{j-1}) \Big\}, \tag{3}$$

where the infimum is taken over all partitions $0 = x_0 < x_1 < \dots < x_n = h$ of $[0, h]$. Further we set $\omega_\eta(0) := 0$.

The fact that ω_η is indeed a modulus will be proved in Lemma 3.4 (i) below.

Remark 3.2. The original motivation for ω_η was that the following statement should hold:

- Let $\eta : (0, \infty) \to (0, \infty)$ be concave, let $\omega \in \mathcal{M}$ and set

$$G := \{(x, y) \in \mathbb{R}^2 : x > 0,\ |y| < \eta(x)\}.$$

  Let $f : G \to \mathbb{R}$ be both ω-semiconvex and ω-semiconcave. Then $f \in C^{1,\omega_\eta}(G)$.

However, we do not need this statement and therefore not prove it. Note also that ω_η is "minimal" in the sense of Proposition 4.4 below which we use substantially at the end of the article.

Remark 3.3. Let $\eta : (0, \infty) \to (0, \infty)$ and $\omega \in \mathcal{M}$. Then

$$\omega_\eta(h) \leq \max \Big\{1, \frac{h}{\eta(h)}\Big\}\, \omega(h), \quad h > 0.$$

Indeed, taking $n = 1$ in the sum in (3), we immediately obtain the estimate.

Lemma 3.4. *Let $\eta : (0,\infty) \to (0,\infty)$ be concave and let $\omega \in \mathcal{M}$. Then the following hold:*

(i) $\omega_\eta \in \mathcal{M}$.

(ii) *For every $x, h > 0$ we have $\omega_\eta(x+h) \le \omega_\eta(x) + \omega_\eta(h)$.*

Proof. (i): Obviously $\omega_\eta(h) \ge 0$ for every $h \ge 0$. By Remark 3.3 and Remark 2.7 (ii), we have

$$\omega_\eta(h) \le \max\Big\{1, \frac{h}{\eta(h)}\Big\}\omega(h) \le \max\Big\{1, \frac{1}{\eta(1)}\Big\}\omega(h), \quad h \in (0,1),$$

and thus $\lim_{h\to 0+} \omega_\eta(h) = \omega_\eta(0) = 0$. To prove that ω_η is non-decreasing, it is sufficient to show

$$\omega_\eta(x) \le \omega_\eta(x+h), \quad x, h > 0. \tag{4}$$

Let $x, h > 0$ and let $0 = x_0 < x_1 < \cdots < x_n = x+h$ be a partition of $[0, x+h]$. Then there exists $m \in \mathbb{N}$ such that $m \le n$ and $x_{m-1} < x \le x_m$. According to Remark 2.7 (i), (ii), we have

$$\frac{x - x_{m-1}}{\eta(x)} = \frac{x}{\eta(x)} - \frac{x_{m-1}}{\eta(x)} \le \frac{x_m}{\eta(x_m)} - \frac{x_{m-1}}{\eta(x_m)} = \frac{x_m - x_{m-1}}{\eta(x_m)}$$

and thus

$$\begin{aligned}
\omega_\eta(x) &\le \max\Big\{1, \frac{x - x_{m-1}}{\eta(x)}\Big\}\omega(x - x_{m-1}) + \sum_{j=1}^{m-1} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) \\
&\le \sum_{j=1}^{m} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) \le \sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}).
\end{aligned}$$

Hence (4) holds.

(ii): Let $x, h > 0$ and $\varepsilon > 0$. Then there exists a partition $0 = x_0 < x_1 < \cdots < x_n = x$ of $[0, x]$ such that

$$\sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\,\omega(x_j - x_{j-1}) \le \omega_\eta(x) + \frac{\varepsilon}{2},$$

and a partition $0 = y_0 < y_1 < \cdots < y_m = h$ of $[0, h]$ such that

$$\sum_{i=1}^{m} \max\Big\{1, \frac{y_i - y_{i-1}}{\eta(y_i)}\Big\}\,\omega(y_i - y_{i-1}) \le \omega_\eta(h) + \frac{\varepsilon}{2}.$$

Hence, for $x_j := x + y_{j-n}$ $(j = n+1, \ldots, n+m)$ we obtain using Remark 2.7 (i)

$$\begin{aligned}
\omega_\eta(x+h) &\le \sum_{j=1}^{n+m} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) + \sum_{i=1}^{m} \max\Big\{1, \frac{y_i - y_{i-1}}{\eta(x + y_i)}\Big\}\omega(y_i - y_{i-1}) \\
&\le \sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) + \sum_{i=1}^{m} \max\Big\{1, \frac{y_i - y_{i-1}}{\eta(y_i)}\Big\}\omega(y_i - y_{i-1}) \\
&\le \omega_\eta(x) + \omega_\eta(h) + \varepsilon.
\end{aligned}$$

Thus (ii) holds. $\square$

In the rest of this section we will prove several properties of ω_η for further use.

Lemma 3.5. *Let $\eta : (0,\infty) \to (0,\infty)$ be non-decreasing and let $\omega \in \mathcal{M}$ be concave. Then*

$$\frac{h\omega(\eta(h))}{\eta(h)} \le \omega_\eta(h), \quad h > 0. \tag{5}$$

Proof. Let $h > 0$ and $\varepsilon > 0$.
Then there exists a partition $0 = x_0 < x_1 < \cdots < x_n = h$ of $[0,h]$ such that

$$\sum_{j=1}^{n} \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\} \omega(x_j - x_{j-1}) \le \omega_\eta(h) + \varepsilon.$$

Set
$$J_1 := \{j \in \{1,\dots,n\} : \eta(x_j) \le x_j - x_{j-1}\},$$
and
$$J_2 := \{j \in \{1,\dots,n\} : x_j - x_{j-1} < \eta(x_j)\}.$$

Using Remark 2.7 (ii) we obtain

$$\begin{aligned}
&(x_j - x_{j-1})\frac{\omega(\eta(h))}{\eta(h)} \le (x_j - x_{j-1})\frac{\omega(\eta(x_j))}{\eta(x_j)} \\
&\le \frac{x_j - x_{j-1}}{\eta(x_j)}\omega(x_j - x_{j-1}) \le \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\}\omega(x_j - x_{j-1}), \quad j \in J_1,
\end{aligned}$$

and
$$\begin{aligned}
&(x_j - x_{j-1})\frac{\omega(\eta(h))}{\eta(h)} \le (x_j - x_{j-1})\frac{\omega(x_j - x_{j-1})}{x_j - x_{j-1}} \\
&\le \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\}\omega(x_j - x_{j-1}), \quad j \in J_2.
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{h\omega(\eta(h))}{\eta(h)} &= \sum_{j\in J_1}(x_j - x_{j-1})\frac{\omega(\eta(h))}{\eta(h)} + \sum_{j\in J_2}(x_j - x_{j-1})\frac{\omega(\eta(h))}{\eta(h)} \\
&\le \sum_{j\in J_1} \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\}\omega(x_j - x_{j-1}) + \sum_{j\in J_2} \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\}\omega(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\}\omega(x_j - x_{j-1}) \le \omega_\eta(h) + \varepsilon.
\end{aligned}$$

Hence (5) holds. □

Lemma 3.6. *Let $\eta : (0,\infty) \to (0,\infty)$ and $\omega \in \mathcal{M}$. Then*

$$\omega_\eta(x+h) \le \omega_\eta(x) + \max\left\{1, \frac{h}{\eta(x+h)}\right\}\omega(h), \quad x, h > 0. \tag{6}$$

Proof. Let $x, h > 0$ and $\varepsilon > 0$.
Then there exists a partition $0 = x_0 < x_1 < \cdots < x_n = x$ of $[0,x]$ such that

$$\sum_{j=1}^{n} \max\left\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\right\}\omega(x_j - x_{j-1}) \le \omega_\eta(x) + \varepsilon.$$

Hence, for $x_{n+1} := x + h$ we have

$$\begin{aligned}\omega_\eta(x+h) &\le \sum_{j=1}^{n+1} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) \\ &= \max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h) + \sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) \\ &\le \omega_\eta(x) + \max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h) + \varepsilon.\end{aligned}$$

Consequently (6) holds. □

Lemma 3.7. *Let $\eta : (0,\infty) \to (0,\infty)$ be non-decreasing and let $\omega \in \mathcal{M}$. Then*

$$\omega_\eta(x+h) \le \omega_\eta(x) + 2\,\frac{h\omega(\eta(x))}{\eta(x)}, \quad x, h > 0,\ h \ge \frac{\eta(x)}{2}. \tag{7}$$

Proof. Let $x, h > 0$, $h \ge \eta(x)/2$, and $\varepsilon > 0$.
Then there exists a partition $0 = x_0 < x_1 < \cdots < x_n = x$ of $[0, x]$ such that

$$\sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\,\omega(x_j - x_{j-1}) \le \omega_\eta(x) + \varepsilon.$$

Denote by m the upper integer part of $h/\eta(x)$ and set $x_j := x + (j-n)h/m$ for each $j = n+1, \dots, n+m$. Then

$$\frac{m}{2} \le \frac{h}{\eta(x)} \le m$$

and $h/m \le \eta(x) \le \eta(x_j)$ for each $j = n+1, \dots, n+m$. Thus

$$\begin{aligned}\omega_\eta(x+h) &\le \sum_{j=1}^{n+m} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \max\Big\{1, \frac{x_j - x_{j-1}}{\eta(x_j)}\Big\}\omega(x_j - x_{j-1}) + \sum_{j=n+1}^{n+m} \max\Big\{1, \frac{\frac{h}{m}}{\eta(x_j)}\Big\}\omega\Big(\frac{h}{m}\Big) \\ &\le \omega_\eta(x) + \varepsilon + \sum_{j=1}^{m} \omega\Big(\frac{h}{m}\Big) \le \omega_\eta(x) + \varepsilon + \sum_{j=1}^{m} \omega(\eta(x)) \\ &= \omega_\eta(x) + m\omega(\eta(x)) + \varepsilon \le \omega_\eta(x) + 2\frac{h\omega(\eta(x))}{\eta(x)} + \varepsilon.\end{aligned}$$

Hence (7) holds. □

4. The case $\mathbb{R}^2$

We begin with three lemmas that we need to prove important Proposition 4.4 below.

Lemma 4.1. *Let $\eta\colon(0,\infty)\to(0,\infty)$ and $\omega\in\mathcal{M}$ be both concave, let $g\colon(0,\infty)\to\mathbb{R}$ be differentiable and let $C>0$. Suppose that*

$$|g(x+h)-g(x)|\le C\max\Big\{1,\frac{h}{\eta(x+h)}\Big\}\omega(h),\quad x,h>0, \tag{8}$$

$$\eta(x)|g'(x)|\le C\omega(\eta(x)),\quad x>0, \tag{9}$$

$$\eta(x)|g'(x)-g'(x+h)|\le C\omega(h),\quad x,h>0, \tag{10}$$

and set

$$G:=\{(x,y)\in\mathbb{R}^2 : x>0,\ |y|<\eta(x)\},$$
$$f(x,y):=g(x)y,\quad (x,y)\in G.$$

Then the function f is both $(5C\omega)$-semiconvex and $(5C\omega)$-semiconcave.

Proof. By [5, Lemma 2.3] it is sufficient to show that

$$\begin{aligned}&|f'(x+h,y+k)[h,k]-f'(x,y)[h,k]|\\ &\le 5C\,|(h,k)|\omega(|(h,k)|),\quad (x,y),(x+h,y+k)\in G,\ h\ge 0.\end{aligned}$$

Let $(x,y),(x+h,y+k)\in G$, $h\ge 0$. Then, according to Remark 2.7 (i), we have

$$|k|\le|y+k|+|y|\le\eta(x+h)+\eta(x)\le 2\eta(x+h) \tag{11}$$

and thus by (9) and Remark 2.7 (ii)

$$\begin{aligned}|k||g'(x+h)|&=|k|\frac{\eta(x+h)|g'(x+h)|}{\eta(x+h)}\le C|k|\frac{\omega(\eta(x+h))}{\eta(x+h)}\\ &\le C|k|\frac{\omega(\frac{|k|}{2})}{\frac{|k|}{2}}\le 2C\omega(|k|).\end{aligned}$$

Further, by (10) we have

$$|y||g'(x+h)-g'(x)|\le\eta(x)|g'(x+h)-g'(x)|\le C\omega(h).$$

Consequently

$$\begin{aligned}|g'(x+h)(y+k)-g'(x)y|&\le|y||g'(x+h)-g'(x)|+|k||g'(x+h)|\\ &\le C\omega(h)+2C\omega(|k|)\le 3C\omega(|(h,k)|).\end{aligned} \tag{12}$$

Since by (11) we have

$$|k|\max\Big\{1,\frac{h}{\eta(x+h)}\Big\}=\max\Big\{|k|,\frac{|k|}{\eta(x+h)}h\Big\}\le 2\max\{|k|,h\},$$

we obtain by (8)

$$\begin{aligned}|k||g(x+h)-g(x)|&\le C|k|\max\Big\{1,\frac{h}{\eta(x+h)}\Big\}\omega(h)\\ &\le 2C\max\{|k|,h\}\omega(h)\le 2C|(h,k)|\omega(|(h,k)|).\end{aligned} \tag{13}$$

Hence (12) and (13) give

$$\begin{aligned}
&|f'(x+h,y+k)[h,k]-f'(x,y)[h,k]| \\
&\le |h|\Big|\frac{\partial f}{\partial x}(x+h,y+k)-\frac{\partial f}{\partial x}(x,y)\Big| + |k|\Big|\frac{\partial f}{\partial y}(x+h,y+k)-\frac{\partial f}{\partial y}(x,y)\Big| \\
&= |h||g'(x+h)(y+k)-g'(x)y| + |k||g(x+h)-g(x)| \\
&\le 3\,C\,|(h,k)|\omega(|(h,k)|) + 2\,C\,|(h,k)|\omega(|(h,k)|) = 5\,C\,|(h,k)|\omega(|(h,k)|). \qquad \square
\end{aligned}$$

Lemma 4.2. *Let $\eta : (0,\infty) \to (0,\infty)$ and $\omega \in \mathcal{M}$ be both concave. Define the function ψ by*

$$\psi(x) := \sup\left\{\lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x_2) : \begin{array}{c} x_1, x_2 \ge 0, \lambda \in [0,1], \\ x = \lambda x_1 + (1-\lambda)x_2 \end{array}\right\}, \quad x \ge 0.$$

Then $\psi \in \mathcal{M}$, ψ is concave, $\omega_\eta \le \psi \le 2\omega_\eta$ and

$$\psi(t+\eta(t)) \le \psi(t) + 2\omega(\eta(t)), \quad t > 0, \eta(t) \le t. \tag{14}$$

Proof. By Remark 2.6 (ii) and Lemma 3.4, there exists a concave $\varphi \in \mathcal{M}$ such that $0 \le \omega_\eta \le \varphi \le 2\omega_\eta$. Note that ψ is the upper concave envelope of ω_η (see [6, Corollary 17.1.5]). In particular, ψ is concave and $0 \le \omega_\eta \le \psi \le \varphi$. Thus $\psi \le 2\omega_\eta$, $\lim_{x\to 0+}\psi(x) = 0$ and it follows from Remark 2.7 (i) that ψ is non-decreasing. Therefore, $\psi \in \mathcal{M}$.

Let $t > 0$, $x_1, x_2 \ge 0$ and $\lambda \in [0,1]$ be such that we have $\eta(t) \le t$, $x_1 \le x_2$ and $\lambda x_1 + (1-\lambda)x_2 = t + \eta(t)$. Then to prove (14) it is sufficient to show

$$\lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x_2) \le \psi(t) + 2\omega(\eta(t)).$$

Note also that η is non-decreasing by Remark 2.7 (i).

If $t \le x_1$, then $\eta(t) \le t \le x_1 \le x_2$ and thus by Lemma 3.6 (with $h = \eta(t)$ and $x = x_1 - \eta(t)$ or $x = x_2 - \eta(t)$)

$$\begin{aligned}
\omega_\eta(x_i) &\le \omega_\eta(x_i - \eta(t)) + \max\Big\{1, \frac{\eta(t)}{\eta(x_i)}\Big\}\omega(\eta(t)) \\
&= \omega_\eta(x_i - \eta(t)) + \omega(\eta(t)), \quad i = 1, 2.
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x_2) &\le \lambda\omega_\eta(x_1 - \eta(t)) + (1-\lambda)\omega_\eta(x_2 - \eta(t)) + \omega(\eta(t)) \\
&\le \lambda\psi(x_1 - \eta(t)) + (1-\lambda)\psi(x_2 - \eta(t)) + \omega(\eta(t)) \\
&\le \psi(\lambda(x_1 - \eta(t)) + (1-\lambda)(x_2 - \eta(t))) + \omega(\eta(t)) \\
&= \psi(t) + \omega(\eta(t)) \le \psi(t) + 2\omega(\eta(t)).
\end{aligned}$$

Now suppose that $x_1 \le t$. Then $\lambda < 1$ and therefore we may set

$$h := \frac{\eta(t)}{1-\lambda},\ x := x_2 - \frac{\eta(t)}{1-\lambda}.$$

Then $(1-\lambda)t = t - \lambda t \le t - \lambda x_1 = (1-\lambda)x_2 - \eta(t) = (1-\lambda)x$ and thus

$$t \le x. \tag{15}$$

In particular, $0 < x$. If $h \le \eta(x+h)$, then by Lemma 3.6 and Remark 2.7 (iii)

$$\begin{aligned}\omega_\eta(x_2) = \omega_\eta(x+h) &\le \omega_\eta(x) + \max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h)\\ &= \omega_\eta(x) + \omega\Big(\frac{\eta(t)}{1-\lambda}\Big) \le \omega_\eta(x) + \frac{\omega(\eta(t))}{1-\lambda}\end{aligned}$$

and thus

$$\begin{aligned}\lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x_2) &\le \lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x) + \omega(\eta(t))\\ &\le \lambda\psi(x_1) + (1-\lambda)\psi(x) + \omega(\eta(t))\\ &\le \psi(\lambda x_1 + (1-\lambda)x) + \omega(\eta(t))\\ &= \psi(t) + \omega(\eta(t)) \le \psi(t) + 2\omega(\eta(t)).\end{aligned}$$

If $\eta(x+h) \le h$, then by Lemma 3.7, Remark 2.7 (ii) and (15) we have

$$\begin{aligned}\omega_\eta(x_2) = \omega_\eta(x+h) &\le \omega_\eta(x) + 2h\frac{\omega(\eta(x))}{\eta(x)}\\ &\le \omega_\eta(x) + 2h\frac{\omega(\eta(t))}{\eta(t)} = \omega_\eta(x) + 2\frac{\omega(\eta(t))}{1-\lambda}\end{aligned}$$

and therefore

$$\begin{aligned}\lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x_2) &\le \lambda\omega_\eta(x_1) + (1-\lambda)\omega_\eta(x) + 2\omega(\eta(t))\\ &\le \lambda\psi(x_1) + (1-\lambda)\psi(x) + 2\omega(\eta(t))\\ &\le \psi(\lambda x_1 + (1-\lambda)x) + 2\omega(\eta(t))\\ &= \psi(t) + 2\omega(\eta(t)).\end{aligned}$$

□

Lemma 4.3. *Let $\eta : (0,\infty) \to (0,\infty)$ and $\psi \in \mathcal{M}$ be both concave, let $\omega \in \mathcal{M}$ and let $C > 0$. Suppose that*

$$\psi(t+\eta(t)) - \psi(t) \le C\omega(\eta(t)), \quad t > 0,\ \eta(t) \le t, \tag{16}$$

$$\psi(h) \le C\max\Big\{1, \frac{h}{\eta(h)}\Big\}\omega(h), \quad h > 0. \tag{17}$$

Then $$\psi(x+h) - \psi(x) \le 2C\max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h), \quad x, h > 0.$$

Proof. Let $x, h > 0$. We will distinguish four cases.

(a) $x \le h$: Then $\eta(x+h) \le \eta(2h) \le 2\eta(h)$ by Remark 2.7 (i), (iii). And so, using Remark 2.6 (i) and (17), we obtain

$$\begin{aligned}\psi(x+h) - \psi(x) &\le \psi(h) \le C\max\Big\{1, \frac{h}{\eta(h)}\Big\}\omega(h)\\ &\le C\max\Big\{1, \frac{2h}{\eta(x+h)}\Big\}\omega(h) \le 2C\max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h).\end{aligned}$$

(b) $h \le \eta(h)$: Then, according to Remark 2.6 (i) and (17), we have

$$\begin{aligned}\psi(x+h)-\psi(x) \le \psi(h) \le C\max\Big\{1,\frac{h}{\eta(h)}\Big\}\omega(h) = C\omega(h)\\ \le 2C\max\Big\{1,\frac{h}{\eta(x+h)}\Big\}\omega(h).\end{aligned}$$

(c) $\eta(x) \le h \le x$: Then $\eta(x+h) \le \eta(2x) \le 2\eta(x)$ by Remark 2.7 (i), (iii). Thus, using the concavity of ψ and (16), we obtain

$$\begin{aligned}\psi(x+h)-\psi(x) &= h\,\frac{\psi(x+h)-\psi(x)}{h} \le h\,\frac{\psi(x+\eta(x))-\psi(x)}{\eta(x)}\\ &\le C\,\frac{h}{\eta(x)}\omega(\eta(x)) \le 2C\,\frac{h}{\eta(x+h)}\omega(h) \le 2C\max\Big\{1,\frac{h}{\eta(x+h)}\Big\}\omega(h).\end{aligned}$$

(d) Suppose that none of cases (a), (b) or (c) holds: Then clearly $h \le x$ and $\eta(h) \le h \le \eta(x)$. Since η is continuous, there exists $t \in [h,x]$ such that $\eta(t) = h$. Hence, using the concavity of ψ and (16), we obtain

$$\begin{aligned}\psi(x+h)-\psi(x) &= \eta(t)\frac{\psi(x+\eta(t))-\psi(x)}{\eta(t)} \le \eta(t)\frac{\psi(t+\eta(t))-\psi(t)}{\eta(t)}\\ &\le C\omega(\eta(t)) \le 2C\max\Big\{1,\frac{h}{\eta(x+h)}\Big\}\omega(h).\end{aligned}$$ □

Now we can prove the crucial proposition of this article, which we will need to prove Theorem 5.5 below. However, we believe that it is also interesting in itself.

Proposition 4.4. *Let $\eta : (0,\infty) \to (0,\infty)$ and $\omega \in \mathcal{M}$ be both concave. Suppose that $\omega(x) > 0$ for all $x > 0$, and set*

$$G := \{(x,y) \in \mathbb{R}^2 : x > 0,\ |y| < \eta(x)\}.$$

Then there exists a concave and strictly positive $g \in C^1((0,\infty))$ such that the function f defined by

$$f(x,y) = g(x)y, \quad (x,y) \in G,$$

has the following properties:

(i) *f is both ω-semiconvex and ω-semiconcave.*

(ii) *If $\varphi \in \mathcal{M}$ satisfies $\liminf_{x\to\infty} \varphi(x)/\omega_\eta(x) = 0$, then there is no $C > 0$ such that*

$$\Big|\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(1,0)\Big| \le C\varphi(|x-1|), \quad x > 0, \tag{18}$$

in particular, $f \notin C^{1,\varphi}(G)$.

Proof. Firstly, note that η is non-decreasing by Remark 2.7 (i). So it follows from Lemma 3.5 that $\omega_\eta(x) > 0$ for all $x > 0$. Hence by Lemma 4.2 there exists a concave $\psi \in \mathcal{M}$ such that (16) holds with $C = 2$ and

$$0 < \omega_\eta(x) \le \psi(x) \le 2\omega_\eta(x), \quad x > 0. \tag{19}$$

Further, it follows from (19) and Remark 3.3 that (17) holds with $C = 2$. Hence by Lemma 4.3 we have

$$\psi(x+h) - \psi(x) \leq 4 \max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h), \quad x, h > 0. \tag{20}$$

Note that since ψ is concave, we have

$$\psi(x) = \int_0^x \psi'_+(s)\, ds, \quad x \geq 0.$$

Since $\psi \in \mathcal{M}$ and $\psi(x) > 0$ for every $x > 0$, there exists $a > 0$ such that

$$\psi'_+(4a) > 0. \tag{21}$$

By Remark 2.7 (ii) we have

$$\frac{\eta(x)}{x} \leq \frac{\eta(a)}{a} \leq q := \max\Big\{1, \frac{\eta(a)}{a}\Big\}, \quad x \geq a. \tag{22}$$

Denote by δ the real function on $[a, \infty)$ that is affine on each $[2^{n-1}a, 2^n a]$ ($n \in \mathbb{N}$) and

$$\delta(2^n a) = \psi'_+(2^n a), \quad n \in \mathbb{N} \cup \{0\}.$$

Then δ is continuous and non-negative. Since ψ'_+ is non-increasing, δ is also non-increasing. Hence for every $n \in \mathbb{N}$ and $x \in [2^{n-1}a, 2^n a]$ we have

$$\psi'_+(2x) \leq \psi'_+(2^n a) = \delta(2^n a) \leq \delta(x) \leq \delta(2^{n-1}a) = \psi'_+(2^{n-1}a) \leq \psi'_+\Big(\frac{x}{2}\Big)$$

and thus
$$\psi'_+(2x) \leq \delta(x) \leq \psi'_+\Big(\frac{x}{2}\Big), \quad x \geq a. \tag{23}$$

We set $b := 1280q^2$ and

$$g(x) := \frac{1}{b}\int_a^{a+x} \delta(s) ds, \quad x > 0.$$

Then g is concave and $g \in C^1((0,\infty))$. Furthermore, for all $x > 0$ we obtain using (23) and (21)

$$\begin{aligned} g(x) &\geq \frac{1}{b}\int_a^{\min\{a+x,2a\}} \delta(s) ds \geq \frac{1}{b}\int_a^{\min\{a+x,2a\}} \psi'_+(2s) ds \\ &\geq \frac{1}{b}\int_a^{\min\{a+x,2a\}} \psi'_+(4a) ds > 0. \end{aligned}$$

(i): Note that g' is non-increasing and, by Remark 2.7 (i), also non-negative. Note also that $8/b \leq 8q/b \leq 1/5$. Hence by Lemma 4.1 (with $C = 1/5$) it is sufficient to show that

$$g(x+h) - g(x) \leq \frac{8}{b} \max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h), \quad x, h > 0, \tag{24}$$

$$\eta(x) g'(x) \leq \frac{8q}{b}\omega(\eta(x)), \quad x > 0, \tag{25}$$

$$\eta(x)(g'(x) - g'(x+h)) \leq \frac{1}{5}\omega(h), \quad x, h > 0. \tag{26}$$

Inequality (24): For all $x, h > 0$ we have by (23), (20) and Remark 2.7 (iii)

$$\begin{aligned}
&g(x+h) - g(x)\\
&= \frac{1}{b}\int_{a+x}^{a+x+h} \delta(s)ds \le \frac{1}{b}\int_{a+x}^{a+x+h} \psi'_+\Big(\frac{s}{2}\Big)ds = \frac{2}{b}\int_{\frac{a+x}{2}}^{\frac{a+x+h}{2}} \psi'_+(t)dt\\
&= \frac{2}{b}\Big(\psi\Big(\frac{a+x+h}{2}\Big) - \psi\Big(\frac{a+x}{2}\Big)\Big) \le \frac{8}{b}\max\Big\{1, \frac{\frac{h}{2}}{\eta(\frac{a+x+h}{2})}\Big\}\omega\Big(\frac{h}{2}\Big)\\
&\le \frac{8}{b}\max\Big\{1, \frac{\frac{h}{2}}{\frac{\eta(a+x+h)}{2}}\Big\}\omega\Big(\frac{h}{2}\Big) \le \frac{8}{b}\max\Big\{1, \frac{h}{\eta(x+h)}\Big\}\omega(h).
\end{aligned}$$

Inequality (25): Let $x > 0$. Then we have by (22)

$$t := \frac{\eta(x)}{2q} \le \frac{\eta(a+x)}{2q} \le \frac{a+x}{2}$$

and by Remark 2.7 (iii)

$$\frac{t}{\eta(\frac{a+x}{2})} = \frac{\eta(x)}{2q\eta(\frac{a+x}{2})} \le \frac{\eta(\frac{x}{2})}{q\eta(\frac{a+x}{2})} \le \frac{1}{q} \le 1.$$

Consequently, using the concavity of ψ, (23) and (20), we obtain

$$\begin{aligned}
\eta(x)g'(x) &= \frac{1}{b}\eta(x)\delta(a+x) \le \frac{1}{b}\eta(x)\psi'_+\Big(\frac{a+x}{2}\Big)\\
&\le \frac{\eta(x)}{b}\,\frac{\psi(\frac{a+x}{2}) - \psi(\frac{a+x}{2} - t)}{t} = \frac{2q}{b}\Big(\psi\Big(\frac{a+x}{2}\Big) - \psi\Big(\frac{a+x}{2} - t\Big)\Big)\\
&\le \frac{8q}{b}\max\Big\{1, \frac{t}{\eta(\frac{a+x}{2})}\Big\}\omega(t) = \frac{8q}{b}\omega\Big(\frac{\eta(x)}{2q}\Big) \le \frac{8q}{b}\omega(\eta(x)).
\end{aligned}$$

Inequality (26): By (25) and Remark 2.7 (iii) we have

$$\begin{aligned}
&\eta(u)(g'(u) - g'(v)) \le \eta(u)g'(u) \le \frac{8q}{b}\omega(\eta(u)) \le \frac{8q}{b}\omega(4q(v-u))\\
&\le \frac{32q^2}{b}\omega(v-u) = \frac{1}{40}\omega(v-u), \quad 0 < u < v,\ \eta(u) \le 4q(v-u). \qquad (27)
\end{aligned}$$

Set
$$x_n := 2^n a - a, \quad n \in \mathbb{N} \cup \{0\}.$$

Then by Remark 2.7 (iii) we have

$$\eta(x_n) \le 2\eta\Big(\frac{x_n}{2}\Big) \le 2\eta\Big(\frac{x_{n-1}+x_n}{2}\Big), \quad n \in \mathbb{N},$$

and by (22)

$$\eta\Big(\frac{x_{n-1}+x_n}{2}\Big) \le \eta(x_n) \le qx_n \le q\cdot 2^n a = 4q\,\frac{x_n - x_{n-1}}{2}, \quad n \in \mathbb{N}.$$

Hence by (27) (for every $n \in \mathbb{N}$ applied with $u = (x_{n-1} + x_n)/2$ and $v = x_n$) we obtain

$$\begin{aligned} q_n := \frac{g'(\frac{x_{n-1}+x_n}{2}) - g'(x_n)}{x_n - \frac{x_{n-1}+x_n}{2}} &\le \frac{1}{40\eta(\frac{x_{n-1}+x_n}{2})} \frac{\omega(\frac{x_n - x_{n-1}}{2})}{\frac{x_n - x_{n-1}}{2}} \\ &\le \frac{1}{20\eta(\frac{x_{n-1}+x_n}{2})} \frac{\omega(x_n - x_{n-1})}{x_n - x_{n-1}} \le \frac{1}{10\eta(x_n)} \frac{\omega(x_n - x_{n-1})}{x_n - x_{n-1}}, \quad n \in \mathbb{N}. \end{aligned} \tag{28}$$

Since g' is affine on $(x_0, x_1]$ and on each $[x_{n-1}, x_n]$ ($n \in \mathbb{N}$, $n > 1$), we obtain by (28) and Remark 2.7 (ii)

$$\begin{aligned} \eta(u)(g'(u) - g'(v)) = \eta(u) q_n (v - u) &\le \frac{\eta(u)}{10\eta(x_n)} \frac{\omega(x_n - x_{n-1})}{x_n - x_{n-1}} (v - u) \\ &\le \frac{\eta(u)}{10\eta(x_n)} \frac{\omega(v - u)}{v - u} (v - u) \le \frac{1}{10} \omega(v - u), \end{aligned} \tag{29}$$

where $n \in \mathbb{N},\ u, v \in [x_{n-1}, x_n],\ 0 < u < v$.

Let $x, h > 0$. Then there exists $n \in \mathbb{N}$ such that $x_{n-1} \le x \le x_n$. If $\eta(x) \le 4qh$ or $x + h \le x_n$, then (26) holds by (27) or (29). Hence we may further suppose that $4qh \le \eta(x)$ and $x_n \le x + h$. Then we have by (22)

$$x + h \le x + \frac{\eta(x)}{4q} \le x_n + \frac{\eta(x_n)}{4q} \le x_n + \frac{x_n}{4} \le 2x_n \le x_{n+1}$$

and thus, using two-times (29), we obtain

$$\begin{aligned} \eta(x)(g'(x) - g'(x+h)) &= \eta(x)(g'(x) - g'(x_n)) + \eta(x)(g'(x_n) - g'(x+h)) \\ &\le \eta(x)(g'(x) - g'(x_n)) + \eta(x_n)(g'(x_n) - g'(x+h)) \\ &\le \frac{1}{10}\omega(x_n - x) + \frac{1}{10}\omega(x + h - x_n) \le \frac{1}{5}\omega(h). \end{aligned}$$

(ii): Let $\varphi \in \mathcal{M}$ satisfy $\liminf_{x\to\infty} \varphi(x)/\omega_\eta(x) = 0$. Suppose to the contrary that there exists $C > 0$ such that (18) holds. Hence

$$|g(x) - g(1)| = \left| \frac{\partial f}{\partial y}(x, 0) - \frac{\partial f}{\partial y}(1, 0) \right| \le C\varphi(|x - 1|), \quad x > 0. \tag{30}$$

If $\varphi(x) = 0$ for all $x > 0$, then (30) implies that g is constant, and thus by (23)

$$\psi'_+(4a) \le \delta(2a) = bg'(a) = 0$$

which is a contradiction with (21). So we may further suppose that $\varphi(x_0) > 0$ for some $x_0 > 0$. Then

$$0 < \frac{\varphi(x_0)}{\omega_\eta(x)} \le \frac{\varphi(x)}{\omega_\eta(x)}, \quad x > x_0,$$

and thus $\liminf_{x\to\infty} \varphi(x_0)/\omega_\eta(x) = 0$. Since ω_η is non-decreasing by Lemma 3.4 (i), it follows that $\lim_{x\to\infty} \omega_\eta(x) = \infty$. Now by (23) and (19) we have

$$\begin{aligned} 2b(g(x) - g(1)) &= 2\int_{a+1}^{a+x} \delta(s)ds \ge 2\int_{a+1}^{a+x} \psi'_+(2s)ds = \int_{2(a+1)}^{2(a+x)} \psi'_+(t)dt \\ &= \psi(2(a+x)) - \psi(2(a+1)) \ge \psi(x) - \psi(2(a+1)) \\ &\ge \omega_\eta(x) - \psi(2(a+1)), \quad x > 1, \end{aligned}$$

and thus by (30)

$$\omega_\eta(x) - \psi(2(a+1)) \le 2b(g(x) - g(1)) \le 2bC\varphi(x-1) \le 2bC\varphi(x), \quad x > 1.$$

Hence
$$1 - \frac{\psi(2(a+1))}{\omega_\eta(x)} \le 2bC\frac{\varphi(x)}{\omega_\eta(x)}, \quad x > 1,$$

which is a contradiction with $\liminf_{x\to\infty} \varphi(x)/\omega_\eta(x) = 0$ and $\lim_{x\to\infty} \omega_\eta(x) = \infty$. This completes the proof. □

5. The case $\mathbb{R}^n$

The main result (Theorem 5.5) of the article works with $\omega \in \mathcal{M}$ that satisfies

$$\inf_{n\in\mathbb{N}} \left(\liminf_{h\to\infty} \frac{\omega(h)}{n\omega\left(\frac{h}{n}\right)} \right) = 0. \tag{*}$$

An immediate consequence of Theorem 5.5 is Theorem 1.1, since we have:

$$\text{If } \alpha \in (0,1) \text{ and } \omega(h) = h^\alpha \text{ for all } h \ge 0, \text{ then (*) holds.} \tag{31}$$

Indeed, if $\alpha \in (0,1)$ and $\omega(h) = h^\alpha$ for all $h \ge 0$, then

$$\liminf_{h\to\infty} \frac{\omega(h)}{n\omega\left(\frac{h}{n}\right)} = \frac{1}{n^{1-\alpha}}, \quad n \in \mathbb{N},$$

and thus (*) holds.

The following remark concerns the validity of (*) for some other $\omega \in \mathcal{M}$.

Remark 5.1.

(i) If $\alpha \in [0,1)$ and $\beta > 0$, then there exist $C \ge 0$, $h_0 > 1$ and a concave $\omega \in \mathcal{M}$ such that $\omega(h) = h^\alpha \log^\beta(h) + C$ for all $h \ge h_0$ and (*) holds.

(ii) If $\omega(h) = h$ for all $h \ge 0$, then (*) doesn't hold.

(iii) If $\beta > 0$, then there exist $C \ge 0$, $h_0 > 1$ and a concave $\omega \in \mathcal{M}$ such that $\omega(h) = h/\log^\beta(h) + C$ for all $h \ge h_0$ and (*) doesn't hold.

All these facts can be easily proved.

The main property of concave $\omega \in \mathcal{M}$ satisfying (*) is that assertion (iii) of the following lemma holds.

Lemma 5.2. *Let $\omega \in \mathcal{M}$ be concave and satisfy (*). Then the following hold:*

(i) $\omega(h) > 0$ *for all* $h > 0$ *and* $\liminf_{h\to 0+} \omega(h)/h > 0$.

(ii) $\lim_{h\to\infty} \omega(h)/h = 0$.

(iii) *If* $\eta : (0,\infty) \to (0,\infty)$ *is non-decreasing and* $\lim_{h\to\infty} \eta(h)/h = 0$, *then*

$$\liminf_{h\to\infty} \frac{\omega(h)}{\omega_\eta(h)} = 0. \tag{32}$$

Proof. (i) This follows easily from (*) and Remark 2.7 (ii).

(ii): Firstly, the limit exists by Remark 2.7 (ii). Further, using (i) and Remark 2.7 (iii), we obtain

$$0 \le \lim_{h\to\infty} \frac{\omega(h)}{h} = \omega(1) \lim_{h\to\infty} \frac{n\omega(h)}{\frac{h}{n}\omega(1)} \le \liminf_{h\to\infty} \frac{n\omega(h)}{\omega\left(\frac{h}{n}\right)}, \quad n \in \mathbb{N},$$

and thus $\lim_{h\to\infty} \omega(h)/h = 0$ by (*).

(iii): Let $\varepsilon > 0$. Then, by (*), there exists $n \in \mathbb{N}$ such that

$$\liminf_{h\to\infty} \frac{\omega(h)}{n\omega\left(\frac{h}{n}\right)} < \varepsilon.$$

Further, there exists $h_0 > 0$ such that $\eta(h)/h \le 1/n$ for every $h \ge h_0$. Hence by (i), Remark 2.7 (ii) and Lemma 3.5, we have

$$0 < \frac{n\omega\left(\frac{h}{n}\right)}{\omega(h)} = \frac{h}{\omega(h)} \frac{\omega\left(\frac{h}{n}\right)}{\frac{h}{n}} \le \frac{h}{\omega(h)} \frac{\omega(\eta(h))}{\eta(h)} \le \frac{\omega_\eta(h)}{\omega(h)}, \quad h \ge h_0,$$

and thus
$$0 \le \liminf_{h\to\infty} \frac{\omega(h)}{\omega_\eta(h)} \le \liminf_{h\to\infty} \frac{\omega(h)}{n\omega\left(\frac{h}{n}\right)} < \varepsilon.$$

Hence (32) holds. □

We will also need the following lemma:

Lemma 5.3. *Let $n \in \mathbb{N}$ and let $G \subset \mathbb{R}^n$ be an open convex set such that we have $1 \le \dim(\operatorname{span}(\operatorname{rec}(G))) < n$. Then there exists a linear surjection $L : \mathbb{R}^n \to \mathbb{R}^2$ such that*

$$(1, 0) \in \operatorname{rec}(L(G)) = L(\operatorname{rec}(G)) \subset \operatorname{span}(\{(1, 0)\}). \tag{33}$$

Proof. We will proceed by induction on n. If $n = 2$, then $\dim(\operatorname{span}(\operatorname{rec}(G))) = 1$ and thus there exists a linear bijection $L : \mathbb{R}^n \to \mathbb{R}^2$ such that

$$(1, 0) \in L(\operatorname{rec}(G)) \subset \operatorname{span}(\{(1, 0)\}).$$

Hence (33) holds by Lemma 2.14.

Now suppose that $n \ge 3$ is given and the assertion holds for $n - 1$. It follows easily from Lemma 2.12 (ii) that $W := \operatorname{rec}(G) \cap (-\operatorname{rec}(G))$ is a linear subspace of $\operatorname{span}(\operatorname{rec}(G))$. Thus

$$m := \dim(W) \le k := \dim(\operatorname{span}(\operatorname{rec}(G)))$$

and there exist $b_1, \dots, b_k \in \mathbb{R}^n$ such that $\{b_1, \dots, b_m\}$ is an algebraic basis of W and $\{b_1, \dots, b_k\}$ an algebraic basis of $\operatorname{span}(\operatorname{rec}(G))$. We will show that there exists a linear surjection $L_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that

$$\operatorname{rec}(G) \cap L_1^{-1}(\{0\}) \subset W,\ 1 \le \dim(L_1(\operatorname{span}(\operatorname{rec}(G)))) < n - 1. \tag{34}$$

We will distinguish three cases:

(a) $k < n-1$: Then it is easy to find a linear surjection $L_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $\operatorname{span}(\operatorname{rec}(G)) \cap L_1^{-1}(\{0\}) = \{0\}$. Then

$$L_1(\operatorname{span}(\operatorname{rec}(G))) = \operatorname{span}(\{L_1(b_1), \dots, L_1(b_k)\})$$

and thus (34) follows.

(b) $m = 0$ and $k = n-1$: Then it is easy (cf. [5, Lemma 5.10] or see [7, Consequence 2]) to find $w \in \operatorname{span}\{b_1, b_2\} \setminus \{0\}$ such that

$$\operatorname{rec}(G) \cap \operatorname{span}(\{w\}) = \{0\}.$$

Then there exists a linear surjection $L_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $L_1^{-1}(\{0\}) = \operatorname{span}(\{w\})$. Since $L_1(w) = 0$ and $w \in \operatorname{span}\{b_1, b_2\}$, it follows that $L_1(b_1)$ and $L_1(b_2)$ are linearly dependent and thus

$$L_1(\operatorname{span}(\operatorname{rec}(G))) = \operatorname{span}(\{L_1(b_2), \dots, L_1(b_k)\}) \tag{35}$$

or
$$L_1(\operatorname{span}(\operatorname{rec}(G))) = \operatorname{span}(\{L_1(b_1), \dots, L_1(b_k)\} \setminus \{L_1(b_2)\}).$$

Hence (34) holds.

(c) $1 \le m \le k = n-1$: Then we choose a linear surjection $L_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ such that $L_1^{-1}(\{0\}) = \operatorname{span}(\{b_1\})$. Then $L_1^{-1}(\{0\}) \subset W$ and (35) holds. Consequently (34) holds.

So there exists L_1 for which (34) holds. Then $L_1(G)$ is an open convex set and

$$L_1(\operatorname{rec}(G)) = \operatorname{rec}(L_1(G)) \tag{36}$$

by Lemma 2.14. Thus

$$\operatorname{span}(\operatorname{rec}(L_1(G))) = \operatorname{span}(L_1(\operatorname{rec}(G))) = L_1(\operatorname{span}(\operatorname{rec}(G)))$$

and hence
$$1 \le \dim(\operatorname{span}(\operatorname{rec}(L_1(G)))) < n-1.$$

Now by induction assumption there exists a linear surjection $L_2 : \mathbb{R}^{n-1} \to \mathbb{R}^2$ such that

$$(1,0) \in \operatorname{rec}(L_2(L_1(G))) = L_2(\operatorname{rec}(L_1(G))) \subset \operatorname{span}(\{(1,0)\}). \tag{37}$$

Then $L := L_2 \circ L_1$ is a linear surjection and it follows from (36) and (37) that (33) holds. $\square$

Remark 5.4. The proof of the preceding lemma is similar to the proof of [5, Proposition 5.13]. Let us note that the proof of [5, Proposition 5.13] is incomplete. Namely, in case (b1) we forgot to consider the case when H is bounded. This can be easily corrected as follows: If H is bounded, then clearly $\dim(\operatorname{span}(\operatorname{rec}(G))) = 1$ and thus it is easy to find a linear surjection $L : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that satisfies [5, (24) and (25)], and so we can finish the proof as in case (b2).

Now we will prove the main theorem of the article. Recall that it implies Theorem 1.1 by (31) and that it can also be applied to other interesting moduli than power type moduli (see Remark 5.1 (i)).

Theorem 5.5. *Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an unbounded open convex set that doesn't contain a translation of a cone with non-empty interior. Let $\omega \in \mathcal{M}$ be concave and satisfy* (*). *Then there exists $f : G \to \mathbb{R}$ that is both ω-semiconvex and ω-semiconcave and $f \notin C^{1,\omega}(G)$.*

Proof. By Lemma 2.12 (iv), (v) and Lemma 5.3 there exists a linear surjection $L : \mathbb{R}^n \to \mathbb{R}^2$ such that

$$(1,0) \in \operatorname{rec}(L(G)) = L(\operatorname{rec}(G)) \subset \operatorname{span}(\{(1,0)\}). \tag{38}$$

Then $L(G)$ is an open convex set. We will further distinguish two cases.

(a) $-(1,0) \notin \operatorname{rec}(L(G))$: Then it follows (see the proof of [5, Proposition 5.13]) from (38) that there exist $b \in \mathbb{R}^2$ and a concave $\eta : (0,\infty) \to (0,\infty)$ such that $\lim_{x\to\infty} \eta(x)/x = 0$ and

$$\{(x,0) \in \mathbb{R}^2 : x > 0\} \subset L(G) + b \subset G_0 := \{(x,y) \in \mathbb{R}^2 : x > 0,\ |y| < \eta(x)\}. \tag{39}$$

By Proposition 4.4 and Lemma 5.2 (iii) there exists a function $f_0 \in C^1(G_0)$ that is both ω-semiconvex and ω-semiconcave and such that f_0' is uniformly continuous on $\{(x,0) \in \mathbb{R}^2 : x > 0\}$ with modulus $C\omega$ for no $C > 0$. Thus it follows from (39) that $L(G) + b \in \mathcal{G}_2(\omega)$. Then $L(G) \in \mathcal{G}_2(\omega)$ by Remark 2.9 (iii), and hence $G \in \mathcal{G}_n(\omega)$ by Lemma 2.13 and (38). Now the assertion follows from Definition 2.8.

(b) $-(1,0) \in \operatorname{rec}(L(G))$: Then it follows from Lemma 2.12 (ii) and (38) that $\operatorname{rec}(L(G)) = \operatorname{span}(\{(1,0)\})$. Hence it is easy to find a bounded open interval $I \subset \mathbb{R}$ such that $L(G) = \mathbb{R} \times I$. Then we find a linear bijection $Q : \mathbb{R}^2 \to \mathbb{R}^2$ and $b \in \mathbb{R}^2$ such that

$$Q(L(G)) + b = \mathbb{R} \times (0,1).$$

Then $Q(L(G)) + b \in \mathcal{G}_2(\omega)$ by Lemma 2.10 and Lemma 5.2 (i), (ii).
Hence $Q(L(G)) \in \mathcal{G}_2(\omega)$ by Remark 2.9 (iii). Now by Lemma 2.14 (applied to Q) and (38) we have

$$\operatorname{rec}(Q(L(G))) = Q(\operatorname{rec}(L(G))) = Q(L(\operatorname{rec}(G)))$$

and thus $G \in \mathcal{G}_n(\omega)$ by Lemma 2.13 (applied to $Q \circ L$). Then the assertion follows from Definition 2.8. $\square$

Acknowledgments. I thank Luděk Zajíček for many helpful suggestions which improved this article.

References

[1] P. Cannarsa, C. Sinestrari: *Semiconcave Functions, Hamilton-Jacobi equations, and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications 58, Birkhäuser, Boston (2004).

[2] J. Duda, L. Zajíček: *Semiconvex functions: Representations as suprema of smooth functions and extensions*, J. Convex Analysis 16 (2009) 239–260.

[3] P. Hájek, M. Johanis: *Smooth Analysis in Banach Spaces*, De Gruyter Series in Nonlinear Analysis and Applications 19, De Gruyter, Berlin (2014).

[4] V. Kryštof: *Generalized versions of Ilmanen lemma: Insertion of $C^{1,\omega}$ or $C^{1,\omega}_{\mathrm{loc}}$ functions*, Comment. Math. Univ. Carolin. 59 (2018) 223–231.

[5] V. Kryštof, L. Zajíček: *Functions on a convex set which are both ω-semiconvex and ω-semiconcave*, J. Convex Analysis 29 (2022) 837–856.

[6] R. T. Rockafellar: *Convex Analysis*, Princeton Mathematical Series 28, Princeton University Press, Princeton (1970).

[7] M. Studený: *Convex cones in finite-dimensional real vector spaces*, Kybernetika 29 (1993) 180–200.