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Unbounded Perturbation of History-Dependent Evolution Inclusion with Difference of Two Subdifferentials

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Dedicated to Ralph Tyrrell Rockafellar on the occasion of his 90th birthday.

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We consider a perturbed hemivariational inequality with an additional negative subdifferential term and history-dependent operators. The perturbation is represented by a multivalued mapping, and its values are not assumed to be convex or bounded. We prove that there is a solution to our problem and establish a relaxation-approximation result for it using a localized version of Hausdorff-Lipschitz continuity adapted to the unbounded case.

Keywords: Unbounded perturbation, truncated Lipschitz condition, difference of subdifferentials, hemivariational inequality, history-dependent operators.

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1. Introduction

In this paper, we consider the evolution inclusion

$$\begin{cases} -\dot{x} \in A(t, x(t), (R_1 x(t))) + \partial\psi^1(t, x(t), R_2 x(t)) - \partial\psi^2(t, x(t), R_3 x(t)) + F(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{1}$$

on the time interval $[0, T]$, where $A : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, is a nonlinear operator, $\psi^1(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}$, $t \in I$, $x \in \mathbb{R}^n$, is a locally Lipschitz function, $\psi^2(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $t \in I$, $x \in \mathbb{R}^n$, is a proper (i.e., not identically $+\infty$)

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convex lower semicontinuous function, $\partial\psi^1$ denotes the Clarke subdifferential of ψ^1, $\partial\psi^2$ is the subdifferential in the sense of convex analysis of ψ^2, R_1, R_2, R_3 are history-dependent operators, $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued mapping with closed values, and x_0 is a given initial vector. The precise conditions under which our problem is considered are stated in the next section. Here, we only mention that the values of the mapping F are not assumed to be convex or bounded.

Evolution inclusions containing the difference of two subdifferentials have been considered in [1] and [19] with a single-valued time dependent perturbation and in [27] and [28] with a multivalued bounded perturbation.

In the case when the subdifferential term $\partial\psi^2 \equiv 0$, Problem (1) is characterized by the Clarke generalized subdifferential $\partial\psi^1$ of locally Lipschitz function ψ^1 and thus represents the so-called hemivariational inequality. Hemivariational inequalities are used to describe nonconvex, nonmonotone and possibly multivalued constitutive laws that govern various processes in nonsmooth mechanics. The mathematical theory of hemivariational inequalities began to emerge in the early 1980s, largely thanks to the pioneering works of Panagiotopoulos (see [20, 21]) and become a subject of intense research since then (see, e.g., [10, 11, 12, 13, 15, 16, 17, 18] and the references therein).

When one studies a mechanical system, one often needs to consider external forces applied to the system. In general, these forces can be included in the model as a multivalued perturbation. Then, based on physical considerations, it occurs reasonable to assume that the perturbation can have nonconvex and unbounded values. Also, the unboundedness of values of the right-hand side of set-valued mapping is a fairly natural property of differential inclusions which appear in optimal control problems, e.g. when dealing with a Mayer problem obtained as the result of reformulation of a problem with integral functional (cf., e.g., [14, 22]). This prompts us to require the values of multivalued perturbation F to be closed only. On the other hand, the numerical analysis of Problem (1) may be ineffective because the values of F are not convex. Therefore, we relax Problem (1) by introducing the following convexified problem:

$$\begin{cases} -\dot{x}(t) \in A(t, x(t), R_1x(t)) + \partial\psi^1(t, x(t), R_2x(t)) \\ \qquad\qquad - \partial\psi^2(t, x(t), R_3x(t)) + \overline{\mathrm{co}}F(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{2}$$

where the symbol $\overline{\mathrm{co}}$ stands for the closed convex hull of a set. This is the intersection of all convex sets containing the set.

We note that in our two problems, (1) and (2), we allow the nonlinear operator and both subdifferentials to depend on history-dependent operators. These are operators that, at a given time moment, depend on the previous history of the solution up to that time. They include operators of the Volterra type and, more generally, various integral-type operators.

The main goal of this paper is to prove the existence of solutions to both Problems (1) and (2). In addition, we demonstrate the relaxation property for solutions of our problems. This property establishes approximation, in a suitable topology, of solutions of Problem (1) by those of Problem (2). In the next section, we will define what a solution is and explain exactly what relaxation means.

A conventional assumption employed when proving relaxation type properties for evolution inclusions with bounded perturbations is the Lipschitz continuity of the perturbation in the Hausdorff distance haus$(\cdot,\cdot)$ between nonempty closed bounded sets:

$$\operatorname{haus}\left(F(t,x),F(t,y)\right) \le k(t)\|x-y\| \tag{3}$$

for $x, y \in \mathbb{R}^n$ and a summable function $k(\cdot)$. However, the condition of Hausdorff-Lipschitzness for F is too strict in the case where the values of F are unbounded and assuming this property essentially reduces the class of unbounded perturbations, because it is not satisfied for half-spaces, for example – a very important case for applications. Therefore, in the unbounded situation, we need a localized or truncated alternative to (3). The $(\rho - H)$-Lipschitzness of a multivalued mapping introduced in the next section proves to be a right candidate for such an alternative. We note that the $(\rho-H)$-Lipschitzness has recently been used to investigate subdifferential inclusions and control problems with unbounded perturbations and control constraints in a number of papers (see, e.g., [9, 25, 26, 29] and references therein). In these papers, with exception of [9], one considers the subdifferentials of convex, lower semicontinuous functions. However, to the best of the authors' knowledge, there have been no contributions so far on unbounded history-dependent problems with the difference of subdifferentials.

The rest of this paper is organized as follows. Section 2 introduces the necessary notation and recalls some results that will be used throughout the paper. In Section 3, we state the assumptions on the data describing Problem (1) and prove the existence result for it using a priori estimates for a solution and a fixed point principle. Finally, in Section 4, we obtain the relaxation result for Problem (1).

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a Banach space, X^* be its dual space and $\langle\cdot,\cdot\rangle_{X^*\times X}$ denote the duality brackets between X^* and X. We use the symbols s–X and w–X to denote the space X endowed with the strong and weak topologies, respectively. The symbols "$\to$" and "$\rightharpoonup$" stand for the strong and weak convergences, respectively. The same notation is adapted for subsets of X. If $\|\cdot\|$ is the norm on X, then the distance from $x \in X$ to $C \subset X$ is $d(x,C) = \inf_{y\in C} \|x-y\|$ (for $C = \emptyset$ we set $d(x,C) = +\infty$) and for $C, D \subset X$ the excess of C over D is defined as $e(C,D) = \sup_{x\in C} d(x,D)$ with the natural convention that $e = 0$ for $C = \emptyset$ (for $C \neq \emptyset$ and $D = \emptyset$ we have $e = +\infty$).

Next, we recall the notion of Clarke generalized gradient, see [5, 16, 24].

Definition 2.1. The *Clarke generalized directional derivative* of a locally Lipschitz function $j\colon X \to \mathbb{R}$ at a point $x \in X$ in the direction v, denoted by $j^0(x;v)$, is defined by

$$j^0(x;v) = \limsup_{y\to x,\, \lambda\to 0^+} \frac{j(y+\lambda v) - j(y)}{\lambda}.$$

The *Clarke generalized gradient* or *Clarke subdifferential* of j at x is then given by

$$\partial j(x) = \left\{x^* \in X^* \mid j^0(x;v) \ge \langle x^*, v\rangle \quad \text{for all} \quad v \in X\right\}. \qquad \square$$

Let B denote the open ball of radius one centered at the origin of the space X and

$$C_\rho = C \cap \rho \bar{B}, \quad \rho \geq 0,$$

be the intersection of the set C with the closed ball of radius ρ centered at the origin. Then, given $\rho \geq 0$ we denote and define the *ρ-Hausdorff* or *$(\rho - H)$-distance* between nonempty sets $C, D \subset X$ (see [2]; also [22] and [24]) as:

$$\mathrm{haus}_\rho(C, D) = \max\left\{e\left(C_\rho, D\right), e\left(D_\rho, C\right)\right\}. \tag{4}$$

For closed sets C, D, $\mathrm{haus}_\rho(C, D) = 0$ for all $\rho \geq 0$ if and only if $C = D$.

Definition 2.2. Let $F : Q \subset [0,T] \times X \to X$ have nonempty values. We call the multivalued mapping F *integrally $(\rho - H)$-Lipschitzian* on Q, if there exist $\beta > 0$ and $l(\cdot) \in L^2(0,T), l \geq 0$, such that

$$\mathrm{haus}_\rho(F(t,x), F(t,y)) \leq (l(t) + \beta\rho)\|x - y\| \quad \text{for all } (t,x),(t,y) \in Q \text{ and } \rho \geq 0. \quad \square$$

The following relationship holds between the usual Hausdorff distance and ρ-Hausdorff distance:

Proposition 2.3. [2, Proposition 1.4] *Let $\rho_0 \geq 0$ be such that C_{ρ_0} and D_{ρ_0} are nonempty for two closed convex sets $C, D \subset X$. Then, for all $\rho > \rho_0$*

$$haus\left(C_\rho, D_\rho\right) \leq \frac{2\rho}{\rho - \rho_0}\,\mathrm{haus}_\rho(C, D).$$

Proposition 2.4. *If $\psi : U \to \mathbb{R}$ is a locally Lipschitz function on an open subset U of X, then the graph of the generalized gradient $\partial\psi$ is closed in $X \times (w^* - X^*)$ topology, i.e., if $x_n \in U$ and $\zeta_n \in X^*$ are sequences such that $\zeta_n \in \partial\psi(x_n)$ and $x_n \to x$ in X and ζ_n weakly* converges to ζ in X^*, then $\zeta \in \partial\psi(x)$.*

For the proof of Proposition 2.4 we refer to [5] (see also [24]). The following definition and a fixed point result (a consequence of the Banach contraction principle) can be found in [23].

Definition 2.5. Let X and Y be normed spaces. An operator $R\colon C([0,T];X) \to C([0,T];Y)$ is called *history-dependent*, if there exists a constant $L_R > 0$ such that

$$\|(Rv_1)(t) - (Rv_2)(t)\|_Y \leq L_R \int_0^t \|v_1(s) - v_2(s)\|_X\, ds \quad \forall v_1, v_2 \in C([0,T];X),\ t \in [0,T].$$

Details on various classes of history-dependent operators, their properties, and applications, can be found in [23].

Lemma 2.6. *Let $F : L^2(0,T;X) \to L^2(0,T;X)$ be an operator such that*

$$\|\left(F\eta_1\right)(t) - \left(F\eta_2\right)(t)\|_X^2 \leq L_F \int_0^t \|\eta_1(s) - \eta_2(s)\|_X^2\, ds$$

for all $\eta_1, \eta_2 \in L^2(0,T;X)$, a.e. $t \in (0,T)$ and a constant $L_F > 0$. Then F has a unique fixed point in $L^2(0,T;X)$, i.e., there exists a unique $\eta^* \in L^2(0,T;X)$ such that $F\eta^* = \eta^*$.

Denote by $\Gamma_0(X)$ the family of all proper, convex, lower semicontinuous function on X with values in $\mathbb{R} \cup \{+\infty\}$.

The Moreau-Yosida regularization of a function $\psi \in \Gamma_0(X)$ is the function

$$\psi_\lambda(x) = \inf\left\{\psi(y) + \frac{1}{2\lambda}\|y-x\|^2; y \in X\right\}, \quad \lambda > 0.$$

The following results can be found in [3, 8].

Lemma 2.7. *Let H be a Hilbert space, $\psi \in \Gamma_0(H)$. Then,*

(a) *the function ψ_λ is finite, convex and continuously differentiable on H with* $\operatorname{dom}\partial\psi_\lambda = H$*;*

(b) $\|\partial\psi_\lambda(x) - \partial\psi_\lambda(y)\| \leq \frac{1}{\lambda}\|x-y\|, \lambda > 0, x, y \in H$*;*

(c) *there exists a constant $L > 0$ such that*

$$-L(\|x\|+1) \leq \psi_\lambda(x) \leq \psi(x),$$

$x \in H$*,* $\lambda > 0$*, and* $\psi_\lambda(x) \uparrow \psi(x)$ *as* $\lambda \downarrow 0$*;*

(d) $\|\partial\psi_\lambda(x)\| \leq \|\partial\psi^0(x)\|$*,* $\lambda > 0$*,* $\partial\psi_\lambda(x) \to \partial\psi^0(x)$ *in H as* $\lambda \downarrow 0$*,* $x \in \operatorname{dom}\partial\psi$*, where $\partial\psi^0(x)$ is the unique element of minimal norm in the set $\partial\psi(x)$ that exists in view of the closedness and convexity of the set* $\partial\psi(x), x \in \operatorname{dom}\partial\psi$*;*

(e) *if* $x_n \in \operatorname{dom}\partial\psi$*,* $v_n \in \partial\psi(x_n)$*,* $n \geq 1$*,* $x_n \to x$ *in H and* $v_n \to v$ *in* $\omega - H$*, then* $v \in \partial\psi(x)$*;*

(f) *if* $\lambda_n \downarrow 0$*,* $x_{\lambda_n} \to x$ *in H, and* $\partial\psi_{\lambda_n}(x_{\lambda_n}) \to x^*$ *in* $\omega - H$*, then* $x^* \in \partial\psi(x)$*.*

Lemma 2.8. *Let* $\psi \in \Gamma_0(H)$*,* $x(\cdot) \in L^2(0,T;H)$*, and*

$$\Phi(x(\cdot)) = \begin{cases} \int_T \psi(x(t))dt \text{ if } \psi(x)(\cdot) \in L^1(0,T;\mathbb{R}), \\ +\infty \text{ otherwise.} \end{cases} \tag{5}$$

Then:

(a) $\Phi(\cdot) \in \Gamma_0\left(L^2(0,T;H)\right)$*;*

(b) $$\Phi_\lambda(x(\cdot)) = \int_T \psi_\lambda(x(t))dt, \lambda > 0; \tag{6}$$

(c) *if* $v(\cdot) \in L^2(0,T;H)$*, then* $v(\cdot) \in \partial\Phi(x(\cdot))$ *if and only if* $v(t) \in \partial\psi(x(t))$ *for a.e.* $t \in [0,T]$*;*

(d) $\partial\Phi_\lambda(x)(t) = \partial\psi_\lambda(x(t))$ *for a.e.* $t \in [0,T]$*,* $\lambda > 0$*.*

Given a Banach space Y, a multivalued mapping $F: X \to Y$ is called lower semicontinuous if for any $x \in X, y \in F(x)$, and any sequence $x_n, n \geq 1$, converging to x, there exists a sequence $y_n \in F(x_n), n \geq 1$, converging to y. Let Σ and $\mathcal{B}(X)$ denote the σ-algebras of Lebesgue measurable subsets of $[0,T]$ and Borel sets from X, respectively. Then, a multivalued mapping $F: [0,T] \times X \to X$ is called $(\Sigma, \mathcal{B}(X))$ measurable if $\{(t,x) \in [0,T] \times X : F(t,x) \cap U \neq \emptyset\} \in \Sigma \otimes \mathcal{B}(X)$ for any open set $U \subset X$.

3. Existence result

Below and in the rest of the paper $\partial\psi^i(t,x,v)$ means $\partial\psi^i(t,x,\cdot)(v)$, $i=1,2$, i.e. the subdifferential with respect to the third variable. We make the following assumptions on the data describing Problem (1).

H(A) **:** The nonlinear operator $A:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the conditions:

(a) A is measurable in the first variable and continuous in the third variable;

(b) there exist two constants $m_A,\bar{m}_A>0$ such that for a.e. $t\in[0,T]$

$$\langle A(t,x_1,v_1)-A(t,x_2,v_2),v_1-v_2\rangle\geq m_A\|v_1-v_2\|^2-\bar{m}_A\|x_1-x_2\|\,\|v_1-v_2\|$$

for all $x_1,x_2,v_1,v_2\in\mathbb{R}^n$;

(c) A satisfies a linear growth condition with respect to the second and third variables, i.e. there exist a function $a_0(\cdot)\in L^2(0,T)$ and two constants $a_1,a_2\geq 0$ such that for a.e. $t\in[0,T]$

$$\|A(t,x,v)\|\leq a_0(t)+a_1\|x\|+a_2\|v\|\quad\text{for all } x,v\in\mathbb{R}^n.$$

H(ψ¹) **:** The function $\psi^1:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is such that

(a) ψ^1 is measurable in the first variable and locally Lipschitz in the third variable;

(b) there exist two constants $m_{\psi^1},\bar{m}_{\psi^1}>0$ such that for a.e. $t\in[0,T]$

$$\langle\xi_1-\xi_2,v_1-v_2\rangle\geq -m_{\psi^1}\|v_1-v_2\|^2-\bar{m}_{\psi^1}\|x_1-x_2\|\|v_1-v_2\|$$

for all $\xi_i\in\partial\psi^1(t,x_i,v_i)\,,x_i,v_i\in\mathbb{R}^n,i=1,2$;

(c) there exist a function $c_0(\cdot)\in L^2(0,T)$ and two constants $c_1,c_2\geq 0$ such that for a.e. $t\in[0,T]$ and for all $x,v\in\mathbb{R}^n$

$$\|\partial\psi^1(t,x,v)\|:=\sup\{\|y\|:y\in\partial\psi^1(t,x,v)\}\leq c_0(t)+c_1\|x\|+c_2\|v\|.$$

H(ψ²) **:** The function $\psi^2:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is such that

(a) ψ^2 is measurable in the first variable and proper convex lower semicontinuous in the third variable;

(b) there exist two constants $m_{\psi^2},\bar{m}_{\psi^2}>0$ such that

$$\langle v_1-v_2,w_1-w_2\rangle\leq m_{\psi^2}\|v_1-v_2\|^2+\bar{m}_{\psi^2}\|x_1-x_2\|\|v_1-v_2\|$$

for any $w_1\in\partial\psi^2(t,x_1,v_1),w_2\in\partial\psi^2(t,x_2,v_2)$;

(c) there exist a function $\bar{c}_0(\cdot)\in L^2(0,T)$ and two constants $\bar{c}_1,\bar{c}_2\geq 0$ such that for a.e. $t\in[0,T]$

$$\|\partial\psi^2(t,x,v)\|:=\sup\{\|y\|:y\in\partial\psi^2(t,x,v)\}\leq\bar{c}_0(t)+\bar{c}_1\|x\|+\bar{c}_2\|v\|$$

for all $x,v\in\mathbb{R}^n$;

(d) there exist two constants $k\in[0,1)$ and $\gamma\geq 0$ such that for a.e. $t\in[0,T]$

$$\left\|\partial\psi^{2^0}(t,x,v)\right\|\leq k\inf\{\|w\|;w\in\partial\psi^1(t,x,v)\}+\gamma\|v\|\quad\text{for all } x,v\in\mathbb{R}^n.$$

***H*(*R*) :** The operators $R_i \colon C([0,T];\mathbb{R}^n) \to C([0,T];\mathbb{R}^n)$, $i = 1,2,3$, satisfy the condition

(a) there exist constants $L_{R_i} > 0$, $i = 1,2,3$ such that

$$\|(R_i v_1)(t) - (R_i v_2)(t)\| \le L_{R_i} \int_0^t \|v_1(s) - v_2(s)\| \, ds$$

for all $v_1, v_2 \in C([0,T];\mathbb{R}^n)$.

***H*(*F*) :** $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ admits closed values and

(a) is $(\Sigma, \mathcal{B}(X))$ measurable;

(b) is integrally $(\rho - H)$-Lipschitzian on $[0,T] \times \mathbb{R}^n$;

(c) there exists a function $k_0(\cdot) \in L^2(0,T)$ such that $d(0, F(t,0)) < k_0(t)$ for a.e. $t \in [0,T]$;

(d) there exist $\beta_0 > 0$ and $l_0(\cdot) \in L^2(0,T)$, $l_0 \ge 0$ such that

$$\overline{\text{co}} F(t,x) \cap \rho(t)\overline{B} \subset \overline{\text{co}}(F(t,x) \cap m(t)B)$$

for $t \in [0,T]$, $x \in \mathbb{R}^n$, $m(t) = l_0(t) + \beta_0 \rho(t)$ and some $\rho(\cdot) \in L^2(0,T)$.

***H*(1) :** $m_A > m_{\psi^1} + m_{\psi^2} + \bar{m}_A C_{R_1} + \bar{m}_{\psi^1} C_{R_2} + \bar{m}_{\psi^2} C_{R_3} + kc_2 + \gamma.$

Definition 3.1. By a *solution* of Problem (1) we mean a function $x(\cdot) \in W^{1,2}(0,T)$, $x(0) = x_0$, for which there exist functions $\xi_1, \xi_2, f \in L^2(0,T)$ such that the following four inclusions hold for a.e. $t \in [0,T]$:

$$-\dot{x}(t) \in A(t, x(t), R_1 x(t)) + \xi_1(t) - \xi_2(t) + f(t),$$

$$\xi_1(t) \in \partial\psi^1(t, x(t), R_2 x(t)), \xi_2(t) \in \partial\psi^2(t, x(t), R_3 x(t)), f(t) \in F(t, x(t)). \qquad \square$$

We fix $\lambda, \eta, \theta \in L^2(0,T)$ and introduce the following evolution inclusion.

Problem 3.2.

$$\begin{cases} -\dot{x}(t) \in \tilde{A}(t,x(t)) + \partial\tilde{\psi}^1(t,x(t)) - \partial\tilde{\psi}^2(t,x(t)) + F(t,x(t)) \ \textit{for a.e.}\ t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{7}$$

where we defined $\tilde{A}(t,x(t)) := A(t,\lambda(t),x(t))$, $\partial\tilde{\psi}^1(t,x(t)) := \partial\psi^1(t,\eta(t),x(t))$, *and* $\partial\tilde{\psi}^2(t,x(t)) := \partial\psi^2(t,\theta(t),x(t))$. *We note that the notion of a solution to Problem* 3.2*, as well as for all the auxiliary problems considered later in the text, extends in a natural way from Definition* 3.1*.*

Next, we consider the following problem.

Problem 3.3.

$$\begin{cases} -\dot{x}(t) \in \tilde{A}(t,x(t)) + \partial\tilde{\psi}^1(t,x(t)) + F(t,x(t)) \ \textit{for a.e.}\ t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{8}$$

Theorem 3.4. *Problem* 3.1 *has a solution.*

Proof. Under our Hypotheses $H(A)(a)$, $H(\psi^1)$, $H(F)$, and $H(1)$, the result follows directly from [9, Theorem 3.1]. $\square$

Now, replace in (7) the multivalued perturbation F with an arbitrary single-valued function $f \in L^2(0,T)$ and consider

Problem 3.5.

$$\begin{cases} -\dot{x}(t) \in \tilde{A}(t,x(t)) + \partial\tilde{\psi}^1(t,x(t)) - \partial\tilde{\psi}^2(t,x(t)) + f(t) \quad \textit{for a.e. } t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{9}$$

Theorem 3.6. *Problem* 3.5 *has a unique solution.*

Proof. For $\lambda > 0$, $f \in L^2(0,T)$, we consider the inclusion

$$\begin{cases} -\dot{x}_\lambda(t) \in \tilde{A}(t,x_\lambda(t)) + \partial\tilde{\psi}^1(t,x_\lambda(t)) - \partial\tilde{\psi}^2_\lambda(t,x_\lambda(t)) + f(t) \quad \text{for a.e. } t \in [0,T], \\ x_\lambda(0) = x_0, \end{cases} \tag{10}$$

where we replaced $\partial\tilde{\psi}^2$ in Problem 3.5 by its Moreau-Yosida regularization $\partial\tilde{\psi}^2_\lambda$. The existence of a solution of inclusion (10) follows from the existence of a solution of the inclusion

$$-\dot{x}(t) \in \tilde{A}(t,x(t)) + \partial\tilde{\psi}^1(t,x(t)) + \tilde{F}(t,x(t)) \text{ a.e.}, \quad x(0) = x_0,$$

with $\tilde{F}(t,x(t)) = -\partial\tilde{\psi}^2_\lambda(t,x(t)) + f(t)$. In view of Lemma 2.7, the latter existence is a consequence of Theorem 3.4. Given $f(\cdot) \in L^2(0,T)$, $\|f\|_{L^2(0,T)} \le R$, for a constant $R > 0$, let

$$f^1_\lambda(t) = -\dot{x}_\lambda(t) + \partial\tilde{\psi}^2_\lambda(t,x_\lambda(t)) - f(t) - \tilde{A}(t,x_\lambda(t)). \tag{11}$$

From (10) we see that

$$f^1_\lambda(t) \in \partial\tilde{\psi}^1(t,x_\lambda(t)). \tag{12}$$

Multiplying (11) by $x_\lambda(t)$ and integrating the result over $[0,t], t \in (0,T]$ we obtain

$$\begin{aligned} &\frac{1}{2}\|\dot{x}_\lambda(t)\| + \int_0^t \langle \tilde{A}(s,x_\lambda(s)), x_\lambda(s)\rangle ds \\ &= \frac{1}{2}\|x_0\| - \int_0^t \langle f(s), x_\lambda(s)\rangle ds - \int_0^t \langle f^1_\lambda(s), x_\lambda(s)\rangle ds + \int_0^t \langle \partial\tilde{\psi}^2_\lambda(s,x_\lambda(s)), x_\lambda(s)\rangle ds. \end{aligned} \tag{13}$$

According to $H(\psi^1)$(a), $H(\psi^1)$(b) we have

$$\begin{aligned} -\langle f^1_\lambda(t), x_\lambda(t)\rangle &= \langle f^1_\lambda(t) - f^1_0(t), -x_\lambda(t)\rangle - \langle f^1_0(t), x_\lambda(t)\rangle \\ &\le m_{\psi^1}\|x_\lambda(t)\|^2 + \|f^1_0(t)\|\|x_\lambda(t)\| \\ &\le m_{\psi^1}\|x_\lambda(t)\|^2 + c_0(t)\|x_\lambda(t)\|, \end{aligned} \tag{14}$$

where $f^1_0(t) \in \partial\tilde{\psi}^1(t,0)$ for a.e. $t \in [0,T]$. From Hypotheses $H(A)$(a), (b) it follows that

$$\begin{aligned} \langle \tilde{A}(t,x_\lambda(t)), x_\lambda(t)\rangle &= \langle \tilde{A}(t,x_\lambda(t)) - \tilde{A}(t,0), x_\lambda(t)\rangle + \langle \tilde{A}(t,0), x_\lambda(t)\rangle \\ &\ge m_A\|x_\lambda(t)\|^2 - a_0(t)\|x_\lambda(t)\|. \end{aligned} \tag{15}$$

According to Lemma 2.7(d), $H(\psi^2)$(d) and $H(\psi^1)$(c), we get

$$\begin{aligned} &\langle \partial\tilde{\psi}^2_\lambda(t,x_\lambda(t)), x_\lambda(t)\rangle \le \|\partial\tilde{\psi}^2_\lambda(t,x_\lambda(t))\|\|x_\lambda(t)\| \le \left\|\partial\tilde{\psi}^{2^0}(t,x_\lambda(t))\right\|\|x_\lambda(t)\| \\ &\le (k\|z\| + \gamma\|x_\lambda(t)\|)\|x_\lambda(t)\| \le (kc_0(t) + kc_2\|x_\lambda(t)\| + \gamma\|x_\lambda(t)\|)\|x_\lambda(t)\|, \end{aligned} \tag{16}$$

where $z \in \partial\tilde{\psi}^1(t, x_\lambda(t))$. Invoking Young's inequality, from (13)–(16) we infer that

$$\frac{1}{2}\|x_\lambda(t)\|^2 + (m_A - m_{\psi^1} - kc_2 - \gamma)\int_0^t \|x_\lambda(s)\|^2 ds$$
$$\leq \frac{1}{2}\|x_0\|^2 + C_1 + C_2\int_0^t \|x_\lambda(s)\|^2 ds$$

for some constants $C_1, C_2 > 0$. Gronwall's inequality and $H(1)$ now imply that $M > 0$ and

$$\|x_\lambda(t)\| \leq M_1, \quad t \in [0, T], \quad \lambda > 0, \tag{17}$$

for a positive constant M_1 independent of λ. Then, taking into account this estimate and (11), (12), $H(A)$(c), $H(\psi^1)$(c), $H(\psi^2)$(c), for a constant $M_2 > 0$ independent of λ we have

$$\|\dot{x}_\lambda\|_{L^2(T,H)} \leq M_2, \quad \lambda > 0. \tag{18}$$

Next, Lemma 2.7(d), Hypotheses $H(\psi^2)$(d), $H(A)$(c) and (11), (12) imply that

$$\|\partial\tilde{\psi}^2_\lambda(t, x_\lambda(t))\|$$
$$\leq k(\|\partial\tilde{\psi}^2_\lambda(t, x_\lambda(t))\| + \|\dot{x}_\lambda(t)\| + \|f(t)\| + \|\tilde{A}(t, x_\lambda(t))\|) + \gamma\|x_\lambda(t)\|. \tag{19}$$

From $H(A)$(c), (17)–(19), we see that

$$\left\|\partial\tilde{\psi}^2_\lambda(t, x_\lambda)\right\|_{L^2(T,H)} \leq N, \quad \lambda > 0,$$
$$\left\|f^1_\lambda\right\|_{L^2(T,H)} \leq N, \quad \lambda > 0,$$

for some constant $N > 0$. The estimates (17), (18) imply, by virtue of the Arzela-Ascoli theorem, that the set $\{x_\lambda; \lambda > 0\}$ is relatively compact in the space $C([0, T])$ and the set $\{\dot{x}_\lambda; \lambda > 0\}$ is relatively compact in the space $\omega - L^2(0, T)$.

Similarly, the sets $\left\{\partial\tilde{\psi}^2_\lambda(t, x_\lambda); \lambda > 0\right\}$ and $\{f^1_\lambda; \lambda > 0\}$ are relatively compact in the space $\omega - L^2(0, T)$. Therefore, there exist a sequence $\lambda_n \downarrow 0$ and functions $x \in W^{1,2}(0, T)$ and $f^1, f^2 \in L^2(0, T)$ such that

$$\begin{aligned} &x_{\lambda_n} \to x \ \text{ in } \ C([0, T]), \quad \dot{x}_{\lambda_n} \to \dot{x} \ \text{ in } \ \omega - L^2(0, T), \\ &f^1_{\lambda_n} \to f^1 \ \text{ in } \ \omega - L^2(0, T), \quad \partial\tilde{\psi}^2_\lambda(t, x_\lambda) \to f^2 \ \text{ in } \ \omega - L^2(0, T). \end{aligned} \tag{20}$$

It follows from (11), (20) and $H(A)$(a) that

$$-\dot{x}(t) = \tilde{A}(t, x(t)) + f^1(t) - f^2(t) + f(t) \text{ for a.e. } t \in [0, T]. \tag{21}$$

Let $\Phi^2, \Phi^2_\lambda : L^2(0, T) \to \mathbb{R}, \lambda > 0$, be the functions defined by equalities (5) and (6) where the functions ψ and ψ_λ are replaced by the functions ψ^2 and $\psi^2_\lambda, \lambda > 0$, respectively. Then, using (20) along with Lemma 2.7(f) applied to the function Φ^2, we obtain the inclusion $f^2 \in \partial\Phi^2(x)$. From (12), (20), Lemma 2.8(c) and Proposition 2.4 we see that

$$f^i(t) \in \partial\tilde{\psi}^i(t, x(t)) \text{ for a.e. } t \in [0, T], \quad i = 1, 2. \tag{22}$$

Now, the existence for Problem 3.5 follows from (21) and (22).

To prove the uniqueness, assume that x_1 and x_2 are two different solutions to Problem 3.5. Then, there exist $\xi_i : [0,T] \to \mathbb{R}^n$, $i = 1, 2$, such that

$$-\dot{x}_i(t) = \tilde{A}(t, x_i(t)) + \xi_i^1(t) - \xi_i^2(t) + f(t),$$

$$x(0) = x_0, \quad \xi_i^1(t) \in \partial\tilde{\psi}^1\left(t, x_i(t)\right), \quad \xi_i^2(t) \in \partial\tilde{\psi}^2\left(t, x_i(t)\right)$$

for a.e. $t \in [0,T]$. Multiplying the last equation by the difference of the two solutions first for $i = 1$ and then for $i = 2$, we obtain

$$\left\langle \dot{x_1}(t) + \tilde{A}\left(t, x_1(t)\right), x_2(t) - x_1(t)\right\rangle + \langle \xi_1^1(t), x_2(t) - x_1(t)\rangle - \langle \xi_1^2(t), x_2(t) - x_1(t)\rangle$$
$$= \langle f(t), x_1(t) - x_2(t)\rangle$$

for a.e. $t \in [0,T]$, and

$$\left\langle \dot{x_2}(t) + \tilde{A}\left(t, x_2(t)\right), x_1(t) - x_2(t)\right\rangle + \langle \xi_2^1(t), x_1(t) - x_2(t)\rangle - \langle \xi_2^2(t), x_1(t) - x_2(t)\rangle$$
$$= \langle f(t), x_2(t) - x_1(t)\rangle$$

for a.e. $t \in [0,T]$. Next, we sum up the last two equalities to get

$$\langle \dot{x_1}(t) - \dot{x_2}(t), x_1(t) - x_2(t)\rangle + \left\langle \tilde{A}\left(t, x_1(t)\right) - \tilde{A}\left(t, x_2(t)\right), x_1(t) - x_2(t)\right\rangle$$
$$+ \left\langle \xi_1^1(t) - \xi_2^1(t), x_1(t) - x_2(t)\right\rangle = \left\langle \xi_1^2(t) - \xi_2^2(t), x_1(t) - x_2(t)\right\rangle$$

for a.e. $t \in [0,T]$. We integrate this equality on $[0,t]$, apply the integration by parts formula to the first term, use Hypotheses $H(A)$(b), $H(\psi^1)$(b), $H(\psi)^2$(b) to deduce that

$$\frac{1}{2}\left\|x_1(t) - x_2(t)\right\|^2 - \frac{1}{2}\left\|x_1(0) - x_2(0)\right\|^2 + m_A \int_0^t \left\|x_1(s) - x_2(s)\right\|^2 ds$$
$$-m_{\psi^1}\int_0^t \left\|x_1(s) - x_2(s)\right\|^2 ds - m_{\psi^2}\int_0^t \left\|x_1(s) - x_2(s)\right\|^2 ds \le 0$$

for all $t \in [0,T]$. By virtue of the fact that $x_1(0) - x_2(0) = 0$, we further have

$$\left(m_A - m_{\psi^1} - m_{\psi^2}\right)\left\|x_1 - x_2\right\|_{L^2(0,t)}^2 \le 0$$

for all $t \in [0,T]$. This finally implies the uniqueness for Problem 3.5. □

Now, in the original problem (1) we replace the multivalued perturbation F with an arbitrary single-valued function $f \in L^2(0,T)$ and consider

Problem 3.7.

$$\begin{cases} -\dot{x} \in A(t, x(t), R_1x(t)) + \partial\psi^1(t, x(t), R_2x(t)) - \partial\psi^2(t, x(t), R_3x(t)) + f(t), \\ x(0) = x_0. \end{cases} \tag{23}$$

Theorem 3.8. *Problem* 3.7 *has a solution.*

Proof. First, we prove the following estimate

$$\|x_1 - x_2\|_{L^2(0,t)} \le C(\|\lambda_1 - \lambda_2\|_{L^2(0,t)} + \|\eta_1 - \eta_2\|_{L^2(0,t)} + \|\theta_1 - \theta_2\|_{L^2(0,t)}) \tag{24}$$

for all $t \in [0,T]$ and a constant $C > 0$, where $x_1 = x_{\lambda_1\eta_1\theta_1}$, $x_2 = x_{\lambda_2\eta_2\theta_2}$ are the unique solutions to Problem (3.5) corresponding to $(\lambda_1, \eta_1, \theta_1)$ and $(\lambda_2, \eta_2, \theta_2)$, respectively, with $\lambda_i, \eta_i, \theta_i \in L^2(0,T)$, $i = 1, 2$. By definition of a solution to Problem 3.5, there exist $\xi_i : [0,T] \to \mathbb{R}^n$, $i = 1, 2$, such that

$$-\dot{x}_i(t) = A(t, x_i(t), \lambda_i(t)) + \xi_i^1(t) - \xi_i^2(t) + f(t),$$
$$x(0) = x_0, \quad \xi_i^1(t) \in \partial\psi^1(t, x_i(t), \eta_i(t)), \quad \xi_i^2(t) \in \partial\psi^2(t, x_i(t), \theta_i(t))$$

for a.e. $t \in [0,T]$. Multiplying this equation by $x_1(t) - x_2(t)$, similarly to above, we obtain

$$\begin{aligned}\langle \dot{x_1}(t) + A(t, x_1(t), \lambda_1(t)), x_2(t) - x_1(t)\rangle + \langle \xi_1^1(t), x_2(t) - x_1(t)\rangle \\ -\langle \xi_1^2(t), x_2(t) - x_1(t)\rangle = \langle f(t), x_1(t) - x_2(t)\rangle\end{aligned}$$

for a.e. $t \in [0,T]$, and

$$\begin{aligned}\langle \dot{x_2}(t) + A(t, x_2(t), \lambda_2(t)), x_1(t) - x_2(t)\rangle + \langle \xi_2^1(t), x_1(t) - x_2(t)\rangle \\ -\langle \xi_2^2(t), x_1(t) - x_2(t)\rangle = \langle f(t), x_2(t) - x_1(t)\rangle\end{aligned}$$

for a.e. $t \in [0,T]$. The addition of these equalities gives

$$\begin{aligned}\langle \dot{x_1}(t) - \dot{x_2}(t), x_1(t) - x_2(t)\rangle + \langle A(t, x_1(t), \lambda_1(t)) - A(t, x_2(t), \lambda_2(t)), x_1(t) - x_2(t)\rangle \\ + \left\langle \xi_1^1(t) - \xi_2^1(t), x_1(t) - x_2(t)\right\rangle = \left\langle \xi_1^2(t) - \xi_2^2(t), x_1(t) - x_2(t)\right\rangle\end{aligned}$$

for a.e. $t \in [0,T]$. Integrating the last inequality on $[0,t]$ and using Hypotheses $H(A)$(b), $H(\psi^1)$(b), and $H(\psi^2)$(b), we obtain

$$\begin{aligned}&\frac{1}{2}\|x_1(t) - x_2(t)\|^2 - \frac{1}{2}\|x_1(0) - x_2(0)\|^2 - \bar{m}_A \int_0^t \|\lambda_1(s) - \lambda_2(s)\| \|x_1(s) - x_2(s)\| \, ds \\ &+ m_A \int_0^t \|x_1(s) - x_2(s)\|^2 ds - \bar{m}_{\psi^1} \int_0^t \|\eta_1(s) - \eta_2(s)\| \|x_1(s) - x_2(s)\| \, ds \\ &- m_{\psi^1} \int_0^t \|x_1(s) - x_2(s)\|^2 ds - \bar{m}_{\psi^2} \int_0^t \|x_1(s) - x_2(s)\| \|\theta_1(s) - \theta_2(s)\| \, ds \\ &- m_{\psi^2} \int_0^t \|x_1(s) - x_2(s)\|^2 ds \le 0\end{aligned}$$

for all $t \in [0,T]$. The condition $x_1(0) - x_2(0) = 0$ and Hölder's inequality further yield

$$\begin{aligned}(m_A - m_{\psi^1} - m_{\psi^2}) \|x_1 - x_2\|^2_{L^2(0,t)} \le \bar{m}_A \|\lambda_1 - \lambda_2\|_{L^2(0,t)} \|x_1 - x_2\|_{L^2(0,t)} \\ + \bar{m}_{\psi^1} \|\eta_1 - \eta_2\|_{L^2(0,t)} \|x_1 - x_2\|_{L^2(0,t)} + \bar{m}_{\psi^2} \|\theta_1 - \theta_2\|_{L^2(0,t)} \|x_1 - x_2\|_{L^2(0,t)}\end{aligned}$$

for all $t \in [0,T]$. Hence, by (H_1), the inequality (24) holds.

Now, define the operator $\Lambda : [L^2(0,T)]^3 \to [L^2(0,T)]^3$ by the rule

$$\Lambda(\lambda, \eta, \theta) = (R_1 x_*, R_2 x_*, R_3 x_*)$$

for all $(\lambda, \eta, \theta) \in [L^2(0,T)]^3 = L^2(0,T) \times L^2(0,T) \times L^2(0,T)$, where x_* denotes the solution to Problem 3.5 corresponding to (λ, η, θ).

Employing Hölder's inequality again, in view of (24), we obtain

$$
\begin{aligned}
&\left\|\Lambda\left(\lambda_1,\eta_1,\theta_1\right)(t)-\Lambda\left(\lambda_2,\eta_2,\theta_2\right)(t)\right\|^2\\
&=\left\|\left(R_1x_1\right)(t)-\left(R_1x_2\right)(t)\right\|^2+\left\|\left(R_2x_1\right)(t)-\left(R_2x_2\right)(t)\right\|^2+\left\|\left(R_3x_1\right)(t)-\left(R_3x_2\right)(t)\right\|^2\\
&\le\left(L_{R_1}\int_0^t\|x_1(s)-x_2(s)\|\,ds\right)^2+\left(L_{R_2}\int_0^t\|x_1(s)-x_2(s)\|\,ds\right)^2\\
&\quad+\left(L_{R_3}\int_0^t\|x_1(s)-x_2(s)\|\,ds\right)^2\\
&\le L\left\|x_1-x_2\right\|^2_{L^2(0,t)}\le L\times C\left(\|\lambda_1-\lambda_2\|^2_{L^2(0,t)}+\|\eta_1-\eta_2\|^2_{L^2(0,t)}+\|\theta_1-\theta_2\|^2_{L^2(0,t)}\right)
\end{aligned}
$$

for some $L>0$. This implies that

$$
\begin{aligned}
&\left\|\Lambda\left(\lambda_1,\eta_1,\theta_1\right)(t)-\Lambda\left(\lambda_2,\eta_2,\theta_2\right)(t)\right\|^2\\
&\le L\times C\int_0^t\left\|\left(\lambda_1,\eta_1,\theta_1\right)(s)-\left(\lambda_2,\eta_2,,\theta_2\right)(s)\right\|^2ds
\end{aligned}
$$

for a.e. $t\in[0,T]$. By Lemma 2.6, we conclude that there exists a unique fixed point $(\lambda^*,\eta^*,\theta^*)\in[L^2(0,T)]^3$ of Λ: $\Lambda(\lambda^*,\eta^*,\theta^*)=(\lambda^*,\eta^*,\theta^*)$.

Let $x_{\lambda^*\eta^*\theta^*}\in W^{1,2}(0,T)$, $x_{\lambda^*\eta^*\theta^*}(0)=x_0$ be the solution to Problem 3.5 corresponding to $(\lambda^*,\eta^*,\theta^*)$. From the definition of operator Λ, we have

$$
\lambda^*=R_1\left(x_{\lambda^*\eta^*\theta^*}\right),\quad \eta^*=R_2\left(x_{\lambda^*\eta^*\theta^*}\right),\quad \theta^*=R_3\left(x_{\lambda^*\eta^*\theta^*}\right),
$$

which finally shows that $x_{\lambda^*\eta^*\theta^*}$ is a solution to Problem 3.7. $\square$

We are now in a position to prove the main result of this section.

Theorem 3.9. *Problem* (1) *has a solution.*

Proof. Let now $f_i\in L^2(0,T)$, $i=1,2$, and x_i, $i=1,2$ be the corresponding solutions to Problem 3.7. From Definition 3.1 it follows immediately that there exist $\xi_i^1,\xi_i^2:[0,T]\to\mathbb{R}^n$, $i=1,2$, such that

$$
\begin{aligned}
-x_i(t)&=A(t,x_i(t),R_1x_i(t))+\xi_i^1(t)-\xi_i^2(t)+f_i(t),\\
x_i(0)=x_0,\quad \xi_i^1(t)&\in\partial\psi^1\left(t,x_i(t),R_2x_i(t)\right),\quad \xi_i^2(t)\in\partial\psi^2\left(t,x_i(t),R_3x_i(t)\right)
\end{aligned}
$$

for a.e. $t\in[0,T]$. Taking the difference of these two equations and multiplying the result by $x_2(t)-x_1(t)$ we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_2(t)-x_1(t)\|^2+\left\langle A\left(t,x_2(t),R_1x_2(t)\right)-A\left(t,x_1(t),R_1x_1(t)\right),x_2(t)-x_1(t)\right\rangle+\\
&\left\langle\xi_2^1(t)-\xi_1^1(t),x_2(t)-x_1(t)\right\rangle=\left\langle f_1(t)-f_2(t),x_2(t)-x_1(t)\right\rangle+\left\langle\xi_1^2(t)-\xi_2^2(t),x_1(t)-x_2(t)\right\rangle.
\end{aligned}
$$

Integrating the last equation from 0 to $t\in(0,T]$, by virtue of Hypotheses $H(A)$(b), $H(\psi^1)$(b) we have

$$
\frac{1}{2}\|x_2(t)-x_1(t)\|^2+\left(m_A-m_{\psi^1}-m_{\psi^2}\right)\int_0^t\|x_2(\tau)-x_1(\tau)\|^2d\tau
$$

$$
\begin{aligned}
&\le \bar{m}_A \int_0^t \|R_1x_1(\tau) - R_1x_2(\tau)\| \, \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\quad + \bar{m}_{\psi^1} \int_0^t \|R_2x_1(\tau) - R_2x_2(\tau)\| \, \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\quad + \bar{m}_{\psi^2} \int_0^t \|R_3x_1(\tau) - R_3x_2(\tau)\| \, \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\quad + \int_0^t \|f_2(\tau) - f_1(\tau)\| \cdot \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\le \bar{m}_A C_{R_1} \int_0^t \int_0^\tau \|x_1(s) - x_2(s)\| ds \, \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\quad + \bar{m}_{\psi^1} C_{R_2} \int_0^t \int_0^\tau \|x_1(s) - x_2(s)\| ds \, \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\quad + \bar{m}_{\psi^2} C_{R_3} \int_0^t \int_0^\tau \|x_1(s) - x_2(s)\| ds \, \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\quad + \int_0^t \|f_2(\tau) - f_1(\tau)\| \cdot \|x_2(\tau) - x_1(\tau)\| \, d\tau \\
&\le \bar{m}_A C_{R_1} \int_0^t \|x_2(\tau) - x_1(\tau)\|^2 d\tau + \bar{m}_{\psi^1} C_{R_2} \int_0^t \|x_2(\tau) - x_1(\tau)\|^2 d\tau \\
&\quad + \bar{m}_{\psi^2} C_{R_3} \int_0^t \|x_2(\tau) - x_1(\tau)\|^2 d\tau + \int_0^t \|f_2(\tau) - f_1(\tau)\| \cdot \|x_2(\tau) - x_1(\tau)\| \, d\tau.
\end{aligned}
$$

From this inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|x_2(t) - x_1(t)\|^2 \\
&\quad + (m_A - m_{\psi^1} - m_{\psi^2} - \bar{m}_A C_{R_1} - \bar{m}_{\psi^1} C_{R_2} - \bar{m}_{\psi^2} C_{R_3}) \int_0^t \|x_2(\tau) - x_1(\tau)\|^2 d\tau \\
&\le \int_0^t \|f_2(\tau) - f_1(\tau)\| \cdot \|x_2(\tau) - x_1(\tau)\| \, d\tau.
\end{aligned}
$$

Neglecting, in view of $H(1)$, the second term on the left, [3, Lemma A5] further implies that

$$
\|x_2(t) - x_1(t)\| \le 2 \int_0^t \|f_2(\tau) - f_1(\tau)\| \, d\tau. \tag{25}
$$

Next, given $f(\cdot) \in L^2(0,T), \|f\|_{L^2(0,T)} \le R$, we derive a priori bounds for a solution of Problem 3.7 depending on R and structural constants of Eq. (3.7) only. To this end, let $x(\cdot)$ be a solution of Problem 3.7 corresponding to $f(\cdot)$, i.e.

$$
-\dot{x}(t) = A(t, x(t), R_1x(t)) + \xi^1(t) - \xi^2(t) + f(t) \tag{26}
$$

with $x(0) = x_0$, for some $\xi^i(t) \in \partial\psi^i$, $i = 1, 2$. Multiplying (26) by $x(t)$ and integrating the result over $[0, t], t \in (0, T]$ we obtain

$$
\begin{aligned}
&\frac{1}{2}\|x(t)\|^2 + \int_0^t \langle A(\tau, x(\tau), R_1x(\tau)), x(\tau)\rangle d\tau \\
&= \frac{1}{2}\|x_0\|^2 - \int_0^t \langle \xi^1(\tau), x(\tau)\rangle d\tau + \int_0^t \langle \xi^2(\tau), x(\tau)\rangle d\tau - \int_0^t \langle f(\tau), x(\tau)\rangle d\tau.
\end{aligned} \tag{27}
$$

From Hypotheses $H(A)$(a), $H(A)$(b) it follows that

$$\begin{aligned}\langle A(t,x(t),R_1x(t)),x(t)\rangle &= \langle A(t,x(t),R_1x(t))-A(t,0,0),x(t)\rangle + \langle A(t,0,0),x(t)\rangle \\ &\geq m_A\|x(t)\|^2 - \bar{m}_A\|(R_1x)(t)\|^2 - a_0(t)\|x(t)\|. \end{aligned} \tag{28}$$

Similarly, from Hypotheses $H(\psi^1)$(a), $H(\psi^1)$(b) we have

$$\begin{aligned}-\langle \xi^1(t),x(t)\rangle &= \left\langle \xi^1(t)-\xi_0^1(t),-x(t)\right\rangle - \left\langle \xi_0^1(t),x(t)\right\rangle \\ &\leq m_{\psi^1}\|x(t)\|^2 + \bar{m}_{\psi^1}\|(R_2x)(t)\|\|x(t)\| + \left\|\xi_0^1(t)\right\|\cdot\|x(t)\| \\ &\leq m_{\psi^1}\|x(t)\|^2 + \bar{m}_{\psi^1}\|(R_2x)(t)\|\|x(t)\| + c_0(t)\|x(t)\|,\end{aligned} \tag{29}$$

where $\xi_0^1(t)\in\partial\psi^1(t,0,0)$ a.e. on $[0,T]$.

From Hypotheses $H(\psi^2)$(b) and $H(\psi^2)$(c), we see that

$$\begin{aligned}\langle \xi^2(t),x(t)\rangle &= \left\langle \xi^2(t)-\xi_0^2(t),x(t)\right\rangle + \left\langle \xi_0^2(t),x(t)\right\rangle \\ &\leq \bar{m}_{\psi^2}\|(R_3x)(t)\|\|x(t)\| + m_{\psi^2}\|x(t)\|^2 + \left\|\xi_0^2(t)\right\|\cdot\|x(t)\| \\ &\leq \bar{m}_{\psi^2}\|(R_3x)(t)\|\|x(t)\| + m_{\psi^2}\|x(t)\|^2 + \bar{c}_0(t)\|x(t)\|,\end{aligned} \tag{30}$$

where $\xi_0^2(t)\in\partial\psi^2(t,0,0)$ a.e. on $[0,T]$. Combining (27)–(30), using Definition 2.5 and invoking Young's inequality we see that

$$\begin{aligned}&\frac{1}{2}\|x(t)\|^2 + (m_A - m_{\psi^1} - m_{\psi^2} - \bar{m}_AC_{R_1} - \bar{m}_{\psi^1}C_{R_2} - \bar{m}_{\psi^2}C_{R_3})\int_0^t \|x(\tau)\|^2d\tau \\ &\leq \frac{1}{2}\left\|x_0\right\|^2 + \bar{C}_1 + \bar{C}_2\int_0^t \|x(\tau)\|^2d\tau\end{aligned} \tag{31}$$

for some constants $\bar{C}_1$ and $\bar{C}_2$ dependent on $a_0, c_0, \bar{c}_0$, $R_i(0)$ and R only. The application of Gronwall's inequality and $H(1)$ yield

$$\|x(t)\| \leq \overline{M}_1 \tag{32}$$

for all $t\in[0,T]$ and some $\overline{M}_1>0$. From this estimate and Eq. (26) with the help of $H(A)$(c), $H(\psi)$(c) we obtain

$$\|\dot{x}\|_{L^2(0,T)} \leq \overline{M}_2 \tag{33}$$

for a constant $\overline{M}_2>0$ dependent on the fixed quantities in $H(A)$(c), $H(\psi)$(c) and R only.

Now, define the solution operator $\mathcal{S}: L^2(0,T)\to C([0,T])$ that to a given function $f(\cdot)\in L^2(0,T)$ associates the unique solution $x=x(f)$ of Problem 3.7:

$$\mathcal{S}(f) = x(f) \tag{34}$$

and prove its weak-strong continuity. To this end, take an arbitrary sequence $f_n(\cdot)\in L^2(0,T), n\geq 1$, weakly convergent in $L^2(0,T)$ to some $f(\cdot)\in L^2(0,T)$ and let $x_n := x\left(f_n\right), n\geq 1$, be the sequence of solutions to Problem 3.7 corresponding to f_n. We show below than x_n converges strongly to $x=x(f)$, the solution of Problem 3.7 corresponding to f, as $n\to\infty$. The uniform estimates (32), (33) extend to the sequence $x_n, n\geq 1$. In particular, (33) implies the equicontinuity of $x_n, n\geq 1$, in

$C([0,T])$. The latter coupled with (32) guarantee, by the Arzela-Ascoli theorem, the existence of a subsequence of x_n (we do not relabel) converging in $C([0,T])$ to some function $y(\cdot) \in C([0,T])$. In addition, $\dot{x}_n \to \dot{y}$ in $\mathrm{w}-L^2(0,T)$. Hence, to prove the required contimuity of $\mathcal{S}$, it remains to verify that $y(\cdot)$ equals $x(\cdot)$.

Similarly as above, we get

$$\begin{aligned}
&\frac{1}{2}\|x_n(t)-x(t)\|^2 + \\
&\quad + (m_A - m_{\psi^1} - m_{\psi^2} - \bar{m}_A C_{R_1} - \bar{m}_{\psi^1} C_{R_2} - \bar{m}_{\psi^2} C_{R_3}) \int_0^t \|x_n(\tau)-x(\tau)\|^2 \, d\tau \\
&\leq \int_0^t \langle x_n(\tau)-y(\tau), f(\tau)-f_n(\tau)\rangle \, d\tau + \int_0^t \langle y(\tau)-x(\tau), f(\tau)-f_n(\tau)\rangle \, d\tau.
\end{aligned}$$

The facts that $x_n \to y$ in $C([0,T])$ and $\dot{x}_n \to \dot{y}$ weakly in $L^2(0,T)$ entail that the right-hand side of this inequality goes to zero as $n \to \infty$ and we, thus, deduce that $x_n(t) \to x(t)$ in $\mathbb{R}^n$ for any $t \in [0,T]$. Since the sequence $x_n, n \geq 1$, is equicontinuous, the Arzela-Ascoli theorem finally implies that $x_n \to x$ in $C([0,T])$.

Let $z(t) \in F(t,0)$, $t \in [0,T]$, be such that $\|z(t)\| = d\,(0, F(t,0))$. Then, from $H(F2)$, $H(F(3))$ we obtain

$$\begin{aligned}
d\,(0, F(t,x)) &\leq d(z(t), F(t,x)) + \|z(t)\| \\
&\leq \big(l(t) + \beta\|z(t)\|\big)\|x\| + \|z(t)\| < k_0(t) + k_1(t)\|x\| \qquad (35)
\end{aligned}$$

with $k_1(t) := l(t) + \beta k_0(t)$. Next, introduce the mapping

$$\Phi(t,x) := \overline{F(t,x) \cap (k_0(t) + k_1(t)\|x\|)B}, \qquad (36)$$

where the bar denotes the closure in $\mathbb{R}^n$. It is a standard matter to show that $\Phi : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is lower semicontinuous in x for a.e. $t \in [0,T]$. The $(\Sigma, \mathcal{B}(X))$ measurability of Φ follows from $H(F1)$ and [7, Corollary 4.2].

Next, consider the set

$$K := \{f \in L^2(0,T) : \ \|f(t)\| \leq k_0(t) + k_1(t) r(t) \quad \text{for a.e. } t \in [0,T]\}, \qquad (37)$$

where the functions $k_0(\cdot)$, $k_1(\cdot)$ are from (35) and $r(\cdot)$ is the unique solution of the differential equation $\dot{r}(t)/2 = k_0(t) + k_1(t) r(t)$, $r(0) = \overline{M}_1$ with the constant $\overline{M}_1$ from (32). Since the set K is compact in the weak topology of $L^2(0,T)$ the weak-strong continuity of the solution operator $\mathcal{S}$ defined in (34) implies that the set

$$\mathcal{K} := \{x \in C([0,T]) : \ x = \mathcal{S}(f), \ f \in K\} \qquad (38)$$

is compact in $C([0,T])$. From (25), (32), and (37) we infer that for any $x(\cdot) \in \mathcal{K}$ we have

$$\|x(t)\| = \|x(f)(t)\| \leq \|x(0)(t)\| + 2\int_0^t \|f(\tau)\| \, d\tau \leq \overline{M}_1 + \int_0^t \dot{r}(\tau) \, d\tau = r(t),$$

for all $t \in T$. Therefore, for every $x(\cdot) \in \mathcal{K}$

$$\sup\{\|z\| : \ z \in \Phi(t, x(t))\} \leq k_0(t) + k_1(t)\|x(t)\| \leq k_0(t) + k_1(t) r(t), \qquad (39)$$

for all $t \in [0,T]$. We remark that $k_0(\cdot) + k_1(\cdot) r(\cdot) \in L^2(0,T)$.

Now, introduce the multifunction $\widetilde{\Phi} : \mathcal{K} \to 2^{L^2[0,T]}$:

$$\widetilde{\Phi}(x) := \{u \in L^2(0,T) : \ u(t) \in \Phi(t, x(t)) \text{ for a.e. } t \in [0,T]\}, \quad x \in \mathcal{K}.$$

Then, the estimate (39) and the properties of Φ established above allow us to apply [6, Corollary 3.3] and to derive the existence of a continuous selection

$$\phi : \mathcal{K} \to L^2(0,T) \text{ of } \widetilde{\Phi} : \quad \phi(x) \in \widetilde{\Phi}(x), \quad x \in \mathcal{K},$$

and we have
$$\phi(x)(t) \in \Phi(t, x(t)) \ \text{ for a.e. } t \in [0,T].$$

Taking now the composition of the solution operator $\mathcal{S}$ with ϕ and restricting the resulting superposition $\phi \circ \mathcal{S} : L^2(0,T) \to L^2(0,T)$ onto the set K defined in (37), from (38)–(39) we infer that $\phi \circ \mathcal{S}$ is a self-mapping on K. Using the facts that $\phi \circ \mathcal{S}$ is continuous from w-K to w-K and w-K is compact and convex, from the Schauder fixed point theorem we conclude that there exists a fixed point of $\phi \circ \mathcal{S}$, i.e. a function $f_* \in K$ such that

$$f_* = \phi \circ \mathcal{S}(f_*) = \phi(\mathcal{S}(f_*)). \tag{40}$$

Finally, defining $x_* := S(f_*)$ from (34), (36), and (40) we see that x_* is a solution to Problem (P). $\square$

4. Relaxation result

In this section, we prove the relaxation result for Problem (1) and Problem (2): solutions of the convexified Problem (2) can be approximated, in a suitable topology, by those of the original one Problem (1). Namely, we have

Theorem 4.1. *Let $y \in W^{1,2}(0,T), y(0) = y_0$, be a solution of Problem (2). Then, there exists a sequence $x_n(t), x_n(0) = y_0, n \geq 1$, of solutions of Problem (1) converging to $y(\cdot)$ in $C([0,T])$. In addition $\dot{x}_n(\cdot) \to \dot{y}(\cdot)$ in $w - L^2(0,T)$.*

Proof. By $H(F(4))$, we have

$$\text{haus}_{\,\rho(t)}(\overline{\text{co}}F(t,x), \overline{\text{co}}F(t,z)) \leq \text{haus}_{m(t)}(F(t,x), F(t,z)), \tag{41}$$

$t \in [0,T], x, z \in \mathbb{R}^n$, with the functions $\rho(\cdot)$ and $m(\cdot)$ as in $H(F)(4)$. For the proof of this fact we refer the reader to [25, Theorem 5.1]. From (41) and $H(F)$(b) we then have

$$\text{haus}_{\,\rho(t)}(\overline{\text{co}}F(t,x), \overline{\text{co}}F(t,z)) \leq (l(t) + \beta m(t))\|x - z\|. \tag{42}$$

From the definition of solution to Problems (2), we see that there exist functions $\xi^1(t) \in \partial\psi^1(t, y(t), R_2 y(t))$, $\xi^2(t) \in \partial\psi^2(t, y(t), R_3 y(t))$, such that

$$-\dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t) \in \overline{\text{co}}F(t, y(t)) \tag{43}$$

for a.e. $t \in [0,T]$. Defining $\rho(t) := \| - \dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t)\|$, $t \in [0,T]$, from (42), (43) we infer that

$$-\dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t) \in \overline{\text{co}}F(t,x) + (l(t) + \beta m(t))\|y(t) - x\|\bar{B},$$

$t \in [0,T]$, $x \in \mathbb{R}^n$. From this inclusion and $H(F)(4)$ we further deduce that

$$\begin{aligned} &\overline{\text{co}}F(t,x) \cap (\| - \dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t)\| \\ &+ \big(l(t) + \beta l_0(t) + \beta\beta_0\| - \dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t)\|\big)\, \|y(t) - x\|\big)\, \bar{B} \neq \emptyset, \end{aligned} \tag{44}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^n$.

Henceforth, we restrict our analysis to the set

$$\mathcal{N}_{y,\alpha} := \{(t,x) \in [0,T] \times \mathbb{R}^n : \|x - y(t)\| \leq \alpha\}$$

for some constant $\alpha > 0$. Define

$$\begin{aligned} \varrho_0(t) &:= \| - \dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t)\| \\ &\quad + \alpha \left(l(t) + \beta l_0(t) + \beta\beta_0 \| - \dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t)\|\right), \\ \varrho(t) &:= 2\varrho_0(t), \end{aligned}$$

where $t \in [0,T]$, and observe from (44) that

$$\overline{\mathrm{co}} F(t,x) \cap \varrho_0(t)\bar{B} \neq \emptyset, \quad \overline{\mathrm{co}} F(t,x) \cap \varrho(t)\bar{B} \neq \emptyset, \quad (t,x) \in \mathcal{N}_{y,\alpha}.$$

Hence, we can apply Proposition 2.3 and infer from (42), in view of $H(F)(4)$, that

$$\begin{aligned} \mathrm{haus}\left(\overline{co} F(t,x)_{\varrho(t)}, \overline{\mathrm{co}} F(t,z)_{\varrho(t)}\right) &\leq \frac{2\varrho(t)}{\varrho(t) - \varrho_0(t)} \left(l(t) + \beta\left(l_0(t) + \beta_0 \varrho(t)\right) \|x - z\|\right. \\ &\leq k(t)\|x - z\|, \end{aligned} \tag{45}$$

$(t,x), (t,z) \in \mathcal{N}_{y,\alpha}$ for $k(t) := 4(l(t) + \beta(l_0(t) + \beta_0 \varrho(t))$. Define $\mathrm{proj}_\alpha(x)$ to be the projection of $x \in \mathbb{R}^n$ onto the set $y(t) + \alpha\bar{B}, t \in [0,T]$ and consider the multivalued mapping $\Psi : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \Psi(t,x) := F(t, \mathrm{proj}\, \alpha(x))$, $t \in [0,T], x \in \mathbb{R}^n$. From (45) and the properties of projection we deduce that

$$\mathrm{haus}\ \left(\overline{\mathrm{co}}\Psi(t,x)_{\varrho(t)}, \overline{\mathrm{co}}\Psi(t,z)_{\varrho(t)}\right) \leq k(t)\|x - z\|, \tag{46}$$

$x, z \in \mathbb{R}^n$. Since $\mathrm{proj}_\alpha(y) = y$, from (43) and the definition of $\varrho(\cdot)$ we have

$$-\dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t) \in \overline{\mathrm{co}}\Psi(t, y(t)) \cap \varrho(t)\bar{B}. \tag{47}$$

Then, the measurability properties of the mapping $t \to \overline{\mathrm{co}}\Psi(t,x) \cap \varrho(t)\bar{B}$ combined with (46) guarantee through the standard measurable selection argument the existence of a Caratheodory function $\eta : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\eta(t,x) \in \overline{\mathrm{co}}\Psi(t,x) \cap \varrho(t)\bar{B} \tag{48}$$

and

$$\begin{aligned} &\| - \dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t) - \eta(t,x)\| \\ &= d(-\dot{y}(t) - A(t, y(t)) - \xi^1(t) + \xi^2(t), \overline{\mathrm{co}}\Psi(t,x) \cap \varrho(t)\bar{B}). \end{aligned} \tag{49}$$

From (47), (49) we deduce that

$$-\dot{y}(t) - A(t, y(t), R_1 y(t)) - \xi^1(t) + \xi^2(t) = \eta(t, y(t)). \tag{50}$$

Moreover, from (48), $H(F)(4)$ and the definition of Ψ we obtain

$$\eta(t,x) \in \mathrm{co}\left(\overline{\Psi(t,x) \cap m_0(t)B}\right), \tag{51}$$

$x \in \mathbb{R}^n$, for $m_0 := l_0(t) + \beta_0 \varrho(t), t \in [0,T]$.

Next, similar to the existence proof, consider the solution operator $\mathcal{S}_0 : L^2(0,T) \to C([0,T])$ that to a given function $f(\cdot) \in L^2(0,T)$ associates the unique solution $x = x(f)$ of Eq. (23) with the initial data x_0 there replaced with y_0 now.

As before, we can show that the operator $\mathcal{S}_0$ is weak-strong continuous and thus the set

$$\mathcal{K}_0 := \{x \in C([0,T]) : x = \mathcal{S}_0(f), f \in K_0\}$$

is compact in $C([0,T])$ for

$$K_0 := \left\{f \in L^2(0,T) : \|f(t)\| \leq m_0(t) \text{ for a.e. } t \in [0,T]\right\}, \tag{52}$$

where the function $m_0(\cdot)$ comes from (51). For $x(\cdot) \in \mathcal{K}_0$ introduce the mapping $\varphi : \mathcal{K}_0 \to L^2(0,T)$ by the rule

$$\varphi(x)(t) := \eta(t, x(t)), \quad t \in [0,T]. \tag{53}$$

From (51) it follows that $\varphi(x)(t) \in \operatorname{co}\left(\overline{\Psi(t,x) \cap m_0(t)B}\right)$, $x(\cdot) \in \mathcal{K}_0$, $t \in [0,T]$.

It is clear that the mapping $\overline{\Psi(t,x) \cap m_0(t)B}$ enjoys the properties established above for the mapping $\Phi(t,x)$. Then, by virtue of [4, Corollary 1.1] and [7, Theorem 5.6] we can show that for any $n \geq 1$, there exists a continuous selection $\varphi_n : K_0 \to L^2(0,T)$ of this mapping, i.e.

$$\varphi_n(x)(t) \in \overline{\Psi(t,x) \cap m_0(t)B}, \tag{54}$$

such that
$$\sup_{0 \leq s \leq t \leq T} \left| \int_s^t (\varphi_n(x)(\tau) - \varphi(x)(\tau))\, d\tau \right| \leq \frac{1}{n}, \quad x(\cdot) \in \mathcal{K}_0. \tag{55}$$

Taking now the composition of the solution operator $\mathcal{S}_0$ with $\varphi_n, n \geq 1$, and restricting the resulting superposition $\varphi_n \circ \mathcal{S}_0 : L^2(0,T) \to L^2(0,T)$ onto the set K_0 defined in (52), from (54) we see that $\varphi_n \circ \mathcal{S}_0$ is a self-mapping on K_0. Therefore, from the Schauder fixed point theorem we see as above that there exists a fixed point $f_n \in K_0$ of $\varphi_n \circ \mathcal{S}_0$:

$$f_n = \varphi_n \circ \mathcal{S}_0(f_n) = \varphi_n(\mathcal{S}_0(f_n)), \quad n \geq 1. \tag{56}$$

The weak compactness of the set $K_0 \subset L^2(0,T)$ implies then that a subsequence of $f_n(\cdot)$ converges weakly in $L^2(0,T)$ to some $f_*(\cdot) \in K_0$. From the weak-strong continuity of $\mathcal{S}_0$ we also infer that

$$x_n = \mathcal{S}_0(f_n) \to x_* = \mathcal{S}_0(f_*) \tag{57}$$

strongly in $C([0,T])$ as $n \to \infty$. From (45) and (46) we obtain

$$\sup_{0 \leq s \leq t \leq T} \left| \int_s^t (f_n(\tau) - \varphi(\mathcal{S}_0(f_n))(\tau))\, d\tau \right| \leq \frac{1}{n}, \quad n \geq 1.$$

Hence, letting $n \to \infty$ and taking into account the continuity of φ and $\mathcal{S}_0$ we see that $f_* = \varphi(\mathcal{S}_0(f_*)) = \varphi(x_*)$. The definition (53) of φ further yields $f_*(t) = \eta(t, x_*(t))$, $t \in [0,T]$.

Consequently, from (57) and the definition of solution operator $\mathcal{S}_0$ we see that there exists $\xi_*^1(t) \in \partial\psi^1(t, x_*(t), R_2x_*(t))$, $\xi_*^2(t) \in \partial\psi^2(t, x_*(t), R_3x_*(t))$, such that

$$\begin{cases} -\dot{x}_*(t) = A(t, R_1x_*(t), x_*(t)) + \xi_*^1(t) - \xi_*^2(t) + \eta(t, x_*(t)) \text{ for a.e. } t \in [0,T], \\ x_*(0) = y_0. \end{cases} \tag{58}$$

Reasoning as in derivation of (25) with the roles of x_1 and x_2 there taken by x_* and y, respectively, now, from (50) and (58) we obtain

$$\|y(t) - x_*(t)\| \leq 2 \int^t \|\eta(\tau, y(\tau)) - \eta\left(\tau, x_*(\tau)\right)\| \, d\tau, \quad t \in T.$$

Furthermore, from (46), (47), (49), and(50) we see that

$$\|\eta(t, y(t)) - \eta(t, x)\| \leq k(t)\|y(t) - x\|, \quad x \in \mathbb{R}^n.$$

Hence, the last two inequalities imply that $x_*(t) = y(t)$, $t \in [0, T]$, by the Gronwall-Bellman lemma, and from (57) we conclude that the sequence $x_n(\cdot)$ converges to $y(\cdot)$ in $C([0, T])$. In particular, this means that there exists $n_0 > 1$ such that $\|x_n(t) - y(t)\| \leq \alpha$, $t \in [0, T]$, for $n \geq n_0$ and α introduced in the definition of $\mathcal{N}_{y,\alpha}$. Then, we see that $\Psi\left(t, x_n(t)\right)$ coincides with $F\left(t, x_n(t)\right)$ for a.e. $t \in [0, T]$ and $n \geq n_0$. Therefore, $x_n(\cdot)$ are solutions of Problem (1) converging to $y(\cdot)$ in $C([0, T])$.

To finish the proof, we need to show the weak convergence of derivatives in $L^2(0, T)$. To do so, we note that the a priori bound (33) extends to the solutions $x_n(\cdot)$, $n \geq n_0$, with the constant $\overline{M}_2$ now depending on the L^2-norm of the function $m_0(\cdot)$ from (52). This bound implies the weak convergence in $L^2(0, T)$ of the derivatives $\dot{x}_n(\cdot), n \geq n_0$, to some function from $L^2(0, T)$. The unique representation of an absolutely continuous function through the integral of its derivative finally ensures that this function coincides a.e. on $[0, T]$ with $\dot{y}$. $\square$

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